Numerical Analysis (089180) A.Y. 2022-2023
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Lab 5

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This lab deals with the numerical resolution of a linear system

$$\mathbb{A}\boldsymbol{x} = \boldsymbol{b}, \quad \text{for } \mathbb{A} \in \mathbb{R}^{n\times n}, \boldsymbol{b} \in \mathbb{R}^n$$

We will focus on direct methods (i.e. methods for which the solution is obtained in a **finite number** of operations) and, in particular on:

- LU factorization with the pivoting technique
- Cholesky decomposition
- Thomas factorization

1 LU with pivoting

When the standard LU factorization fails, dealing with non-singular matrices, we can resort to the **LU factorization with pivoting**. Pivoting is a technique which changes the order of the rows of a matrix in order to avoid null pivot elements during the LU factorization.

LU decomposition with pivoting

$$\left.\begin{array}{l} \mathbb{A}\boldsymbol{x} = \boldsymbol{b} \\ \mathbb{P}\mathbb{A} = \mathbb{L}\mathbb{U} \end{array}\right\} \longrightarrow \mathbb{L}\mathbb{U}\boldsymbol{x} = \mathbb{P}\boldsymbol{b} \longrightarrow \left\{\begin{array}{l} \mathbb{L}\boldsymbol{y} = \mathbb{P}\boldsymbol{b} \\ \mathbb{U}\boldsymbol{x} = \boldsymbol{y} \end{array}\right.$$

Notice that the rigth-hand side of the lower triangular system is modified with respect to the standard LU factorization

Exercise 5.1. Consider the linear system $\mathbb{E}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{E} = \begin{bmatrix} 4 & 1 & 1 & 1 & 5 \\ 4 & 1 & 2 & 0 & 0 \\ 1 & 0 & 15 & 5 & 1 \\ 0 & 2 & 4 & 10 & 2 \\ 3 & 1 & 2 & 4 & 20 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 12 \\ 19 \\ 22 \\ 18 \\ 30 \end{bmatrix}$$

1. Verify with Matlab that the sufficient conditions for the existence and uniqueness of the LU decomposition without pivoting are not fulfilled by matrix $\mathbb{E}$.
2. Compute the LU of matrix $\mathbb{E}$ with the Matlab command `lu` and verify if pivoting takes place with the Matlab command `spy`.
3. Solve the system $\mathbb{E}\mathbf{x} = \mathbf{b}$ by using the LU decomposition at the previous item.

2 Cholesky decomposition

Let $\mathbb{A} \in \mathbb{R}^{n\times n}$ be a **symmetric** and **positive definite** matrix: then there exits a unique **lower triangular** matrix $\mathbb{H}$, with $h_{ii} > 0$, s.t.

$$\mathbb{A} = \mathbb{H}\mathbb{H}^T.$$

For any symmetric matrix $\mathbb{A}$, $\quad \mathbb{A}$ pos. def. $\iff$

- $\boldsymbol{x}^T\mathbb{A}\,\boldsymbol{x} > 0, \quad \forall \boldsymbol{x} \neq \mathbf{0}$
- $\lambda_i > 0, \quad \forall i = 1, \ldots, n, \quad \lambda_i$ eigenvalue of $\mathbb{A}$
- $\det(\mathbb{A}_i) > 0, \quad \forall i = 1, \ldots, n$

The elements of $\mathbb{H}$ can be computed by the following algorithm:

Cholesky decomposition
We set $h_{11} = \sqrt{a_{11}}$ and for $i = 2, ..., n$,

$$h_{ij} = \frac{1}{h_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} h_{ik}h_{jk}\right) \qquad j = 1, \cdots i-1$$

$$h_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} h_{ik}^2}$$

Cholesky factorization is available in MATLAB via the command `R=chol(A)`, where `R` is the upper triangular matrix $\mathbb{H}^T$. Notice that the Matlab command `chol` **does not** check the symmetry of the matrix!

Cholesky decomposition

$$\left.\begin{array}{l} \mathbb{A}\boldsymbol{x} = \boldsymbol{b} \\ \mathbb{A} = \mathbb{H}\mathbb{H}^T \end{array}\right\} \longrightarrow \mathbb{H}\mathbb{H}^T\boldsymbol{x} = \boldsymbol{b} \longrightarrow \left\{\begin{array}{l} \mathbb{H}\boldsymbol{y} = \boldsymbol{b} \\ \mathbb{H}^T\boldsymbol{x} = \boldsymbol{y} \end{array}\right.$$

Exercise 5.2. Consider the matrix

$$\mathbb{A} = \begin{bmatrix} 44 & 15 & 29 & 26 & 119 \\ 15 & 33 & 32 & 18 & 15 \\ 29 & 32 & 252 & 112 & 73 \\ 26 & 18 & 112 & 124 & 90 \\ 119 & 15 & 73 & 90 & 430 \end{bmatrix}$$

1. Verify that the Cholesky decomposition can be applied to matrix $\mathbb{A}$ (the command `eig`);
2. Write a Matlab function with signature [`H`] = `MyChol(A)` to implement the Cholesky decomposition of the matrix $\mathbb{A}$;
3. Compute the Cholesky decomposition of the matrix $\mathbb{A}$ by means of both the Matlab command `chol` and the function `MyChol` and compare the results;
4. Solve the linear system with $\mathbb{A}\mathbf{x} = \mathbf{b}$, $\mathbf{b} = [1, 1, 1, 1, 1]^T$

3 Thomas algorithm

Consider the *tridiagonal* matrix

$$\mathbb{A} = \begin{bmatrix} a_1 & c_1 & & 0 \\ e_2 & a_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & e_n & a_n \end{bmatrix}$$

If the $\mathbb{LU}$ decomposition exists, then factors $\mathbb{L}$ and $\mathbb{U}$ are *bidiagonal*, namely

$$\mathbb{L} = \begin{bmatrix} 1 & & & 0 \\ \beta_2 & 1 & & \\ & \ddots & \ddots & \\ 0 & & \beta_n & 1 \end{bmatrix}, \quad \mathbb{U} = \begin{bmatrix} \alpha_1 & c_1 & & 0 \\ & \alpha_2 & \ddots & \\ & & \ddots & c_{n-1} \\ 0 & & & \alpha_n \end{bmatrix}$$

The unknown coefficients α_i and β_i can be determined by imposing the equality $\mathbb{LU} = \mathbb{A}$. This yields

$$\alpha_1 = a_1, \quad \beta_i = \frac{e_i}{\alpha_{i-1}}, \quad \alpha_i = a_i - \beta_i c_{i-1}, \quad i = 2, \ldots, n.$$

Moreover, due to the bidiagonal structure of $\mathbb{L}$ and $\mathbb{U}$, a special version of the forward and backward substitution algorithms can be applied, so that we obtain

$$(\mathbb{L}\boldsymbol{y} = \boldsymbol{b}) \quad y_1 = b_1, \quad y_i = b_i - \beta_i y_{i-1}, \quad i = 2, \ldots, n$$

$$(\mathbb{U}\boldsymbol{x} = \boldsymbol{y}) \quad x_n = \frac{y_n}{\alpha_n}, \quad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i}, \quad i = n-1, \ldots, 1$$

Exercise 5.3. Consider the tridiagonal matrix $\mathbb{A} \in \mathbb{R}^{10\times 10}$ defined as

$$\mathbb{A} = \begin{bmatrix} 1 & 11 & & & & \\ 102 & 2 & 12 & & & \\ & 103 & 3 & 13 & & \\ & \cdots & \cdots & \cdots & & \\ & & 109 & 9 & 19 & \\ & & & 110 & 10 & \end{bmatrix}.$$

Then consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ such that $\boldsymbol{x} =$`ones(10,1)`.

a. Use the Matlab commands `spdiags` and `sparse` to store the matrix in sparse format, and compare the effect of these commands with respect to the commands `diag` and `full`. Read carefully the documentation of the command `spdiags` using the `help` command.

b. Implement the Thomas algorithm and solve the linear system.

A Homework

Homework 5.1. Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 10 & 4 & 0 \\ 1 & 0 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ 10 \\ 1 \end{bmatrix}.$$

a. Compute the solution of the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ using the $\mathbb{LU}$ decomposition with pivoting of matrix $\mathbb{A}$.

b. Find the determinant of $\mathbb{A}$ (use the Matlab command `det` only to compute the determinant of $\mathbb{P}$).

Homework 5.2. Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 10 & 0 & 3 & 0 \\ 0 & 5 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

a. Compute the solution of the system. Use the Cholesky decomposition of matrix $\mathbb{A}$ if possible (check the hypotheses).

b. Find the determinant of $\mathbb{A}$ without using the Matlab commnand `det`.

Homework 5.3. Consider the tridiagonal matrix $\mathbb{A}_n \in \mathbb{R}^{n\times n}$ defined as

$$\mathbb{A}_n = \begin{bmatrix} a & b & & & \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ & & & b & a \end{bmatrix}$$

for $a = 2$ and $b = 1$.

a. Verify with MATLAB that $\mathbb{A}_{10}$ is a symmetric positive definite matrix.

b. Provide the form of the matrix $\mathbb{V}_{10}$ such that $\mathbb{A}_{10} = \mathbb{V}_{10}^T \mathbb{V}_{10}$

Homework 5.4. Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 10^{10} & 1 & 1 \\ 10^{10} & 1 & 1 & 10^{10} \\ 1 & 1 & 10^{-10} & 1 \\ 1 & 10^{10} & 1 & 10^{10} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 10^{10}+3 \\ 2\cdot 10^{10}+2 \\ 3+10^{-10} \\ 2\cdot 10^{10}+2 \end{bmatrix}$$

such that the exact solution is $\boldsymbol{x} = [1\ 1\ 1\ 1]^T$.

a. Solve the system using LU decomposition with pivoting

b. Compute the infinity norm of the error associated with the solution of the previous item

c. Can we apply the Cholesky decomposition to matrix $\mathbb{A}$?

d. Is the normalized residual $||\boldsymbol{b} - \mathbb{A}\boldsymbol{x}||/||\boldsymbol{b}||$ a good estimator for the relative error? Motivate your answer.