Lecture #4: Pixels and Filters

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1 Image Sampling and Quantization

1.1 Image Types

Binary Images contain pixels that are either black (0) or white (1).

Grayscale Images have a wider range of intensity than black and white. Each pixel is a shade of gray with pixel values ranging between 0 (black) and 255 (white).

Color Images have multiple color channels; each color image can be represented in different color models (e.g., RGB, LAB, HSV). For example, an image in the RGB model consists of red, green, and blue channel. Each pixel in a channel has intensity values ranging from 0-255. Please note that this range depends on the choice of color model. A 3D *tensor* usually represents color images (Width x Length x 3), where the 3 channels can represent the color model such as RGB (Red-Green-Blue), LAB (Lightness-A-B), and HSV (Hue-Saturation-Value).

1.2 Sampling and Resolution

Images are **samples**: they are not continuous; they consist of discrete pixels of a certain size and density. This can lead to errors (or graininess) because pixel intensities can only be measured with a certain resolution and must be approximated.

Resolution is a sampling parameter, defined in dots per inch (DPI). The standard DPI value for screens is 72 DPI.

Pixels are quantized (i.e, all pixels (or channels of a pixel) have one of a set numbers of values (usually [0, 255]). Quantization and sampling loses information due to a finite precision.

2 Image Histograms

Histograms measure the frequency of brightness within the image: how many times does a particular pixel value appear in an image.

Figure 1: Illustrations of different pixel densities. Taken from the accompanying lecture slides. (Slide 14, slide credit Ulas Bagci)

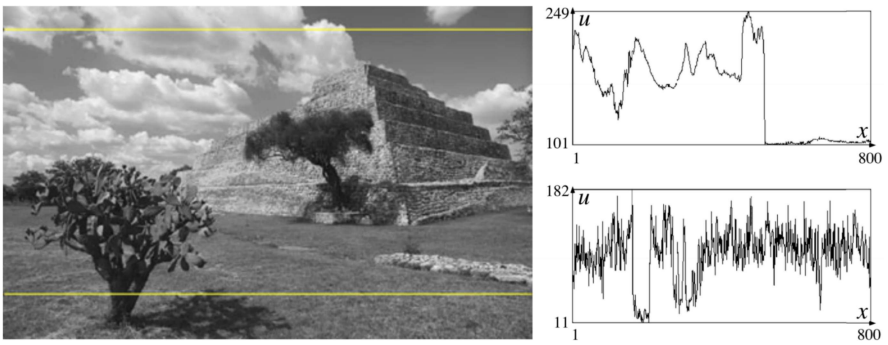


Figure 2: The image is sampled at two vertical positions, sampling a patch of sky and sampling a patch of grass. The corresponding histograms are shown to the right. Adapted from the accompanying lecture slide (Slide 23, slide credit Dr. Mubarak Shah

Histograms help us detect particular features in images, for example:

- Sky: smooth coloration denotes consistency in image, consistent with the image of a sky.
- Grass: a jagged histogram shows wide ranging variety in coloration, consistent with the shadows of a grass field.
- Faces: the color composition of a face will be displayed in a histogram.

An image histogram measures the frequency of certain grayscale intensities in an image. Histograms can be used to provide a quantifiable description of what things look like; this can be used as input to classifiers.

3 Images as Functions

Most images that we deal with in computer vision are digital, which means that they are discrete representations of the photographed scenes. This discretization is achieved through the sampling of 2-dimensional space onto a regular grid, eventually producing a representation of the image as a matrix of integer values.

When dealing with images, we can imagine the image matrix as infinitely tall and wide. However, the displayed image is only a finite subset of this infinite matrix. Having employed such definition of images, we can write them as coordinates in a matrix

$$\begin{bmatrix} \ddots & & \vdots & & \iddots \\ & f[-1,1] & f[0,1] & f[1,1] & \\ \dots & f[-1,0] & f[0,0] & f[0,1] & \dots \\ & f[-1,-1] & f[0,-1] & f[1,-1] & \dots \\ \iddots & & \vdots & & \ddots \end{bmatrix}$$

Figure 3: Graphical representation of a system's mapping of f to g

An Image can also be treated as a function $f : \mathbb{R}^2 \to \mathbb{R}^N$. When we do so, $f[n, m]$ where $f[n, m]$ is the intensity of a pixel at position (m, n). Note that we use square brackets, rather than the typical parentheses, to denote discrete functions.

When we treat an image as a function, it is defined over a rectangle with finite range. For example, the following function f returns the (grayscale) intensity of a single pixel in an image located between a and b horizontally and c and d vertically.

$$f : [a, b] \times [c, d] \to [0, 255] \qquad \text{(Grayscale Pixel Intensity)}$$

The set of values $[a, b] \times [c, d]$ is known as the *domain support*, and contains all values that are valid inputs to the function f, while $[0, 255]$ (in this case) is the range defining the set of possible outputs.

An Image can also be treated as a function mapping $\mathbb{R}^2 \to \mathbb{R}^3$. For example the RGB intensities of a given pixel can be written as the function g below

$$g[x, y] = \begin{bmatrix} r[x, y] \\ g[x, y] \\ b[x, y] \end{bmatrix} \qquad \text{(Color Pixel Intensity)}$$

where $r, g, b : [a, b] \times [c, d] \to [0, 255]$.

4 Linear Systems (Filters)

The term *filtering* refers to a process that forms a new image, the pixel values of which are transformations of the original pixel values. In general, the purpose in applying filters is the extraction of useful information (e.g., edge detection) or the adjustment of an image's visual properties (e.g., de-noising).

Filters are examples of *systems*, which are units that convert an input function $f[m, n]$ to an output (or response) function $g[m, n]$ where m, n are the independent variables. When dealing with images, m, n are the representation of a spatial position in the image.

Notationally, $\mathcal{S}$ is referred to as the *system operator*, which maps a member of the set of possible outputs $g[m, n]$ to a member of the set of possible inputs $f[m, n]$. When using notation involving $\mathcal{S}$, we can write that

$$\mathcal{S}[g] = f$$
$$\mathcal{S}\{f[m, n]\} = g[m, n]$$
$$f[m, n] \xrightarrow{\mathcal{S}} g[m, n]$$

4.1 Examples of Filters

Moving Average

One intuitive example of a filter is the *moving average*. This filter sets the value of a pixel to be the average of its neighboring pixels (e.g., the nine pixels in a 3×3 radius, when applying a 3×3 filter). Mathematically, we can represent this as

$$g[m, n] = \frac{1}{9} \sum_{i=-1}^{1} \sum_{j=-1}^{1} f[m - i, n - j] \qquad \text{(Weighted Average)}$$

This weighted average filter serves to smooth out the sharper edges of the image, creating a blurred or smoothed effect.

Image Segmentation

We can also use filters to perform rudimentary *image segmentation* based on a simple threshold system. In this case, the filter sets the value of a pixel either to an extremely high or an extremely low value, depending on whether or not it meets the threshold t. Mathematically, we write this as

$$g[m,n] = \begin{cases} 255 & f[m,n] \geq t \\ 0 & \text{otherwise} \end{cases} \qquad \text{(Threshold)}$$

This basic image segmentation filter divides an image's pixels into binary classifications of bright regions and dark regions, depending on whether or not the $f[m,n] \geq t$.

4.2 Properties of Systems

When discussing specific systems, it is useful to describe their properties. The following includes a list of properties that a system *may* possess. However, not all systems will have all (or any) of these properties. In other words, these are potential characteristics of individual systems, not traits of systems in general.

Amplitude Properties

- *Additivity*: A system is additive if it satisfies the equation

$$\mathcal{S}[f_i[m,n] + f_j[m,n]] = \mathcal{S}[f_i[m,n]] + \mathcal{S}[f_j[m,n]]$$

- *Homogeneity*: A system is homogeneous if it satisfies the equation

$$\mathcal{S}[\alpha f_i[n,m]] = \alpha \mathcal{S}[f_i[n,m]$$

- *Superposition*: A system has the property of superposition if it satisfies the equation

$$\mathcal{S}[\alpha f_i[n,m]] + \beta f_j[n,m]] = \alpha \mathcal{S}[f_i[n,m]] + \beta \mathcal{S} f_j[n,m]]$$

- *Stability*: A system is stable if it satisfies the inequality

$$|f[n,m]| \leq k \implies |g[n,m]| \leq ck$$

  for some c.

- *Invertibility*: A system is invertible if it satisfies the equation

$$\mathcal{S}^{-1}[\mathcal{S}[f[n,m]]] = f[n,m]$$

Spatial Properties

- *Causality*: A system is causal if for $m < m_0$ and $n < n_0$

$$f[m,n] = 0 \implies g[m,n] = 0$$

- *Shift Invariance*: A system is shift invariant if

$$f[m - m_0, n - n_0] \xrightarrow{\mathcal{S}} g[m - m_0, n - n_0]$$

4.3 Linear Systems

A *linear system* is a system that satisfies the property of superposition. When we employ a linear system for filtering; we create a new image whose pixels are weighted sums of the original pixel values, using the same set of weights for each pixel. A *linear shift-invariant system* is a linear system that is also shift invariant.

Linear systems also have what is known as an *impulse response*. To determine the impulse response of a system $\mathcal{S}$, consider first $\delta_2[m,n]$. This is a function defined as follows

$$\delta_2[m,n] = \begin{cases} 1 & m = 0 \text{and} n = 0 \\ 0 & \text{otherwise} \end{cases}$$

The impulse response r is then simply

$$r = \mathcal{S}[\delta_2]$$

A simple linear shift-invariant system is a system that shifts the pixels of an image, based on the shifting property of the delta function.

$$f[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i,j]\delta_2[m-i, n-j]$$

We can then use the superposition property to write *any* linear shift-invariant system as a weighted sum of such shifting system

$$\alpha_1 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i,j]\delta_{2,1}[m-i, n-j] + \alpha_{2,2} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i,j]\delta_{2,3}[m-i, n-j] + \ldots$$

We then define the filter h of a linear shift-invariant system as

$$h[m,n] = \alpha_1\delta_{2,1}[m-i, n-j] + \alpha_2\delta_{2,2}[m-i, n-j] + \ldots$$

Impulse response (for all linear systems) Delta[n,m] function: has value that is 1 specifically at one pixel, gives back a response, h[n,m] A shifted delta function gives back a shifted response

Linear shift invariant systems (LSI) Example: a moving average filter is a summation of impulse responses

- Systems that satisfy the superposition property
- Have an *impulse response*: $\mathcal{S}[\delta_2[n,m]] = \delta_2[n,m]$
- Discrete convolution: $f[n,m] * h[n,m]$ (multiplication of shifted-version of impulse response by original function)

5 Convolution and Correlation

5.1 Convolution

The easiest way to think of convolution is as a system that uses information from neighboring pixels to filter the target pixel. A good example of this is a moving average, or a box filter discussed earlier.

Convolution is represented by *, for example:
$f[n,m]$*$h[n,m]$ represents a function being multiplied by a shifted impulse response

Convolution allows us to compute the output of passing any input signal through a system simply by considering the impulse response of the system. To understand what this means, we must first understand how to break up a signal into a set of impulse functions.

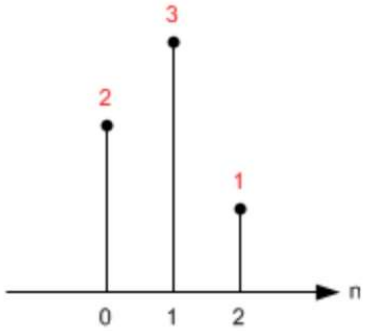
The impulse (delta) function, $\delta[n]$, is defined to be 1 at $n = 0$ and 0 elsewhere. As seen in the image below, any signal can be decomposed as the weighted sum of impulse functions.

More generally, an arbitrary signal x can be written as $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$.

The impulse response of a system, $h[n]$, is defined as the output resulting from passing an impulse function into a system. When a system is linear, scaling the impulse function results in the scaling of the impulse response by the same amount. Moreover, when the system is shift-invariant, shifting the impulse function shifts the impulse response by the same amount. The image below illustrates these properties for a signal consisting of 3 components.

More generally, for an arbitrary input signal $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ passed into a linear, shift-invariant system, the output is $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, i.e. the convolution of the signal x with the impulse response h.

Convolution can also be done in 2 dimensions. For example, when an arbitrary, 2D signal $x[n,m] = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x[i,j]\delta[n-i, m-j]$ is passed into a linear, shift-invariant system, the output is $y[n,m] = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x[i,j]h[n-i, m-j]$, i.e. the convolution of the signal x with the impulse response h in 2 dimensions.



$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]$$

Figure 4: An example decomposition of a signal into an impulse function. (Adapted from http://www.songho.ca/dsp/convolution/convolution.html)

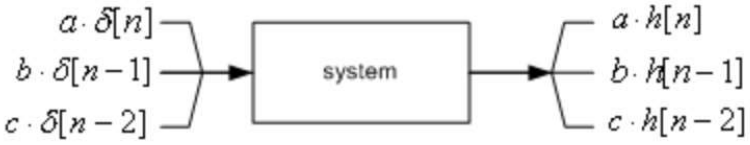


Figure 5: An impulse function is sent through a system to create an impulse response function. Adapted from http://www.songho.ca/dsp/convolution/convolution.html

5.2 Correlation

Cross correlation is the same as convolution, except that the filter kernel is not flipped. Two-dimensional cross correlation is represented as:
$r[k,l] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m+k, n+l]g[m,n]$
It can be used to find known features in images by using a kernel that contains target features.

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