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# Numerical Analysis

# Lab 8

Luca Liverotti

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# Polynomial interpolation

Given a function $f(x)$, an **interpolating function** is generally defined as

$$g(x) = \sum_{k=0}^{n} \alpha_k \phi_k(x) \qquad \text{s.t.} \qquad g(x_i) = f(x_i) \quad i = 0, 1, \ldots, n$$

where $\{x_i\}_{i=0}^n$ and $\{\phi_i(x)\}_{i=0}^n$ are respectively sets of $n+1$ **nodes** and **basis functions**, and the coefficients $\alpha_k$ are called **weights**.

**Definition 8.1 (Lagrange interpolating polynomial)**

Characteristic polynomials are defined as

$$\varphi_k(x) = \frac{\prod_{i \neq k} (x - x_i)}{\prod_{i \neq k} (x_k - x_i)}$$

and they are s.t. $\varphi_k(x_i) = \delta_{ik}$. Using them as basis functions, it holds $\alpha_i = f(x_i)$ so that we obtain the Lagrange

$$\Pi_n f(x) = \sum_{k=0}^{n} f(x_k) \varphi_k(x).$$

# Polynomial interpolation

**Theorem 8.1 (Interpolation error)**

Let $\Pi_n f(x)$ be the interpolating polynomial of order $n$ at the nodes $x_i \in [a, b]$ . If $f(x) \in \mathcal{C}^{n+1}([a, b])$ then

$$E_n f(x) = f(x) - \Pi_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \qquad \xi \in [a, b],\ \forall x \in [a, b].$$

If the nodes are equally spaced with step $h$, it holds

$$\|E_n f(x)\|_{L^\infty([a,b])} \leq \frac{\|f^{(n+1)}(x)\|_{L^\infty([a,b])}}{4(n+1)} h^{n+1}$$

This does not ensure a priori the uniform convergence of $\Pi_n f(x)$ to $f(x)$ for $n \to \infty$.

# Polynomial interpolation

**Definition 8.2 (Piecewise linear interpolation)**

Given of nodes: $x_0 < x_1 < \cdots < x_n$, we denote by $I_i$ the interval $[x_i, x_{i+1}]$ and by $H$ the maximum length of these intervals.
We denote by $\Pi_1^H f$ the piecewise linear interpolating polynomial of $f$ given by:

$$\Pi_1^H f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) \quad \forall x \in I_i.$$

**Theorem 8.2 (Interpolation error)**

If $f \in C^2(I)$, where $I = [x_0, x_n]$, then

$$\|f(x) - \Pi_1^H f(x)\|_{L^\infty(I)} \leqslant \frac{H^2}{8}\|f''(x)\|_{L^\infty(I)}$$

This bound allows us to ensure the uniform convergence of the interpolating polynomial to $f$.

# Polynomial interpolation

## MATLAB commands

Lagrange interpolation:

```
>> help polyfit
>> help polyval
```

Piecewise polynomial interpolation:

```
>> help interp1
```

Cubic spline:

```
>> help spline
```

# Polynomial interpolation

Notice that the commands polyfit and polyval can be used to compute both the interpolation and a least-square approximation

```
polyfit(x_nodes, y_nodes, N)
```

$$
\begin{aligned}
N+1 &= \#x\_nodes \implies \text{Lagrange interpolation;} \\
N+1 &!= \#x\_nodes \implies \text{least square interpolation.}
\end{aligned}
$$

# Polynomial interpolation

## Exercise 8.1

Approximate the function $f(x) = x\sin(x)$ in the interval $[-2, 6]$.

a. Consider a set of equally spaced nodes: find the interpolating polynomial using Lagrange basis polynomials (`polyfit`, `polyval` commands) of degree $4, 6, 8$. Plot the function $f(x)$ and the associate interpolating polynomials.

b. Estimate the interpolation error when resorting to a quadratic interpolation (n=2), using the theoretical results. Then evaluate the interpolating polynomial in 1000 equally spaced points and plot the interpolation error. Motivate the result.

c. Compute with Matlab the piecewise linear interpolant (`interp1` command) and the cubic spline (`spline` command) associated with $n+1 = 6, 11, 21$ equally spaced nodes.

## Exercise 8.2 (Runge's counterexample)

Consider the function
$$f(x) = \frac{1}{1+x^2} \qquad -5 \le x \le 5.$$

a. Verify with Matlab that the interpolating polynomials $\Pi_n f(x)$ using equally spaced nodes are such that
$$\lim_{n\to\infty} |f(x) - \Pi_n f(x)| \neq 0.$$
Check this statement graphycally and by computing the infinity norm

b. Compute the piecewise linear interpolation (`interp1` command) with $n = 1, 2, 4, 8, 16, 32$ subintervals and plot the result. Find the error with the respect to the infinity norm and verify that it converges quadratically as a function of the distance between two nodes.

c. Interpolate the Runge function using a piecewise cubic spline (`spline` command) with $n = 1, 2, 4, 8, 16, 32$ subintervals. Plot the result.

d. Compute $\Pi_n f(x)$ using the Chebyshev nodes:
$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\hat{x}_i, \quad \hat{x}_i = -cos(\pi i/n), i = 0, \cdots, n$$

## Exercise 8.3

Consider the function

$$f(x) = \left| x - \frac{\pi}{12} \right| \qquad -1 \leq x \leq 1.$$

- a Verify that the interpolating polynomials $\Pi_n f(x)$ based on equally spaced nodes exibit the Runge's phenomenon.
- b Compute the piecewise linear interpolation (`interp1` command) based on $n = 1, 2, 4, 8, 16, 32$ intervals and compute the associated error with the respect to infinity norm. Is the quadratic convergence ensured in such a case?
- c Interpolate the function using a piecewise cubic spline (`spline` command) with $n = 1, 2, 4, 8, 16, 32$ subintervals. Plot the result.
- d Interpolate the function using the least square approach bu setting the number of nodes $n = 10$ and varying the degree of the interpolant. Plot the results and check when the least-square approximation exactly matches the values y_nodes in correspondence with the x_nodes.

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# Homework

## Homework 8.1

Consider the function $f(x) = e^x$ in the interval $[-1, 1]$ and the associated piecewise linear interpolating polynomial on a set of nodes $\{x_i\}$.

a. Determine how many equally spaced nodes $x_i$ are needed in order to ensure an interpolation error with infinity norm smaller than $10^{-3}$.
b. Construct the interpolating polynomial based on such a number of nodes, compute the associated error and check if the request on accuracy is satisfied.

# Homework

## Homework 8.2

Consider the function

$$f(x) = \frac{1}{(x-0.3)^2+0.01} + \frac{1}{x^2+0.04} - 6.$$

a. Sample $f(x)$ at 10 equally spaced nodes in the interval $[-1, 3]$ and construct the minimum degree interpolating polynomial.
b. Numerically compute the error. Provide a method to improve the result.

# Homework

## Homework 8.3

Let $f(x) = x^5 + 1$ and $x_i = 0.2 \cdot i$, with $i = 0, 1, 2, 3$.

- a. Let $\Pi_n f(x)$ be the interpolating polynomial of $f(x)$ at the nodes $x_i$. Compute $\Pi_n f(x)$ and provide an upper bound for $|E_n(x)| = |\Pi_n f(x) - f(x)|$.
- b. Let $\Pi_1^H f(x)$ be the piecewise linear interpolation of $f(x)$ at the nodes $x_i$. Compute an upper bound for $\left|E_1^H(x)\right| = \left|\Pi_1^H f(x) - f(x)\right|$.

# Homework

### Homework 8.4

Consider the following function

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x \in [1, 2]. \end{cases}$$

First interpolate the function using the Lagrange polynomial with an increasing number of nodes. Then repeat the procedure using the piecewise linear and a least-squares approximation. What do you observe? Justify the answer.

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