Numerical Analysis (089180)
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Lab 2

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This lab deals with the numerical approximation of the zeros of a real valued function of one variable, that is

given $f : \mathcal{I} = (a, b) \subset \mathbb{R} \to \mathbb{R}$, find $\xi \in \mathbb{R}$ such that $f(\xi) = 0$.

Methods for the numerical approximations of a zero of f are usually iterative; the aim is to generate a sequence of values $x^{(k)}$ such that

$$\lim_{k\to\infty} x^{(k)} = \xi$$

1 Bisection method

Method 2.1 (Bisection method). *Let* $x^{(0)}_{left} = a$, $x^{(0)}_{right} = b$; *at step* $k \in \mathbb{N}$ *compute the midpoint* $\overline{x}^{(k)} = (x^{(k)}_{left} + x^{(k)}_{right})/2$ *and* $f(\overline{x}^{(k)})$:

- *if* $f(x^{(k)}_{left})f(\overline{x}^{(k)}) < 0$, *set* $x^{(k+1)}_{left} = x^{(k)}_{left}$ *and* $x^{(k+1)}_{right} = \overline{x}^{(k)}$;
- *else set* $x^{(k+1)}_{left} = \overline{x}^{(k)}$ *and* $x^{(k+1)}_{right} = x^{(k)}_{right}$.

and stop when the number of iteration $n = N_{Max} = \left\lceil \frac{\log \frac{b-a}{tol}}{\log 2} \right\rceil$

Exercise 2.1. Consider the following function

$$f(x) = x^3 - (2 + e)x^2 + (2e + 1)x + (1 - e) - cosh(x - 1), x \in [0.5, 5.5].$$

- Plot the function f and determine two intervals that contain its roots (use the Matlab command `grid on`).
- For which roots the bisection method can be used? Compute the number of needed iterations for the bisection method to converge with a tolerance of 10^{-3}, when the interval $[3, 5]$ is choosen as starting interval.
- Implement the bisection method: `function [x,x_iter]=bisection(f,a,b,tol)` where `x` is the solution, `x_iter` is the vector of the approximations at each iteration, `f` is the function, defined as handle function, `a,b` are the end points of the interval, `tol` is the required tolerance.
- Employ the bisection method to approximate the roots with a tolerance equal to 10^{-3}.

2 Newton method

Method 2.2 (Newton method). *Newton method consists in approximating the zero of $f(x)$ with the sequence*

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, k \geq 0, f'(x^{(k)}) \neq 0$$

Theorem 2.1. *If $f \in \mathcal{C}^2([a,b])$ and $f'(x) \neq 0$ in an open interval containing ξ, then $\exists \delta > 0$ s.t. $\forall x^{(0)} : \left|x^{(0)} - \xi\right| < \delta$ the Newton method converges* ***quadratically*** *to ξ.*

Exercise 2.2. Consider the following function in the interval $[-0.5, 1.5]$

$$f(x) = sin(x)(1-x)^2.$$

- Plot f in order to find some intervals containing the roots.
- Implement the Newton method by using a stopping criterion based on the error estimator $|x^k - x^{k-1}|$. The signature of the function is: `function [x,x_iter]=newton(f,df,x0,tol,Nmax)` where `x` is the approximate, `x_iter` is the vector of the approximations at each iteration, `f, df` are the function and its first derivative, defined as handle functions, `x0` is the initial guess, `tol` is the tolerance demanded by user and `Nmax` is the maximum number of allowed iterations.
- Use Newton method to find the roots with a tolerance equal to 10^{-6}, by considering as initial guess $x_0 = 0.3$ and $x_0 = 0.5$.
- Compute an estimate of the convergence rate. Is it the expected one?

A Homework

Homework 2.1 (Bisection method). Consider the following equation

$$\cot(x) = \frac{x^2 - 1}{2x}.$$

a. Define three intervals containing the three smallest positive roots.

b. Find the biggest root among the ones at item a), using bisection method with tolerance equal to 10^{-6}.

c. Plot the evolution of the error as a function of the number of the iterations.

Homework 2.2 (Bisection method). Consider the following equation

$$e^{-(x-2)^2} = 1 - e^{x-4}$$

a. Plot fuction $f(x) = 0$ obtained from the above equation by writing all the therms in the first member and determine the number of real solutions.

b. Apply the bisection method by choosing a tolerance equal to 10^{-3} and initial interval $I_1 = [0,5], I_2 = [1,6], I_3 = [1,5]$ and $I_4 = [-1,5]$. Can you find all the roots?

Homework 2.3 (Newton method). a. Approximate the root x^* of $\tan(x) = 2x$ in the interval $\left(0, \frac{\pi}{2}\right)$ up to a tolerance equal to 10^{-15}.

b. Approximate the zero of $\sin(x)$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ using Newton method and starting from the value $x^{(0)} = x^*$. What do you expect? What do you observe?

Homework 2.4 (Newton method). Consider the following function

$$f(x) = e^{ax} - 1 = 0, \qquad a \neq 0.$$

For any value of the parameter a the unique solution is clearly $\xi = 0$.

a. Consider the case of $a = 200$ and apply Newton method starting from $x^{(0)} = 1$ with tolerance equal to 10^{-3} and 10^{-12}.

b. Repeat item a) for $a = 10^{-3}$.

c. For both items b) and c), compare the absolute value of the residual with the error. Justify the obtained results.

Homework 2.5 (Newton method). The equation $x^3 - x^2 - x + 1 = 0$ has a double root at $\xi = 1$. Given $x^{(0)} = 2$, does Newton method converge to ξ? How can you improve the method for approximating ξ?