1 Maximum degree of $4\Delta_{avg}$

1.1 Algorithm

For each node i, let's denote :

- α_i its intial degree
- β_i the number of edges this node can help
- γ_i its final degree

We also note L and H the subsets of low-degree and high-degree nodes. The low degree-nodes are the $\frac{n}{4}$ nodes with the lowest degree. We note HO and HI the subsets of high-out and high-in-degree nodes. A node $\in HO$ (HI) if its out-degree (in-degree) is greater than $2\Delta_{avg}$.

Let's choose β_i that way:

- if $i \in L$, then $\beta_i = \lceil \frac{2}{3}\Delta_{avg} - \frac{1}{2}\alpha_i \rceil \in \mathbb{N}$
- if $i \in H$, then $\beta_i = max(\lceil \frac{2}{3}\Delta_{avg} - \frac{1}{6}\alpha_i \rceil, 0) \in \mathbb{N}$

Thanks to helping nodes, we remove all the initial connexions between high-degree nodes. Once it is done we do melhorn trees with the incoming edges of an high-in-degree node and with outgoing edges of an high-out-degree node. Some high-in and high-out-degree nodes may have helped two high degree nodes. In that case we do not include these two nodes in the melhorn tree of the helper.

1.2 Analysis

Let's first prove that if $i \in L$, $\beta_i \in \mathbb{N}$. By definition, β_i is clearly an integer but we can imagine that it could be strictly negative. But we have an upper bound on the degree of i

$$\alpha_i <= \frac{n\Delta_{avg}}{n - n_l}$$

with n_l the number of low degree nodes. In this case $n_l = \frac{n}{4}$ so

$$\alpha_i <= \frac{4}{3}\Delta_{avg}$$

And

$$\frac{2}{3}\Delta_{avg} - \frac{1}{2}\alpha_i >= 0$$

So

$$\beta_i >= 0$$

Let's note B the number of edges to remove and A the number of edges in which a low degree node is involved. So we have :

$$m = A + B$$

Furthermore we have

$$A <= \sum_{i \in L} \alpha_i <= 2A$$

Because the sum on low-degree nodes is between two extreme cases :

- no edges between low degree nodes so $\sum_{i \in L} \alpha_i = A$
- no edges between low and high-degree nodes so $\sum_{i \in L} \alpha_i = 2A$

Then it leads to

$$B <= m - \frac{1}{2} \sum_{i \in L} \alpha_i$$

It means that beeing able to help more than $m - \frac{1}{2} \sum_{i \in L} \alpha_i$ is sufficient to remove all the initial edges between high-degree nodes. Let's find a lower bound on $\sum_i \beta_i$:

- if $i \in L$,

  $$\beta_i >= \frac{2}{3} \Delta_{avg} - \frac{1}{2} \alpha_i$$

  so

  $$\sum_{i \in L} \beta_i >= \frac{2}{3} * \frac{n}{4} \Delta_{avg} - \frac{1}{2} \sum_{i \in L} \alpha_i$$

  $$\sum_{i \in L} \beta_i >= \frac{m}{3} - \frac{1}{2} \sum_{i \in L} \alpha_i$$

- if $i \in H$,

  $$\beta_i >= \frac{2}{3} \Delta_{avg} - \frac{1}{6} \alpha_i$$

  so

  $$\sum_{i \in L} \beta_i >= \frac{2}{3} * \frac{3n}{4} \Delta_{avg} - \frac{1}{6} \sum_{i \in H} \alpha_i$$

  $$\sum_{i \in L} \beta_i >= m - \frac{1}{6} \sum_{i \in H} \alpha_i$$

  And because $\sum_{i \in H} \alpha_i <= 2m$ We have

  $$\sum_{i \in L} \beta_i >= m - \frac{1}{3} m$$

So

$$\sum_{i} \beta_i >= m - \frac{1}{2} \sum_{i \in L} \alpha_i$$

And that way we are able to help enough edges.

Let's evaluate the maximal final degree of the nodes

- if $i \in L$,

$$\gamma_i <= 3(2\beta_i + \alpha_i)$$

  and

$$\beta_i < \frac{2}{3}\Delta_{avg} - \frac{1}{2}\alpha_i + 1$$

  So

$$\gamma_i < 4\Delta_{avg} + 6$$

$$\gamma_i <= 4\Delta_{avg} + 5$$

- if $i \in HO \cap HI$,

$$\gamma_i <= 2 + 6\beta_i$$

  if $\beta_i = 0$, $\gamma_i <= 2$ else

$$\gamma_i < 2 + 6(\frac{2}{3}\Delta_{avg} - \frac{1}{6}\alpha_i + 1) = 8 + 4\Delta_{avg} - \alpha_i$$

  But

$$\alpha_i > 4\Delta_{avg}$$

  So

$$\gamma_i < 8$$

  So

$$\gamma_i <= 7$$

- if $i \in HO \setminus HI$ (same reasoning for $i \in HI \setminus HO$)

  Let's note α_i^1 the initial out-degree of i and α_i^2 the initial in-degree We have :

  1. $\alpha_i = \alpha_i^1 + \alpha_i^2$
  2. $\alpha_i^1 >= 2\Delta_{avg}$
  3. $\alpha_i^2 < 2\Delta_{avg}$

$$\gamma_i <= 1 + \alpha_i^2 + 6\beta_i$$

  if $\beta_i = 0$, $\gamma_i <= 2 + \alpha_i^2 < 2 + 2\Delta_{avg}$ else

$$\gamma_i < 1 + \alpha_i^2 + 6(\frac{2}{3}\Delta_{avg} - \frac{1}{6}\alpha_i + 1) = 8 + 4\Delta_{avg} - \alpha_i^1$$

  But

$$\alpha_i^1 > 2\Delta_{avg}$$

  So

$$\gamma_i < 7 + 2\Delta_{avg}$$

  So

$$\gamma_i <= 6 + 2\Delta_{avg}$$

- if $i \in H \setminus (HI \cup HO)$

$$\gamma_i <= \alpha_i + 6\beta_i$$

because $\alpha_i < 4\Delta_i$, $\beta_i = \lceil \frac{2}{3}\Delta_{avg} - \frac{1}{6}\alpha_i \rceil > 0$ so

$$\gamma_i < \alpha_i + 6(\frac{2}{3}\Delta_{avg} - \frac{1}{6}\alpha_i + 1)$$

$$\gamma_i < 4\Delta_{avg} + 6$$

$$\gamma_i <= 4\Delta_{avg} + 5$$