Numerical Analysis (089180)
prof. Simona Perotto
simona.perotto@polimi.it

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Luca Liverotti
luca.liverotti@polimi.it

Lab 4

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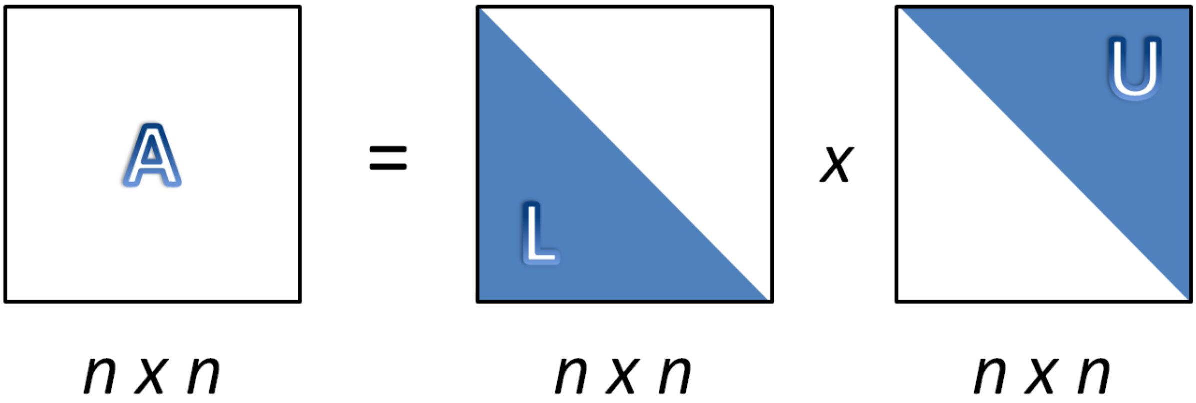
This lab deals with the numerical resolution of a linear system

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{for } \mathbb{A} \in \mathbb{R}^{n\times n}, \boldsymbol{b} \in \mathbb{R}^n.$$

1 Backward and forward substitution methods



Let $\mathbb{A} \in \mathbb{R}^{n\times n}$. We assume to have LU factorization of $\mathbb{A}$, $\mathbb{A} = \mathbb{L}\mathbb{U}$, with $\mathbb{L}$ **lower triangular** and $\mathbb{U}$ **upper triangular** matrix.

How to use $\mathbb{L}$ and $\mathbb{U}$ to solve thelinear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$

$$\left.\begin{array}{l} \mathbb{A}\boldsymbol{x} = \boldsymbol{b} \\ \mathbb{A} = \mathbb{L}\mathbb{U} \end{array}\right\} \longrightarrow \mathbb{L}\mathbb{U}\boldsymbol{x} = \boldsymbol{b} \longrightarrow \left\{\begin{array}{l} \mathbb{L}\boldsymbol{y} = \boldsymbol{b} \\ \mathbb{U}\boldsymbol{x} = \boldsymbol{y} \end{array}\right.$$

Forward substitution

$$\begin{array}{cccccc}
L_{1,1}y_1 & & & = & b_1 \\
L_{2,1}y_1 + & L_{2,2}y_2 & & = & b_2 \\
\vdots & \vdots\ddots & & & \vdots \\
L_{m,1}y_1 + & L_{m,2}y_2 + \cdots + & L_{m,m}y_m & = & b_m
\end{array}$$

$$
\begin{aligned}
y_1 &= \frac{b_1}{L_{1,1}}, \\
y_2 &= \frac{b_2 - L_{2,1}y_1}{L_{2,2}}, \\
&\vdots \\
y_m &= \frac{b_m - \sum_{i=1}^{m-1} L_{m,i}y_i}{L_{m,m}}.
\end{aligned}
$$

Backward substitution

$$
\begin{array}{cccc}
U_{1,1}x_1 + \cdots + & U_{1,m-1}x_{m-1}+ & U_{1,m}x_m = & y_1 \\
 & \vdots & \vdots & \vdots \\
 & U_{m-1,m-1}x_{m-1}+ & U_{m-1,m}x_m = & y_{m-1} \\
 & & U_{m,m}x_m = & y_m
\end{array}
$$

$$
\begin{aligned}
x_m &= \frac{y_m}{U_{m,m}} \\
x_{m-1} &= \frac{y_{m-1} - U_{m-1,m}x_m}{U_{m-1,m-1}}, \\
&\vdots \\
x_1 &= \frac{y_1 - \sum_{j=2}^{m} U_{1,j}x_j}{U_{1,1}}.
\end{aligned}
$$

Exercise 4.1. a. Write a function which implements the backward substitution method for solving a generic upper triangular system. Apply such a function to solve the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$
\mathbb{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
$$

b. Write a function to implement the forward substitution method to solve lower triangular systems. Apply this function to the system $\mathbb{A}^T\boldsymbol{x} = \boldsymbol{b}$. with $\mathbb{A}$ and $\boldsymbol{b}$ defined as above.

c. Use the backward and the forward algorithms to solve the system $\mathbb{A}^T\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$.

2 LU decomposition

Let $\mathbb{A} \in \mathbb{R}^{n\times n}$: the LU decomposition of $\mathbb{A}$ allows to write

$$\mathbb{A} = \mathbb{L}\mathbb{U}$$

with $\mathbb{L}$ **lower triangular** matrix s.t. $l_{ii} = 1,\ \forall i = 1\ldots n$, and $\mathbb{U}$ **upper triangular** matrix.

The existence and the uniqueness of such decomposition is related to the principal submatrices of order i of the matrix $\mathbb{A}$, with $i = 1, \cdots, n-1$, which are demanded to be non singular (necessary and sufficient condition). As an alternative, the sufficient condition holds:

- $\mathbb{A}$ is a strictly **diagonally dominant** by row OR by column matrix
- $\mathbb{A}$ is a **symmetric positive definite** matrix.

LU decomposition in MATLAB/Octave

```
>> help lu
lu   lu factorization
[L,U,P] = lu(A)
returns unit lower triangular matrix L, upper triangular matrix U, and permutation
matrix P so that P*A = L*U.

[L,U] = lu(A)
stores an upper triangular matrix in U and a "psychologically lower triangular matrix
" (i.e. the product of a lower triangular and a permutation matrix) in L, so that A =
 L*U, A can be rectangular.
```

LU decomposition with no pivoting

$$\left.\begin{array}{l}\mathbb{A}\boldsymbol{x} = \boldsymbol{b} \\ \mathbb{A} = \mathbb{L}\mathbb{U}\end{array}\right\} \longrightarrow \mathbb{L}\mathbb{U}\boldsymbol{x} = \boldsymbol{b} \longrightarrow \left\{\begin{array}{l}\mathbb{L}\boldsymbol{y} = \boldsymbol{b} \\ \mathbb{U}\boldsymbol{x} = \boldsymbol{y}\end{array}\right.$$

This approach is ideal to solve multiple systems $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}_i$ with different right hand-sides $\boldsymbol{b}_i$, and all sharing the same coefficient matrix. Actually it sufficies to compute the decomposition $\mathbb{L}\mathbb{U}$ just **once**. Computational cost: $\mathcal{O}(n^3)$.

Exercise 4.2. Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 2 & 10 & 4 & 0 \\ 1 & 0 & 2 & 2 \\ 1 & 4 & 0 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 10 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

a. Compute the solution of the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ using the $\mathbb{L}\mathbb{U}$ decomposition of matrix $\mathbb{A}$.

b. Find the determinant of $\mathbb{A}$ without using the command `det`.

Exercise 4.3. Let us consider the matrix $\mathbb{A} = \begin{bmatrix} 50 & 1 & 3 \\ 1 & 6 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

1. Apply the Matlab® command `lu` and compute the *LU* factorization of the matrix $\mathbb{A}$.
2. Solve the system $\mathbb{A}\mathbf{x} = \mathbf{b}$. Choose the vector $\mathbf{b}$, such that the solution of the system is $\mathbf{x}_{ex} = [1,\ 1,\ 1]^T$.
3. Compute the solution of the system $\mathbb{A}\mathbf{x} = \mathbf{b}$, by using the backward and forward substitution the functions previously implemented.

Inverse Matrix

We can define the inverse of a squared matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$ as the matrix $\mathbb{X} = \mathbb{A}^{-1} \in \mathbb{R}^{n\times n}$ such that $\mathbb{A}\mathbb{X} = \mathbb{X}\mathbb{A} = \mathbb{I}$. It is possible to determine $\mathbb{A}^{-1}$ by solving the following n linear systems:

$$\mathbb{A}\mathbf{v}_i = \mathbf{e}_i\ , \quad i = 1, \cdots, n\ ,$$

where $\mathbf{e}_i$ denote the consecutive columns of the matrix $\mathbb{I}$ (i.e the vectors of the standard basis of $\mathbb{R}^n$). Thus it turns out that

$$\mathbb{A}^{-1} = [\mathbf{v}_1|\mathbf{v}_2|...|\mathbf{v}_n]$$

Exercise 4.4.
1. Write a function `[InvA]=MyInv(A)` that computes the inverse `InvA` of a generic square matrix `A`.
2. Use the function `MyInv` to compute the inverse of the matrix $\mathbb{A}$ in the previous exercise. Compare the result with the output provided by Matlab with command `inv`.

A Homework

Homework 4.1 (LU decomposition). Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 12 & 5 & -8 & -5 \\ -4 & -4 & 8 & -6 \\ 4 & 2 & -3 & 0 \\ 0 & -1 & 2 & -4 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \\ 3 \\ -3 \end{bmatrix}.$$

a. Use the $\mathbb{LU}$ decomposition of matrix $\mathbb{A}$ to solve the system. Use both the functions implemented and the `lu` MATLAB/Octave command.

b. Compute the determinant of $\mathbb{A}$ using the $\mathbb{LU}$ decomposition.

c. Consider now a different right-hand side vector $\boldsymbol{c} = [-22, -12, -1, -11]^T$. Use an approach which allow us to contain the computational cost.