Numerical Analysis (089180)
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A.Y. 2022-2023
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# Lab 9 – Homework Solutions

December 16, 2022

## A Homework

**Homework 9.1.** Consider the quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{4}\left[f(0) + 2f\Big(\frac{1}{2}\Big) + f(1)\right] + E(f).$$

a. Which is the degree of exactness of this rule?

b. Is there any quadrature formula with the same degree of exactness that needs a smaller number of functional evaluations?

**Solution Homework 9.1.**

hw_9_1.m

```
% a)
% Check the solution to ex_11_3.
% b)
% Yes, the midpoint formula needs just one functional evaluation.
```

**Homework 9.2.** Determine the coefficients $\alpha_1, \alpha_2$ and $\alpha_3$ in the quadrature below so that the quadrature formula has a degree of exactness equal to 2

$$I(f) = \int_0^1 f(x)\,dx \simeq \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f^{'}(0) = Q(f).$$

**Solution Homework 9.2.**

hw_9_2.m

```
% We require that I(x^i) = Q(x^i) for i = 0, 1, 2: this condition gives the
% following linear system:
%
% alpha_1 + alpha_2 = 1
% alpha_2 + alpha_3 = 1/2
% alpha_2 = 1/3
%
% so
%
% Q(f) = 2/3 f(0) + 1/3 f(1) + 1/2 f'(0)
```

**Homework 9.3.** Consider an interpolary quadrature formula for the integration of a generic function $f$ in the interval $[-1, 1]$ based on two nodes.

a. What is the maximum achievable degree of exactness? Determine the quadrature nodes $x_j$ and the related weights so that such degree of exactness is ensured.

**Solution Homework 9.3.**

hw_9_3.m

```
% a)
%
% The maximum achievable degree of exactness is 2n+1 = 3, being n+1 = 2 the number of nodes.
%
% We thus define
%
% Q(f) = alpha_1 f(x_1) + alpha_2 f(x_2)
%
% and require that I(x^i) = Q(x^i) for i = 0, 1, 2, 3:
% this entails the following NON-linear system
%
% alpha_1 + \alpha_2 = 2
% alpha_1 x_1 + alpha_2 x_2 = 0
% alpha_1 x_1^2 + alpha_2 x_2^2 = 2/3
% alpha_1 x_1^3 + alpha_2 x_2^3 = 0
%
% From the second equation and fourth equations we get that either
%
% alpha_2 = 0 (unacceptable, because we want two non-null weights)
%
% or
%
% x_2 = 0
% (unacceptable, because again from the second equation either alpha_1 = 0 (unacceptable)
% or x1 = 0(= x_2, unacceptable)
%
% or
%
% x_1 = x_2 (unacceptable, because we want two different nodes)
%
% or
%
% x_1 = -x_2 \quad \text{acceptable}
%
% Thus, from the second and first equations,
% we get alpha_1 = alpha_2 = 1 and, finally, from the third equation x_{1,2} = +/- sqrt(3)/3.
%
% b)
%
% The quadrature rule defined on the interval (alpha, beta) = (-1,1)
% can be employed to approximate an integral over (a,b) = (2,3)
% provided that the original nodes are mapped from (alpha, beta) to (a,b) with
%
% x -> (b-a)(beta-alpha) (x - alpha) + a
%
% and weights are rescaled as
%
% w -> (b-a)/(beta-alpha) w.
```