Numerical Analysis (089180)
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Lab 3

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1 Modified Newton Method

Method 3.1 (Newton method). *Newton method consists in approximating the solution of $f(x) = 0$ with the sequence*

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \quad \textit{with} \quad k \geq 0 \quad \textit{and} \quad f'(x^{(k)}) \neq 0$$

If the root ξ is not simple the Newton method converges with order one. If m is the multiplicity of the root, then the modification

$$x^{(k+1)} = x^{(k)} - m\frac{f(x^{(k)})}{f'(x^{(k)})} \quad \text{with} \quad k \geq 0 \quad \text{and} \quad f'(x^{(k)}) \neq 0$$

allow us to recover the second order of convergence.

Remark 3.1 (Exam tips!). Complete exercise 2.2 and homework 2.5. ■

2 Fixed point methods

Method 3.2 (Fixed point method). *A **fixed point method** consists in the sequence*

$$x^{(k+1)} = \phi(x^{(k)})$$

*with **consistency** and **convergence** properties, that means ϕ is s.t.*

$$\xi = \phi(\xi) \qquad \textit{and} \qquad \lim_{k\to\infty} \left| x^{(k)} - \xi \right| = 0.$$

Theorem 3.1 (Local convergence, Ostrowski theorem). *Let ξ be a fixed point of a function ϕ which is continuous and differentiable in a neighborhood of ξ. If $|\phi'(\xi)| < 1$, then there exists $\delta > 0$ s.t., for any $x^{(0)} \in (\xi - \delta, \xi + \delta)$, the sequence $x^{(k+1)} = \phi(x^{(k)})$ converges to ξ. Moreover, the following limit holds*

$$\lim_{k\to\infty} \frac{x^{(k+1)} - \xi}{x^{(k)} - \xi} = \phi'(\xi)$$

Remark 3.2. Let ξ be a fixed point of the a function ϕ which is continuous and differentiable in a neighborhood of ξ.

- if $|\phi'(\xi)| > 1$, then the sequence $x^{(k+1)} = \phi(x^{(k)})$ will not converge to ξ;
- if $|\phi'(\xi)| = 1$, then no general conclusion can be drawn both convergence and divergence become possibile. ■

Remark 3.3 (Geometrical interpretation). Solving the fixed point problem $x = \phi(x)$ is equivalent to solve the system

$$\begin{cases} y = x \\ y = \phi(x) \end{cases}$$

i.e. to determine the intersections between $\phi(x)$ and the bisector line $y = x$ of the first and the third quadrants. ■

Theorem 3.2. *If ϕ is $\mathcal{C}^p$ in a suitable neighborhood of ξ, and if*

$$\phi^{(i)}(\xi) = 0 \qquad i = 1, \dots, p-1, \quad \phi^{(p)}(\xi) \neq 0,$$

then the fixed-point method $x^{(k+1)} = \phi(x^{(k)})$ has order p, i.e.

$$\lim_{k \to \infty} \frac{x^{(k+1)} - \xi}{(x^{(k)} - \xi)^p} = \frac{\phi^{(p)}(\xi)}{p!}$$

Remark 3.4 (How to estimate the rate of convergence p). Consider the limit of the above-mentioned quantity for two consecutive steps and, for $k \to \infty$, analyze the corresponding error defining $e^k = x^k - \xi$:

$$\frac{e^{(k+1)}}{(e^{(k)})^p} = \frac{e^{(k)}}{(e^{(k-1)})^p},$$

so that

$$\frac{e^{(k+1)}}{e^{(k)}} = \left(\frac{e^{(k)}}{e^{(k-1)}}\right)^p,$$

and applying the logarithm

$$p = \frac{\log \frac{e^{(k+1)}}{e^{(k)}}}{\log \frac{e^{(k)}}{e^{(k-1)}}} = \frac{\log e^{(k+1)} - \log e^{(k)}}{\log e^{(k)} - \log e^{(k-1)}}.$$ ■

Exercise 3.1. Solve the equation $x^2 - 5 = 0$ to find the root $\xi = \sqrt{5}$ using the following iterative methods:

1. $x^{(k+1)} = 5 + x^{(k)} - (x^{(k)})^2$;
2. $x^{(k+1)} = \frac{5}{x^{(k)}}$;
3. $x^{(k+1)} = 1 + x^{(k)} - \frac{1}{5}(x^{(k)})^2$;
4. $x^{(k+1)} = \frac{1}{2}\left(x^{(k)} + \frac{5}{x^{(k)}}\right)$.

a. Are these iterations consistent and convergent? Motivate your answer.

b. Implement a function for the generic fixed point iteration with function ϕ.

c. Assess numerically the convergence of the proposed iterative methods.

3 Bisection - Newton methods

Theorem 3.3. *If $f \in \mathcal{C}^2([a,b])$ and $f'(x) \neq 0$ in an open interval containing ξ, then $\exists \delta > 0$ **s.t.** $\forall x^{(0)} : \left|x^{(0)} - \xi\right| < \delta$ the Newton method converges quadratically to ξ.*

The convergence is guaranteed only if the initial guess $x^{(0)}$ is close enough to the root ξ, and for this reason the Newton method is a **locally** convergent method. A simple solution to overcome this issue consists in employing the bisection method to predict the inital guess $x^{(0)}$, as shown in the next exercise.

Exercise 3.2. Consider the following function in the interval $[-1, 6]$

$$f(x) = \arctan\left[7\left(x - \frac{\pi}{2}\right)\right] + \sin\left[\left(x - \frac{\pi}{2}\right)^3\right].$$

a. Plot f in order to find an interval containing a root. What is the multiplicity of the root?

b. Use the Newton method to find the root with a tolerance of 10^{-10} and initial guess $x^{(0)} = 1.5$. Compute the error.

c. Use the Newton method to find the root with a tolerance of 10^{-10} and initial guess $x^{(0)} = 4$. Compute the error.

d. If possible, apply the bisection method on the interval $[a, b] = [-1, 6]$ and tolerance $\frac{b-a}{2^{30}}$. Compute the error.

e. Write a function `bisection_newton.m` to find ξ using the Newton method starting from an initial guess obtained after few iterations of a bisection method. Test with $[a, b] = [-1, 6]$, 5 iterations of the bisection method and tolerance 10^{-10} for the Newton method.

A Homework

Homework 3.1 (Fixed point method).

a. Let $\phi(x) = x - x^3$, which admits $\xi = 0$ as a fixed point. Compute $\phi'(\xi)$ and test the convergence of the sequence $x^{(k+1)} = \phi(x^{(k)})$ for $x^{(0)} \in [-1, 1]$.

b. Let $\phi(x) = x + x^3$, which admits $\xi = 0$ as a fixed point. Compute $\phi'(\xi)$ and test the convergence of the sequence $x^{(k+1)} = \phi(x^{(k)})$ for $x^{(0)} \in [-1, 1]$.

Homework 3.2 (Fixed point method, Newton method). Given the equation

$$f(x) = e^{x^2} \log(x + 1) = 1$$

consider the following three iterative methods

1. $x^{(k+1)} = \sqrt{-\log\log(x^{(k)} + 1)}$;
2. $x^{(k+1)} = x^{(k)}\, e^{(x^{(k)})^2} \log(x^{(k)} + 1)$;
3. Newton method.

Verify the consistency of the schemes and apply them to compute the root ξ with a tolerance equal to 10^{-3}. Comment on the results.