1 Strong Form of Elasticity using Taylor-Hood elements

The strong form is posed as:

$$\left\{\begin{array}{rcl} \nabla\cdot\sigma+\mathbf{b} & = & 0 \\ & & \\ -\nabla\cdot\mathbf{u} & = & \frac{1}{\kappa}p \end{array}\right. \quad \text{in} \;\; \Omega_3 \tag{1}$$

where $\mathbf{u}$ is the displacement, $\mathbf{b}$ is the body force, κ is the material bulk modulus and σ is the Cauchy stress tensor, which can be decomposed in its deviatoric and hydrostatic counterparts as:

$$\sigma = \sigma' - p\mathbf{I},$$

where p is the hydrostatic pressure, $\mathbf{I}$ is the indentity matrix, ϵ is the strain tensor defined as:

$$\epsilon(\mathbf{u}) = \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2}$$

σ' is the deviatoric stress tensor that depends on the constitutive model adopted. Let's consider three different cases: for 3D elasticity, 2D elasticity and Stokes flows.

$$\left\{\begin{array}{rclc} \sigma' & = & 2\mu\epsilon-\frac{2}{3}(\mu+\lambda)tr\epsilon\mathbf{I} & \text{for 3D elasticity} \\ \sigma' & = & 2\mu\epsilon-\mu tr\epsilon\mathbf{I} & \text{for 2D elasticity} \\ \sigma' & = & 2\mu\epsilon & \text{for 2D/3D stokes} \end{array}\right. \tag{2}$$

with μ and λ standing for Lamé parameters, computed as:

$$\mu = \frac{E}{2(1+\nu)}$$
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

The problem boundary conditions can be stated as:

$$\begin{array}{rclc} \mathbf{u} & = & \mathbf{u}_D & \text{on } \partial\Omega_D \\ \sigma\mathbf{n} & = & \mathbf{g} & \text{on } \partial\Omega_N \end{array} \tag{3}$$

where $\mathbf{u}_D$ is the imposed displacemente and $\mathbf{g}$ is the imposed traction.

It is noted that the material bulk modulus might be different for different cases as defined next:

$$\left\{\begin{array}{rclc} \kappa & = & \lambda+\frac{2}{3}\mu & \text{for 3D elasticity} \\ \kappa & = & \lambda+\mu & \text{for plane strain} \\ \kappa & = & \mu\frac{(2\mu+3\lambda)}{\lambda+2\mu} & \text{for plane stress} \\ \kappa & = & \infty & \text{for 2D/3D stokes} \end{array}\right. \tag{4}$$

One observes that when $\nu \to 0.5$, $\lambda \to \infty$, so $\kappa \to \infty$ for all the aforementioned cases, and a full incompressible formulation is recovered.

2 Weak Form

Define the following spaces:

$$\mathcal{V} = \{\mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \partial\Omega_D\}$$

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega) \mid \mathbf{u} = \mathbf{u}_D \text{ on } \partial\Omega_D\}$$

We multiply the strong form by test functions $v \in \mathcal{V}$ and $q \in L^2(\Omega)$ and pose to find $\mathbf{u} \in \mathcal{U}$ and $p \in L^2(\Omega)$, such that:

$$
\begin{cases} \int_\Omega \mathbf{v}\nabla\cdot\sigma\ \partial\Omega + \int_\Omega \mathbf{v}\cdot\mathbf{b}\ \partial\Omega &= 0 \qquad \forall\mathbf{v}\in\mathcal{V} \\ -\int_\Omega q\nabla\cdot\mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega &= 0 \qquad \forall q\in L^2(\Omega) \end{cases} \tag{5}
$$

Integrating by parts:

$$
\begin{cases} \int_\Omega \nabla\mathbf{v}\cdot\sigma\ \partial\Omega &= \int_{\partial\Omega_N}\mathbf{v}\cdot\mathbf{g}\ d\partial\Omega + \int_\Omega \mathbf{v}\cdot\mathbf{b}\ \partial\Omega \qquad \forall\mathbf{v}\in\mathcal{V} \\ -\int_\Omega q\nabla\cdot\mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega &= 0 \qquad \forall q\in L^2(\Omega) \end{cases} \tag{6}
$$

Plugging the formula for σ:

$$
\begin{cases} \int_\Omega \nabla\mathbf{v}\cdot(2\mu\epsilon - p\mathbf{I})\ \partial\Omega &= \int_{\partial\Omega_N}\mathbf{v}\cdot\mathbf{g}\ d\partial\Omega + \int_\Omega \mathbf{v}\cdot\mathbf{b}\ \partial\Omega \qquad \forall\mathbf{v}\in\mathcal{V} \\ -\int_\Omega q\nabla\cdot\mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega &= 0 \qquad \forall q\in L^2(\Omega) \end{cases} \tag{7}
$$

And rearranging:

$$
\begin{cases} \int_\Omega \nabla\mathbf{v}\cdot 2\mu\epsilon - \nabla\mathbf{v}\cdot p\mathbf{I}\ \partial\Omega &= \int_{\partial\Omega_N}\mathbf{v}\cdot\mathbf{g}\ d\partial\Omega + \int_\Omega \mathbf{v}\cdot\mathbf{b}\ \partial\Omega \qquad \forall\mathbf{v}\in\mathcal{V} \\ -\int_\Omega q\nabla\cdot\mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega &= 0 \qquad \forall q\in L^2(\Omega) \end{cases} \tag{8}
$$

Or, using the strain tensor:

$$
\begin{cases} \int_\Omega \epsilon(\mathbf{v})\cdot 2\mu\epsilon(\mathbf{u}) - p\nabla\cdot\mathbf{v}\ \partial\Omega &= \int_{\partial\Omega_N}\mathbf{v}\cdot\mathbf{g}\ d\partial\Omega + \int_\Omega \mathbf{v}\cdot\mathbf{b}\ \partial\Omega \qquad \forall\mathbf{v}\in\mathcal{V} \\ -\int_\Omega q\nabla\cdot\mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega &= 0 \qquad \forall q\in L^2(\Omega) \end{cases} \tag{9}
$$

Note also that the space of p could be $H^1(\Omega)$ since $H^1(\Omega) \in L^2(\Omega)$.