Numerical Analysis
Lab 7

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A.Y. 2022-2023

November 25, 2022

Iterative methods for linear systems

This lab deals with the numerical solution of a linear system

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{for } \mathbb{A} \in \mathbb{R}^{n \times n}, \boldsymbol{b} \in \mathbb{R}^n$$

We will focus on iterative (stationary and dynamic) methods.

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Richardson Method

Richardson methods consist in computing the sequence

$$\boldsymbol{x}^{(k)} = \boldsymbol{x}^{(k-1)} + \alpha\, \mathbb{P}^{-1}\, \boldsymbol{r}^{(k-1)}$$

where $\mathbb{P}$ is a suitable non singular **preconditioning matrix**, $\boldsymbol{r}^{(k-1)} = \boldsymbol{b} - \mathbb{A}\boldsymbol{x}^{(k-1)}$ is the **residual** vector associated with the $(k-1)$-th iteration and $\alpha > 0$ is the **acceleration parameter** of the method.

Richardson schemes are a generalization of the methods already presented, where the iteration matrix is

$$\mathbb{B}(\alpha) = \mathbb{I} - \alpha\mathbb{P}^{-1}\mathbb{A}$$

Convergence properties depend on α, $\mathbb{A}$ and $\mathbb{P}$.
Taking $\alpha = 1$, with $\mathbb{P} = \mathbb{D}$ and $\mathbb{P} = \mathbb{D} - \mathbb{E}$ Jacobi and Gauss-Seidel methods are obtained respectively.

Richardson Methods

Exercise 7.1

Consider the linear $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ system obtained with the following MATLAB commands:

```
>> A = diag(8*ones(8,1)) + diag(2*ones(7,1),1)
...
>>  + diag(2*ones(7,1),-1);
>> b = A*ones(8,1);
```

a. Analyze the convergence of the Jacobi and Gauss-Seidel methods.
b. How many iterations are demanded by the two methods to obtain an approximate solution with tolerance 10^{-12} and initial guess $\boldsymbol{x}^{(0)} = \mathbf{0}$?
c. Write a function that implements Richardson methods.
d. Verify experimentally the answers at items *a* and *b* using the function implemented at the previous item.

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Choice of the parameter α of the Richardson method

Theorem 7.1

Let $\mathbb{A}$ and $\mathbb{P}$ be symmetric positive definite matrices. Then stationary Richardson methods are convergent for any initial guess if and only if the parameter α is selected in the $[0, 2/\lambda_{\text{max}}]$ interval, being λ_{max} the maximum eigenvalue of $\mathbb{P}^{-1}\mathbb{A}$. Moreover the spectral radius of the iteration matrix is minimized for $\alpha = \alpha_{opt}$, where

$$\alpha_{opt} = \frac{2}{\lambda_{\text{min}} + \lambda_{\text{max}}}.$$

and λ_{min} the minimum eigenvalue of $\mathbb{P}^{-1}\mathbb{A}$.

Choice of the parameter α of the Richardson method

Exercise 7.2

Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{b} = 0.2*$`ones(50,1)` and

$$\mathbb{A} = \begin{bmatrix} 4 & -1 & -1 & & & \\ -1 & 4 & -1 & -1 & & \\ -1 & -1 & 4 & -1 & -1 & \\ & \dots & \dots & \dots & \dots & \\ & & -1 & -1 & 4 & -1 \\ & & & -1 & -1 & 4 \end{bmatrix}, \quad \mathbb{T} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & \dots & \dots & \dots & \dots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

- a Are $\mathbb{A}$ and $\mathbb{T}$ symmetric positive definite matrices?
- b Apply the Richardson method starting from $\boldsymbol{x}^{(0)} = \mathbf{0}$, with a tolerance equal to 10^{-6} and $\mathbb{P} = \mathbb{I}$. Compare the required number of iterations (if the method is convergent) for $\alpha = 0.2$, $\alpha = 0.33$ and $\alpha = \alpha_{opt}$.
- c Apply the Richardson method starting from $\boldsymbol{x}^{(0)} = \mathbf{0}$, with a tolerance equal to 10^{-6}, $\mathbb{P} = \mathbb{T}$, $\alpha = \alpha_{opt}$. Compare the required number of iterations with the one associated with the non-preconditioned case. Compare the condition numbers of $\mathbb{A}$ and $\mathbb{P}^{-1}\mathbb{A}$.

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Gradient method

Solving a linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbb{A}$ is a **symmetric positive definite** matrix, is equivalent to solve the **minimization problem** of the quadratic form associated with

$$\Phi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\mathbb{A}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b}.$$

The idea of the **preconditioned gradient method** is to compute the solution $\boldsymbol{x}$ with the sequence

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \alpha^{(k)}\mathbb{P}^{-1}\boldsymbol{d}^{(k)}.$$

taking $\boldsymbol{d}^{(k)}$ to be the **steepest descent** direction for $\Phi(\boldsymbol{x})$ at $\boldsymbol{x}^{(k)}$ and the step length $\alpha^{(k)}$ such that $\Phi(\boldsymbol{x}^{(k+1)})$ is minimized. This is an example of **dynamic** Richardson method with

Gradient method (II)

$$\boldsymbol{d}^{(k)} = \boldsymbol{r}^{(k)} = \boldsymbol{b} - \mathbb{A}\boldsymbol{x}^{(k)}, \quad \mathbb{P}\boldsymbol{z}^{(k)} = \boldsymbol{r}^{(k)}, \quad \alpha^{(k)} = \frac{\boldsymbol{z}^{(k)^T}\boldsymbol{r}^{(k)}}{\boldsymbol{z}^{(k)^T}\mathbb{A}\,\boldsymbol{z}^{(k)}}.$$

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Homework 7.1

Consider the Hilbert matrix $\mathbb{H}_n$ of order n

$$\mathbb{H}_n = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & \cdots & 1/(2n-1) \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad n = 10, 20, 40.$$

- a Estimate $\mathcal{K}_2(\mathbb{H}_n)$ by choosing the rigth hand side term $\boldsymbol{b}$ of $\mathbb{H}_n\boldsymbol{x} = \boldsymbol{b}$ such that $x = ones(n, 1)$ and a perturbation of the right hand side $\boldsymbol{\delta b}$ randomly generated via MATLAB command `randn` and then multiplied by the scalar number $\alpha = 1e - 6$
- b Compute $\mathcal{K}_2(\mathbb{H}_n)$ using the MATLAB command `cond`.
- c What is the maximum value for n such that the relative error on the solution is equal to 1?

Homework 7.2

Consider the following matrix

$$\mathbb{A} = \begin{bmatrix} 1.001 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1.001 & 1 & 1 \\ 1 & 1.001 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1.001 \end{bmatrix}.$$

a. Compute the condition number $\mathcal{K}_\infty(\mathbb{A})$ using the MATLAB command `cond`
b. Then compute $\mathcal{K}_\infty(\mathbb{A})$ using the explicit definition of the conditioning number
c. Finally estimate $\mathcal{K}_\infty(\mathbb{A})$ by introducing a perturbation of the right hand side $\boldsymbol{\delta b}$ randomly generated via MATLAB command `randn` and then multiplied by the scalar number $\alpha = 1e-6$

Homework 7.3

Consider a linear system whose coefficient matrix is obtained with the commands

```
» B = rand(5) + diag(10*ones(5,1));
» A = B*B';
```

a. Analyze the convergence of the Jacobi and of the Gauss-Seidel methods.

b. Check the results using assuming the exact solution $xex = [1, 1, 1, 1, 1]^T$ and setting $x_0 = [0, 0, 0, 0, 0]^T$, tol=1.e-12, maxit=100

Homework 7.4

Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

a. Study the convergence of the Jacobi and of the Gauss-Seidel methods when applied to the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$.

b. Compute the spectral radius of the iteration matrix for both the methods.

c. How many iterations are required to obtain the solution $\boldsymbol{x}$ with a tolerance equal to 10^{-8} and choosing as initial guess $\boldsymbol{x}^{(0)} = \mathbf{0}$?

Homework 7.5

Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 50 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 7 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

a. Compute $\|\mathbb{A}\|_\infty$ and $\|\mathbb{A}\|_1$.
b. Is $\mathbb{A}$ a symmetric and positive definite matrix?
c. Compute the $\mathbb{L}\mathbb{U}$ factorization of $\mathbb{A}$ using the Matlab command `lu`.
d. Compute $\mathcal{K}_2(\mathbb{A})$. Is $\mathbb{A}$ a well-conditioned matrix $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$?
e. What methods (direct or iterative) can be employed to solve the system? Motivate your choice. If the Jacobi method belongs to this list, estimate both the convergence rate and the error after 4 iterations starting from the initial guess $\boldsymbol{x}^{(0)} = \mathbf{0}$. Validate numerically the results.