Numerical Analysis (089180) | A.Y. 2022-2023
prof. Simona Perotto | Luca Liverotti
simona.perotto@polimi.it | luca.liverotti@polimi.it

Lab 6 – Homework Solutions

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A Homework

Homework 6.1. Consider the Hilbert matrix $\mathbb{H}_n$ of order n

$$\mathbb{H}_n = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & \cdots & 1/(2n-1) \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad n = 10, 20, 40.$$

a. Estimate $\mathcal{K}_2(\mathbb{H}_n)$ using the technique of perturbing a linear system with $\mathbb{H}_n$ as coefficient matrix. Consider the right hand side term b such that $x = ones(n, 1)$.
b. Compute $\mathcal{K}_2(\mathbb{H}_n)$ using MATLAB command `cond`.
c. What can you say about the **accuracy of the solution** of the generic linear system $\mathbb{H}_n\boldsymbol{x} = \boldsymbol{b}$? How large can you take n before the relative error is 1? (i.e. there are **no significant digits** in the solution)

Solution Homework 6.1.

hw_6_1.m

```
clc
clear all
close all

% We can invert the relation
% norm(deltax)/norm(x) <= norm(deltab)/norm(b)
% to get an estimate for Kp(A):
% Kp(A) >= norm(deltax)/norm(x) * norm(b)/norm(deltab)

for (n = [10 20 40])
   pert_mag = 1e-6; % Perturbation magnitude
   n_trials = 100;
   H = hilb(n);
   x = ones(n, 1);
   b = H*x;
   KH_est = 0;
   for (ii = 1:n_trials)
     deltab = randn(n, 1) * pert_mag;
     deltax = H \ deltab;
     KH_est_ii = norm(deltax)/norm(x) * norm(b)/norm(deltab);
     KH_est = max(KH_est_ii, KH_est);
   end
   KH_est
end

cond( hilb(10) )
cond( hilb(20) )
cond( hilb(40) )

% The estimates obtained at the point a) are lower. This is expected, since
```

```
% the condition number represents the maximum
% possible amplification of the relative error on data. With the first
% approach we might not have considered the worst condition,
% hence the estimate is lower than the result with cond command.

n = [1:20];

for (ii = n)
 H = hilb(ii);
 x_ex = ones(ii, 1);
 b = H*x_ex;
 x = H\b;
 KH(ii) = cond(H);
 err(ii) = norm(x - x_ex) / norm(x);
end

figure
semilogy(n, KH, 'xb-', 'LineWidth',2, 'MarkerSize',8)
hold on
semilogy(n, err, 'xr-', 'LineWidth',2, 'MarkerSize',8)
semilogy(n, ones(n(end),1), 'k--', 'LineWidth',2, 'MarkerSize',8)
set(gca, 'FontSize',16)
set(gca, 'LineWidth', 1.5)
axis([0 n(end)+1 1e-17 1e20])
xlabel('n')
ylabel('K_2(H_n) and err')
legend('K_2(H_n)', 'relative error', 'Location','SouthEast')


% For n > 12 the relative error is 1. This means that the solution is completely meaningless.
```

Homework 6.2. Compute the condition number and the determinant of the following matrices » `A = diag(0.1*ones(15,1));`
» `B = triu(rand(15),14)*1e5 + diag(0.1*ones(15,1));`
Is, in general, the determinant a good measure of the condition number of the matrix?

Solution Homework 6.2.

hw_6_2.m

```
clc
clear all
close all

A = diag(0.1*ones(15,1));
B = triu(rand(15),14)*1e5 + diag(0.1*ones(15,1));

det(A)
cond(A)

det(B)
cond(B)

% The two matrices have the same determinant but B is very badly
% conditioned. Thus, we cannot say that the determinant is a good indicator
% for the condition number of a matrix.
```

Homework 6.3. Consider the following matrix

$$\mathbb{A} = \begin{bmatrix} 1.001 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1.001 & 1 & 1 \\ 1 & 1.001 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1.001 \end{bmatrix}.$$

a. Compute the condition number $\mathcal{K}_\infty(\mathbb{A})$ using MATLAB command `cond`.

b. Then compute $\mathcal{K}_\infty(\mathbb{A})$ using its definition.

c. Finally estimate $\mathcal{K}_\infty(\mathbb{A})$ by **perturbing a suitable linear system** having $\mathbb{A}$ as coefficient matrix.

Solution Homework 6.3.

hw_6_3.m

```
clc
clear all
close all

A = ones(5,5);
A(1,1) = 1.001;
A(3,2) = 1.001;
A(2,3) = 1.001;
A(5,5) = 1.001;
A
KA = cond(A, inf)

KA_def = norm(A, inf) * norm(inv(A), inf)

n_trials = 100;
KA_store = zeros(n_trials,1);
pert_mag = 1e-6; % Perturbation magnitude
x = ones(5,1);
b = A*x;

for (i=1:n_trials)
  deltab = pert_mag * randn(5,1);
  deltax = A\deltab;
  KA = norm(deltax,inf)/norm(x,inf) * norm(b,inf)/norm(deltab,inf);
  KA_store(i) = KA;
end
KA_est = max(KA_store)

figure
set(gcf, 'Position', [100 100 1000 500])
plot([1:n_trials], KA_store, 'bx-','LineWidth',2, 'MarkerSize',8)
hold on, box on
plot([1:n_trials], cond(A,inf)*ones(n_trials,1), 'r--', 'LineWidth',2)
axis([0 n_trials+1 0 1.1*cond(A,inf)])
set(gca,'LineWidth',1.5)
set(gca,'FontSize',16)
xlabel('trials')
ylabel('K_{est}(A)')
```

Homework 6.4. Consider the linear system $\mathbb{A}_3\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A}_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b}_3 = \mathbb{A}_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

a. Apply Jacobi method to compute the solution with a tolerance of 10^{-5} and 10^{-8}. What do you observe?

b. Compute the iteration matrix $\mathbb{B}_J$. Evaluate its spectral radius $\rho(\mathbb{B}_J)$ and $(\mathbb{B}_J)^3$. Relying on the results, motivate what you found at the previous point.

Solution Homework 6.4.

hw_6_4.m

```
clc
clear all
close all

A = [1 2 -2;
    1 1 1;
    2 2 1];
b = A * [1 2 3]';

D = diag(diag(A));
L = tril(A, -1);
U = triu(A, 1);
Bj = -D \ (L+U);
gj = D \ b;

x0 = zeros(3,1);

iter = 0;
%tol  = 1e-5;
tol  = 1e-8;
maxit = 100;

[xj, iterj, incrj] = stationary_method(Bj, gj, x0, tol, maxit)

% Four iterations are performed in both cases, independently on the tolerance values we
considered.
% Notice that the last increment is actually zero!

% The spectral radius is very small.

eig(Bj)
rhoBj = max( abs( eig(Bj) ) )

% Indeed, computing by hand the eigenvalues we see that
% lambda_1 = lambda_2 = lambda_3 = 0, so the spectral radius is actually 0.

% Compute the cube of Bj
Bj^2
Bj^3     % At the third step we get the null matrix.

% Thus, for all k > 3
% Jacobi method has converged to the exact solution
```

Homework 6.5. Consider the tridiagonal matrix $\mathbb{A} \in \mathbb{R}^{10\times 10}$ defined as

$$\mathbb{A} = \begin{bmatrix} 3 & -2 & & & \\ -1 & 3 & -2 & & \\ & \dots & \dots & \dots & \\ & & -1 & 3 & -2 \\ & & & -1 & 3 \end{bmatrix}.$$

Consider also the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ such that $\boldsymbol{x} =$`ones(10,1)`.

a. Do both the Jacobi and the Gauss-Seidel methods converge? Numerically confirm the relation between their spectral radii.

b. Apply both methods, with $\boldsymbol{x}^{(0)} = \boldsymbol{0}$ and with a tolerance of 10^{-12}, and compare the required number of iterations.

Solution Homework 6.5.

hw_6_5.m

```
clc
clear all
close all
```

```
n = 10;
A = 3*eye(n) - 2*diag(ones(n-1, 1), 1) - diag(ones(n-1, 1), -1);
b = A*ones(n, 1);

% Iteration matrices.
D = diag( diag(A) );
L = tril(A, -1);
U = triu(A, 1);

Bj = -D \ (L+U);  % Jacobi method
Bgs = -(D+L) \ U; % Gauss-Seidel method

% Compute the spectral radii of both matrices to determine whether the
% methods are convergent or not.

rhoBj = max( abs( eig(Bj) ) )
rhoBgs = max( abs( eig(Bgs) ) )

% It can be noted that both spectral radii are less than 1, so that
% both method are convergent, and that the spectral radius of the Gauss-Seidel
% method is the square of the spectral radius of the Jacobi method.

% Iteration vector
gj = D \ b;
ggs = (D+L) \ b;

x0 = zeros(n, 1);
tol = 1e-12;
maxit = 1000;

% Jacobi method
[xj, iterj, incrj] = stationary_method(Bj, gj, x0, tol, maxit)

% Gauss-Seidel method
[xgs, itergs, incrgs] = stationary_method(Bgs, ggs, x0, tol, maxit)

% The number of iterations of the Gauss-Seidel method is approximately half
% of the ones of the Jacobi method.
```

Homework 6.6. Consider a linear system whose coefficient matrix is obtained with the commands `» B = rand(5) + diag(10*ones(5,1)); » A = B*B';` Analyze the convergence of Jacobi and Gauss-Seidel methods.

Solution Homework 6.6. Not provided.

Homework 6.7. Consider a system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

a. Study the convergence of Jacobi and Gauss-Seidel methods if applied to the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$.

b. Compute the spectral radii of the iteration matrices of both methods.

c. How many iterations are needed to obtain the solution with a tolerance of 10^{-8} starting from $\boldsymbol{x}^{(0)} = \boldsymbol{0}$?

Solution Homework 6.7.

hw_6_7.m

```
clc
clear all
close all

A = diag(3*ones(6,1)) - diag(ones(5,1), 1) - diag(ones(5,1), -1);
A(1, 6) = -1;
A(2, 5) = -1;
A(5, 2) = -1;
A(6, 1) = -1;
A

b = [1 0 1 1 0 1]';

% The matrix is clearly symmetric, and the computation of the eigenvalues
% shows that is positive definite, so the Gauss-Seidel method converges.
% We cannot conclude anything on the convergence of the Jacobi method because
% the matrix is only weakly diagonally dominant.

D = diag(diag(A));
L = tril(A, -1);
U = triu(A, 1);

Bj = -D \ (L+U);
rhoBj = max(abs(eig(Bj)))

Bgs = -(D + L) \ U;
rhoBgs = max(abs(eig(Bgs)))

% Both Jacobi and Gauss-Seidel method will converge.

x0 = zeros(6,1);
tol = 1e-8;

% Jacobi
normBj = norm(Bj, 2) % Choose 2-norm becuase the implementation ofstationary_method provides
stopping criterion in 2-norm. This is actually the spectral radius itself, since matrix A is
symmetric.
gj = D \ b;
x1j = Bj*x0 + gj;
kjmin_2 = log(tol*(1 - normBj)/norm(x1j-x0)) / log(normBj)

% Rounding up to the first integer the estimate says that tolerance
% will be satisfied in no more than 91 iterations.

% Gauss-Seidel
normBgs = norm(Bgs)
ggs = (D + L) \ b;
x1gs = Bgs*x0 + ggs;
kgsmin_2 = log(tol*(1 - normBgs)/norm(x1gs-x0)) / log(normBgs)

% The estimate says that tolerance will be satisfied in no more than 89 iterations

[xj,niterj] = stationary_method(Bj, gj, x0, tol, 1000)
[xgs,nitergs] = stationary_method(Bgs, ggs, x0, tol, 1000)

% We get convergence in respectively 83 and 45 iteration, in accordance
% with what we found from the estimates.
```

Homework 6.8. Consider a system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 50 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 7 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

a. Compute $\|\mathbb{A}\|_\infty$ and $\|\mathbb{A}\|_1$.

b. Is $\mathbb{A}$ symmetric and positive definite?

c. Compute the $\mathbb{LU}$ decomposition using `lu_decomposition`.

d. Compute $\mathcal{K}_2(\mathbb{A})$. Is $\mathbb{A}$ well-conditioned?

e. Which methods (direct and iterative) can be employed for solving the system? Motivate. If Jacobi method is in that list, estimate both the convergence rate and the error after 4 iterations starting with $\boldsymbol{x}^{(0)} = \mathbf{0}$. Numerically validate the results.

Solution Homework 6.8.

hw_6_8.m

```
clc
clear all
close all

A = [50 1 2;
     1 5 2;
     2 2 7];
b = [1 0 0]';

norm(A,1)
norm(A,inf)

% Yes, since A is symmetric and strictly diagonally dominant, it is
% positive definite. We can also check with the eigenvalues: all positive.
eig(A)

[L1, U1] = lu_decomposition(A)

% The condition number is quite small, so the matrix is well conditioned.
cond(A,2)

% All the methods we saw are suitable for the solution of the linear system,
% since the matrix is symmetric and positive definite.

D = diag(diag(A));
L = tril(A, -1);
U = triu(A, 1);
Bj = -D \ (L+U);
rhoBj = max(eig(Bj))

% Asymptotic convergence rate
rate_jacobi = -log10(rhoBj)

% We now compute the estimate of the error after 4 iterations
x0 = zeros(3,1);
x1 = Bj*x0 + D\b;

esterr4 = norm(Bj)^4/(1-norm(Bj))*norm(x1-x0)

% We now compute the approximated solution after four steps.
xj = x0;
for i=1:4
   xj = Bj*xj + D\b;
end
x4 = xj
xtrue = A\b;

norm(xtrue-x4)
esterr4

% The error is lower than the estimate as we expected it to be.
```

Homework 6.9 (Exam February 14, 2012). Let $\mathbb{A} \in \mathbb{R}^{n\times n}$, $n > 3$ be defined as

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & (1/\alpha) \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}, \qquad 0 < \alpha < 1.$$

a. Write the iteration matrix $\mathbb{J}$ for the Jacobi method applied to the solution of the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ and compute $\mathbb{J}^2$ and $\mathbb{J}^3$.

b. Let $\alpha = 10^{-4}$, $\boldsymbol{x}^{(0)} = \boldsymbol{b} = [1, 1, \ldots, 1]^T$. Will the Jacobi method converge? How many iterations will be required to ensure that the infinity norm of the error is less than 10^{-9}?

c. Let $\mathbb{B} = \mathbb{A}^T\mathbb{A}$ and consider the linear system $\mathbb{B}\boldsymbol{x} = \boldsymbol{b}$. Compute the iteration matrix $\mathbb{G}$ for the Gauss-Seidel method. Will the Gauss-Seidel method converge?

d. Given $\alpha = 10^{-4}$, $\boldsymbol{x}^{(0)} = \boldsymbol{b} = [1, 1, \ldots, 1]^T$, how many iterations of the Gauss-Seidel method will be required to compute $\boldsymbol{x}$ with infinity norm of the error less than 10^{-9}?

Solution Homework 6.9.

hw_6_9.m

```
clc
clear all
close all

n = 4;

alpha = 1e-4;

A = eye(n);
A(1, n) = 1/alpha;

D = diag( diag(A) );
L = tril(A,-1);
U = triu(A,1);

J = -D \ (L+U);
J^2
J^3

% Since J^2 is null the method will converge after the first iteration
% to the exact solution.

b = ones(n, 1);
x = b;
gJ = D \ b;
dx = b;
counter = 0;
while norm(dx, inf) >= 1e-9
  xold = x;
  x = J*x + gJ;
  dx = x - xold;
  norm(dx);
  counter = counter + 1;
end
counter
x

B = A'*A

DD = diag( diag(B) );
LL = tril(B,-1);
```

```
UU = triu(B,1);

G = -(DD+LL) \ UU;

% B is built to be symmetric and positive definite.
% Gauss -Seidel method converges

format long e
rhoG = max(abs(eig(G)))

% However the spetral radius is very close to 1!!
% We expect extremely slow convergence!
```