Numerical Analysis (089180) A.Y. 2022-2023
prof. Simona Perotto Luca Liverotti
simona.perotto@polimi.it luca.liverotti@polimi.it

Lab 5 – Homework Solutions

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A Homework

Homework 5.1. Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 4 & 0 & 2 \\ 2 & 10 & 4 & 0 \\ 1 & 0 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ 10 \\ 1 \end{bmatrix}.$$

a. Compute the solution of the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ using the $\mathbb{LU}$ decomposition with pivoting of matrix $\mathbb{A}$.

b. Find the determinant of $\mathbb{A}$ (use the Matlab command `det` only to compute the determinant of $\mathbb{P}$).

Solution Homework 5.1.

hw_5_1.m

```
clc
clear all
close all

A = [1 2 1 1;
    1 4 0 2;
    2 10 4 0;
    1 0 2 2];

b = [3 3 10 1]';

[L, U] = lu(A);
figure()
spy(L) % L is not lower triangular => pivoting is needed

[L, U, P] = lu(A);
figure()
spy(L) % OK!

y = L\(P*b);
x = U\y

xex = A\b

detA = prod(diag(U))/det(P)
det(A)

% Thanks to the usual Binet formula, we have det(PA) = det(P)det(A) =
% det(LU) = det(L)det(U)
```

Homework 5.2. Consider the system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 10 & 0 & 3 & 0 \\ 0 & 5 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

a. Compute the solution of the system. Use the Cholesky decomposition of matrix $\mathbb{A}$ if possible (check the hypotheses).

b. Find the determinant of $\mathbb{A}$ without using the Matlab commnand `det`.

Solution Homework 5.2.

hw_5_2.m

```
clc
clear all
close all

A = [10 0 3 0;
    0 5 0 -2;
    3 0 5 0;
    0 -2 0 2];

eig(A)>0

% All the eigenvalues are positive => A is spd (the simmetry is trivial).

b = [2 2 2 2]';

H = chol(A)';

% Alternatively you can use your own function MyChol.

y = H\b;
x = H'\y

% Alternatively you can use your own functions for the forward and backward substitution.

xex = A\b

detA = (prod(diag(H)))^2
det(A)

% Since A = H * H^T and det(H) = det(H^T) = product of the diagonal entries of H,
% we have det(A) = det(H) * det(H^T) = det(H)^2 = (product of the diagonal
% entries of H)^2.
```

Homework 5.3. Consider the tridiagonal matrix $\mathbb{A}_n \in \mathbb{R}^{n\times n}$ defined as

$$\mathbb{A}_n = \begin{bmatrix} a & b & & & \\ b & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & b & a & b \\ & & & b & a \end{bmatrix}$$

for $a = 2$ and $b = 1$.

a. Verify with MATLAB that $\mathbb{A}_{10}$ is a symmetric positive definite matrix.

b. Provide the form of the matrix $\mathbb{V}_{10}$ such that $\mathbb{A}_{10} = \mathbb{V}_{10}^T\mathbb{V}_{10}$

Solution Homework 5.3.

hw_5_3.m

```
clc
clear all
close all

a = 2;
b = 1;
n = 10;
A = diag(a * ones(n, 1)) + diag(b * ones(n-1,1), -1) + diag(b * ones(n-1,1), +1);
eig(A)
% All the eigenvalues are positive and it is trivial to prove the simmetry.

V = chol(A)
format short e
V'*V - A
% V is bidiagonal since Cholesky decomposition preserves the pattern of the
% matrix.
```

Homework 5.4. Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 10^{10} & 1 & 1 \\ 10^{10} & 1 & 1 & 10^{10} \\ 1 & 1 & 10^{-10} & 1 \\ 1 & 10^{10} & 1 & 10^{10} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 10^{10}+3 \\ 2\cdot 10^{10}+2 \\ 3+10^{-10} \\ 2\cdot 10^{10}+2 \end{bmatrix}$$

such that the exact solution is $\boldsymbol{x} = [1\ 1\ 1\ 1]^T$.

a. Solve the system using LU decomposition with pivoting

b. Compute the infinity norm of the error associated with the solution of the previous item

c. Can we apply the Cholesky decomposition to matrix $\mathbb{A}$?

d. Is the normalized residual $||\boldsymbol{b} - \mathbb{A}\boldsymbol{x}||/||\boldsymbol{b}||$ a good estimator for the relative error? Motivate your answer.

Solution Homework 5.4.

hw_5_4.m

```
clc
clear all
close all

A = [1  1e10 1   1;
    1e10 1  1    1e10;
    1   1   1e-10 1;
    1   1e10 1   1e10];
x = [1 1 1 1]';
b = [1e10 + 3; 2e10 + 2; 3 + 1e-10; 2e10 + 2];

% solution with LU decomposition without pivoting
[L1, U1] = lu_decomposition(A);
y1 = L1 \ b;
x1 = U1 \ y1

% The computed solution is incorrect! The third component is not 1.

% solution with LU decomposition with pivoting
[L2, U2, P2] = lu(A);
y2 = L2 \ (P2*b);
x2 = U2 \ y2
% The computed solution is ok.

% infinity norm of the error
norm(x - x1, inf)
norm(x - x2, inf)
% As already noted, the best solution is that obtained with the pivoting.

% Cholesky decomposition? -> no, because A is symmetric but not definite
% positive definite, since the eigenvalues are not all positive.
eig(A)

% normalized residual
norm(b - A*x1, inf) / norm(b, inf)
norm(b - A*x2, inf) / norm(b, inf)

% In both cases the normalized residuals are very small.
% Nevertheless the relative error is not small in both cases.
% The normalized residual is a good estimate of the relative error
% only when the condition number of the matrix is small!

cond(A, inf)
```