Numerical Analysis
Lab 6

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A.Y. 2022-2023

November 11, 2022

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Condition number

Definition 6.1

Let V be a vector space associated with the field $K = \mathbb{R}$ or $\mathbb{C}$. The map $\|\cdot\|$ from V into $\mathbb{R}$ is a **norm** in V if

1. $\|\boldsymbol{v}\| \geq 0 \ \ \forall \boldsymbol{v} \in V$ and $\|\boldsymbol{v}\| = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$;
2. $\|\alpha \boldsymbol{v}\| = |\alpha| \, \|\boldsymbol{v}\| \ \ \forall \alpha \in K, \ \forall \boldsymbol{v} \in V$;
3. $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\| \ \ \forall \boldsymbol{v}, \boldsymbol{w} \in V$.

In $V = \mathbb{R}^n$ we can define the ***p*-norm**

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad 1 \leq p < \infty;$$

when $p = 2$ we recover the **Euclidean norm**. For $p = \infty$ we define the so called **infinity norm**

$$\|\boldsymbol{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| \, .$$

Condition number

Definition 6.2

Consider $\mathbb{A} \in \mathbb{R}^{m\times n}$. The mapping $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ is a **matrix norm** if it satisfies the same properties as for vector norms. Moreover, a matrix norm is said **submultiplicative** if it satisfies

4. $\|\mathbb{A}\mathbb{B}\| \leq \|\mathbb{A}\| \, \|\mathbb{B}\| \quad \forall \mathbb{A}, \mathbb{B} \in \mathbb{R}^{m\times n}$.

The matrix 1-norm and matrix ∞-norm are defined by

$$\|\mathbb{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{and} \quad \|\mathbb{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$$

namely by the so called **column sum** and **row sum norm**, respectively. The 2-norm is called **spectral norm** and is defined by $\|\mathbb{A}\|_2 = \sqrt{\lambda_{\max}(\mathbb{A}^T\mathbb{A})}$.

Condition number

Definition 6.3

The **condition number** with respect to generic norm p is defined as

$$\mathcal{K}_p(A) = \|\mathbb{A}\|_p \left\|\mathbb{A}^{-1}\right\|_p$$

Given a system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ we consider the problem where the right-hand side term is **perturbed** by vector $\boldsymbol{\delta b}$

$$\mathbb{A}(\boldsymbol{x} + \boldsymbol{\delta x}) = \boldsymbol{b} + \boldsymbol{\delta b}$$

$\mathcal{K}_p(\mathbb{A})$ is a **measure of the sensitivity** of the solution $\boldsymbol{x}$ to changes in $\boldsymbol{b}$ and it holds

$$\frac{\|\boldsymbol{\delta x}\|_p}{\|\boldsymbol{x}\|_p} \leq \mathcal{K}_p(\mathbb{A}) \frac{\|\boldsymbol{\delta b}\|_p}{\|\boldsymbol{b}\|_p}$$

If $\mathcal{K}_p(\mathbb{A}) = 10^k$ it means that you **may loose** up to k digits of accuracy in the solution. In `MATLAB`: `cond(A,p)` computes the condition number with respect to the p-norm, see also, the commands `cond(A)`, `condest` and `rcond`.

Condition number

Exercise 6.1

Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $0 < \varepsilon \ll 1$.

a. Compute $\mathcal{K}_p(\mathbb{A})$ for $p = 1, 2, \infty$ without using MATLAB. Check the results in `MATLAB` for some values of ε.

b. Consider the perturbation term $\boldsymbol{\delta b} = [0\ 0\ \alpha]^T$, with $|\alpha| \ll 1$. Compute the associated perturbation on the solution $\boldsymbol{x}$

c. Consider now $\boldsymbol{\delta b} = [\alpha\ 0\ 0]^T$, $|\alpha| \ll 1$. Compute the associated perturbation on the solution $\boldsymbol{x}$

d. Verify with `MATLAB` the obtained results for $p = \infty$ with $\varepsilon = 10^{-6}$ and with $\alpha = 10^{-12}, 10^{-6}$.

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Iterative schemes

According to a generic iterative **method**, the solution $\boldsymbol{x}$ to the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ is approximated by

$$\boldsymbol{x}^{(k+1)} = \mathbb{B}\boldsymbol{x}^{(k)} + \boldsymbol{g}$$

such that

$$\boldsymbol{x} = \mathbb{B}\boldsymbol{x} + \boldsymbol{g} \qquad \text{and} \qquad \boldsymbol{x} = \lim_{k\to\infty} \boldsymbol{x}^{(k)}$$

(consistency) (convergence)

Usually $\boldsymbol{x}$ is not reached in a finite number of iterations, so a **stopping criterion** is applied

- **increment** $\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\| \leq \varepsilon$
- **residual** $\left\|\boldsymbol{b} - \mathbb{A}\boldsymbol{x}^{(k)}\right\| \leq \varepsilon$.

If the method is consistent

$$\boldsymbol{x}^{(k+1)} = \mathbb{B}\boldsymbol{x}^{(k)} + \boldsymbol{g} \;\text{ converges} \quad \Longleftrightarrow \quad \rho(\mathbb{B}) < 1$$

$R(\mathbb{B}) = -\log \rho(\mathbb{B})$ **asymptotic convergence rate**.

Jacobi and Gauss-Seidel methods

The square matrix $\mathbb{A}$ can be written as $\mathbb{A} = -\mathbb{E} + \mathbb{D} + (\mathbb{A} - \mathbb{D} + \mathbb{E})$ where $-\mathbb{E}$ is the lower triangular part with null entries on the main diagonal, and $\mathbb{D}$ the diagonal matrix extracted from $\mathbb{A}$.

Jacobi method

$$\mathbb{B} = \mathbb{D}^{-1}(\mathbb{D} - \mathbb{A})$$
$$\boldsymbol{g} = \mathbb{D}^{-1}\boldsymbol{b}$$

Gauss-Seidel method

$$\mathbb{B} = (\mathbb{D} - \mathbb{E})^{-1}(\mathbb{D} - \mathbb{E} - \mathbb{A})$$
$$\boldsymbol{g} = (\mathbb{D} - \mathbb{E})^{-1}\boldsymbol{b}$$

Jacobi and Gauss-Seidel methods

Exercise 6.2

Consider the linear systems $\mathbb{A}_i \boldsymbol{x} = \boldsymbol{b}_i,\ i = 1, \dots, 4$ with

$$\mathbb{A}_1 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \mathbb{A}_2 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\mathbb{A}_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \mathbb{A}_4 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

with $\boldsymbol{x} = [1\ 2\ 3]^T$ and $\boldsymbol{b}_i = \mathbb{A}_i \boldsymbol{x}$.

- a) Write a MATLAB function that implements the generic iterative scheme $\boldsymbol{x}^{(k+1)} = \mathbb{B}\boldsymbol{x}^{(k)} + \boldsymbol{g}$. Use the stopping criterion based on the residual
- b) Study the convergence for Jacobi and Gauss-Seidel methods. Check the results using the function at the previous item, setting $x_0 = [0, 0, 0]^T$, tol=1.e-6, maxit=100

Jacobi and Gauss-Seidel methods

Exercise 6.3

Let the 10×10 square matrix $\mathbb{A}$ be

$$\mathbb{A}=\begin{bmatrix} 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^T$.

1. Write a MATLAB function to define the matrix and calculate $\boldsymbol{b}$
2. Study the convergence of the Jacobi and Gauss-Seidel methods by investigating the relation between spectral radius of the two iteration matrices
3. Check the results using the functions implemented in 6.2 and choosing $x_0 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$, tol=1.e-12, maxit=500

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Homework 6.1

Consider the Hilbert matrix $\mathbb{H}_n$ of order n

$$\mathbb{H}_n = \begin{bmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & \cdots & 1/(2n-1) \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad n = 10, 20, 40.$$

a. Estimate $\mathcal{K}_2(\mathbb{H}_n)$ using the technique of perturbing a linear system with $\mathbb{H}_n$ as coefficient matrix. Consider the right hand side term b such that $x = ones(n, 1)$.

b. Compute $\mathcal{K}_2(\mathbb{H}_n)$ using MATLAB command `cond`.

c. What can you say about the **accuracy of the solution** of the generic linear system $\mathbb{H}_n \boldsymbol{x} = \boldsymbol{b}$? How large can you take n before the relative error is 1? (i.e. there are **no significant digits** in the solution)

Homework 6.2

Compute the condition number and the determinant of the following matrices

```
» A = diag(0.1*ones(15,1));
» B = triu(rand(15),14)*1e5 + diag(0.1*ones(15,1));
```

Is, in general, the determinant a good measure of the condition number of the matrix?

Homework 6.3

Consider the following matrix

$$\mathbb{A} = \begin{bmatrix} 1.001 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1.001 & 1 & 1 \\ 1 & 1.001 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1.001 \end{bmatrix}.$$

a. Compute the condition number $\mathcal{K}_\infty(\mathbb{A})$ using MATLAB command `cond`.
b. Then compute $\mathcal{K}_\infty(\mathbb{A})$ using its definition.
c. Finally estimate $\mathcal{K}_\infty(\mathbb{A})$ by **perturbing a suitable linear system** having $\mathbb{A}$ as coefficient matrix.

Homework

Homework 6.4

Consider the linear system $\mathbb{A}_3 \boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A}_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b}_3 = \mathbb{A}_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

a. Apply Jacobi method to compute the solution with a tolerance of 10^{-5} and 10^{-8}. What do you observe?
b. Compute the iteration matrix $\mathbb{B}_J$. Evaluate its spectral radius $\rho(\mathbb{B}_J)$ and $(\mathbb{B}_J)^3$. Relying on the results, motivate what you found at the previous point.

Homework 6.5

Consider the tridiagonal matrix $\mathbb{A} \in \mathbb{R}^{10\times 10}$ defined as

$$\mathbb{A} = \begin{bmatrix} 3 & -2 & & & \\ -1 & 3 & -2 & & \\ & \dots & \dots & \dots & \\ & & -1 & 3 & -2 \\ & & & -1 & 3 \end{bmatrix}.$$

Consider also the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ such that $\boldsymbol{x} =$ **`ones(10,1)`**.

- a Do both the Jacobi and the Gauss-Seidel methods converge? Numerically confirm the relation between their spectral radii.
- b Apply both methods, with $\boldsymbol{x}^{(0)} = \mathbf{0}$ and with a tolerance of 10^{-12}, and compare the required number of iterations.

Homework 6.6

Consider a linear system whose coefficient matrix is obtained with the commands

```
» B = rand(5) + diag(10*ones(5,1));
» A = B*B';
```

Analyze the convergence of Jacobi and Gauss-Seidel methods.

Homework 6.7

Consider a system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

a. Study the convergence of Jacobi and Gauss-Seidel methods if applied to the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$.
b. Compute the spectral radii of the iteration matrices of both methods.
c. How many iterations are needed to obtain the solution with a tolerance of 10^{-8} starting from $\boldsymbol{x}^{(0)} = \mathbf{0}$?

Homework 6.8

Consider a system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 50 & 1 & 2 \\ 1 & 5 & 2 \\ 2 & 2 & 7 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- a Compute $\|\mathbb{A}\|_\infty$ and $\|\mathbb{A}\|_1$.
- b Is $\mathbb{A}$ symmetric and positive definite?
- c Compute the $\mathbb{LU}$ decomposition using `lu_decomposition`.
- d Compute $\mathcal{K}_2(\mathbb{A})$. Is $\mathbb{A}$ well-conditioned?
- e Which methods (direct and iterative) can be employed for solving the system? Motivate. If Jacobi method is in that list, estimate both the convergence rate and the error after 4 iterations starting with $\boldsymbol{x}^{(0)} = \mathbf{0}$. Numerically validate the results.

Homework 6.9 (Exam February 14, 2012)

Let $\mathbb{A} \in \mathbb{R}^{n \times n}$, $n > 3$ be defined as

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 & (1/\alpha) \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}, \qquad 0 < \alpha < 1.$$

a) Write the iteration matrix $\mathbb{J}$ for the Jacobi method applied to the solution of the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ and compute $\mathbb{J}^2$ and $\mathbb{J}^3$.

b) Let $\alpha = 10^{-4}$, $\boldsymbol{x}^{(0)} = \boldsymbol{b} = [1, 1, \ldots, 1]^T$. Will the Jacobi method converge? How many iterations will be required to ensure that the infinity norm of the error is less than 10^{-9}?

c) Let $\mathbb{B} = \mathbb{A}^T\mathbb{A}$ and consider the linear system $\mathbb{B}\boldsymbol{x} = \boldsymbol{b}$. Compute the iteration matrix $\mathbb{G}$ for the Gauss-Seidel method. Will the Gauss-Seidel method converge?