1 Strong Form

In the case of incompressible material, i.e., $\nu = 0.5$, the strong form is posed as:

$$\left\{ \begin{array}{rcl} \nabla \cdot \sigma + \mathbf{b} & = & 0 \\ & & \\ -\nabla \cdot \mathbf{u} & = & \frac{1}{\kappa} p \end{array} \right. \quad \text{in} \quad \Omega_3 \tag{1}$$

where $\mathbf{u}$ is the displacement, $\mathbf{b}$ is the body force, κ is the material bulk modulus, σ is the Cauchy stress tensor defined as:

$$\sigma = 2\mu\epsilon - p\mathbf{I},$$

where λ and $\mu$are the Lame parameters, $\mathbf{I}$ is the indentity matrix, the pressure p is defined as $p = \frac{1}{3}Tr(\sigma)$ and ϵ is the strain tensor defined as:

$$\epsilon(\mathbf{u}) = \frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}$$

The problem boundary conditions can be stated as:

$$\begin{array}{rclc} \mathbf{u} & = & \mathbf{u}_D & \text{on } \partial\Omega_D \\ \sigma \mathbf{n} & = & \mathbf{g} & \text{on } \partial\Omega_N \end{array} \tag{2}$$

where $\mathbf{u}_D$ is the imposed displacemente and $\mathbf{g}$ is the imposed traction.

It is noted the material bulk modulus might be different for differente dimensions as defined next:

$$\left\{ \begin{array}{rclc} \kappa & = & \lambda + \frac{2}{3}\mu & \text{for 3D} \\ \kappa & = & \lambda + \mu & \text{for plane strain} \\ \kappa & = & \mu\frac{(2\mu+3\lambda)}{\lambda+2\mu} & \text{for plane stress} \end{array} \right. \tag{3}$$

2 Weak Form

Define the following spaces:

$$\mathcal{V} = \{\mathbf{v} \in H^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \partial\Omega_D\}$$

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega) \mid \mathbf{u} = \mathbf{u}_D \text{ on } \partial\Omega_D\}$$

We multiply the strong form by test functions $v \in \mathcal{V}$ and $q \in L^2(\Omega)$ and pose to find $\mathbf{u} \in \mathcal{U}$ and $p \in L^2(\Omega)$, such that:

$$\left\{ \begin{array}{rclc} \int_\Omega \mathbf{v} \nabla \cdot \sigma \, \partial\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{b} \, \partial\Omega & = & 0 & \forall \mathbf{v} \in \mathcal{V} \\ -\int_\Omega q \nabla \cdot \mathbf{u} \, \partial\Omega - \int_\Omega q \frac{1}{\kappa} p \, d\Omega & = & 0 & \forall q \in L^2(\Omega) \end{array} \right. \tag{4}$$

Integrating by parts:

$$\left\{ \begin{array}{rclc} \int_\Omega \nabla \mathbf{v} \cdot \sigma \, \partial\Omega & = & \int_{\partial\Omega_N} \mathbf{v} \cdot \mathbf{g} \, d\partial\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{b} \, \partial\Omega & \forall \mathbf{v} \in \mathcal{V} \\ -\int_\Omega q \nabla \cdot \mathbf{u} \, \partial\Omega - \int_\Omega q \frac{1}{\kappa} p \, d\Omega & = & 0 & \forall q \in L^2(\Omega) \end{array} \right. \tag{5}$$

Plugging the formula for σ:

$$\left\{ \begin{array}{rclc} \int_\Omega \nabla \mathbf{v} \cdot (2\mu\epsilon - p\mathbf{I}) \, \partial\Omega & = & \int_{\partial\Omega_N} \mathbf{v} \cdot \mathbf{g} \, d\partial\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{b} \, \partial\Omega & \forall \mathbf{v} \in \mathcal{V} \\ -\int_\Omega q \nabla \cdot \mathbf{u} \, \partial\Omega - \int_\Omega q \frac{1}{\kappa} p \, d\Omega & = & 0 & \forall q \in L^2(\Omega) \end{array} \right. \tag{6}$$

And rearranging:

$$\left\{ \begin{array}{rclc} \int_\Omega \nabla \mathbf{v} \cdot 2\mu\epsilon - \nabla \mathbf{v} \cdot p\mathbf{I} \, \partial\Omega & = & \int_{\partial\Omega_N} \mathbf{v} \cdot \mathbf{g} \, d\partial\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{b} \, \partial\Omega & \forall \mathbf{v} \in \mathcal{V} \\ -\int_\Omega q \nabla \cdot \mathbf{u} \, \partial\Omega - \int_\Omega q \frac{1}{\kappa} p \, d\Omega & = & 0 & \forall q \in L^2(\Omega) \end{array} \right. \tag{7}$$

Or, using the strain tensor:

$$
\begin{cases}
\int_\Omega \epsilon(\mathbf{v}) \cdot 2\mu\epsilon(\mathbf{v}) - p\nabla \cdot \mathbf{v}\ \partial\Omega & = & \int_{\partial\Omega_N} \mathbf{v} \cdot \mathbf{g}\ d\partial\Omega + \int_\Omega \mathbf{v} \cdot \mathbf{b}\ \partial\Omega & \forall \mathbf{v} \in \mathcal{V} \\
-\int_\Omega q\nabla \cdot \mathbf{u}\ \partial\Omega - \int_\Omega q\frac{1}{\kappa}p\ d\Omega & = & 0 & \forall q \in L^2(\Omega)
\end{cases}
\tag{8}
$$

Note also that the space of p could be $H^1(\Omega)$ since $H^1(\Omega) \in L^2(\Omega)$.