Numerical Analysis
Lab 9

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Quadrature formulas

The definite integral
$$I(f) = \int_a^b f(x)dx$$
can be computed by **replacing** f with an approximation f_n. Using polynomial interpolation, we get **quadrature formula** of the form
$$I_n(f) = \sum_{j=0}^{n} \alpha_j f(x_j)$$
where x_j are the so called quadrature **nodes** and α_j the related **weights**.

A quadrature formula is said to have **degree of exactness** p if it exactly integrates the polynomial of degree $\leq p$.
A formula with $n+1$ nodes has at most degree of exactness equal to $2n+1$ (hint: think about the Gaussian quadrature rule).

Simple Quadrature formulas

Midpoint quadrature rule

$$I_0(f) = (b-a)\, f\left(\frac{a+b}{2}\right).$$

If $f \in \mathcal{C}^2([a,b])$, $\quad E_0(f) = \dfrac{(b-a)^3}{24} f''(\xi), \quad \xi \in (a,b).$

Trapezoidal quadrature rule

$$I_1(f) = \frac{b-a}{2}\Big[f(a) + f(b)\Big].$$

If $f \in \mathcal{C}^2([a,b])$, $\quad E_1(f) = -\dfrac{(b-a)^3}{12} f''(\xi), \quad \xi \in (a,b).$

Simpson quadrature rule

$$I_2(f) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

If $f \in \mathcal{C}^4([a,b])$, $\quad E_2(f) = -\dfrac{(b-a)^5}{2880} f^{(4)}(\xi), \quad \xi \in (a,b).$

Composite quadrature formulas

Similarly to the case of polynomial interpolation we can divide the integration interval into **subintervals** and apply on each of them a quadrature formula with a **low number** of nodes. In this way the generic quadrature formula becomes

$$I_{n,m}(f) = \sum_{k=0}^{m-1} \sum_{j=0}^{n} \alpha_j^{(k)} f(x_j^{(k)})$$

with $x_j^{(k)}$ and $\alpha_j^{(k)}$ the quadrature nodes and weights on the k^{th} subinterval respectively. The order of accuracy of a composite quadrature formula coincides with the rate of the convergence of the error.

Composite midpoint rule

$$I_{0,m}(f) = h \sum_{k=0}^{m-1} f(\bar{x}_k), \qquad h = \frac{b-a}{m}.$$

If $f \in \mathcal{C}^2([a,b])$, $\quad E_{0,m}(f) = \dfrac{(b-a)}{24} h^2 f''(\xi), \quad \xi \in (a,b).$

Composite quadrature formulas

Composite trapezoidal rule

$$
\begin{aligned}
I_{1,m}(f) &= \frac{h}{2}\sum_{k=0}^{m-1}\Big(f(x_k)+f(x_{k+1})\Big) = \\
&= \frac{h}{2}\left[f(x_0)+f(x_m)\right]+\sum_{k=1}^{m-1} f(x_k)), \quad h=\frac{b-a}{m}.
\end{aligned}
$$

If $f \in \mathcal{C}^2([a,b])$, $\quad E_{1,m}(f) = -\frac{b-a}{12}h^2 f''(\xi), \quad \xi \in (a,b).$

Composite Simpson rule

$$
I_{2,m}(f) = \frac{h}{6}\left[f(x_0)+2\sum_{r=1}^{m-1} f(x_{2r})+4\sum_{s=0}^{m-1} f(x_{2s+1})+f(x_{2m})\right], \quad h=\frac{b-a}{m}.
$$

If $f \in \mathcal{C}^4([a,b])$, $\quad E_{2,m}(f) = -\frac{b-a}{2880}h^4 f^{(4)}(\xi), \quad \xi \in (a,b).$

Exercises

Exercise 9.1

Consider the following integrals

$$I_1 = \int_0^1 x^3 \, dx = \frac{1}{4} \qquad \text{and} \qquad I_2 = \int_0^1 x^5 \, dx = \frac{1}{6}.$$

a. Implement the simple midpoint, trapezoidal and Simpson rule in three different MATLAB functions.
b. Using the code at the previous question, implement composite version of the midpoint, trapezoidal and Simpson rule.
c. Compute the integral I_1 using the midpoint rule with 1 and 10 nodes and the composite trapezoidal rule with 2 and 10 nodes. Comment on the obtained results.
d. Compute I_1 and I_2 with the composite Simpson rule with 3 and 7 nodes. Comment on the results.
e. Estimate the number of nodes needed to compute I_2 with a tolerance of 10^{-3} using the composite trapezoidal and Simpson rule.
f. Approximate the order of accuracy of the three composite methods.

Exercises

Exercise 9.2

Consider the following integral

$$I = \int_{-1}^{1} \sqrt{1 - x^2}\, dx = \frac{\pi}{2}.$$

a. Compute I using the composite trapezoidal and Simpson rules with a number of nodes $n = 3^k, \quad k = 1, \ldots, 9$.
b. Estimate the order of accuracy of the two methods and comment on the results.

Exercises

Exercise 9.3

Consider the integral

$$I = \int_{-1}^{1} f(x)\, dx$$

and the following quadrature formulas

$$Q_1 = \frac{2}{3}\left[2f\left(-\frac{1}{2}\right) - f(0) + 2f\left(\frac{1}{2}\right)\right],$$

$$Q_2 = \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right].$$

a. Which is the degree of exactness of these formulas? Justify the answer.

b. Use the quadrature formulas above to approximate the integral

$$\int_{1}^{3} \log(x)\, dx.$$

Which is the best one giving in term of accuracy ?

Exercises

Exercise 9.4

Consider the integral

$$I = \int_0^1 x^\alpha \, dx$$

with $\alpha = 1/2,\ 3/2,\ 5/2,\ 7/2$. Consider the composite midpoint, trapezoidal and Simpson rules. For each value of α check if the order of accuracy predicted by the theory of each quadrature formula is obtained and, if not, motivate the different behavior.

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Derivative approximation

1. Compute the exact value of the first order derivative of the function $f(x) = e^{-x}$ at $x_0 = 0.25$.
2. Compute the value of the first order derivative of $f(x)$ at $x_0 = 0.25$ by exploiting the forward, backward and centered finite difference schemes with step $h = 0.05$
3. For each of the above methods compute the corresponding error. What method does provide the smallest error?
4. Consider now the discretization steps $h = 0.2, 0.1, 0.05, 0.025, 0.0125$. Compute the error provided by the three schemes at item 2. Is the numerical trend consistent with the theory?

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Homework 9.1

Consider the quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{4}\left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right] + E(f).$$

a. Which is the degree of exactness of this rule?
b. Is there any quadrature formula with the same degree of exactness that needs a smaller number of functional evaluations?

Homeworks

Homework 9.2

Determine the coefficients α_1, α_2 and α_3 in the quadrature below so that the quadrature formula has a degree of exactness equal to 2

$$I(f) = \int_0^1 f(x)\,dx \simeq \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f^{'}(0) = Q(f).$$

Homeworks

Homework 9.3

Consider an interpolary quadrature formula for the integration of a generic function f in the interval $[-1, 1]$ based on two nodes.

a. What is the maximum achievable degree of exactness? Determine the quadrature nodes x_j and the related weights so that such degree of exactness is ensured.