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MULTISCALE HYBRID-MIXED METHODS FOR THE STOKES AND BRINKMAN EQUATIONS – A PRIORI ANALYSIS

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ABSTRACT. The multiscale hybrid-mixed (MHM) method for the Stokes operator was formally introduced in Araya at al. (2017) and numerically validated. The method has face degrees of freedom associated with multiscale basis functions computed from local Neumann problems driven by discontinuous polynomial spaces on skeletal meshes. The two-level MHM version approximates the multiscale basis using a stabilized finite element method. This work proposes the first numerical analysis for the one- and two-level MHM methods applied to the Stokes/Brinkman equations within a new abstract framework relating MHM methods to discrete primal hybrid formulations. As a result, we generalize the two-level MHM method to include general second-level solvers and continuous polynomial interpolation on faces and establish abstract conditions to have those methods well-posed and optimally convergent on natural norms. We apply the abstract setting to analyze the MHM methods using stabilized and stable finite element methods as second-level solvers with (dis)continuous interpolation on faces. Also, we find that the discrete velocity and pressure variables preserve the balance of forces and conservation of mass at the element level. Numerical benchmarks assess theoretical results.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded polyhedron domain with Lipschitz boundary $\partial\Omega$. We consider the generalized Stokes problem, also called Brinkman model, which corresponds to finding the velocity $\boldsymbol{u}$ and the pressure p such that

$$\begin{aligned} -\nu\,\Delta\boldsymbol{u} + \boldsymbol{\theta}\,\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^d$ with $\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{s} = 0$. The viscosity ν is a positive constant, and the reaction coefficient $\boldsymbol{\theta} = \boldsymbol{\theta}(\boldsymbol{x})$ is a semidefinite positive, symmetric tensor which is uniformly elliptic, i.e., there exist constants $c_{min} \geq 0$ and $c_{max} > 0$ such that

$$0 \leq c_{min}\, |\boldsymbol{\xi}|^2 \leq \theta_{min}(\boldsymbol{x})\, |\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^T \boldsymbol{\theta}(\boldsymbol{x})\, \boldsymbol{\xi} \leq \theta_{max}(\boldsymbol{x})\, |\boldsymbol{\xi}|^2 \leq c_{max}\, |\boldsymbol{\xi}|^2, \tag{1.2}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\boldsymbol{x}$ a.e. in Ω, where θ_{min} and θ_{max} are the smallest and largest eigenvalues of $\boldsymbol{\theta}$ ($c_{min} > 0$ if $\boldsymbol{\theta}$ is a definite positive matrix). Also, $\boldsymbol{\theta}$ may contain multiscale geometrical features of the media. We recognize the Stokes problem in (1.1) if $\boldsymbol{\theta} = \mathbf{0}$. The Dirichlet boundary condition in (1.1) is chosen for the sake of the presentation, Neumann or Robin boundary conditions can be easily accommodated in what follows. The Stokes-Brinkman's unique weak solution $(\boldsymbol{u}, p) \in H^1(\Omega)^d \times L^2_0(\Omega)$, with $\boldsymbol{u}\,|_{\partial\Omega} = \boldsymbol{g}$, satisfies

$$\begin{aligned} \int_\Omega (\nu \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + \boldsymbol{\theta}\, \boldsymbol{u} \cdot \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} - \int_\Omega p\, \nabla \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \quad \text{for all } \boldsymbol{v} \in H^1_0(\Omega)^d, \\ \int_\Omega q\, \nabla \cdot \boldsymbol{u}\, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } q \in L^2_0(\Omega), \end{aligned} \tag{1.3}$$

and there exists positive C, dependent on ν, $\boldsymbol{\theta}$ and Ω, such that

$$\|\boldsymbol{u}\|_{1,\Omega} + \|p\|_{0,\Omega} \,\leq\, C\, \left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{1/2,\partial\Omega} \right).$$

Given any measurable set $D \subset \overline{\Omega}$ and any integer $m \geq 0$, we respectively denote by $|\cdot|_{m,D}$ and $\|\cdot\|_{m,D}$ the standard semi-norm and norm in $H^m(D)$. The $L^2_0(D)$ space stands for the functions in $L^2(D) := H^0(D)$ with zero mean value in D. We use the convention of denoting vector-valued functions and spaces in bold. Hereafter, we shall denote by C a positive constant, which is independent of any partition (mesh) parameter but can change in each occurrence.

When attempting to approximate solutions to problem (1.3), one must deal with some (potential) numerical instabilities. First, one must either choose compatible approximation spaces (in the sense of the inf–sup condition [19]) or use stabilized numerical methods that overcome this incompatibility [53]. Furthermore, in the case of a dominant reaction term, numerical methods must be robust with respect to vanishing diffusion coefficient due to the singularly perturbed nature of the solutions characterized by boundary layers. Without such care, spurious non-physical oscillations may plague the numerical approximation. Beyond these, solutions may exhibit poor approximation properties arising from the insufficient resolution of the physics occurring at multiple scales coming through heterogeneous and or high contrast coefficients. Finally, it is desirable that numerical methods for the Stokes/Brinkman problem produce discrete velocity fields respecting the local mass balance.

Theoretically, the issue involving multiple scales and boundary layers may be overcome by choosing a mesh whose size is smaller than the smallest relevant scale for the physics.

Practically, this can lead to problem statements that are intractable computationally. For this reason, multi-scale methods are proposed to correctly incorporate the effect of subscales (including boundary layers) into a numerical scheme proposed on a relatively coarse partition. Such methods use a form of upscaling to incorporate sub-scales within the scheme on the coarse mesh. This idea underlies the Variational Multiscale Method (VMS) [52], Residual-Free Bubbles (RFB) [62, 23] and other enriched finite element methods as the Residual Local Projection (RELP) method [15] and the Petrov-Galerkin Enriched (PGEM) method [4]. See [33] for an overview. Some of these multiscale methods are directly related to stabilized finite element methods, as pointed out in [22, 23, 14], for example, and they deal with issues related to the inf-sup condition and the approximation of boundary layers at the same time.

The idea of multiscale finite element methods goes back to [12] where the multiscale basis concept was introduced and analyzed first in one dimension for a highly oscillatory coefficient problem. This seminal work was later extended to higher dimensions in the form of the MsFEM [51]. Following such an idea, other multiscale methods have been proposed for various operators, including the heterogeneous multiscale method [36], localizable orthogonal decomposition [57], the sub-grid finite element method [9], and the generalized multiscale finite element method [37]. More recently, multiscale methods have been devised for Stokes and Brinkmann problems [1, 5, 44], with discontinuous methods which are proving effective in highly heterogeneous media context [24, 50, 54].

The Multiscale Hybrid-Mixed (MHM) finite element method fits with relatively recent attempts to use hybridization to resolve multiple scales [10, 38, 64]. First analyzed in the context of the Poisson problem [60], hybrid methods for various operators are intrinsically related to the domain decomposition methods [3, 25, 26], the discontinuous enriched method (DEM) [40], and strategies to reduce computational cost related to saddle point problems [11, 32]. As for the MHM method, it was proposed for the first time for the Poisson (Darcy) equation in [47] with an a priori and a posteriori error analysis developed in [6, 16, 58, 59]. The methodology was extended to other operators, such as the linear elasticity model in [46, 45], the complete transport equation in [48], and wave problems in [55, 30]. Furthermore, the MHM method is closely related to other multiscale methods, such as the MsHHO [28] and the multiscale version of the Crouzeix-Raviart element [56], and retrieves the lowest-order Raviart-Thomas element in the homogeneous coefficient case [47]. In its two-level version, it is related to the FETI domain decomposition approach [41] and can also be seen as the dual version of the MsFEM and the multiscale mortar methods [10] (see also [20] for a recent primal version of the multiscale mortar method).

In a companion paper [7] to the present work, the MHM method was extended to the Stokes/Brinkman problem (1.1), and a residual a posteriori error estimator was proposed in [8]. The philosophy underlying the MHM method involves using a hybrid form of (1.1) posed on a coarse partition. The approach decomposes the exact solution into local and global coupled systems, and discretization decouples the system as follows: the global formulation is responsible for the degrees of freedom over the skeleton of the coarse partition, and the local problems driven by upscaling operators T and $\widehat{T}$ provide the multiscale basis functions resolving sub-scales. To fix ideas, consider $\mathscr{P}_{\mathcal{H}}$ a general partition of a domain Ω with characteristic length $\mathcal{H}$ decomposed into elements K with contour ∂K. The skeleton of $\mathscr{P}_{\mathcal{H}}$, denoted by $\partial\mathscr{P}_{\mathcal{H}}$, is partitioned with elements of diameter H. Remember (c.f. [7]) that the one-level MHM method for the Stokes equation consists of finding a flow $\boldsymbol{\lambda}_H$ in a polynomial space $\boldsymbol{\Lambda}_H$ defined on the skeleton of $\mathscr{P}_{\mathcal{H}}$, a velocity $\boldsymbol{u}_0^H$ in $\mathbf{V}_0$, the piecewise constant space over $\mathscr{P}_{\mathcal{H}}$, and $\rho \in \mathbb{R}$ such that

$$
\begin{aligned}
\sum_{K\in\mathscr{P}_{\mathcal{H}}} \int_{\partial K} \boldsymbol{\lambda}_H \cdot \boldsymbol{v}_0 \, \mathrm{d}\boldsymbol{s} &= \sum_{K\in\mathscr{P}_{\mathcal{H}}} \int_K \boldsymbol{f} \cdot \boldsymbol{v}_0 \, \mathrm{d}\boldsymbol{x} \quad \text{for all } \boldsymbol{v}_0 \in \mathbf{V}_0, \\
\sum_{K\in\mathscr{P}_{\mathcal{H}}} \int_{\partial K} \boldsymbol{\mu}_H \cdot \boldsymbol{u}_H \, \mathrm{d}\boldsymbol{s} &= \int_{\partial\Omega} \boldsymbol{\mu}_H \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{s} \quad \text{for all } \boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H, \\
\xi \int_\Omega p_H \, \mathrm{d}\boldsymbol{x} &= 0 \quad \text{for all } \xi \in \mathbb{R},
\end{aligned}
\tag{1.4}
$$

where the discrete velocity and pressure variables $(\boldsymbol{u}_H, p_H)$ in (1.4) depend on the coarse scale variables $(\boldsymbol{u}_0^H, \boldsymbol{\lambda}_H, \rho)$ in the form

$$
(\boldsymbol{u}_H, p_H) := (\boldsymbol{u}_0^H, 0) + T(\boldsymbol{\lambda}_H, \rho) + \widehat{T}(\boldsymbol{f}). \tag{1.5}
$$

The fine scales are resolved via the locally defined bounded mappings T and $\widehat{T}$, which have finite-dimensional images in the local product space $[H^1(K) \cap L^2_0(K)]^d \times L^2(K)$. Specifically, the mappings T and $\widehat{T}$ are inverses of well-posed Stokes problems with prescribed Neuman boundary conditions on each $K \in \mathscr{P}_{\mathcal{H}}$. As a result, the MHM method (1.4) is non-conforming in $H^1(\Omega)$ as the discrete velocity in (1.5) does not belong to the $H^1_0(\Omega)$ space. Note that the second equation in (1.4) reinforces the weak continuity of the velocity field in the boundary elements, while the third equation guarantees the uniqueness of the pressure. The first equation in (1.4) imposes that the numerical flux is in local equilibrium with external force, which is a consequence of the spatial decomposition adopted in the construction of the MHM method (see [7] and Section 3 for more details). Furthermore, the discrete velocity field $\boldsymbol{u}_H$ is locally divergent free, i.e.,

$$\nabla \cdot \boldsymbol{u}_H = 0 \quad \text{in all } K \in \mathscr{P}_{\mathcal{H}}.$$

In [7], the MHM method was presented along with numerical results involving a stabilized finite element method to approximate local problems, but numerical analysis was absent. The use of these local numerical schemes is viewed as forming approximations of multiscale basis functions and leads to two-level methods. In the context of [7], the choice of the unusual stabilized finite element method (USFEM) [14] intended to take advantage of its well-known robustness to approximate boundary layers that can be presented in the multiscale basis while making the pair of polynomials spaces of the same order available to approximate both pressure and velocity multiscale basis functions. The efficiency of this choice has been extensively verified in [7] numerically.

In the present paper, we fill this theoretical gap and propose the first numerical analysis for the one- and two-level MHM methods applied to the Stokes/Brinkman equations. For example, we prove that (1.4) is well-posed and the error due to exact flow approximation $\boldsymbol{\lambda}_H$ of $\boldsymbol{\lambda} := \nu \nabla \boldsymbol{u} \cdot \boldsymbol{n}^K - p\, \boldsymbol{n}^K$ on ∂K turns out to be an upper bound for the error associated with velocity and pressure approximations, where $\boldsymbol{n}^K$ is the outward normal vector on ∂K for all $K \in \mathscr{P}_{\mathcal{H}}$. Notably, we prove (see Theorem 3.2 in the context of the one-level MHM method)

$$\|\boldsymbol{u} - \boldsymbol{u}_H\|_{1,\mathscr{P}_{\mathcal{H}}} + \|p - p_H\|_{0,\Omega} \leq C \inf_{\substack{\boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H \\ (\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \boldsymbol{v}_0)_{\partial \mathscr{P}_{\mathcal{H}}} = 0 \ \forall \boldsymbol{v}_0 \in \mathbf{V}_0}} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_H\|_{\boldsymbol{\Lambda}},$$

where $\| \cdot \|_{\boldsymbol{\Lambda}}$ is a norm in $\boldsymbol{\Lambda}$ and $(\cdot, \cdot)_{\partial \mathscr{P}_{\mathcal{H}}}$ means a dual product on the skeleton $\partial \mathscr{P}_{\mathcal{H}}$ (see Section 2.2 for precisions). So, convergence arises by bringing approximability properties to the definition of the finite-dimensional space $\boldsymbol{\Lambda}_H \subset \boldsymbol{\Lambda}$. This will be the subject of Section 4. Furthermore, under local regularity assumptions for the exact velocity and pressure variables, we prove that the two-level MHM solution in [7] converges in the energy norm with rate $O(H^{\ell+1} + h^k)$, where $\ell \geq 0$ is the polynomial degree of interpolation on faces used in $\boldsymbol{\Lambda}_H$, and h is the local submesh diameter and $k \geq 1$ is the polynomial degree of interpolation on sub-meshes (see Theorem 5.2). This way, convergence can be achieved by keeping the macroelement mesh fixed (i.e., $\mathcal{H}$ remains fixed). This property, in addition to its practical interest in avoiding remeshing of complex geometries to improve accuracy, leads to super-convergence with additional convergence order $O(H^{1/2})$ under smoother local exact solution assumption (see Remark 4.3). This is also verified numerically in Section 6.

In a broad sense, the proof strategy is as follows: we first establish a relationship between the primal-hybrid version of (1.3) and the MHM method in an abstract general sense, showing that the well-posedness of both are related at the fully discrete level and also establish how their stability constants are related. Then, the numerical analysis of the MHM method proposed in [7] arises from fulfilling the conditions to have the primal-hybrid version of (1.3) well-posed. Thereby, the overall analysis in this work follows a totally different path from that used in previous MHM works, which is fundamental to prove the existence and uniqueness and the best approximation property of the two-level MHM method with the USFEM method used to approximate T and $\widehat{T}$ in (1.5). In a sense, such a perspective is also presented in previous works on MHM-type methods applied to the Poisson problem [49, 28, 13].

The correspondence between the solution of the discrete primal-hybrid methods and the two-level MHM methods also produces an interesting reinterpretation of the two-level MHM methods. When the stabilized finite element method (USFEM) is used as a second-level solver on one-element sub-meshes, the corresponding discrete primal-hybrid method can be seen as a *new member* of the class of non-conforming stabilized finite element methods. Stabilization then acts at the element level, which has a global impact by weakly imposing continuity on the boundary elements (see [27, 2] for other examples of non-conforming stabilized methods). As a result, local mass is conserved, which is a property missing from conforming-stabilized methods in general (see [15] for a post-processing strategy to recover this property in the conforming scenario). Furthermore, the use of refined sub-meshes in this context can be interpreted as a *multiscale version* of non-conforming stabilized methods. A similar interpretation arises if we replace the stabilized second-level solver with the Galerkin method based on stable pairs of spaces. For example, when adopting the lowest-degree Taylor-Hood element into the scope of the two-level MHM method with single-element sub-meshes, this corresponds to nothing more than the non-conforming Crouzeix-Raviart method [35]. Thus, with the adoption of refined sub-meshes in the context of the stable two-level MHM method, a non-conforming multiscale version of the Crouzeix-Raviart method emerges.

In addition to providing a numerical analysis for the one- and two-level MHM methods introduced in [7], in this work we

- (i) establish abstract conditions for the well-posedness and optimality of MHM methods for the Stokes/Brinkman model in natural norms. Such conditions allow extending the analysis to more general second-level solvers than the stabilized method used in [7];

(*ii*) extend the two-level MHM analysis to include a stable second-level solver and continuous polynomial interpolation for the flux variable. This version of the two-level MHM method is new, and we prove that the method also achieves optimal convergence using the abstract conditions of item (*i*). This version of the two-level MHM method produces point-wise divergence-free exact discrete velocity when it adopts one-element sub-meshes;

(*iii*) validate numerically the MHM method using continuous interpolation on faces. The results indicate that interpolating the flux continuously improves convergence when compared to the discontinuous case, at least for regular exact solutions. The use of continuous interpolation within the MHM methodology is new, even in the context of other operators.

We also provide some numerical verification about the dependence of the constant in the error estimates in terms of the physical coefficients, but we leave this theoretical question outside the scope of this work.

The remainder of the paper is organized as follows. We present the hybrid framework for problem (1.1) in Section 2, which is leveraged in Section 3 to present and analyze an abstract MHM method. This setting is used in Section 4 to establish that the MHM method, on closed sub-spaces, is well-posed and provides best approximation results under the assumption that the second level is exact. In Section 5, we apply the abstraction in the context of stable and stabilized finite element methods used at the second level to approximate multiscale basis functions. Numerical tests assessing theoretical results are presented in Section 6, with conclusions in Section 7. A technical result is included in the appendix section.

2. Hybridization

The MHM methods are built on reformulating a hybridized version of (1.1). In a broad sense, given spaces $\mathbf{V}$, Q, and $\mathbf{\Lambda}$ and (bi)linear forms $f : \mathbf{V} \times Q \to \mathbb{R}$, $g : \mathbf{\Lambda} \times \mathbb{R} \to \mathbb{R}$, $a : (\mathbf{V} \times Q) \times (\mathbf{V} \times Q) \to \mathbb{R}$, and $b : (\mathbf{\Lambda} \times \mathbb{R}) \times (\mathbf{V} \times Q) \to \mathbb{R}$, the hybrid formulation takes the form: Find $(\boldsymbol{u}, p) \in \mathbf{V} \times Q$ and $(\boldsymbol{\lambda}, \rho) \in \mathbf{\Lambda} \times \mathbb{R}$ such that

$$\begin{aligned} a(\boldsymbol{u}, p; \boldsymbol{v}, q) + b(\boldsymbol{\lambda}, \rho; \boldsymbol{v}, q) &= f(\boldsymbol{v}, q), \\ b(\boldsymbol{\mu}, \xi; \boldsymbol{u}, p) &= g(\boldsymbol{\mu}, \xi), \end{aligned} \tag{2.1}$$

for all $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$ and $(\boldsymbol{\mu}, \xi) \in \mathbf{\Lambda} \times \mathbb{R}$. The hybrid forms in (2.1) arise working in the space $\mathbf{V}$, which contains $H^1(\Omega)^d$, as well as functions that satisfy more flexible conditions. The burden of enforcing the missing conditions is then borne by the action of the bilinear form $b(\cdot;\cdot)$ defined over spaces appropriate to enforce $\boldsymbol{u} \in H^1(\Omega)^d$. The remainder of this section is dedicated to defining this setting precisely and establishing the equivalence of (1.3) and (2.1). This will provide a solid basis for defining the MHM formulation of (1.3) in Section 3.

2.1. **Partitions.** Consider a family of partitions $\{\mathscr{P}_{\mathcal{H}}\}_{\mathcal{H}>0}$ of Ω parameterized by $\mathcal{H} := \max_{K\in\mathscr{P}_{\mathcal{H}}} h_K$, where h_K is the diameter of simplex elements K. The collection of all element boundaries ∂K is denoted $\partial\mathscr{P}_{\mathcal{H}}$. Without loss of generality, we shall use hereafter the terminology employed for three-dimensional domains. The collection of all faces E in the triangulations, with diameter h_E, is denoted $\mathcal{E}$. This set is decomposed into the set of faces on $\partial\Omega$ denoted $\mathcal{E}_\partial$, and its complement $\mathcal{E}_0$. To each $E \in \mathcal{E}$, a normal $\boldsymbol{n}$ is associated, taking care to ensure this is directed outward on $\partial\Omega$. For each $K \in \mathscr{P}_{\mathcal{H}}$ we collect local faces of $E \subset \partial K$ in the set $\mathcal{E}^K$, while denoting the outward normal on ∂K by $\boldsymbol{n}^K$ and defining $\boldsymbol{n}^K_E := \boldsymbol{n}^K|_E$ for each $E \subset \mathcal{E}^K$. Also, we denote by $\{\mathcal{E}_H\}_{H>0}$ a regular family of simplicial partitions of $\mathcal{E}$, where $H := \max_{F\in\mathcal{E}_H} h_F$ and h_F is the diameter of $F \in \mathcal{E}_H$. We collect the faces of $F \subset \partial K$ in the set $\mathcal{E}^K_H$.

2.2. **Broken spaces and norms.** Given a partition $\mathscr{P}_{\mathcal{H}}$ in $\{\mathscr{P}_{\mathcal{H}}\}_{\mathcal{H}>0}$, we adopt the notation

$$Q := L^2(\Omega) \quad \text{and} \quad \mathbf{V} := \{\boldsymbol{v} \in L^2(\Omega)^d \;:\; \boldsymbol{v}_K \in H^1(K)^d \quad \forall K \in \mathscr{P}_{\mathcal{H}}\}, \tag{2.2}$$

where we denote $\boldsymbol{v}_D := \boldsymbol{v}|_D$ with $D \subset \overline{\Omega}$ a measurable set, and

$$\boldsymbol{\Lambda} := \left\{\boldsymbol{\mu} \in \Pi_{K\in\mathscr{P}_{\mathcal{H}}} H^{-1/2}(\partial K)^d \;:\; \boldsymbol{\mu}_{\partial K} = \boldsymbol{\sigma}_K\, \boldsymbol{n}^K|_{\partial K} \quad \forall K \in \mathscr{P}_{\mathcal{H}} \text{ and } \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)\right\}, \tag{2.3}$$

and

$$\boldsymbol{\Sigma} := \left\{\boldsymbol{\zeta} \in \Pi_{K\in\mathscr{P}_{\mathcal{H}}} H^{1/2}(\partial K)^d \;:\; \boldsymbol{\zeta}_{\partial K} = \boldsymbol{v}_K|_{\partial K} \quad \forall K \in \mathscr{P}_{\mathcal{H}} \text{ and } \boldsymbol{v} \in H^1_0(\Omega)^d\right\}. \tag{2.4}$$

Also, we define the broken gradient operator $\nabla_{\mathcal{H}} : \mathbf{V} \to L^2(\Omega)^{d\times d}$ as such, for all $\boldsymbol{v} \in \mathbf{V}$,

$$(\nabla_{\mathcal{H}}\boldsymbol{v})|_K := \nabla\boldsymbol{v}_K \quad \text{for all } K \in \mathscr{P}_{\mathcal{H}}.$$

Given $\boldsymbol{w}$, $\boldsymbol{v} \in L^2(\Omega)^d$, and owing to the notation

$$(\boldsymbol{w}, \boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}} := \sum_{K\in\mathscr{P}_{\mathcal{H}}} (\boldsymbol{w}_K, \boldsymbol{v}_K)_K,$$

where $(\cdot,\cdot)_D$ stands for the $L^2(D)^d$ inner product, we equip $\mathbf{V}$ with the norm $\|\cdot\|_{\mathbf{V}}$ induced by the inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{\mathbf{V}} := d_\Omega^{-2}(\boldsymbol{w}, \boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}} + (\nabla_{\mathcal{H}}\boldsymbol{w}, \nabla_{\mathcal{H}}\boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}}, \tag{2.5}$$

where d_Ω is the diameter of Ω. Further, we equip Q with the induced norm

$$\|\cdot\|_Q^2 := (\cdot,\cdot)_Q = (\cdot,\cdot)_{\mathscr{P}_\mathcal{H}}.$$

We denote $(\cdot,\cdot)_{\mathbf{V}\times Q} := (\cdot,\cdot)_\mathbf{V} + (\cdot,\cdot)_Q$ the inner-product on the product space $\mathbf{V}\times Q$, which induces the following norm in $\mathbf{V}\times Q$

$$\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}^2 := \|\boldsymbol{v}\|_\mathbf{V}^2 + \|q\|_Q^2 \quad \text{for all } (\boldsymbol{v},q)\in\mathbf{V}\times Q. \tag{2.6}$$

We equip the space $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{\Lambda}$ with the norms,

$$\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div};\Omega)}^2 := \sum_{K\in\mathscr{P}_\mathcal{H}} \big(\|\boldsymbol{\sigma}\|_{0,K}^2 + d_\Omega^2\,\|\nabla\cdot\boldsymbol{\sigma}\|_{0,K}^2\big), \tag{2.7}$$

$$\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} := \inf_{\substack{\boldsymbol{\sigma}\in\boldsymbol{H}(\mathrm{div};\Omega)\\ \boldsymbol{\sigma}_{\partial K}\boldsymbol{n}^K|_{\partial K}=\boldsymbol{\mu}_{\partial K},\,K\in\mathscr{P}_\mathcal{H}}} \|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\mathrm{div};\Omega)}, \tag{2.8}$$

and the space $\boldsymbol{\Sigma}$ with the norm

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{\Sigma}} := \inf_{\substack{\boldsymbol{v}\in H_0^1(\Omega)^d\\ \boldsymbol{v}_{\partial K}=\boldsymbol{\zeta}_{\partial K},\,K\in\mathscr{P}_\mathcal{H}}} \|\boldsymbol{v}\|_{1,\Omega}.$$

Also, we define the following product norm

$$\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}^2 := \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}^2 + \|\xi\|_Q^2 \quad \text{for all } (\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}\times\mathbb{R}. \tag{2.9}$$

The duality pairing between $H^{-1/2}(\partial K)^d$ and $H^{1/2}(\partial K)^d$ is denoted by $\langle\cdot,\cdot\rangle_{\partial K}$, and we define, for $\boldsymbol{\mu}\in\Pi_{K\in\mathscr{P}_\mathcal{H}}H^{-1/2}(\partial K)$ and $\boldsymbol{\zeta}\in\Pi_{K\in\mathscr{P}_\mathcal{H}}H^{1/2}(\partial K)$,

$$(\boldsymbol{\mu},\boldsymbol{\zeta})_{\partial\mathscr{P}_\mathcal{H}} := \sum_{K\in\mathscr{P}_\mathcal{H}} \langle\boldsymbol{\mu}_{\partial K},\boldsymbol{\zeta}_{\partial K}\rangle_{\partial K}.$$

Observe that if $\boldsymbol{\mu}\in\boldsymbol{\Lambda}$ and $\boldsymbol{\zeta}\in\boldsymbol{\Sigma}$, then $(\boldsymbol{\mu},\boldsymbol{\zeta})_{\partial\mathscr{P}_\mathcal{H}}=0$ (c.f. [7, Lemma 4]). Also, consider closed (not necessarily finite) subspaces

$$\boldsymbol{\Lambda}_s := \sum_{K\in\mathscr{P}_\mathcal{H}}\boldsymbol{\Lambda}_s(\partial K), \quad \mathbf{V}_s := \sum_{K\in\mathscr{P}_\mathcal{H}}\mathbf{V}_s(K) \quad \text{and} \quad Q_s := \sum_{K\in\mathscr{P}_\mathcal{H}} Q_s(K),$$

of spaces $\boldsymbol{\Lambda}$, $\mathbf{V}$ and Q, respectively, where $\boldsymbol{\Lambda}_s(\partial K)$, $\mathbf{V}_s(K)$ and $Q_s(K)$ are respectively closed subspaces of $H^{-1/2}(\partial K)^d$, $H^1(K)^d$ and $L^2(K)$.

2.3. **(Bi)linear Forms.** We now define the bilinear forms $a(\cdot;\cdot)$ and $b(\cdot;\cdot)$. Given, $\boldsymbol{w}$, $\boldsymbol{v} \in \mathbf{V}$ and q, $r \in Q$, and $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}$

$$
\begin{aligned}
a(\boldsymbol{w}, r;\, \boldsymbol{v}, q) \;&:=\; \sum_{K \in \mathscr{P}_{\mathcal{H}}} a_K(\boldsymbol{w}_K, r_K;\, \boldsymbol{v}_K, q_K),\\
b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q) \;&:=\; \sum_{K \in \mathscr{P}_{\mathcal{H}}} b_K(\boldsymbol{\mu}_{\partial K}, \xi_K;\, \boldsymbol{v}_{\partial K}, q_K),
\end{aligned}
$$

where,

$$
\begin{aligned}
a_K(\boldsymbol{v}_K, q_K;\, \boldsymbol{w}_K, r_K) \;&:=\; (\nu\, \nabla \boldsymbol{v}_K, \nabla \boldsymbol{w}_K)_K + (\boldsymbol{\theta}\, \boldsymbol{v}_K, \boldsymbol{w}_K)_K\\
&\qquad - (q_K, \nabla \cdot \boldsymbol{w}_K)_K + (\nabla \cdot \boldsymbol{v}_K, r_K)_K,\\
b_K(\boldsymbol{\mu}_{\partial K}, \xi_K;\, \boldsymbol{v}_{\partial K}, q_K) \;&:=\; \langle \boldsymbol{\mu}_{\partial K}, \boldsymbol{v}_{\partial K} \rangle_{\partial K} + (\xi_K, q_K)_K.
\end{aligned}
\tag{2.10}
$$

Furthermore, the linear forms $f(\cdot)$ and $g(\cdot)$ are defined by

$$
f(\boldsymbol{v}, q) := (\boldsymbol{f}, \boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}} \quad \text{and} \quad g(\boldsymbol{\mu}, \xi) := (\boldsymbol{\mu}, \boldsymbol{g})_{\partial\Omega}. \tag{2.11}
$$

Observe that since $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^d$, there exist constants $C > 0$ independent of the mesh $\mathscr{P}_{\mathcal{H}}$, such that $f(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}} \le C \|\boldsymbol{v}, q\|_{\mathbf{V}\times Q}$ for all $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$ and $g(\boldsymbol{\mu}, \xi) = (\boldsymbol{\mu}, \boldsymbol{g})_{\partial\Omega} \le C \|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q}$ for all $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}$. We denote by $\|f\|$ and $\|g\|$ the smallest possible of such constants. Similarly, bilinear form $a(\cdot;\cdot)$ is uniformly bounded over all $(\boldsymbol{v}, q)$, $(\boldsymbol{w}, r) \in \mathbf{V} \times Q$, and then

$$
\|a\| := \sup_{(\boldsymbol{v},q)\in\mathbf{V}\times Q}\; \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{a(\boldsymbol{v}, q;\, \boldsymbol{w}, r)}{\|\boldsymbol{v}, q\|_{\mathbf{V}\times Q} \|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} < \infty. \tag{2.12}
$$

Above and hereafter, we lighten notation and understand the supremum to be taken over sets excluding the zero function, even though this is not specifically indicated. The nullspace of $a(\cdot;\cdot)$ will play a critical role in the analysis. We define

$$
\mathscr{N}_a := \{(\boldsymbol{v}, 0) \in \mathbf{V}_s \times Q_s \,:\, (\boldsymbol{v}_K, 0) \in \mathscr{N}_a(K) \quad \forall K \in \mathscr{P}_{\mathcal{H}}\},
$$

where

$$
\mathscr{N}_a(K) := \left\{(\boldsymbol{v}, q) \in H^1(K)^d \times L^2(K) \,:\, a_K(\boldsymbol{v}, q;\, \boldsymbol{w}, r) = 0 \quad \forall (\boldsymbol{w}, r) \in H^1(K)^d \times L^2(K)\right\},
$$

and their orthogonal complements are

$$
\begin{aligned}
\mathscr{N}_a(K)^{\perp} &:= \left\{(\boldsymbol{v}, q) \in H^1(K)^d \times L^2(K) \,:\, (\boldsymbol{v}, q;\, \boldsymbol{w}, r)_{\mathbf{V}\times Q} = 0 \quad \forall (\boldsymbol{w}, r) \in \mathscr{N}_a(K)\right\},\\
\mathscr{N}_a^{\perp} &:= \left\{(\boldsymbol{v}, q) \in \mathbf{V} \times Q \,:\, (\boldsymbol{v}_K, q_K) \in \mathscr{N}_a(K)^{\perp} \quad \forall K \in \mathscr{P}_{\mathcal{H}}\right\}.
\end{aligned}
\tag{2.13}
$$

We notice that, based on the space $\mathscr{N}_a(K)$, the following characterisations hold

$$\underline{\text{Stokes model}}: \quad \mathscr{N}_a(K) = \mathbb{P}_0(K)^d \times \{0\} \text{ and } \mathscr{N}_a^{\perp}(K) = [H^1(K) \cap L^2_0(K)]^d \times L^2(K);$$

$$\underline{\text{Brinkman model}}: \quad \mathscr{N}_a(K) \text{ is trivial and } \mathscr{N}_a^{\perp}(K) = H^1(K)^d \times L^2(K),$$

where $\mathbb{P}_0(K)^d$ is the constant vector function space in $K \in \mathscr{P}_{\mathcal{H}}$.

2.4. **The well-posedness of hybrid problem** (2.1)**.** With the previous definitions, we establish that hybrid formulation (2.1) of (1.1) is well-posed.

Lemma 2.1. *Let* $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}$ *and* $(\boldsymbol{v}, q) \in \mathscr{N}_b$, *where*

$$\mathscr{N}_b := \{(\boldsymbol{v}, q) \in \mathbf{V} \times Q \,:\, b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q) = 0 \quad \text{for all } (\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}\}\,. \tag{2.14}$$

Then, there exists a positive constant α_b, *independent of* $\mathcal{H}$, *such that*

$$\begin{aligned} \alpha_b \|\boldsymbol{v}, q\|_{\mathbf{V}\times Q} &\le \sup_{(\boldsymbol{w}, r)\in\mathscr{N}_b} \frac{a(\boldsymbol{v}, q;\, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} \quad \text{for all } (\boldsymbol{v}, q) \in \mathscr{N}_b, \\ \|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q} &= \sup_{(\boldsymbol{w}, r)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu}, \xi;\, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} \quad \text{for all } (\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}. \end{aligned} \tag{2.15}$$

Moreover, hybrid formulation (2.1) *is well-posed and*

$$\begin{aligned} \|\boldsymbol{u}, p\|_{\mathbf{V}\times Q} &\le \qquad \frac{1}{\alpha_b}\|f\| + \left(1 + \frac{\|a\|}{\alpha_b}\right)\|g\|, \\ \|\boldsymbol{\lambda}, \rho\|_{\boldsymbol{\Lambda}\times Q} &\le \left(1 + \frac{\|a\|}{\alpha_b}\right)\Big(\|f\| + \|a\|\|g\|\Big). \end{aligned} \tag{2.16}$$

Proof. First observe that $\mathscr{N}_b = H^1_0(\Omega)^d \times L^2_0(\Omega)$ (c.f. [7, Lemma 4]), and then the first result in (2.15) stems from the classical well-posedness of (1.3) over $H^1_0(\Omega)^d \times L^2_0(\Omega)$. Next, let $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}$. From Green's Theorem and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q) &= \sum_{K\in\mathscr{P}_{\mathcal{H}}} [(\boldsymbol{\sigma}, \nabla \boldsymbol{v})_K + (\nabla\cdot \boldsymbol{\sigma}, \boldsymbol{v})_K] + (\xi, q)_Q \\ &\le \|\boldsymbol{v}, q\|_{\mathbf{V}\times Q} \left(\|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega)} + \|\xi\|^2_Q\right)^{1/2}, \end{aligned}$$

for all $(\boldsymbol{v}, q) \in \mathbf{V} \times Q$ and all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)$ with the property $\boldsymbol{\sigma}_K\, \boldsymbol{n}^K\,|_{\partial K} = \boldsymbol{\mu}_{\partial K}$ for each $K \in \mathscr{P}_{\mathcal{H}}$. It follows that

$$\sup_{(\boldsymbol{v}, q)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q)}{\|\boldsymbol{v}, q\|_{\mathbf{V}\times Q}} \le \|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q},$$

for all $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda} \times \mathbb{R}$. Next, for each $K \in \mathscr{P}_{\mathcal{H}}$, note that for each component $\mu_{\partial K,i}$ of $\boldsymbol{\mu}_{\partial K}$, $1 \le i \le d$, there exists a unique $v^{\star}_{K,i} \in H^1(K)$ such that

$$(\nabla v^{\star}_{K,i}, \nabla z_K)_K + d_{\Omega}^{-2}(v^{\star}_{K,i}, z_K)_K = \langle \mu_{\partial K,i}, z_K\rangle_{\partial K},$$

for all $z_K \in H^1(K)$. It follows that $\nabla\cdot \nabla v^\star_{K,i} = d_\Omega^{-2} v^\star_{K,i} \in L^2(K)$ and $\nabla v^\star_{K,i} \cdot \boldsymbol{n}^K|_{\partial K} = \mu_{\partial K,i}$. Defining $\boldsymbol{v}^\star \in \mathbf{V}$ by $\boldsymbol{v}^\star|_K = \boldsymbol{v}^\star_K$, where $\boldsymbol{v}^\star_K := (v^\star_{K,1}, \dots, v^\star_{K,d})$, it holds that $\nabla_{\mathcal{H}} \boldsymbol{v}^\star \in \boldsymbol{H}(\mathrm{div};\Omega)$ and,

$$\begin{aligned}\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q} &\le \left(\|\nabla_{\mathcal{H}}\boldsymbol{v}^\star\|^2_{0,\Omega} + d_\Omega^2 \|\nabla\cdot\nabla_{\mathcal{H}}\boldsymbol{v}^\star\|^2_{0,\Omega} + \|\xi\|^2_Q\right)^{1/2}\\ &= \left(\|\nabla_{\mathcal{H}}\boldsymbol{v}^\star\|^2_{0,\Omega} + \frac{1}{d_\Omega^2}\|\boldsymbol{v}^\star\|^2_{0,\Omega} + \|\xi\|^2_Q\right)^{1/2}\\ &= \frac{\langle \boldsymbol{\mu}, \boldsymbol{v}^\star\rangle_{\partial\mathscr{P}_{\mathcal{H}}} + (\xi,\xi)_Q}{\|\boldsymbol{v}^\star,\xi\|_{\mathbf{V}\times Q}}.\end{aligned}$$

So, $\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q} \le \sup_{(\boldsymbol{v},q)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q)}{\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}}$, thereby proving the second condition in (2.15). Finally, the stability of the solutions follows by classical results (see [39, Theorem 2.34]). $\square$

Remark 2.2 (Equivalence with primal-hybrid formulation)**.** *The solution of the classical weak formulation* (1.3) *and it's hybrid form* (2.1) *coincides and* $\boldsymbol{\lambda}_{\partial K} = (\nu\,\nabla \boldsymbol{u}_K - p_K\,\mathbb{I})\boldsymbol{n}^K$ *on* ∂K*, with* $\mathbb{I}$ *being the* $d\times d$ *identity operator, and* $\rho = 0$ *(c.f.* [7, Theorem 1]*).* $\square$

3. MHM's abstract setting

This section presents the MHM formulation in general terms, which will be used to analyze specific cases in the following sections. It also establishes necessary and sufficient conditions for the MHM method to be well-posed and with best approximation properties.

To this end, let $a_{s,K}(\cdot;\cdot)$ and $f_{s,K}(\cdot)$ be bounded (bi)linears form over closed subspace $\mathbf{V}_s(K)\times Q_s(K)$. For all instances below, we assume the nullspace of $a_{s,K}(\cdot;\cdot)$ equals $\mathscr{N}_a(K)$ and $f_{s,K}(\cdot)$ coincides with $f_K(\cdot)$ over $\mathscr{N}_a(K)$. However, we distinguish its orthogonal complement from (2.13) as follows

$$\begin{aligned}\mathscr{N}_{a,s}(K)^\perp &:= \{(\boldsymbol{v},q)\in \mathbf{V}_s(K)\times Q_s(K) \,:\, (\boldsymbol{v},q;\,\boldsymbol{w},r)_{\mathbf{V}\times Q} = 0 \quad \forall (\boldsymbol{w},r)\in\mathscr{N}_a(K)\}\,,\\ \mathscr{N}_{a,s}^\perp &:= \left\{(\boldsymbol{v},q)\in \mathbf{V}\times Q \,:\, (\boldsymbol{v},q)|_K \in \mathscr{N}_{a,s}(K)^\perp \quad \forall K\in\mathscr{P}_{\mathcal{H}}\right\}.\end{aligned}$$

Next, we define two global mappings T_s and $\widehat{T}_s$ from their local counterpart, namely, $T_{s,K} : \boldsymbol{\Lambda}_s(\partial K)\times\mathbb{R} \to \mathscr{N}_{a,s}^\perp(K)$ and $\widehat{T}_{s,K} : L^2(K)^d \to \mathscr{N}_{a,s}^\perp(K)$ defined by

$$a_{s,K}(T_{s,K}(\boldsymbol{\mu},\xi);\,\boldsymbol{v},q) \;=\; -b_K(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q) \quad\text{and}\quad a_{s,K}(\widehat{T}_{s,K}(\boldsymbol{q});\,\boldsymbol{v},q) \;=\; l^{\boldsymbol{q}}_{s,K}(\boldsymbol{v},q), \tag{3.1}$$

for all $(\boldsymbol{v},q)\in\mathscr{N}_{a,s}^\perp(K)$, where $l^{\boldsymbol{q}}_{s,K}(\cdot)$ is a given bounded linear form over $\mathbf{V}_s(K)\times Q_s(K)$ associated with a $\boldsymbol{q}\in L^2(\Omega)^d$, and $b_K(\cdot;\cdot)$ is given in (2.10). Global versions of $a_{s,K}(\cdot;\cdot)$, $f_s(\cdot)$ and $l^{\boldsymbol{q}}_{s,K}(\cdot)$ are defined by

$$a_s(\boldsymbol{v},q;\boldsymbol{w},r) := \sum_{K\in\mathscr{P}_{\mathcal{H}}} a_{s,K}(\boldsymbol{v},q;\boldsymbol{w},r),\;\; f_s(\boldsymbol{v},q) := \sum_{K\in\mathscr{P}_{\mathcal{H}}} f_{s,K}(\boldsymbol{v},q),\;\; l^{\boldsymbol{q}}_s(\boldsymbol{v},q) := \sum_{K\in\mathscr{P}_{\mathcal{H}}} l^{\boldsymbol{q}}_{s,K}(\boldsymbol{v},q),$$

for all $(\boldsymbol{v},q)$, $(\boldsymbol{w},r)\in\mathbf{V}_s\times Q_s$, where it is assumed that $l^{\boldsymbol{q}}_{s,K}(\cdot)$ are such that $\|l^{\boldsymbol{q}}_s\|\le C\|\boldsymbol{q}\|_{0,\Omega}$.

Given those definitions, the MHM abstract formulation is: *Find* $(\boldsymbol{u}_0^s,0)\in\mathscr{N}_a$ *and* $(\boldsymbol{\lambda}_s,\rho_s)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$ *such that*

$$\begin{aligned}
b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\lambda}_s,\rho_s))+b(\boldsymbol{\mu},\xi;\,\boldsymbol{u}_0^s,0)&=(\boldsymbol{\mu},\boldsymbol{g})_{\partial\Omega}-b(\boldsymbol{\mu},\xi;\,\widehat{T}_s(\boldsymbol{f})),\\
b(\boldsymbol{\lambda}_s,\rho_s;\,\boldsymbol{v}_0,0)&=f(\boldsymbol{v}_0,0),
\end{aligned}\tag{3.2}$$

for all $(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$ and $(\boldsymbol{v}_0,0)\in\mathscr{N}_a$.

Remark 3.1 (Simplified MHM formulation for the Brinkman case)**.** *In the case of the Brinkman equation ($\boldsymbol{\theta}\neq\mathbf{0}$), the nullspace $\mathscr{N}_a$ contains only the zero element. It follows that for this case, the MHM formulation* (3.2) *reduces to find $(\boldsymbol{\lambda}_s,\rho_s)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$ such that*

$$b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\lambda}_s,\rho_s))=(\boldsymbol{\mu},\boldsymbol{g})_{\partial\Omega}-b(\boldsymbol{\mu},\xi;\,\widehat{T}_s(\boldsymbol{f}))\quad\textit{for all } (\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}.$$

□

3.1. **Well-posedness of the abstract MHM** (3.2)**.** Here, we establish conditions under which the MHM formulation (3.2) is well-posed. To this end, define

$$\mathscr{N}_{b,s}:=\left\{(\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s\,:\,b(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q)=0\quad\text{for all }(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}\right\},\tag{3.3}$$

and assume the following conditions hold:

Assumption. *Given $(\boldsymbol{v}_K,q_K)\in\mathscr{N}_{a,s}^{\perp}(K)$, $(\boldsymbol{z},m)\in\mathscr{N}_{b,s}$ and $(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$,*

$$\alpha_{a,s}\|\boldsymbol{v}_K,q_K\|_{H^1(K)\times L^2(K)}\le\sup_{(\boldsymbol{w}_K,r_K)\in\mathscr{N}_{a,s}^{\perp}(K)}\frac{a_{s,K}(\boldsymbol{v}_K,q_K;\,\boldsymbol{w}_K,r_K)}{\|\boldsymbol{w}_K,r_K\|_{H^1(K)\times L^2(K)}},\tag{H1}$$

$$\alpha_{b,s}\|\boldsymbol{z},m\|_{\mathbf{V}\times Q}\le\sup_{(\boldsymbol{w},r)\in\mathscr{N}_{b,s}}\frac{a_s(\boldsymbol{z},m;\,\boldsymbol{w},r)}{\|\boldsymbol{w},r\|_{\mathbf{V}\times Q}},\tag{H2}$$

$$\beta_s\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}\le\sup_{(\boldsymbol{w},r)\in\mathbf{V}_s\times Q_s}\frac{b(\boldsymbol{\mu},\xi;\,\boldsymbol{w},r)}{\|\boldsymbol{w},r\|_{\mathbf{V}\times Q}},\tag{H3}$$

where positive constants $\alpha_{a,s}$, $\alpha_{b,s}$ and β_s are independent of mesh parameters.

Remark 3.2 (Stability of T_s and $\widehat{T}_s$)**.** *Condition* (H1) *ensures computations may be localized to each $K\in\mathscr{P}_{\mathcal{H}}$, thereby ensuring that operators T_s and $\widehat{T}_s$ are well-defined. Indeed, using* (H1) *and appendix A, we establish that*

$$\alpha_{a,s}\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}\le\sup_{(\boldsymbol{w},r)\in\mathscr{N}_{a,s}^{\perp}}\frac{a_s(\boldsymbol{v},q;\,\boldsymbol{w},r)}{\|\boldsymbol{w},r\|_{\mathbf{V}\times Q}}\quad\textit{for all } (\boldsymbol{v},q)\in\mathscr{N}_{a,s}^{\perp},\tag{3.4}$$

from which we have the operators T_s and $\widehat{T}_s$ are bounded as follows

$$\begin{aligned}
\alpha_{a,s}\|T_s(\boldsymbol{\mu},\xi))\|_{\mathbf{V}\times Q} &\leq \sup_{(\boldsymbol{w},\,q)\in\mathscr{N}_{a,s}^{\perp}} \frac{a_s(T_s(\boldsymbol{\mu},\xi);\,\boldsymbol{w},q)}{\|\boldsymbol{w},q\|_{\mathbf{V}\times Q}} \\
&= -\sup_{(\boldsymbol{w},q)\in\mathscr{N}_a^{\perp}} \frac{b(\boldsymbol{\mu},\xi;\,\boldsymbol{w},q)}{\|\boldsymbol{w},q\|_{\mathbf{V}\times Q}} \leq \|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q},
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\alpha_{a,s}\|\widehat{T}_s(\boldsymbol{q})\|_{\mathbf{V}\times Q} &\leq \sup_{(\boldsymbol{w},\,q)\in\mathscr{N}_{a,s}^{\perp}} \frac{a_s(\hat{T}_s(\boldsymbol{q});\,\boldsymbol{w},q)}{\|\boldsymbol{w},q\|_{\mathbf{V}\times Q}} \\
&= \sup_{(\boldsymbol{w},q)\in\mathscr{N}_a^{\perp}} \frac{l_s^{\boldsymbol{q}}(\boldsymbol{w},q)}{\|\boldsymbol{w},q\|_{\mathbf{V}\times Q}} \leq \|l_s^{\boldsymbol{q}}\| \leq C\,\|\boldsymbol{q}\|_{0,\Omega}.
\end{aligned} \tag{3.6}$$

□

Remark 3.3 (Well-posedness of hybrid problem (2.1) on closed spaces)**.** *In* (H2)-(H3) *we recognize the necessary and sufficient condition for hybrid problem* (2.1) *to be well-posed over the spaces $\mathbf{V}_s \times Q_s$ and $\boldsymbol{\Lambda}_s \times \mathbb{R}$ with $a(\cdot;\cdot)$ replaced by $a_s(\cdot;\cdot)$ and $f(\cdot)$ by $f_s(\cdot)$. If $(\boldsymbol{u}_s, p_s, \boldsymbol{\lambda}_s, \rho_s)$ denotes such a unique solution of* (2.1) *in $\mathbf{V}_s \times Q_s \times \boldsymbol{\Lambda}_s \times \mathbb{R}$, then following the argument in Lemma 2.1*

$$\begin{aligned}
\|\boldsymbol{u}_s,p_s\|_{\mathbf{V}\times Q} &\leq \frac{1}{\alpha_{b_s}} \sup_{(\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s} \frac{f_s(\boldsymbol{v},q)}{\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}} + \frac{1}{\beta_s}\left(1+\frac{\|a_s\|}{\alpha_{b,s}}\right) \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s} \frac{g(\boldsymbol{\mu},0)}{\|\boldsymbol{\mu},0\|_{\boldsymbol{\Lambda}\times Q}}, \\
\|\boldsymbol{\lambda}_s,\rho_s\|_{\boldsymbol{\Lambda}\times Q} &\leq \frac{1}{\beta_s}\left(1+\frac{\|a_s\|}{\alpha_{b,s}}\right)\left(\sup_{(\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s} \frac{f_s(\boldsymbol{v},q)}{\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}} + \frac{\|a_s\|}{\beta_s} \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s} \frac{g(\boldsymbol{\mu},0)}{\|\boldsymbol{\mu},0\|_{\boldsymbol{\Lambda}\times Q}}\right).
\end{aligned} \tag{3.7}$$

□

The following result is central to the analysis of the MHM methods that will follow. We establish that (3.2) is indeed well-posed under the assumptions (H1)-(H3). We also see that the unique solution of (2.1), when taken over the subspaces $\mathbf{V}_s \times Q_s$ and $\boldsymbol{\Lambda}_s \times \mathbb{R}$, can be constructed using the unique solution of (3.2), establishing thus an equivalence between both formulations.

Theorem 3.1. *Consider*

$$\mathscr{N}_s := \{(\boldsymbol{\mu},\xi) \in \boldsymbol{\Lambda}_s \times \mathbb{R} \,:\, b(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q) = 0 \quad \textit{for all } (\boldsymbol{v},q) \in \mathscr{N}_a\}\,. \tag{3.8}$$

Under assumptions (H1)–(H3)*, it holds*

$$
\begin{aligned}
\frac{\alpha_{b,s}\beta_s^2}{(\alpha_{b,s}+\|a_s\|)\|a_s\|}\,\|\boldsymbol{\gamma},\tau\|_{\boldsymbol{\Lambda}\times Q} &\le \sup_{(\boldsymbol{\mu},\xi)\in\mathscr{N}_s}\frac{b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\gamma},\tau))}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}},\\
\frac{\alpha_{b,s}\beta_s}{\alpha_{b,s}+\|a_s\|}\,\|\boldsymbol{v}_0,0\|_{\mathbf{V}\times Q} &\le \sup_{(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}}\frac{b(\boldsymbol{\mu},\xi;\,\boldsymbol{v}_0,0)}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}},
\end{aligned}
\tag{3.9}
$$

for all $(\boldsymbol{\gamma},\tau)\in\mathscr{N}_s$ *and* $(\boldsymbol{v}_0,0)\in\mathscr{N}_a$*. Thus, the abstract MHM formulation* (3.2) *admits a unique solution, and*

$$
\begin{aligned}
\|\boldsymbol{\lambda}_s,\rho_s\|_{\boldsymbol{\Lambda}\times Q} &\le C\left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a}\frac{f(\boldsymbol{v}_0,0)}{\|\boldsymbol{v}_0,0\|_{\mathbf{V}\times Q}}+\|\widehat{T}_s(\boldsymbol{f})\|_{\mathbf{V}\times Q}+\sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s}\frac{g(\boldsymbol{\mu},0)}{\|\boldsymbol{\mu},0\|_{\boldsymbol{\Lambda}\times Q}}\right),\\
\|\boldsymbol{u}_0^s,0\|_{\mathbf{V}\times Q} &\le C\left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a}\frac{f(\boldsymbol{v}_0,0)}{\|\boldsymbol{v}_0,0\|_{\mathbf{V}\times Q}}+\|\widehat{T}_s(\boldsymbol{f})\|_{\mathbf{V}\times Q}+\sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s}\frac{g(\boldsymbol{\mu},0)}{\|\boldsymbol{\mu},0\|_{\boldsymbol{\Lambda}\times Q}}\right).
\end{aligned}
$$

Furthermore, the solution $(\boldsymbol{u}_s,p_s)$ *of hybrid problem* (2.1) *in* $\mathbf{V}_s\times Q_s$ *can be written in terms of the solution* $(\boldsymbol{u}_0^s,0)$ *and* $(\boldsymbol{\lambda}_s,\rho_s)$ *of* (3.2) *as follows*

$$
(\boldsymbol{u}_s,p_s)\;=\;(\boldsymbol{u}_0^s,0)\;+\;T_s(\boldsymbol{\lambda}_s,\rho_s)+\widehat{T}_s(\boldsymbol{f}).
\tag{3.10}
$$

Proof. Let $(\boldsymbol{\gamma},\tau)\in\mathscr{N}_s$ and define $T_s(\boldsymbol{\gamma},\tau)$ by the first equation of (3.1), observing the need for assumption (H1). Then, define $(\boldsymbol{w},r)\in\mathbf{V}_s\times Q_s$ and $(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$ the unique solution (from (H2)-(H3)) of

$$
\begin{aligned}
a_s^T(\boldsymbol{w},r;\,\boldsymbol{v},q)+b(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q) &= (T_s(\boldsymbol{\gamma},\tau);\,\boldsymbol{v},q)_{\mathbf{V}\times Q} && \text{for all } (\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s,\\
b(\boldsymbol{\zeta},\phi;\,\boldsymbol{w},r) &= 0 && \text{for all } (\boldsymbol{\zeta},\phi)\in\boldsymbol{\Lambda}_s\times\mathbb{R},
\end{aligned}
\tag{3.11}
$$

where $a_s^T(\cdot,\cdot)$ stands for the adjoint operator of $a_s(\cdot,\cdot)$. Picking $(\boldsymbol{v},q):=T_s(\boldsymbol{\gamma},\tau)$ in (3.11),

$$
\begin{aligned}
\|T_s(\boldsymbol{\gamma},\tau)\|^2_{\mathbf{V}\times Q} &= a^T(\boldsymbol{w},r;\,T_s(\boldsymbol{\gamma},\tau))+b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\gamma},\tau))\\
&= a(T_s(\boldsymbol{\gamma},\tau);\,\boldsymbol{w},r)+b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\gamma},\tau))\\
&= -b(\boldsymbol{\gamma},\tau;\,\boldsymbol{w},r)+b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\gamma},\tau))\\
&= b(\boldsymbol{\mu},\xi;\,T_s(\boldsymbol{\gamma},\tau)),
\end{aligned}
$$

where we used the second equation in (3.11). The term $\|T_s(\boldsymbol{\gamma},\tau)\|^2_{\mathbf{V}\times Q}$ may be bounded from below. On one hand, from (3.7) it holds

$$
\begin{aligned}
\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q} &\le \frac{1}{\beta_s}\left(1+\frac{\|a_s\|}{\alpha_{b,s}}\right)\sup_{(\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s}\frac{(T_s(\boldsymbol{\gamma},\tau);\,\boldsymbol{v},q)_{\mathbf{V}\times Q}}{\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}}\\
&\le \frac{1}{\beta_s}\left(1+\frac{\|a_s\|}{\alpha_{b,s}}\right)\|T_s(\boldsymbol{\gamma},\tau)\|_{\mathbf{V}\times Q}.
\end{aligned}
$$

On the other hand, from (H3) and the first equation in (3.1) (which we use owing to the definition of $\mathscr{N}_s$), and the boundedness of $a_s(\cdot;\cdot)$ that

$$\frac{\beta_s}{\|a_s\|}\,\|\boldsymbol{\gamma},\tau\|_{\boldsymbol{\Lambda}\times Q} \leq \|T_s(\boldsymbol{\gamma},\tau)\|_{\mathbf{V}\times Q} \quad \text{for all } (\boldsymbol{\gamma},\tau)\in\mathscr{N}_s.$$

As a result, the first inequality of (3.9) holds since the above three equations imply

$$\frac{\beta_s^2\,\alpha_{b,s}}{\|a_s\|(\alpha_{b,s}+\|a_s\|)}\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}\|\boldsymbol{\gamma},\tau\|_{\boldsymbol{\Lambda}\times Q} \leq b(\boldsymbol{\mu},\xi;\, T_s(\boldsymbol{\gamma},\tau)).$$

We follow the same strategy to establish the second inequality of (3.9), this time using $(\boldsymbol{v}_0,0)\in\mathscr{N}_a$ and defining $(\boldsymbol{w},r)\in\mathbf{V}_s\times Q_s$ and $(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}$ as the unique solution of

$$\begin{aligned} a_s^T(\boldsymbol{w},r;\,\boldsymbol{v},q)+b(\boldsymbol{\mu},\xi;\,\boldsymbol{v},q) &= (\boldsymbol{v}_0,0;\,\boldsymbol{v},q)_{\mathbf{V}\times Q} && \text{for all } (\boldsymbol{v},q)\in\mathbf{V}_s\times Q_s,\\ b(\boldsymbol{\zeta},\tau;\,\boldsymbol{w},r) &= 0 && \text{for all } (\boldsymbol{\zeta},\tau)\in\boldsymbol{\Lambda}_s\times\mathbb{R}. \end{aligned}$$

Taking $(\boldsymbol{v},q):=(\boldsymbol{v}_0,0)$ we have

$$\begin{aligned} \|\boldsymbol{v}_0,0\|^2_{\mathbf{V}\times Q} &= a_s^T(\boldsymbol{w},r;\,\boldsymbol{v}_0,0)+b(\boldsymbol{\mu},\xi;\,\boldsymbol{v}_0,0)\\ &= a_s(\boldsymbol{v}_0,0;\,\boldsymbol{w},r)+b(\boldsymbol{\mu},\xi;\,\boldsymbol{v}_0,0)\\ &= b(\boldsymbol{\mu},\xi;\,\boldsymbol{v}_0,0). \end{aligned}$$

By (3.7), we get,

$$\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q} \leq \frac{1}{\beta_s}\left(1+\frac{\|a_s\|}{\alpha_{b,s}}\right)\|\boldsymbol{v}_0,0\|_{\mathbf{V}\times Q},$$

which establishes the second inequality since the above two equations imply

$$\frac{\beta_s\,\alpha_{b,s}}{\alpha_{b,s}+\|a_s\|}\|\boldsymbol{v}_0,0\|_{\mathbf{V}\times Q}\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q} \leq b(\boldsymbol{\mu},\xi;\,\boldsymbol{v}_0,0).$$

Hence, the abstract MHM method (3.2) has a unique $(\boldsymbol{u}_0^s,\boldsymbol{\lambda}_s,\rho_s)$ from standard saddle-point theory. Finally, it is straightforward to verify that $(\boldsymbol{u}_0^s,0)+T_s(\boldsymbol{\lambda}_s,\rho_s)+\widehat{T}_s(\boldsymbol{f})$ satisfies the hybrid formulation (2.1) when restricted to subspaces $\mathbf{V}_s\times Q_s$. Then, the equivalence (3.10) follows from the uniqueness of solution of (2.1) on $\mathbf{V}_s\times Q_s$ using (H2)–(H3). ☐

Remark 3.4 (The continuous MHM formulation)**.** *We consider the MHM formulation* (3.2) *for the case* $\mathbf{V}_s=\mathbf{V}$*,* $Q_s=Q$*, and* $\boldsymbol{\Lambda}_s=\boldsymbol{\Lambda}$*, i.e., we seek* $(\boldsymbol{u}_0,0)\in\mathscr{N}_a$ *and* $(\boldsymbol{\lambda},\rho)\in\boldsymbol{\Lambda}\times\mathbb{R}$

$$\begin{aligned} b(\boldsymbol{\mu},\xi;\,T(\boldsymbol{\lambda},\rho))+b(\boldsymbol{\mu},\xi;\,\boldsymbol{u}_0,0) &= (\boldsymbol{\mu},\boldsymbol{g})_{\partial\Omega}-b(\boldsymbol{\mu},\xi;\,\widehat{T}(\boldsymbol{f})),\\ b(\boldsymbol{\lambda},\rho;\,\boldsymbol{v}_0,0) &= f(\boldsymbol{v}_0,0), \end{aligned} \tag{3.12}$$

for all $(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}\times\mathbb{R}$ *and* $(\boldsymbol{v}_0,0)\in\mathscr{N}_a$*. Since* (3.1) *is solved using* $a_s(\cdot;\cdot):=a(\cdot;\cdot)$ *and* $l_s^{\boldsymbol{f}}(\boldsymbol{v},q):=(\boldsymbol{f},\boldsymbol{v})_{\mathscr{P}_{\mathcal{H}}}$*, we adopt the notation* $T_s=T:\,\boldsymbol{\Lambda}\times\mathbb{R}\to\mathbf{V}\times Q$ *and* $\widehat{T}_s=\widehat{T}:\,L^2(\Omega)^d\to\mathbf{V}\times Q$*. Theorem 3.1 asserts this problem is well-posed under the assumptions* (H1)–(H3)*.*

Note that Lemma 2.1 establishes directly (H2)–(H3)*, and* (H1) *also holds. It follows that* (3.12) *provides a unique solution, which we use to characterize the (unique) solution to the hybrid formulation* (2.1)

$$(\boldsymbol{u}, p) \,=\, (\boldsymbol{u}_0, 0) \,+\, T(\boldsymbol{\lambda}, \rho) + \widehat{T}(\boldsymbol{f}) \quad \text{and} \quad \boldsymbol{\lambda}_{\partial K} = (\nu\,\nabla \boldsymbol{u}_K - p_K\,\mathbb{I})\boldsymbol{n}^K \text{ on } \partial K. \tag{3.13}$$

Note that ρ vanishes (see [7, Theorem 1] *for details). Also, following Remark 3.2, T and $\widehat{T}$ are bounded as follows*

$$\alpha_a \|T(\boldsymbol{\mu}, \xi)\|_{\mathbf{V}\times Q} \le \|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q} \quad \text{and} \quad \alpha_a \|\widehat{T}(\boldsymbol{q})\|_{\mathbf{V}\times Q} \le C\|\boldsymbol{q}\|_{0,\Omega}. \tag{3.14}$$

□

3.2. **Best approximation results.** Having established conditions for the MHM formulation to be well posed, we can explore best approximation properties. These will find application in later sections using interpolation results to establish convergence for the various versions of the MHM methods to be considered. Let

$$\boldsymbol{\Lambda}_s^\star := \{\boldsymbol{\mu} \in \boldsymbol{\Lambda}_s \,:\, b(\boldsymbol{\mu}, 0; \boldsymbol{v}_0, 0) = f(\boldsymbol{v}_0, 0) \quad \forall (\boldsymbol{v}_0, 0) \in \mathscr{N}_a\}\,, \tag{3.15}$$

and note that $\boldsymbol{\Lambda}_s^\star = \boldsymbol{\Lambda}_s$ in the Brinkman case.

Theorem 3.2 (Best approximation)**.** *Assume* (H1)–(H3) *hold. Let $(\boldsymbol{\lambda}_s, \rho_s) \in \boldsymbol{\Lambda}_s \times \mathbb{R}$ and $(\boldsymbol{u}_0^s, 0) \in \mathscr{N}_a$ be the solution of* (3.2)*, $(\boldsymbol{\lambda}, \rho) \in \boldsymbol{\Lambda} \times \mathbb{R}$ and $(\boldsymbol{u}_0, 0) \in \mathscr{N}_a$ the exact solution of* (3.12)*, and $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_s, p_s)$ given in* (3.13) *and* (3.10)*, respectively. Then, there exist positive constants C, depending only on constants in* (H1)–(H3)*, such that*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_s\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^s\|_{\mathbf{V}} \le C\left(\inf_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s^\star} \|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} + \|(T - T_s)(\boldsymbol{\lambda}, 0) + (\widehat{T} - \widehat{T}_s)(\boldsymbol{f})\|_{\mathbf{V}\times Q}\right),$$
$$\|\boldsymbol{u} - \boldsymbol{u}_s, p - p_s\|_{\mathbf{V}\times Q} \le C\left(\inf_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s^\star} \|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} + \|(T - T_s)(\boldsymbol{\lambda}, 0) + (\widehat{T} - \widehat{T}_s)(\boldsymbol{f})\|_{\mathbf{V}\times Q}\right).$$

Proof. First, recall that $\rho = \rho_s = 0$. Given $\boldsymbol{\mu}_s \in \boldsymbol{\Lambda}_s^\star$, and using $\boldsymbol{\lambda}_s$ satisfies the second equation in (3.2), it holds for all $(\boldsymbol{v}, 0) \in \mathscr{N}_a$,

$$\sum_{K\in\mathscr{P}_\mathcal{H}} \langle \boldsymbol{\mu}_s\,|_{\partial K} - \boldsymbol{\lambda}_s\,|_{\partial K}, \boldsymbol{v}_{\partial K}\rangle_{\partial K} = 0,$$

and then $(\boldsymbol{\mu}_s - \boldsymbol{\lambda}_s, 0) \in \mathscr{N}_s$, where $\mathscr{N}_s$ is defined in (3.8). Therefore, from Theorem 3.1 there exists $\alpha := \frac{\alpha_{b,s}\beta_s^2}{(\alpha_{b,s}+\|a_s\|)\|a_s\|}$ such that

$$\begin{aligned}
\alpha\,\|\boldsymbol{\mu}_s - \boldsymbol{\lambda}_s, 0\|_{\boldsymbol{\Lambda}\times Q} &\le \sup_{(\boldsymbol{\mu},\xi)\in\mathscr{N}_s} \frac{b(\boldsymbol{\mu},\xi;\, T_s(\boldsymbol{\mu}_s - \boldsymbol{\lambda}_s, 0))}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}} \\
&\le \sup_{(\boldsymbol{\mu},\xi)\in\mathscr{N}_s} \frac{-b(\boldsymbol{\mu},\xi;\, T(\boldsymbol{\lambda} - \boldsymbol{\mu}_s, 0)) - b(\boldsymbol{\mu},\xi;\, (T-T_s)(\boldsymbol{\mu}_s,0) + (\widehat{T}-\widehat{T}_s)(\boldsymbol{f}))}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}} \\
&\le \frac{1}{\alpha_a}\|\boldsymbol{\lambda} - \boldsymbol{\mu}_s\|_{\boldsymbol{\Lambda}} + \|(T-T_s)(\boldsymbol{\mu}_s,0) + (\widehat{T}-\widehat{T}_s)(\boldsymbol{f})\|_{\mathbf{V}\times Q} \\
&\le \frac{\alpha_a + 2\,\alpha_{a,s}}{\alpha_a\alpha_{a,s}}\|\boldsymbol{\lambda} - \boldsymbol{\mu}_s\|_{\boldsymbol{\Lambda}} + \|(T-T_s)(\boldsymbol{\lambda},0) + (\widehat{T}-\widehat{T}_s)(\boldsymbol{f})\|_{\mathbf{V}\times Q},
\end{aligned}$$

where we used (3.14), (3.5) and $\|(T-T_s)(\boldsymbol{\lambda} - \boldsymbol{\mu}_s, 0)\|_{\mathbf{V}\times Q} \le \frac{\alpha_a+\alpha_{a,s}}{\alpha_a\alpha_{a,s}}\|(\boldsymbol{\lambda} - \boldsymbol{\mu}_s, 0)\|_{\boldsymbol{\Lambda}\times Q}$. Therefore, the triangle inequality yields

$$\begin{aligned}
\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_s\|_{\boldsymbol{\Lambda}} &\le \|\boldsymbol{\lambda} - \boldsymbol{\mu}_s\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{\mu}_s - \boldsymbol{\lambda}_s\|_{\boldsymbol{\Lambda}} \\
&\le C\left(\inf_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_s^\star}\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} + \|(T-T_s)(\boldsymbol{\lambda},0) + (\widehat{T}-\widehat{T}_s)(\boldsymbol{f})\|_{\mathbf{V}\times Q}\right).
\end{aligned}$$

Next, observe that from Theorem 3.1 there exists $\beta := \frac{\alpha_{b,s}\beta_s}{\alpha_{b,s}+\|a_s\|}$ such that

$$\begin{aligned}
\beta\|\boldsymbol{u}_0 - \boldsymbol{u}_0^s, 0\|_{\mathbf{V}\times Q} &\le \sup_{(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}} \frac{b(\boldsymbol{\mu},\xi; \boldsymbol{u}_0 - \boldsymbol{u}_0^s, 0)}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}} \\
&\le \sup_{(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}_s\times\mathbb{R}} \frac{-b(\boldsymbol{\mu},\xi;\, (\widehat{T}-\widehat{T}_s)(\boldsymbol{f}) + (T-T_s)(\boldsymbol{\lambda}_s,0) + T(\boldsymbol{\lambda} - \boldsymbol{\lambda}_s, 0))}{\|\boldsymbol{\mu},\xi\|_{\boldsymbol{\Lambda}\times Q}} \\
&\le \|(\widehat{T}-\widehat{T}_s)(\boldsymbol{f}) + (T-T_s)(\boldsymbol{\lambda}_s,0)\|_{\mathbf{V}\times Q} + \frac{1}{\alpha_a}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_s\|_{\boldsymbol{\Lambda}} \\
&\le \|(\widehat{T}-\widehat{T}_s)(\boldsymbol{f}) + (T-T_s)(\boldsymbol{\lambda},0)\|_{\mathbf{V}\times Q} + \frac{\alpha_a + 2\,\alpha_{a,s}}{\alpha_a\alpha_{a,s}}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_s\|_{\boldsymbol{\Lambda}},
\end{aligned}$$

we used again (3.14), (3.5), and the first estimate follows. The second estimate is a simple consequence of the characterization of $(\boldsymbol{u}, p)$ in (3.13) and $(\boldsymbol{u}_s, p_s)$ in (3.10), using the first estimate. □

4. The one-level MHM method

For the one-level MHM method, we define a finite-dimensional space $\boldsymbol{\Lambda}_H^\ell$ of $\boldsymbol{\Lambda}$, and choose $\mathbf{V}_s = \mathbf{V}$ and $Q_s = Q$. The latter two choices correspond to assuming that the action of the mappings $T_{s,K}$ and $\widehat{T}_{s,K}$ can be calculated exactly using (3.1).

Specifically, let $0 \leq \ell \in \mathbb{N}$ and define the space of (dis)continuous piecewise polynomial functions on $\mathcal{E}_H$ of degree up to ℓ given by

$$\mathbb{P}_\ell^{dis}(\mathcal{E}_H^K)^d := \left\{ \boldsymbol{\mu} \in L^2(\partial K)^d \,:\, \boldsymbol{\mu}_F \in \mathbb{P}_\ell^d(F) \quad \text{for all } F \in \mathcal{E}_H^K \right\},$$

and

$$\mathbb{P}_\ell^{con}(\mathcal{E}_H^K)^d := \left\{ \boldsymbol{\mu} \in C^0(E)^d \quad \text{for all } E \subset \partial K \,:\, \boldsymbol{\mu}_F \in \mathbb{P}_\ell^d(F) \quad \text{for all } F \in \mathcal{E}_H^K \right\}.$$

Their global counterparts are

$$\mathbb{P}_\ell^{dis}(\mathcal{E}_H)^d := \left\{ \boldsymbol{\mu} \in L^2(\mathcal{E})^d \,:\, \boldsymbol{\mu}_F \in \mathbb{P}_\ell^d(F) \quad \text{for all } F \in \mathcal{E}_H \right\},$$

and

$$\mathbb{P}_\ell^{con}(\mathcal{E}_H)^d := \left\{ \boldsymbol{\mu} \in C^0(E)^d \quad \text{for all } E \subset \mathcal{E} \,:\, \boldsymbol{\mu}_F \in \mathbb{P}_\ell^d(F) \quad \text{for all } F \in \mathcal{E}_H \right\}.$$

Then, we consider the following finite-dimensional subspace of $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda}_s = \boldsymbol{\Lambda}_H^\ell := \left\{ \boldsymbol{\mu} \in \boldsymbol{\Lambda} \,:\, \boldsymbol{\mu}_{\partial K} \in \mathbb{P}_\ell^{dis}(\mathcal{E}_H^K)^d \text{ or } \mathbb{P}_\ell^{con}(\mathcal{E}_H^K)^d \quad \text{for all } K \in \mathscr{P}_\mathcal{H} \right\}. \tag{4.1}$$

The <u>one-level MHM method</u> consists of finding $(\boldsymbol{u}_0^H, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\lambda}_H, \rho_H) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$ such that

$$\begin{aligned} b(\boldsymbol{\mu}_H, \xi;\, T(\boldsymbol{\lambda}_H, \rho_H)) + b(\boldsymbol{\mu}_H, \xi;\, \boldsymbol{u}_0^H, 0) &= (\boldsymbol{\mu}_H, \boldsymbol{g})_{\partial\Omega} - b(\boldsymbol{\mu}_H, \xi;\, \widehat{T}(\boldsymbol{f})), \\ b(\boldsymbol{\lambda}_H, \rho_H;\, \boldsymbol{v}_0, 0) &= f(\boldsymbol{v}_0, 0), \end{aligned} \tag{4.2}$$

for all $(\boldsymbol{v}_0, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\mu}_H, \xi) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$. According to Theorem 3.1, we form

$$(\boldsymbol{u}_H, p_H) \;=\; (\boldsymbol{u}_0^H, 0) + T(\boldsymbol{\lambda}_H, \rho_H) + \widehat{T}(\boldsymbol{f}), \tag{4.3}$$

the discrete solution to the hybrid formulation (2.1) over $\mathbf{V} \times Q$ and $\boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$. Note that although $(\boldsymbol{u}_H, p_H)$ is searched in $\mathbf{V} \times Q$, they belong to a finite-dimensional subspace induced by the basis functions of $\boldsymbol{\Lambda}_H$ and $\mathscr{N}_a$ plus $\widehat{T}(\boldsymbol{f})$. Moreover, it also provides the discrete dual variable $\boldsymbol{\sigma}_H \in \boldsymbol{H}(\text{div};\Omega)$ defined by

$$\boldsymbol{\sigma}_H \;:=\; \nu \nabla_\mathcal{H} \boldsymbol{u}_H - p_H \mathbb{I}. \tag{4.4}$$

We analyze the one-level MHM method (4.2) by establishing the assumptions (H1)–(H3) as well as best approximation results for $\boldsymbol{\Lambda}_H^\ell$. In this setting we may understand convergence properties without consistency errors related to approximations of the operators T and $\widehat{T}$.

4.1. **Well-posedness of** (4.2)**.** Here, we set $s = H$ and observe that

$$\mathscr{N}_{b,s} = \mathscr{N}_{b,H} := \left\{(\boldsymbol{v}, q) \in \mathbf{V} \times Q \,:\, b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q) = 0 \quad \text{for all } (\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda}^{\ell}_H \times \mathbb{R}\right\}, \tag{4.5}$$

$$\mathscr{N}_{s} = \mathscr{N}_{H} := \left\{(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda}^{\ell}_H \times \mathbb{R} \,:\, b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}, q) = 0 \quad \text{for all } (\boldsymbol{v}, q) \in \mathscr{N}_a\right\}. \tag{4.6}$$

The one-level MHM method is well-posed as shown in the next theorem.

Theorem 4.1 (Well-posedness of (4.2))**.** *Let $(\boldsymbol{\gamma}, \tau) \in \mathscr{N}_H$ and $(\boldsymbol{v}_0, 0) \in \mathscr{N}_a$. The following inequalities hold*

$$\begin{aligned} \frac{\alpha_b \beta_s^2}{(\alpha_b + \|a\|)\|a_s\|}\, \|\boldsymbol{\gamma}, \tau\|_{\boldsymbol{\Lambda}\times Q} &\leq \sup_{(\boldsymbol{\mu},\xi)\in\mathscr{N}_H} \frac{b(\boldsymbol{\mu}, \xi;\, T(\boldsymbol{\gamma}, \tau))}{\|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q}}, \\ \frac{\alpha_{b,s}\beta_s}{\alpha_{b,s} + \|a_s\|}\, \|\boldsymbol{v}_0, 0\|_{\mathbf{V}\times Q} &\leq \sup_{(\boldsymbol{\mu},\xi)\in\boldsymbol{\Lambda}^{\ell}_H\times\mathbb{R}} \frac{b(\boldsymbol{\mu}, \xi;\, \boldsymbol{v}_0, 0)}{\|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times Q}}. \end{aligned} \tag{4.7}$$

Then, there exists unique solution $(\boldsymbol{u}_0^H, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\lambda}_H, \rho_H) \in \boldsymbol{\Lambda}^{\ell}_H \times \mathbb{R}$ of (4.2)*, and*

$$\|\boldsymbol{\lambda}_H, \rho_H\|_{\boldsymbol{\Lambda}\times Q} \leq C \left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a} \frac{f(\boldsymbol{v}_0, 0)}{\|\boldsymbol{v}_0, 0\|_{\mathbf{V}\times Q}} + \|\widehat{T}(\boldsymbol{f})\|_{\mathbf{V}\times Q} + \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_H} \frac{g(\boldsymbol{\mu}, 0)}{\|\boldsymbol{\mu}, 0\|_{\boldsymbol{\Lambda}\times Q}} \right),$$

$$\|\boldsymbol{u}_0^H, 0\|_{\mathbf{V}\times Q} \leq C \left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a} \frac{f(\boldsymbol{v}_0, 0)}{\|\boldsymbol{v}_0, 0\|_{\mathbf{V}\times Q}} + \|\widehat{T}(\boldsymbol{f})\|_{\mathbf{V}\times Q} + \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_H} \frac{g(\boldsymbol{\mu}, 0)}{\|\boldsymbol{\mu}, 0\|_{\boldsymbol{\Lambda}\times Q}} \right).$$

Proof. We shall prove the conditions (H1)–(H3) needed in Theorem 3.1. Indeed, (H1) and (H3) hold since $\mathbf{V}_s \times Q_s = \mathbf{V} \times Q$, meaning these are inherited from Lemma 2.1. For condition (H2), consider $(\boldsymbol{v}, q) \in \mathscr{N}_{b,H}$ in (4.5). Since $q \in L^2_0(\Omega)$, there exists $\widetilde{\boldsymbol{v}} \in H^1_0(\Omega)^d$ such that $(q, \nabla \cdot \widetilde{\boldsymbol{v}})_\Omega = \|q\|^2_{0,\Omega}$ and $\|\widetilde{\boldsymbol{v}}\|_{1,\Omega} \leq C\,\|q\|_{0,\Omega}$. We set $(\boldsymbol{w}, r) := (\boldsymbol{v} - \varepsilon\,\widetilde{\boldsymbol{v}}, q)$ and observe that $(\boldsymbol{w}, r) \in \mathscr{N}_{b,H}$, where ε is a positive constant. Then, there exist positive constants C_1 and C_2 independent of mesh parameters, such that

$$\begin{aligned} a(\boldsymbol{v}, q;\, \boldsymbol{w}, q) &= a(\boldsymbol{v}, q;\, \boldsymbol{v}, q) - \varepsilon\, a(\boldsymbol{v}, q;\, \widetilde{\boldsymbol{v}}, 0) \\ &\geq C_1\,\|\boldsymbol{v}\|^2_{\mathbf{V}} - \varepsilon \left[\nu(\nabla_{\mathcal{H}}\boldsymbol{v}, \nabla_{\mathcal{H}}\tilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}} + (\boldsymbol{\theta}\,\boldsymbol{v}, \widetilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}} - (q, \nabla \cdot \tilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}}\right] \\ &\geq C_1\,\|\boldsymbol{v}\|^2_{\mathbf{V}} + \varepsilon \left[\|q\|^2_{0,\Omega} - C_2\,\|\boldsymbol{v}\|_{\mathbf{V}}\|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}}\right] \\ &\geq C_1\,\|\boldsymbol{v}\|^2_{\mathbf{V}} + \varepsilon \left[(1 - CC_2\,\frac{\gamma}{2})\|q\|^2_{0,\Omega} - C_2\,\frac{1}{2\gamma}\|\boldsymbol{v}\|^2_{\mathbf{V}}\right] = C_3\,\|\boldsymbol{v}, q\|^2_{\mathbf{V}\times Q}, \end{aligned}$$

where C_3 is a positive constant independent of mesh parameters by taking the positive constants γ and ε small enough. Next, observe that

$$\|\boldsymbol{w}, q\|_{\mathbf{V}\times Q} \leq \|\boldsymbol{v}, q\|_{\mathbf{V}\times Q} + \varepsilon\,\|\widetilde{\boldsymbol{v}}, 0\|_{\mathbf{V}\times Q} \leq \|\boldsymbol{v}, q\|_{\mathbf{V}\times Q} + C\,\varepsilon\,\|\mathbf{0}, q\|_{\mathbf{V}\times Q} \leq C_4\,\|\boldsymbol{v}, q\|_{\mathbf{V}\times Q},$$

where C_4 is a positive constant independent of mesh parameters. Combining this with previous estimates gives condition (H2), therefore Theorem 3.1 holds. $\square$

4.2. **Convergence.** In light of Theorem 3.2, the convergence properties will follow from an interpolation result for $\mathbf{\Lambda}_H^\ell$, and focus on convergence with respect to the mesh parameters $\mathcal{H}$, H. We follow the strategy of [16] proposed for the Poisson problem and in [45] for elasticity, which extended the approach of using interpolation mappings in one-element sub-meshes (c.f. [60]) to more general sub-mesh cases. Here, we adapt the proof given in [45] to deal with the Stokes/Brinkman operator. We do not prove spectral convergence in terms of the polynomial degree ℓ in this work, which would be expected based on preliminary theoretical and numerical results in [29].

For the present approach, we require a submesh Ξ_H^K, which denotes a regular minimal simplicial partition for each $K \in \mathscr{P}_{\mathcal{H}}$ and extends the partition on $\mathcal{E}_H$ to the interior of K. That is, for each $F \in \mathcal{E}_H$, there exists $\tau_F \in \Xi_H^K$ with diameter h_{τ_F} such that $\partial\tau_F \cap \partial K = F$, and for simplicity, we assume that for two different F, $F' \in \mathcal{E}_H$, we have $\tau_F \neq \tau_{F'}$. We note that by mesh regularity there exists a uniform, positive global constant C, independent of H and $\mathcal{H}$ and physical parameters, such that $h_{\tau_F} \leq C h_F$.

Lemma 4.1. *Suppose* $(\boldsymbol{v}, q) \in H^{m+1}(\mathscr{P}_{\mathcal{H}})^d \times H^m(\mathscr{P}_{\mathcal{H}})$*,* $1 \leq m \leq \ell+1$*, and* $(\nu\nabla_{\mathcal{H}}\boldsymbol{v} - q\mathbb{I}) \in \boldsymbol{H}(\mathrm{div};\Omega)$*, and* $\ell \geq 1$ *if* $\mathbf{\Lambda}_H^\ell$ *is the continuous space and* $\ell \geq 0$ *otherwise, and let* $\boldsymbol{\mu} \in \mathbf{\Lambda}$ *be such that* $\boldsymbol{\mu}_E^K := (\nu\nabla\boldsymbol{v} - q\mathbb{I})\,\boldsymbol{n}_E^K|_E$ *for each* $E \in \mathcal{E}$*. Then, there exists* $\boldsymbol{\mu}_\ell \in \mathbf{\Lambda}_H^\ell$ *and* C *such that*

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}_\ell\|_{\mathbf{\Lambda}} \leq C\, H^m \left(|\nu\nabla_{\mathcal{H}}\boldsymbol{v}|_{m,\mathscr{P}_{\mathcal{H}}} + |q|_{m,\mathscr{P}_{\mathcal{H}}}\right), \tag{4.8}$$

and $b(\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, 0; \boldsymbol{w}_0, 0) = 0$ *for all* $\boldsymbol{w}_0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$*.*

Proof. For sake of completeness, we describe the main steps following closely [45] for the discontinuous case and [59] for the continuous case. To begin, let $K \in \mathscr{P}_{\mathcal{H}}$ and $E \in \mathcal{E}$ be a face of ∂K. Define $\boldsymbol{\chi}_E^K := (\nu\nabla\boldsymbol{v} - q\mathbb{I})\boldsymbol{n}_E^K \in H^m(\Xi_H^K)^d$, where $\boldsymbol{n}_E^K$ is the extension of the outward unit normal vector $\boldsymbol{n}^K|_E$ to the constant function defined on the interior of K, and observe $\boldsymbol{\mu}_E^K = \boldsymbol{\chi}_E^K|_E$. Define the function $\boldsymbol{\mu}_K \in L^2(\partial K)^d$ by $\boldsymbol{\mu}_K|_E = \boldsymbol{\mu}_E^K$, and collect each of these across all K to yield $\boldsymbol{\mu} \in \mathbf{\Lambda}$.

We first analyze the discontinuous interpolant case. Consider the orthogonal projection $\Pi_{\partial K}^\ell(\boldsymbol{\mu})\,|_F := \Pi_F^\ell(\boldsymbol{\mu}_K)$ where $F \in \mathcal{E}_H$ satisfies $F \subset E \subset \partial K$ and Π_F^ℓ is the $L^2(F)^d$-orthogonal projector onto $\mathbb{P}_\ell(F)^d$, $\ell \geq 0$. Define $\boldsymbol{\mu}_{\partial K}^\ell := \Pi_{\partial K}^\ell\,(\boldsymbol{\mu})$. Using the regularity of the mesh, and following closely the proof in [45, Lemma 5.1], for each $K \in \mathscr{P}_{\mathcal{H}}$ we get

$$\langle \boldsymbol{\mu} - \boldsymbol{\mu}^{\ell}_{\partial K}, \boldsymbol{w} \rangle_{\partial K} \leq C \sum_{F \subset \partial K} h^m_{\tau_F} \, |\nu \nabla \boldsymbol{v}_{\tau_F} - q_{\tau_F} \, \mathbb{I}|_{m,\tau_F} \, |\boldsymbol{w}_{\tau_F}|_{1,\tau_F} \quad \text{for all } \boldsymbol{w} \in H^1(K)^d.$$

Define $\boldsymbol{\mu}_\ell \in \boldsymbol{\Lambda}^{\ell}_H$ by $\boldsymbol{\mu}_\ell|_{\partial K} := \boldsymbol{\mu}^{\ell}_{\partial K}$ and observe that $b(\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, 0; \boldsymbol{w}_0, 0) = 0$ for all $\boldsymbol{w}_0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$. Summing up over $K \in \mathscr{P}_{\mathcal{H}}$ and using mesh regularity, it holds

$$\begin{aligned} b(\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, 0; \, \boldsymbol{w}, r) = (\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, \boldsymbol{w})_{\partial \mathscr{P}_{\mathcal{H}}} &\leq C \sum_{K \in \mathscr{P}_{\mathcal{H}}} \sum_{F \subset \partial K} h^m_F \, |\nu \nabla \boldsymbol{v}_{\tau_F} - q_{\tau_F} \, \mathbb{I}|_{m,\tau_F} \, |\boldsymbol{w}_{\tau_F}|_{1,\tau_F} \\ &\leq C \, H^m (|\nu \nabla_{\mathcal{H}} \boldsymbol{v}|_{m,\mathscr{P}_{\mathcal{H}}} + |q|_{m,\mathscr{P}_{\mathcal{H}}}) \, |\boldsymbol{w}|_{1,\mathscr{P}_{\mathcal{H}}} \, , \end{aligned}$$

for all $(\boldsymbol{w}, r) \in \mathbf{V} \times Q$, which immediately leads to

$$\sup_{(\boldsymbol{w},r) \in \mathbf{V} \times Q} \frac{b(\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, 0; \, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V} \times Q}} \leq C \, H^m (|\nu \nabla_{\mathcal{H}} \boldsymbol{v}|_{m,\mathscr{P}_{\mathcal{H}}} + |q|_{m,\mathscr{P}_{\mathcal{H}}}). \tag{4.9}$$

The result (4.8) follows using the second equation in (2.15).

Now, we analyze a continuous interpolant, which is constructed using a vector version of the Scott-Zhang interpolation operator $\mathcal{P}^{\ell}_K$ on the space of piecewise continuous polynomials $\mathbb{P}_\ell(\Xi^K_H)^d$ of degree ℓ with respect to Ξ^K_H. We recall that this interpolation operator is a projection, i.e., $\mathcal{P}^{\ell}_K(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{P}_\ell(\Xi^K_H)^d$. Taking $\boldsymbol{\mu}^{\ell}_{\partial K} := \mathcal{P}^{\ell}_K(\boldsymbol{\chi}^K_E)|_{\partial K}$ and defining $\boldsymbol{\mu}_\ell \in \boldsymbol{\Lambda}^{\ell}_H$ by $\boldsymbol{\mu}_\ell|_{\partial K} := \boldsymbol{\mu}^{\ell}_{\partial K}$, we follow closely [16] to find

$$\|\boldsymbol{\mu} - \boldsymbol{\mu}_\ell\|_{\boldsymbol{\Lambda}} \leq C \, H^m \left(|\nu \nabla_{\mathcal{H}} \boldsymbol{v}|_{m,\mathscr{P}_{\mathcal{H}}} + |q|_{m,\mathscr{P}_{\mathcal{H}}} \right). \tag{4.10}$$

Observing that $b(\boldsymbol{\mu} - \boldsymbol{\mu}_\ell, 0; \boldsymbol{w}_0, 0) \neq 0$ for all $\boldsymbol{w}_0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$, we introduce $\widetilde{\boldsymbol{\mu}}_\ell \in \boldsymbol{\Lambda}^{\ell}_H$ defined by $\widetilde{\boldsymbol{\mu}}_\ell|_E := \boldsymbol{\mu}_\ell|_E + \pi^1_E(\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E)$ for each $E \subset \partial K, \forall K \in \mathscr{P}_{\mathcal{H}}$. Here, π^1_E is the $L^2(E)^d$-orthogonal projector onto $[\mathbb{P}_1(E) \cap C^0(E)]^d$. It follows immediately that that $b(\boldsymbol{\mu} - \widetilde{\boldsymbol{\mu}}_\ell, 0; \boldsymbol{w}_0, 0) = 0$ for all $(\boldsymbol{w}_0, 0) \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$. Moreover, letting π^1_K be the $L^2(K)^d$-orthogonal projector onto $\mathbb{P}_1(\Xi^K_H)^d$, for all $\boldsymbol{w} \in \mathbf{V}$, we get

$$\begin{aligned} &(\boldsymbol{\mu}_{\partial K} - \widetilde{\boldsymbol{\mu}}_\ell|_{\partial K}, \boldsymbol{w}_{\partial K})_{\partial K} = \sum_{E \subset \partial K} (\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E - \pi^1_E(\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E), \boldsymbol{w}_E)_E \\ &= \sum_{E \subset \partial K} (\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E - \pi^1_E(\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E), \boldsymbol{w}_E - \pi^1_K(\boldsymbol{w}_K)|_E)_E \\ &\leq \sum_{E \subset \partial K} \Big(\|\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E\|_{0,E} + \|\pi^1_E(\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E)\|_{0,E} \Big) \|\boldsymbol{w}_E - \pi^1_K(\boldsymbol{w}_K)|_E\|_{0,E} \\ &\leq C H^{1/2} \|\boldsymbol{\mu}_{\partial K} - \boldsymbol{\mu}^{\ell}_{\partial K}\|_{0,\partial K} |\boldsymbol{w}_K|_{1,K}, \end{aligned}$$

where we used the standard L^2 stability result $\|\pi^1_E(\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E)\|_{0,E} \leq C \, \|\boldsymbol{\mu}_E - \boldsymbol{\mu}_\ell|_E\|_{0,E}$ and approximability property $\|\boldsymbol{w}_E - \pi^1_K(\boldsymbol{w}_K)|_E\|_{0,E} \leq C H^{1/2} |\boldsymbol{w}_K|_{1,K}$ (c.f. [39]). Then, summing up for all $K \in \mathscr{P}_{\mathcal{H}}$, we get

$$b(\boldsymbol{\mu} - \widetilde{\boldsymbol{\mu}}_\ell, 0; \boldsymbol{w}, r) = (\boldsymbol{\mu} - \widetilde{\boldsymbol{\mu}}_\ell, \boldsymbol{w})_{\partial \mathscr{P}_{\mathcal{H}}} \leq C H^{1/2} \|\boldsymbol{\mu} - \boldsymbol{\mu}_\ell\|_{0,\partial\mathscr{P}_{\mathcal{H}}} \|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}.$$

Following closely the proof in [59, Lemma 5]), we get $\|\boldsymbol{\mu} - \boldsymbol{\mu}_\ell\|_{0,\partial\mathscr{P}_{\mathcal{H}}} \leq C H^{m-1/2} |\nu \nabla_{\mathcal{H}} \boldsymbol{v} - q\,\mathbb{I}|_{m,\mathscr{P}_{\mathcal{H}}}$ which yields (4.9) for continuous interpolation. □

Remark 4.2 (Conservation properties)**.** *Note from* (4.3) *and the definitions of local mapping* T *and* $\widehat{T}$ *that the one-level discrete velocity field is divergence-free, i.e.,*

$$\nabla \cdot \boldsymbol{u}_H \;=\; 0 \quad \text{in } K \in \mathscr{P}_{\mathcal{H}}.$$

Also, from the local problems (3.1)*, the discrete stress variable* $\boldsymbol{\sigma}_H$ *in* (4.4) *is in point-wisely local equilibrium with external force, i.e.,*

$$\boldsymbol{\theta}\, \boldsymbol{u}_H + \nabla \cdot \boldsymbol{\sigma}_H \;=\; \boldsymbol{f} \quad \text{in } K \in \mathscr{P}_{\mathcal{H}}. \tag{4.11}$$

Those properties are intrinsically related to the fact that exact second-level solutions are assumed to be known and readily available. In general, this assumption is not valid and second-level discrete solvers are needed to approximate the T *and* $\widehat{T}$ *operators. So, the next challenge is to propose two-level versions of the MHM method that preserve the optimal convergence rates and conservative properties of the one-level MHM method. This is precisely the subject of Section 5.* □

We are ready to present the main convergence result.

Theorem 4.2. *Assume* $(\boldsymbol{u}, p) \in H^{m+1}(\mathscr{P}_{\mathcal{H}})^d \times H^m(\mathscr{P}_{\mathcal{H}})$, $(\nu\nabla \boldsymbol{u} - p\mathbb{I}) \in \boldsymbol{H}(\mathrm{div};\Omega)$, *with* $1 \leq m \leq \ell + 1$ *and* $\ell \geq 1$ *if* $\boldsymbol{\Lambda}_H^\ell$ *is the continuous space and* $\ell \geq 0$ *otherwise. If* $(\boldsymbol{u}_H, p_H)$ *and* $\boldsymbol{\sigma}_H$ *are defined in* (4.3) *and* (4.4)*, respectively, then there exists* C *such that*

$$\begin{aligned}
\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_{\mathbf{V}} &\leq C\, H^m \left(|\nu \nabla_{\mathcal{H}} \boldsymbol{u}|_{m,\mathscr{P}_{\mathcal{H}}} + |p|_{m,\mathscr{P}_{\mathcal{H}}} \right), \\
\|\boldsymbol{u} - \boldsymbol{u}_H, p - p_h\|_{\mathbf{V}\times Q} &\leq C\, H^m \left(|\nu \nabla_{\mathcal{H}} \boldsymbol{u}|_{m,\mathscr{P}_{\mathcal{H}}} + |p|_{m,\mathscr{P}_{\mathcal{H}}} \right), \\
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|_{\boldsymbol{H}(\mathrm{div};\Omega)} &\leq C\, H^m \left(|\nu \nabla_{\mathcal{H}} \boldsymbol{u}|_{m,\mathscr{P}_{\mathcal{H}}} + |p|_{m,\mathscr{P}_{\mathcal{H}}} \right).
\end{aligned} \tag{4.12}$$

Proof. Observing $(T - T_s)(\boldsymbol{\lambda}_H)$ and $(\widehat{T} - \widehat{T}_s)(\boldsymbol{f})$ are identically zero, the results are a direct consequence of Theorem 3.2, Lemma 4.1 and (4.11). □

Remark 4.3 (Super-convergence)**.** *When the exact solution* $(\boldsymbol{u}, p)$ *is regular "enough" the MHM method* (4.2) *super-converges with an additional* $O(H^{1/2})$ *for a given quasi-uniform coarse mesh* $\mathscr{P}_{\mathcal{H}}$ *(e.g.,* $\mathcal{H}$ *fixed). The proof follows the strategy proposed in* [29] *for the discontinuous* $\boldsymbol{\Lambda}_H^\ell$ *case. Notably, let* $\Pi_K^0 : L^2(K)^d \to \mathbb{P}_0(K)^d$ *be the* L^2 *projection operator* *on the space of constant functions on* $K \in \mathscr{P}_{\mathcal{H}}$*, and* $\Pi^0(\cdot)$ *its global counterpart defined by* $\Pi^0(\cdot)|_K = \Pi^0_K(\cdot)$*. Note that if* $\boldsymbol{\mu}_\ell \in \boldsymbol{\Lambda}_H^\ell$ *is the function obtained from Lemma 4.1 then* $(\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell, \boldsymbol{v}_0)_{\partial\mathscr{P}_{\mathcal{H}}} = 0$*, for all* $\boldsymbol{v}_0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$ *and* $\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell\|_{\partial\mathscr{P}_{\mathcal{H}}} \leq CH^m\|\boldsymbol{\lambda}\|_{m,\partial\mathscr{P}_{\mathcal{H}}}$ *with* $1 \leq m \leq \ell+1$ *and* $\ell \geq 0$*, and using (2.15) it holds*

$$\begin{aligned}
\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell\|_{\boldsymbol{\Lambda}} = \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell, 0; \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} &= \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{(\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell, \boldsymbol{w} - \Pi^0(\boldsymbol{w}))_{\partial\mathscr{P}_{\mathcal{H}}}}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} \\
&\leq \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell\|_{\partial\mathscr{P}_{\mathcal{H}}}\|\boldsymbol{w} - \Pi^0(\boldsymbol{w})\|_{\partial\mathscr{P}_{\mathcal{H}}}}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} \\
&\leq CH^{1/2}\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell\|_{\partial\mathscr{P}_{\mathcal{H}}} \\
&\leq CH^{m+1/2}\|\boldsymbol{\lambda}\|_{m,\partial\mathscr{P}_{\mathcal{H}}},
\end{aligned}$$

where we used Cauchy-Schwarz inequality and the estimate $\|\boldsymbol{w} - \Pi^0(\boldsymbol{w})\|_{\partial\mathscr{P}_{\mathcal{H}}} \leq C\,H^{1/2}|\boldsymbol{w}|_{1,\mathscr{P}_{\mathcal{H}}}$*. Now, using* $\boldsymbol{\lambda} = (\nu\nabla\boldsymbol{u} - p\mathbb{I})\boldsymbol{n}^K$ *in* ∂K *for all* $K \in \mathscr{P}_{\mathcal{H}}$*, assuming additional regularity such that* $\|\boldsymbol{u}\|_{m+2,\mathscr{P}_{\mathcal{H}}}$ *and* $\|p\|_{m+1,\mathscr{P}_{\mathcal{H}}}$ *are limited, and from trace inequalities (see [29] for details), it results that* $\boldsymbol{\mu}_\ell \in \boldsymbol{\Lambda}_H^\ell$ *satisfies*

$$\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell\|_{\boldsymbol{\Lambda}} \leq C(\mathcal{H})\,H^{m+1/2}\left(\|\boldsymbol{u}\|_{m+2,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m+1,\mathscr{P}_{\mathcal{H}}}\right), \tag{4.13}$$

where $C(\mathcal{H})$ *is a positive constant dependent on a negative power of* $\mathcal{H}$ *(which is fixed). Then, from Theorem 3.2 the MHM solution converges as follows*

$$\|\boldsymbol{u} - \boldsymbol{u}_H, p - p_h\|_{\mathbf{V}\times Q} \leq C(\mathcal{H})\,H^{m+1/2}\left(\|\boldsymbol{u}\|_{m+2,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m+1,\mathscr{P}_{\mathcal{H}}}\right). \tag{4.14}$$

For the continuous case $\boldsymbol{\Lambda}_H^\ell$*, the idea is to select* $\widetilde{\boldsymbol{\mu}}_\ell \in \boldsymbol{\Lambda}_H^\ell$ *used in the proof of Lemma 4.1 such that* $(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\mu}}_\ell, \boldsymbol{v}_0)_{\partial\mathscr{P}_{\mathcal{H}}} = 0$*, for all* $\boldsymbol{v}_0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})^d$*. Then, following steps analogous to those in the discontinuous case*

$$\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\mu}}_\ell\|_{\boldsymbol{\Lambda}} \leq CH^{1/2}\|\boldsymbol{\lambda} - \widetilde{\boldsymbol{\mu}}_\ell\|_{\partial\mathscr{P}_{\mathcal{H}}} \leq CH^{1/2}\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell^\star\|_{\partial\mathscr{P}_{\mathcal{H}}},$$

where here $\boldsymbol{\mu}_\ell^\star$ *is related to the Scott-Zhang interpolator (see proof of Lemma 4.1). The rest of the proof follows closely that for the discontinuous case using* $\|\boldsymbol{\lambda} - \boldsymbol{\mu}_\ell^\star\|_{\partial\mathscr{P}_{\mathcal{H}}} \leq CH^m\|\boldsymbol{\lambda}\|_{m,\partial\mathscr{P}_{\mathcal{H}}}$ *with* $1 \leq m \leq \ell+1$ *and* $\ell \geq 1$ *and thus (4.14) is valid.* □

5. Two-level MHM methods

In this section, we introduce practical strategies to approximate the solutions of the local Stokes/Brinkman problems (3.1) in order to seed the global problem of the form (3.2). In principle, second-level solvers may be quite general. Here, we consider two approaches:

- (i) a stabilized finite element method with equal-order nodal approximation for velocity and pressure (the USFEM, [14]);

(ii) the Galerkin method using stable pairs of spaces (Taylor Hood elements, c.f. [19]).

The first level of a two-level MHM method is built on partitions $\mathscr{P}_{\mathcal{H}}$ of Ω and $\mathcal{E}_H$ of $\mathcal{E}$, using $\boldsymbol{\Lambda}_s = \boldsymbol{\Lambda}_H^\ell$ in (3.2) and (3.1) defined by (4.1) (as with the one-level method). The second level involves building numerical solutions to these local problems using local shape-regular, simplicial triangulations $\left\{\mathcal{T}_h^K\right\}_{h>0}$ of each $K \in \mathscr{P}_{\mathcal{H}}$, where we denote the collection of such triangulations

$$\mathcal{T}_h := \bigcup_{K \in \mathscr{P}_{\mathcal{H}}} \mathcal{T}_h^K.$$

Here, $h = \max_{\tau \in \mathcal{T}_h} h_\tau$ and h_τ is the diameter of an element $\tau \in \mathcal{T}_h$. We denote by $\mathcal{E}_h^K$ the set of faces on $\mathcal{T}_h^K$, and $\mathcal{E}_0^K$ the set of internal faces. To each face $\gamma \in \mathcal{E}_h^K$, we associate a normal vector $\boldsymbol{n}_\gamma^\tau$, taking care to ensure this is facing outward on $\partial\tau$.

Since methods will be defined by choosing spaces and forms with notation dependent of mesh parameter h, e.g., all subscripts s in previous sections will be replaced by h. Notably, the methods are then defined using specific choices of spaces $\mathbf{V}_s = \mathbf{V}_h$ and $Q_s = Q_h$ and (bi)linear forms $a_{s,K}(\cdot;\cdot) = a_{h,K}(\cdot;\cdot)$ and $l_{s,K}^{\boldsymbol{f}}(\cdot) = l_{h,K}^{\boldsymbol{f}}(\cdot)$. These will be introduced and analyzed in Sections 5.1 and 5.2, respectively, which define the approximate upscaling mappings T_h and $\widehat{T}_h$.

Owing to those discrete (local) operators, the <u>two-level MHM method</u> reads: Find $(\boldsymbol{u}_0^{H,h}, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\lambda}_{H,h}, \rho_{H,h}) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$ such that

$$\begin{aligned} b(\boldsymbol{\mu}_H, \xi;\, T_h(\boldsymbol{\lambda}_{H,h}, \rho_{H,h})) + b(\boldsymbol{\mu}_H, \xi;\, \boldsymbol{u}_0^{H,h}, 0) &= (\boldsymbol{\mu}_H, \boldsymbol{g})_{\partial\Omega} - b(\boldsymbol{\mu}_H, \xi;\, \widehat{T}_h(\boldsymbol{f})), \\ b(\boldsymbol{\lambda}_{H,h}, \rho_H;\, \boldsymbol{v}_0, 0) &= f(\boldsymbol{v}_0, 0), \end{aligned} \tag{5.1}$$

for all $(\boldsymbol{v}_0, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\mu}_H, \xi) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$. We form the two-level discrete solution $(\boldsymbol{u}_{H,h}, p_{H,h}) \in \mathbf{V}_h \times Q_h$ as follows

$$(\boldsymbol{u}_{H,h}, p_{H,h}) \;=\; (\boldsymbol{u}_0^{H,h}, 0) + T_h(\boldsymbol{\lambda}_{H,h}, \rho_{H,h}) + \widehat{T}_h(\boldsymbol{f}), \tag{5.2}$$

which also corresponds to the solution to the hybrid formulation (2.1) over $\mathbf{V}_h \times Q_h$ and $\boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$ from Theorem 3.1.

5.1. **Analysis of a *stabilized* two-level MHM method.** The unusual stabilized finite element method (USFEM) proposed in [14] was first adopted in [7] as a consistent second-level solver for local problems (3.1) to solve MHM formulation (5.1). For this method, we take

$$\mathbf{V}_h := \bigoplus_{K \in \mathscr{P}_{\mathcal{H}}} \mathbf{V}_h(K) \quad \text{and} \quad Q_h := \bigoplus_{K \in \mathscr{P}_{\mathcal{H}}} Q_h(K),$$

where, given $k \geq 1$ and $K \in \mathscr{P}_{\mathcal{H}}$,

$$\mathbf{V}_h(K) := \left\{ \boldsymbol{v} \in C^0(K)^d \,:\, \boldsymbol{v}_\tau \in \mathbb{P}_k(\tau)^d \quad \forall \tau \in \mathcal{T}_h^K \right\},$$
$$Q_h(K) := \left\{ q \in C^0(K) \,:\, q_\tau \in \mathbb{P}_k(\tau) \quad \forall \tau \in \mathcal{T}_h^K \right\}.$$

The global operators $T_h \in \mathcal{L}(\mathbf{\Lambda}_H^\ell \times \mathbb{R}, \mathscr{N}_{a,h}^\perp)$ and $\widehat{T}_h \in \mathcal{L}(L^2(\Omega)^d, \mathscr{N}_{a,h}^\perp)$ are defined locally by using the USFEM in (3.1), i.e., taking $(\boldsymbol{u}, p)$, $(\boldsymbol{v}, q) \in \mathbf{V}_h(K) \times Q_h(K)$ and the (bi)linear forms

$$\begin{aligned} a_{h,K}(\boldsymbol{u}, p; \boldsymbol{v}, q) :=& (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})_K + (\boldsymbol{\theta}\, \boldsymbol{u}, \boldsymbol{v})_K - (p, \nabla \cdot \boldsymbol{v})_K + (q, \nabla \cdot \boldsymbol{u})_K \\ & - \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau (-\nu \Delta \boldsymbol{u} + \boldsymbol{\theta} \boldsymbol{u} + \nabla p, -\nu \Delta \boldsymbol{v} + \boldsymbol{\theta} \boldsymbol{v} - \nabla q)_\tau, \end{aligned} \tag{5.3}$$

$$l_{h,K}^{\boldsymbol{f}}(\boldsymbol{v}, q) := (\boldsymbol{f}, \boldsymbol{v})_K - \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau (\boldsymbol{f}, -\nu \Delta \boldsymbol{v} + \boldsymbol{\theta}\, \boldsymbol{v} - \nabla q)_\tau. \tag{5.4}$$

The stabilization parameter δ_τ reads

$$\delta_\tau := \frac{h_\tau^2}{\theta_{max}^\tau\, h_\tau^2 \max\{1, Pe_\tau\} + \dfrac{4\,\nu}{m_k}} \quad \text{with} \quad Pe_\tau := \frac{4\nu}{\theta_{max}^\tau\, h_\tau^2\, m_k},$$

where $m_k := \min\left\{\frac{1}{3}, C_k\right\}$, the non-negative constant $\theta_{max}^\tau := \max_{\boldsymbol{x} \in \tau} \theta_{max}(\boldsymbol{x})$ and C_k is a positive constant, independent of h_τ, such that,

$$C_k\, h_\tau \, \|\Delta \boldsymbol{v}\|_{0,\tau} \leq \|\nabla \boldsymbol{v}\|_{0,\tau} \quad \text{for all } \boldsymbol{v} \in \mathbf{V}_h(K).$$

The USFEM has the property (see [14, Lemma 4.2] for details), given $(\boldsymbol{v}, q) \in \mathbf{V}_h(K) \times Q_h(K)$,

$$\alpha_{a,h} \|\boldsymbol{v}, q\|_{H^1(K) \times L^2(K)} \leq \sup_{(\boldsymbol{w}, r) \in \mathscr{N}_{a,h}^\perp(K)} \frac{a_{h,K}(\boldsymbol{v}, q;\, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{H^1(K) \times L^2(K)}}, \tag{5.5}$$

where $\alpha_{a,h}$ depends on physical parameters but is independent of the partition.

Remark 5.1. *Following closely Remark 3.2, using* (5.5)*, mappings T_h and $\widehat{T}_h$ are bounded*

$$\|T_h(\boldsymbol{\mu}, \xi)\|_{\mathbf{V} \times Q} \leq \alpha_{a,h}^{-1} \, \|\boldsymbol{\mu}, \xi\|_{\mathbf{\Lambda} \times \mathbb{R}} \quad \textit{and} \quad \|\widehat{T}_h(\boldsymbol{q})\|_{\mathbf{V} \times Q} \leq C\, \alpha_{a,h}^{-1} \, \|l_h^{\boldsymbol{q}}\|, \tag{5.6}$$

where $l_h^{\boldsymbol{q}}(\cdot)$ is the linear form such that $l_h^{\boldsymbol{q}}(\cdot)\,|_K = l_{h,K}^{\boldsymbol{q}}(\cdot)$ for all $K \in \mathscr{P}_{\mathcal{H}}$. Furthermore, there exists C dependent only on physical coefficients, such that

$$\|l_h^{\boldsymbol{q}}\| \leq C\, \|\boldsymbol{q}\|_{0,\Omega} \Rightarrow \|\widehat{T}_h(\boldsymbol{q})\|_{\mathbf{V} \times Q} \leq C\, \alpha_{a,h}^{-1} \, \|\boldsymbol{q}\|_{0,\Omega},$$

as a result of definition of the dual norm (2.12)*, the linear form $l_{h,K}^{\boldsymbol{q}}(\cdot)$ in* (5.4)*, and inverse inequalities.* $\square$

Remark 5.2 (The MHM method as a reduced modeling technique)**.** *When restricted to an element $K \in \mathscr{P}_{\mathcal{H}}$ the two-level MHM solution $(\boldsymbol{u}_{H,h}, p_{H,h})$ belongs to the polynomial spaces $\mathbf{V}_h(K)$ and $Q_h(K)$ of degree $k \geq 1$. However, note that $(\boldsymbol{u}_{H,h}, p_{H,h})|_K$ effectively belongs to a much smaller subspace of $\mathbf{V}_h(K) \times Q_h(K)$ driven by the dimension of $\boldsymbol{\Lambda}_H^\ell$ and $\mathscr{N}_a$ in K. For example, consider the simplest case of sub-mesh of an element in two dimensions for Stokes problems ($\mathscr{N}_a$ is non-trivial) and define $\ell = 0$ and $k = 2$. The product space dimension $\mathbf{V}_h(K) \times Q_h(K)$ is 18, but by (5.2) $(\boldsymbol{u}_{H,h}, p_{H,h})|_K$ is a linear combination of 9 space functions arising from 2 constant vector functions of $\mathscr{N}_a$, 6 functions induced by the local basis in $\boldsymbol{\Lambda}_H^0$ through the operator $T_{h,K}$, and one basis function associated to $\widehat{T}_{h,K}\boldsymbol{f}$. Such reduction may be even more prominent for sub-meshes with higher k values or with refinement due to the approximation of problems with multiple scales. The burden of the overhead computational cost related to $T_{h,K}$ and $\widehat{T}_{h,K}$ is "embarrassingly parallelizable" and can be done in an off-line stage.* $\square$

The remainder of this subsection is dedicated to the analysis of the method (5.1) with T_h and $\widehat{T}_h$ defined throught (5.3)-(5.4). Observe that by (5.5), assumption (H1) is satisfied. It remains to establish (H2)-(H3). Before these may be tackled, some preliminary results must be established.

5.1.1. *Preliminary results.* The analysis for the stabilized two-level MHM method requires a geometrical assumption between meshes. Specifically, letting κ_F be an element of minimal partition Ξ_H^K of $K \in \mathscr{P}_{\mathcal{H}}$ such that $\partial\kappa_F \cap \partial K = F$, choose $\tau \in \mathcal{T}_h^K$ such that $\tau \subset \kappa_F$. We assume that there exists a positive constant C such that

$$h_{\kappa_F} \leq C\, h_\tau. \tag{5.7}$$

Remark 5.3 (Interpretation of (5.7))**.** *Since $h_F \leq h_{\kappa_F}$, we see that the assumption (5.7) is equivalent to the requirement that the collection $\mathcal{T}_h$ cannot contain elements of size "too small" relative to the elements in $\mathcal{E}_H$. This indicates the need for a sufficiently small partition of faces in areas of the domain with strong small-scale physics, which fits in with practice.* $\square$

Let $\mathscr{C}_h : H^1(\mathscr{P}_{\mathcal{H}})^d \to \mathbf{V}_h$ be the Clément interpolation operator defined locally. In other words, for every $\boldsymbol{v} \in \mathbf{V}$ we define $\mathscr{C}_h(\boldsymbol{v})|_K := \mathscr{C}_h^K(\boldsymbol{v})$ where $\mathscr{C}_h^K : H^1(K)^d \to \mathbf{V}_h(K)$ is the usual Clément interpolation operator. It is well-known that the operator $\mathscr{C}_h^K$ satisfies the following two properties (see [39, Lemma 1.127]):

(i) there exists $C > 0$ such that

$$\|\mathscr{C}_h^K(\boldsymbol{v}_K)\|_{1,K} \leqslant C\,\|\boldsymbol{v}_K\|_{1,K}, \tag{5.8}$$

(ii) for m and s satisfying $0 \leqslant m \leqslant s$, with $s = 0,\,1$, there is $C > 0$ such that

$$\|\boldsymbol{v}_\tau - \mathscr{C}_h^K(\boldsymbol{v}_K)\,|_\tau\|_{m,\tau} \leq c\,h_\tau^{s-m}\,\|\boldsymbol{v}_\tau\|_{s,\omega_\tau^K}, \tag{5.9}$$

for all $\boldsymbol{v}_\tau \in H^s(\omega_\tau^K)^d$ and all $\tau \in \mathcal{T}_h^K$, where $\omega_\tau^K := \{\tau \in \mathcal{T}_h^K \,:\, \tau \cap \tau' \neq \emptyset\}$. The constants C depend only on k and d. The next lemma recalls the Fortin operator proposed in [45], (which is based on an argument in [16]), adding some additional information about it.

Lemma 5.4. *Assume integers $k \geq 1$ and $\ell \geq 0$ satisfy $k - \ell \geq d$, and let $F \in \mathcal{E}_H$. Then, there exists a mapping $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ such that, for all $\boldsymbol{v} \in \mathbf{V}$, it holds*

$$\begin{aligned} (\Pi_h(\boldsymbol{v})\,|_F, \boldsymbol{\mu}_\ell)_F \;&=\; (\boldsymbol{v}_F, \boldsymbol{\mu}_\ell)_F \quad \text{for all } \boldsymbol{\mu}_\ell \in \mathbb{P}_\ell(F)^d, \\ \int_K \nabla\cdot\Pi_h(\boldsymbol{v})\,|_K \;d\boldsymbol{x} \;&=\; \int_K \nabla\cdot\boldsymbol{v}_K \;d\boldsymbol{x} \quad \text{for all } K \in \mathscr{P}_{\mathcal{H}}, \end{aligned} \tag{5.10}$$

and $(\Pi_h(\boldsymbol{v}), 0) \in \mathscr{N}_{0,h}$ given in (3.3) if $\boldsymbol{v} \in H_0^1(\Omega)^d$. Moreover, there exist constants C such that

$$\|\Pi_h(\boldsymbol{v})\|_{\mathbf{V}} \;\leq\; C\,\|\boldsymbol{v}\|_{\mathbf{V}} \quad \text{and} \quad \sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{\tau\in\mathcal{T}_h^K} h_\tau^{-2}\|\boldsymbol{v}_\tau - \Pi_h\,(\boldsymbol{v})\,|_\tau\|_{0,\tau}^2 \;\leq\; C\,\|\boldsymbol{v}\|_{\mathbf{V}}^2. \tag{5.11}$$

Proof. From [45], it follows there exists a mapping $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ that satisfies the first equality in (5.10) and the left inequality in (5.11). The second equality in (5.10) follows from the first one after integration by parts and $\boldsymbol{n}^K|_F \in \mathbb{P}_0(F)^d$. As for the second inequality in (5.11), we recall the definition of $\Pi_h(\cdot)$ defined locally on each $K \in \mathscr{P}_{\mathcal{H}}$ as follows (see [45, Lemma 4.2])

$$\Pi_h(\boldsymbol{v})\,|_K := \mathscr{C}_h^K(\boldsymbol{v}_K) + \sum_{F\subset\partial K} \rho_F^K(\boldsymbol{v}_K - \mathscr{C}_h^K(\boldsymbol{v}_K)), \tag{5.12}$$

where ρ_F^K is a mapping from $H^1(K)^d$ to $\mathbf{V}_h(K)$ (see [45, Lemma 4.1] for details). Mapping ρ_F^K is bounded, for each $\tau \in \mathcal{T}_h^K$, as follows

$$\|\rho_F^K\boldsymbol{v}_K\,|_\tau\|_{0,\tau} + h_\tau\,|\rho_F^K\boldsymbol{v}_K\,|_\tau\;|_{1,\tau} \leq C\,(\|\boldsymbol{v}_\tau\|_{0,\tau} + h_\tau\,|\boldsymbol{v}_\tau|_{1,\tau}).$$

Using this, mesh regularity, and (5.8)-(5.9), we get

$$\begin{aligned}
&\sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{\tau\in\mathcal{T}_h^K} h_\tau^{-2}\|\boldsymbol{v}_\tau-\Pi_h(\boldsymbol{v})\,|_\tau\|_{0,\tau}^2\\
&\qquad\le \sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{\tau\in\mathcal{T}_h^K}\Big(h_\tau^{-2}\|\boldsymbol{v}_\tau-\mathscr{C}_h^K(\boldsymbol{v}_K)\,|_\tau\|_{0,\tau}^2+h_\tau^{-2}\sum_{F\subset\partial K\cap\partial\tau}\|\rho_F^K(\boldsymbol{v}_K-\mathscr{C}_h^K(\boldsymbol{v}_K))\,|_\tau\|_{0,\tau}^2\Big)\\
&\qquad\le C\sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{\tau\in\mathcal{T}_h^K}\Big(h_\tau^{-2}\|\boldsymbol{v}_\tau-\mathscr{C}_h^K(\boldsymbol{v}_K)\,|_\tau\|_{0,\tau}^2+\|\boldsymbol{v}_\tau-\mathscr{C}_h^K(\boldsymbol{v}_K)\,|_\tau\|_{1,\tau}^2\Big)\le C\,\|\boldsymbol{v}\|_{\mathbf{V}}^2.
\end{aligned}$$

Finally, if one restricts $\Pi_h(\cdot)$ to the space $H_0^1(\Omega)^d$, then it holds from (5.10) that

$$\begin{aligned}
\sum_{K\in\mathscr{P}_{\mathcal{H}}}\int_{\partial K}\Pi_h(\boldsymbol{v})\,|_{\partial K}\,\boldsymbol{\mu}_\ell\,|_{\partial K}\;\mathrm{d}\boldsymbol{s}&=\sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{F\subset\partial K}\int_F\Pi_h(\boldsymbol{v})\,|_F\,\boldsymbol{\mu}_\ell\,|_F\;\mathrm{d}\boldsymbol{s}\\
&=\sum_{K\in\mathscr{P}_{\mathcal{H}}}\sum_{F\subset\partial K}\int_F\boldsymbol{v}_F\,\boldsymbol{\mu}_\ell\,|_F\;\mathrm{d}\boldsymbol{s}=\sum_{K\in\mathscr{P}_{\mathcal{H}}}\int_{\partial K}\boldsymbol{v}_{\partial K}\,\boldsymbol{\mu}_\ell\,|_{\partial K}\;\mathrm{d}\boldsymbol{s}=0,
\end{aligned}$$

for all $\boldsymbol{\mu}_\ell\in\boldsymbol{\Lambda}_H^\ell$, and then $(\Pi_h(\boldsymbol{v}),0)$ belongs to $\mathscr{N}_{b,h}$. □

The next result generalizes [43, Lemma 3.3].

Lemma 5.5. *Let $(0,q)\in\mathscr{N}_{b,h}$ given in* (3.3). *There exist positive constants C_1 and C_2, independent of mesh parameters, such that for all $q\in Q$ we have*

$$\sup_{(\boldsymbol{v},0)\in\mathscr{N}_{b,h}}\frac{(\nabla\cdot\boldsymbol{v},q)_{\mathscr{P}_{\mathcal{H}}}}{\|\boldsymbol{v},0\|_{\mathbf{V}\times Q}}\;\ge\;C_1\|q\|_Q-C_2|q|_h,\tag{5.13}$$

where $|q|_h^2:=\sum_{K\in\mathscr{P}_{\mathcal{H}}}|q_K|_{h,K}^2$ with $|q_K|_{h,K}:=\Big(\sum_{\tau\in\mathcal{T}_h^K}\delta_\tau\|\nabla q_\tau\|_{0,\tau}^2\Big)^{1/2}$ is a semi-norm on Q.

Proof. Let $(0,q)\in\mathscr{N}_{b,h}$ and $\boldsymbol{v}\in H_0^1(\Omega)^d$ be such that

$$\nabla\cdot\boldsymbol{v}=q\quad\text{and}\quad\|\boldsymbol{v}\|_{1,\Omega}\le C\|q\|_Q.$$

Define $(\widetilde{\boldsymbol{v}},0):=(\Pi_h(\boldsymbol{v}),0)\in\mathscr{N}_{b,h}$ where $\Pi_h(\cdot)$ is the Fortin operator available from Lemma 5.4. Given $K\in\mathscr{P}_{\mathcal{H}}$, let κ_F be an element of Ξ_H^K such that $\partial\kappa_F\cap\partial K=F$. Set $q_0:=\frac{1}{|\kappa_F|}\int_{\kappa_F}q_{\kappa_F}\;\mathrm{d}\boldsymbol{x}$, and define h_{κ_F} the diameter of κ_F. From the trace and Poincaré inequalities

$$\|q_F-q_0\,|_F\|_{0,F}\le C\Big(\frac{1}{h_{\kappa_F}^{1/2}}\|q_{\kappa_F}-q_0\|_{0,\kappa_F}+h_{\kappa_F}^{1/2}\|\nabla(q_{\kappa_F}-q_0)\|_{0,\kappa_F}\Big)\le Ch_{\kappa_F}^{1/2}\|\nabla q_{\kappa_F}\|_{0,\kappa_F}.\tag{5.14}$$

Then, by integration by parts, using (5.10), the trace inequality (5.14), (5.7) and (5.11), and the fact that $\delta_\tau^{-1} \leq C \frac{\theta_{max}^\tau + 4\nu}{h_\tau^2}$ (c.f. [14, estimate (39)]), we obtain

$$
\begin{aligned}
&(\nabla \cdot \widetilde{\boldsymbol{v}}, q)_{\mathscr{P}_\mathcal{H}} = (\nabla \cdot \boldsymbol{v}, q)_{\mathscr{P}_\mathcal{H}} + (\nabla \cdot (\widetilde{\boldsymbol{v}} - \boldsymbol{v}), q)_{\mathscr{P}_\mathcal{H}} \\
&= \|q\|_Q^2 + (\nabla \cdot (\widetilde{\boldsymbol{v}} - \boldsymbol{v}), q)_{\mathscr{P}_\mathcal{H}} \\
&= \|q\|_Q^2 - \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} (\widetilde{\boldsymbol{v}}_\tau - \boldsymbol{v}_\tau, \nabla q_\tau)_\tau + \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} (\widetilde{\boldsymbol{v}}_{\partial\tau} - \boldsymbol{v}_{\partial\tau}, q_{\partial\tau}\, \boldsymbol{n}_\tau)_{\partial\tau} \\
&= \|q\|_Q^2 - \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} (\widetilde{\boldsymbol{v}}_\tau - \boldsymbol{v}_\tau, \nabla q_\tau)_\tau + \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \sum_{\gamma \subset \partial K \cap \partial\tau} (\widetilde{\boldsymbol{v}}_\gamma - \boldsymbol{v}_\gamma, q_\gamma\, \boldsymbol{n}_\gamma^K)_\gamma \\
&= \|q\|_Q^2 - \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} (\widetilde{\boldsymbol{v}}_\tau - \boldsymbol{v}_\tau, \nabla q_\tau)_\tau + \sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \sum_{\gamma \subset \partial K \cap \partial\tau} (\widetilde{\boldsymbol{v}}_\gamma - \boldsymbol{v}_\gamma, (q_\gamma - q_0|_\gamma)\boldsymbol{n}_\gamma^K)_\gamma \\
&\geq \|q\|_Q^2 - \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau^{-1} \|\boldsymbol{v}_\tau - \widetilde{\boldsymbol{v}}_\tau\|_{0,\tau}^2 \Big)^{1/2} \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} \\
&- \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \sum_{\gamma \subset \partial K \cap \partial\tau} h_\gamma^{-1} \|\widetilde{\boldsymbol{v}}_\gamma - \boldsymbol{v}_\gamma\|_{0,\gamma}^2 \Big)^{1/2} \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \sum_{\gamma \subset \partial K \cap \partial\tau} h_\gamma \,\|q_\gamma - q_0|_\gamma\|_{0,\gamma}^2 \Big)^{1/2} \\
&\geq \|q\|_Q^2 - C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} h_\tau^{-2} \|\boldsymbol{v}_\tau - \widetilde{\boldsymbol{v}}_\tau\|_{0,\tau}^2 \Big)^{1/2} \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} \\
&- C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} (h_\tau^{-2} \|\boldsymbol{v}_\tau - \widetilde{\boldsymbol{v}}_\tau\|_{0,\tau}^2 + \|\nabla(\boldsymbol{v}_\tau - \widetilde{\boldsymbol{v}}_\tau)\|_{0,\tau}^2) \Big)^{1/2} \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \sum_{\gamma \subset \partial K \cap \partial\tau} h_\gamma \,\|q_\gamma - q_0|_\gamma\|_{0,\gamma}^2 \Big)^{1/2} \\
&\geq \|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}} \Big[C_1 \,\|q\|_Q - C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} - C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{F \subset \partial K} \sum_{\tau \subset \mathcal{T}_h^{\kappa_F}} h_\tau^2 \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} \Big] \\
&\geq \|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}} \Big[C_1 \,\|q\|_Q - C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} - C \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{F \subset \partial K} \sum_{\tau \subset \mathcal{T}_h^{\kappa_F}} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} \Big] \\
&\geq \|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}} \Big[C_1 \,\|q\|_Q - C_2 \Big(\sum_{K \in \mathscr{P}_\mathcal{H}} \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\nabla q_\tau\|_{0,\tau}^2 \Big)^{1/2} \Big],
\end{aligned}
$$

and the result follows. □

5.1.2. *Well-posedness.* We prove that the two-level MHM method (5.1) with the stabilized finite element method (5.3)-(5.4) is well-posed in the following theorem.

Theorem 5.1 (Well-posedness of (5.1) with (5.3)-(5.4))**.** *Assume that $k-\ell \geq d$ with $\ell \geq 0$ if $\boldsymbol{\Lambda}_h^\ell$ is discontinous and $\ell \geq 1$ otherwise, and assume that (5.7) is satisfied. Then, there exists a unique solution of (5.1) with the stabilized method (5.3)-(5.4) denoted by $(\boldsymbol{u}_0^{H,h}, 0) \in \mathscr{N}_a$ and $(\boldsymbol{\lambda}_{H,h}, \rho_{H,h}) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$, such that*

$$\|\boldsymbol{\lambda}_{H,h}, \rho_{H,h}\|_{\boldsymbol{\Lambda}\times Q} \leq C \left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a} \frac{(\boldsymbol{f};\, \boldsymbol{v}_0, 0)_{\mathbf{V}\times Q}}{\|\boldsymbol{v}_0, 0\|_{\mathbf{V}\times Q}} + \|\widehat{T}_h(\boldsymbol{f})\|_{\mathbf{V}\times Q} + \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_H^\ell} \frac{g(\boldsymbol{\mu}, 0)}{\|\boldsymbol{\mu}, 0\|_{\boldsymbol{\Lambda}\times Q}} \right),$$
$$\|\boldsymbol{u}_0^{H,h}, 0\|_{\mathbf{V}\times Q} \leq C \left(\sup_{(\boldsymbol{v}_0,0)\in\mathscr{N}_a} \frac{(\boldsymbol{f};\, \boldsymbol{v}_0, 0)_{\mathbf{V}\times Q}}{\|\boldsymbol{v}_0, 0\|_{\mathbf{V}\times Q}} + \|\widehat{T}_h(\boldsymbol{f})\|_{\mathbf{V}\times Q} + \sup_{\boldsymbol{\mu}\in\boldsymbol{\Lambda}_H^\ell} \frac{g(\boldsymbol{\mu}, 0)}{\|\boldsymbol{\mu}, 0\|_{\boldsymbol{\Lambda}\times Q}} \right).$$

Proof. We prove conditions (H2)–(H3) as (H1) holds from (5.5). First, condition (H3) follows from the Fortin operator in Lemma 5.4 under the condition $k \geq \ell + d$, and using the L^2 orthogonal projection $\Pi_0(r) := \frac{1}{|\Omega|}\int_\Omega r \, \mathrm{d}\boldsymbol{x}$. In fact, let $(\boldsymbol{\mu}, \xi) \in \boldsymbol{\Lambda}_H^\ell \times \mathbb{R}$ then

$$\begin{aligned}
\|\boldsymbol{\mu}, \xi\|_{\boldsymbol{\Lambda}\times\mathbb{R}} &= \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu}, \xi;\, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} = \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu}, \xi;\, \Pi_h(\boldsymbol{w}), \Pi_0(r))}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}} \\
&\leq C \sup_{(\boldsymbol{w},r)\in\mathbf{V}\times Q} \frac{b(\boldsymbol{\mu}, \xi;\, \Pi_h(\boldsymbol{w}), \Pi_0(r))}{\|\Pi_h(\boldsymbol{w}), \Pi_0(r)\|_{\mathbf{V}\times Q}} \leq C \sup_{(\boldsymbol{w},r)\in\mathbf{V}_h\times Q_h} \frac{b(\boldsymbol{\mu}, \xi;\, \boldsymbol{w}, r)}{\|\boldsymbol{w}, r\|_{\mathbf{V}\times Q}},
\end{aligned}$$

where we used (2.15). As for condition (H2), we adapt the strategy of the proof of [14, Lemma 4.2]. Let $(\boldsymbol{v}, q) \in \mathscr{N}_{b,h}$ and set $\boldsymbol{w} := \boldsymbol{v} - \varepsilon\, \boldsymbol{z}$, where $\boldsymbol{z}$ achieves the supremum in Lemma 5.5 with properties $\|\boldsymbol{z}\|_{\mathbf{V}} = \|q\|_{0,\Omega}$, and $(\boldsymbol{z}, 0) \in \mathscr{N}_{b,h}$. First, [14] establishes

$$a_{h,K}(\boldsymbol{v}_K, q_K; \boldsymbol{v}_K, q_K) \geq \frac{3\nu}{4}\|\nabla \boldsymbol{v}_K\|_{0,K}^2 + |q_K|_{h,K}^2 \quad \text{for all } K \in \mathscr{P}_{\mathcal{H}}.$$

Further, given $K \in \mathscr{P}_{\mathcal{H}}$ define $\theta_{max}^K := \max_{\tau\in\mathcal{T}_h^K} \theta_{max}^\tau$. Since $\delta_\tau\theta_{max}^\tau \leq 1$ and $\delta_\tau\nu\|\boldsymbol{\Delta}\boldsymbol{v}_\tau\|_{0,\tau}^2 \leq \frac{1}{4}\|\nabla\boldsymbol{v}_\tau\|_{0,\tau}^2$, and using $ab \leq \frac{1}{2\gamma}a^2 + \frac{\gamma}{2}b^2$ we get

$$
\begin{aligned}
a_{h,K}(\boldsymbol{v}_K, q_K; -\boldsymbol{z}_K, 0) \geq{}& -\theta_{max}^K \|\boldsymbol{v}_K\|_{0,K} \|\boldsymbol{z}_K\|_{0,K} - \nu \|\nabla \boldsymbol{v}_K\|_{0,K} \|\nabla \boldsymbol{z}_K\|_{0,K} + (\nabla \cdot \boldsymbol{z}_K, q_K)_K \\
& - \sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \left(\|\boldsymbol{\theta} \boldsymbol{v}_\tau - \nu \Delta \boldsymbol{v}_\tau\|_{0,\tau} + \|\nabla q_K\|_{0,\tau} \right) \|\boldsymbol{\theta} \boldsymbol{z}_\tau - \nu \Delta \boldsymbol{z}_\tau\|_{0,\tau} \\
\geq{}& - \theta_{max}^K \|\boldsymbol{v}_K\|_{0,K} \|\boldsymbol{z}\|_{0,K} - \nu \|\nabla \boldsymbol{v}_K\|_{0,K} \|\nabla \boldsymbol{z}_K\|_{0,K} + (\nabla \cdot \boldsymbol{z}_K, q_K)_K \\
& - \left[\left(\sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\boldsymbol{\theta} \boldsymbol{v}_\tau - \nu \Delta \boldsymbol{v}_\tau\|_{0,\tau}^2 \right)^{1/2} + |q_K|_{h,K} \right] \left(\sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\boldsymbol{\theta} \boldsymbol{z}_\tau - \nu \Delta \boldsymbol{z}_\tau\|_{0,\tau}^2 \right)^{1/2} \\
\geq{}& - \left(\theta_{max}^K \|\boldsymbol{v}_K\|_{0,K}^2 + \nu \|\nabla \boldsymbol{v}_K\|_{0,K}^2 \right)^{1/2} \left(\theta_{max}^K \|\boldsymbol{z}_K\|_{0,K}^2 + \nu \|\nabla \boldsymbol{z}_K\|_{0,K}^2 \right)^{1/2} + (\nabla \cdot \boldsymbol{z}_K, q_K)_K \\
& - \left[\left(2 \sum_{\tau \in \mathcal{T}_h^K} \theta_{max}^\tau \|\boldsymbol{v}_\tau\|_{0,\tau}^2 + \frac{\nu}{4} \|\nabla \boldsymbol{v}_\tau\|_{0,\tau}^2 \right)^{1/2} + |q_K|_{h,K} \right] \left(\sum_{\tau \in \mathcal{T}_h^K} \delta_\tau \|\boldsymbol{\theta} \boldsymbol{z}_\tau - \nu \Delta \boldsymbol{z}_\tau\|_{0,\tau}^2 \right)^{1/2} \\
\geq{}& - 3 \left(\theta_{max}^K \|\boldsymbol{v}_K\|_{0,K}^2 + \nu \|\nabla \boldsymbol{v}_K\|_{0,K}^2 \right)^{1/2} \left(\theta_{max}^K \|\boldsymbol{z}_K\|_{0,K}^2 + \nu \|\nabla \boldsymbol{z}_K\|_{0,K}^2 \right)^{1/2} + (\nabla \cdot \boldsymbol{z}_K, q_K)_K \\
& - |q_K|_{h,K} \left(2 \sum_{\tau \in \mathcal{T}_h^K} \theta_{max}^\tau \|\boldsymbol{z}_\tau\|_{0,\tau}^2 + \frac{\nu}{4} \|\nabla_{\mathcal{H}} \boldsymbol{z}_\tau\|_{0,\tau}^2 \right)^{1/2} \\
\geq{}& - \frac{3}{2\gamma_1} \left(\theta_{max}^K \|\boldsymbol{v}_K\|_{0,K}^2 + \nu \|\nabla \boldsymbol{v}_K\|_{0,K}^2 \right) - \frac{3\gamma_1 \widetilde{C}}{2} \Big(\frac{1}{d_\Omega^2} \|\boldsymbol{z}_K\|_{1,K}^2 + \|\nabla \boldsymbol{z}_K\|_{0,K}^2 \Big) + (\nabla \cdot \boldsymbol{z}_K, q_K)_K \\
& - \frac{1}{2\gamma_2} |q_K|_{h,K}^2 - \frac{2\gamma_2 \widetilde{C}}{2} \Big(\frac{1}{d_\Omega^2} \|\boldsymbol{z}_K\|_{1,K}^2 + \|\nabla \boldsymbol{z}_K\|_{0,K}^2 \Big),
\end{aligned}
$$

where $\widetilde{C} := \max\{\theta_{max}^K d_\Omega^2, \nu\}$.

Therefore, using the lower bound on $(\nabla \cdot \boldsymbol{z}_K, q_K)_K$ from Lemma 5.5, and $\|\boldsymbol{z}\|_{\mathbf{V}} = \|q\|_Q$, we get

$$
\begin{aligned}
a_h(\boldsymbol{v},q;\boldsymbol{w},q) \geq & \frac{3\nu}{2}\left(\frac{1}{2}-\frac{\varepsilon}{\gamma_1}\right)\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} - \frac{3\,c_{max}\varepsilon}{2\gamma_1}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \varepsilon(q,\nabla\cdot\boldsymbol{z})_{\mathscr{P}_{\mathcal{H}}} \\
& - \frac{\widetilde{C}\varepsilon}{2}(3\gamma_1+2\gamma_2)\|\boldsymbol{z}\|^2_{\mathbf{V}} + \left(1-\frac{\varepsilon}{2\gamma_2}\right)|q|^2_h \\
\geq & \frac{3\nu}{2}\left(\frac{1}{2}-\frac{\varepsilon}{\gamma_1}\right)\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} - \frac{3\,c_{max}\varepsilon}{2\gamma_1}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \varepsilon(C_1\|q\|^2_Q - C_2|q|_h\|q\|_Q) \\
& - \frac{\widetilde{C}\varepsilon}{2}(3\gamma_1+2\gamma_2)\|q\|^2_Q + \left(1-\frac{\varepsilon}{2\gamma_2}\right)|q|^2_h \\
\geq & \frac{3\nu}{2}\left(\frac{1}{2}-\frac{\varepsilon}{\gamma_1}\right)\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} - \frac{3\,c_{max}\varepsilon}{2\gamma_1}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \frac{1}{2}\left(2-\frac{\varepsilon}{\gamma_2}-\frac{C_2\varepsilon}{\gamma_3}\right)|q|^2_h \\
& + \frac{\varepsilon}{2}\left(2C_1-\widetilde{C}\,(3\gamma_1+2\gamma_2)-C_2\gamma_3\right)\|q\|^2_Q \\
= & \frac{3\nu}{2}\left(\frac{1}{2}-12\widetilde{C}\frac{\varepsilon}{C_1}\right)\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} - 18\,c_{max}\widetilde{C}\frac{\varepsilon}{C_1}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \frac{\varepsilon C_1}{2}\|q\|^2_Q \\
& + \left(1-\frac{\varepsilon}{C_1}\left(4\widetilde{C}+C_2^2\right)\right)|q|^2_h,
\end{aligned}
$$

where we used $\gamma_1 = \frac{C_1}{12\widetilde{C}}$, $\gamma_2 = \frac{C_1}{8\widetilde{C}}$, and $\gamma_3 = \frac{C_1}{2C_2}$. Now, since $(\boldsymbol{v},q) \in \mathscr{N}_{b,h}$, we have the Poincaré inequality [21]

$$
\frac{C_P}{d_\Omega^2}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} \leq \|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}},
$$

where C_P is a positive constant independent of mesh parameters, from which we find

$$
\begin{aligned}
a_{h,K}(\boldsymbol{v},q;\boldsymbol{w},q) \geq & \frac{3\nu}{4}\left(\frac{1}{2}-12\widetilde{C}\frac{\varepsilon}{C_1}\right)\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \frac{3}{8}\left(\frac{C_P\nu}{d_\Omega^2}-48c_{max}\widetilde{C}\frac{\varepsilon}{C_1}\right)\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \frac{\varepsilon C_1}{2}\|q\|^2_Q \\
& + \left(1-\frac{\varepsilon}{C_1}\left(4\widetilde{C}+C_2^2\right)\right)|q|^2_h \\
\geq & \frac{3\nu}{16}\left(\|\nabla_{\mathcal{H}}\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}} + \frac{C_P}{d_\Omega^2}\|\boldsymbol{v}\|^2_{\mathscr{P}_{\mathcal{H}}}\right) + \frac{\varepsilon C_1}{2}\|q\|^2_Q \geq C\,\|\boldsymbol{v},q\|^2_{\mathbf{V}\times Q},
\end{aligned}
$$

where we took $\frac{\varepsilon}{C_1} = \min\{\frac{1}{48\widetilde{C}}, \frac{C_P\nu}{96c_{max}\widetilde{C}d_\Omega^2}, \frac{1}{4\widetilde{C}+C_2^2}\}$. Finally, condition (H2) follows observing that

$$
\|\boldsymbol{w},q\|_{\mathbf{V}\times Q} \leq \|\boldsymbol{v},q\|_{\mathbf{V}\times Q} + \varepsilon\,\|\boldsymbol{z},0\|_{\mathbf{V}\times Q} \leq \|\boldsymbol{v},q\|_{\mathbf{V}\times Q} + C\,\varepsilon\,\|\mathbf{0},q\|_{\mathbf{V}\times Q} \leq C\,\|\boldsymbol{v},q\|_{\mathbf{V}\times Q},
$$

and the result follows from Theorem 3.1. □

Remark 5.6 (Relaxing $k-\ell \geq d$ constraint)**.** *The constraint $k-d \geq \ell \geq 0$ in Theorem 5.1 leads to the well-posedness of eq.* (5.1) *in two and three-dimensional problems on simplicial* *meshes. In the two-dimensional case, other possibilities exist to ensure the well-posedness of eq.* (5.1)*, such as*

- $\mathcal{T}_h = \mathscr{P}_{\mathcal{H}}$*, and even polynomial degree* ℓ *under the constraint* $k = \ell + 1$*. The proof follows from* [61, Lemma 4] *applied to each component of the vector function* $\boldsymbol{\mu}_H^\ell \in \boldsymbol{\Lambda}_H^\ell$*;*
- *meshes with "enough" refined local meshes* $\mathcal{T}_h^K$ *under the conditions* $k = \ell + 1$ *with* $\ell \geq 0$ *or* $k = \ell$ *with* $\ell \geq 1$*. The proof follows from* [16, Lemmas 4 and 5] *applied to each component of the vector function* $\boldsymbol{\mu}_H^\ell \in \boldsymbol{\Lambda}_H^\ell$. □

5.1.3. *Convergence.* We recall from [14, Theorem 4.1] that the mapping T_h and $\widehat{T}_h$ carry approximation properties. In other words, adopted element-wisely in each $K \in \mathscr{P}_{\mathcal{H}}$, they approximate T and $\widehat{T}$ with sharp constants which are independent of mesh parameters. Specifically, assume that $(\boldsymbol{u}, p) \in H^{m+1}(\mathscr{P}_{\mathcal{H}})^d \times H^m(\mathscr{P}_{\mathcal{H}})$, for $1 \leq m \leq k$ and $k \geq 1$, it holds

$$\|(T - T_h)(\boldsymbol{\lambda}, \rho) + (\widehat{T} - \widehat{T}_h)(\boldsymbol{f})\|_{\mathbf{V}\times Q} \leq C\, h^m \left(|\boldsymbol{u}|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}}\right), \tag{5.15}$$

where T_h and $\widehat{T}_h$ are defined from (3.1) with (5.3)-(5.4). We are ready to present the main convergence result.

Theorem 5.2 (Convergence of (5.1) with (5.3)-(5.4))**.** *Assume* $k - \ell \geq d$*, with* $\ell \geq 0$ *(*$\ell \geq 1$*) if* $\boldsymbol{\Lambda}_H^\ell$ *is discontinuous (continuous), and* $(\boldsymbol{u}, p) \in H^{m+1}(\mathscr{P}_{\mathcal{H}})^d \times H^m(\mathscr{P}_{\mathcal{H}})$*, with* $1 \leq m \leq \ell + 1$*, and* $(\nu \nabla \boldsymbol{u} - p\mathbb{I}) \in \boldsymbol{H}(\mathrm{div};\Omega)$*. Then, there exist positive constants* C_i*,* $i = 1, ...4$*, independent of* $\mathcal{H}$*,* H*,* h*, such that*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{H,h}\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^{H,h}\|_{\mathbf{V}} \leq C_1 H^m \Big(|\nu \nabla_{\mathcal{H}} \boldsymbol{u}|_{m,\mathscr{P}_{\mathcal{H}}} + |p|_{m,\mathscr{P}_{\mathcal{H}}}\Big) + C_2 h^m \Big(|\boldsymbol{u}|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}}\Big),$$

$$\|\boldsymbol{u} - \boldsymbol{u}_{H,h}, p - p_{H,h}\|_{\mathbf{V}\times Q} \leq C_3 H^m \Big(|\nu \nabla_{\mathcal{H}} \boldsymbol{u}|_{m,\mathscr{P}_{\mathcal{H}}} + |p|_{m,\mathscr{P}_{\mathcal{H}}}\Big) + C_4 h^m \Big(|\boldsymbol{u}|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}}\Big),$$

where $(\boldsymbol{\lambda}_{H,h}, \boldsymbol{u}_0^{H,h})$ *solves* (5.1) *with* (5.3)-(5.4)*, and* $(\boldsymbol{u}_{H,h}, p_{H,h})$ *is given in* (5.2).

Proof. The result is a direct consequence of Theorem 3.2, Lemma 4.1, and (5.15). □

Remark 5.7 (Local conservation)**.** *The discrete velocity* $\boldsymbol{u}_{H,h}$ *built according to* (5.2) *preserves the divergence-free constraint weakly. Indeed, taking* $(\boldsymbol{w}_h, q_h)|_K = (0, 1_K) \in \mathscr{N}_{a,s}^{\perp}(K)$ *in* (5.3)–(5.4) *and using* $\rho_{H,h} = 0$*, it holds*

$$\int_K \nabla \cdot \boldsymbol{u}_{H,h}\; d\boldsymbol{x} = 0 \quad \text{for all } K \in \mathscr{P}_{\mathcal{H}}\,.$$

In addition, the local equilibrium between the numerical traction $\boldsymbol{\lambda}_{H,h}$ and the external force is satisfied, i.e.,

$$\int_K \boldsymbol{\theta}\, \boldsymbol{u}_{H,h}\; d\boldsymbol{x} + \int_{\partial K} \boldsymbol{\lambda}_{H,h}\; d\boldsymbol{s} \;=\; \int_K \boldsymbol{f}\; d\boldsymbol{x} \quad \textit{for all } K \in \mathscr{P}_{\mathcal{H}}, \tag{5.16}$$

where we used the second equation in (5.1) *if $\boldsymbol{\theta} = \mathbf{0}$, and the definition of $\boldsymbol{u}_{H,h}$ in* (5.2) *and local problems* (3.1) *with* (5.3)–(5.4) *tested with $(\boldsymbol{w}_h, q_h)|_K = (\mathbf{1}_K, 0) \in \mathbf{V}_h(K) \times Q_h(K)$ otherwise.* □

5.2. **Analysis of a *stable* two-level MHM method.** Stable pairs of spaces may be used in local problems (3.1) to solve MHM formulation (5.1). Notably, we analyze here a two-level method based on the Taylor-Hood element (c.f. [19]). Recall that the Taylor-Hood element is such that given $k \geq 2$ and $K \in \mathscr{P}_{\mathcal{H}}$,

$$\begin{aligned} \mathbf{V}_h(K) &:= \left\{ \boldsymbol{v}_h \in \mathbf{V}(K) \cap C^0(K)^d \,:\, \boldsymbol{v}_h\,|_\tau \in \mathbb{P}_k(\tau)^d \quad \forall \tau \in \mathcal{T}_h^K \right\}, \\ Q_h(K) &:= \left\{ q_h \in Q(K) \cap C^0(K) \,:\, q_h\,|_\tau \in \mathbb{P}_{k-1}(\tau) \quad \forall \tau \in \mathcal{T}_h^K \right\}, \end{aligned}$$

and velocity and pressure are approximated in the corresponding global finite-dimensional spaces

$$\mathbf{V}_h := \bigoplus_{K \in \mathscr{P}_{\mathcal{H}}} \mathbf{V}_h(K) \quad \text{and} \quad Q_h := \bigoplus_{K \in \mathscr{P}_{\mathcal{H}}} Q_h(K). \tag{5.17}$$

We then have a fully defined MHM method by letting $a_{h,K}(\cdot;\cdot) := a_K(\cdot;\cdot)$ and $l^{\boldsymbol{f}}_{h,K}(\cdot) = (\boldsymbol{f}, \cdot)_K$ as defined in (2.10) and (2.11), respectively.

To see that the method (5.1) is well-posed and converges optimally using these definitions, we first note that assumption (H1) is met when using spaces $\mathbf{V}_h(K)$ and $Q_h(K)$ (c.f. [17, 18]), with a constant independent of K, when the sub-mesh $\mathcal{T}_h^K$ satisfies the following mild conditions:

Assumption (M)**.**

<u>*2D case:*</u> $\mathcal{T}_h^K$ *contains at least one internal vertex;*

<u>*3D case:*</u> *Every tetrahedron in $\mathcal{T}_h^K$ has at least one internal vertex.*

Lemma 5.8. *Given $K \in \mathscr{P}_{\mathcal{H}}$, assume that $\mathcal{T}_h^K$ satisfies Assumption (M). Then, condition* (H1) *holds, and*

$$\|(T - T_h)(\boldsymbol{\lambda}, \rho) + (\widehat{T} - \widehat{T}_h)(\boldsymbol{f})\|_{\mathbf{V}\times Q} \leq C\, h^m \Big(\|\boldsymbol{u}\|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}} \Big), \tag{5.18}$$

where $1 \leq m \leq k$, and $k \geq 2$. Moreover, if $\mathcal{T}_h^K$ is a one-element mesh, then condition (H1) *and estimate* (5.18) *also hold.*

Proof. We use the standard technique for mixed problems to prove (H1) on each $K \in \mathscr{P}_{\mathcal{H}}$ (see [19]). Note that

$$(\nu\, \nabla \boldsymbol{v}_h, \nabla \boldsymbol{v}_h)_K + (\boldsymbol{\theta}\, \boldsymbol{v}_h, \boldsymbol{v}_h)_K \geq C\, \|\boldsymbol{v}_h\|_{1,K}^2 \tag{5.19}$$

for all $(\boldsymbol{v}_h, 0) \in \mathscr{N}_{a,h}^{\perp}(K)$ where we used Poincaré inequality for the Stokes case ($\boldsymbol{\theta} = \mathbf{0}$). It remains to prove that velocity and pressure spaces are compatible in the sense of Babuska-Brezzi, i.e., given $(0, q_h) \in \mathscr{N}_{a,h}^{\perp}(K)$, there exists $(\boldsymbol{v}_h, 0) \in \mathscr{N}_{a,h}^{\perp}(K)$ such that

$$(\nabla \cdot \boldsymbol{v}_h, q_h)_K \geq C\, \|q_h\|_{0,K} \|\boldsymbol{v}_h\|_{1,K}. \tag{5.20}$$

First, let the case $\boldsymbol{\theta} \neq \mathbf{0}$, e.g., $\mathscr{N}_{a,h}^{\perp}(K) = \mathbf{V}_h(K) \times Q_h(K)$. From the assumption on the partition $\mathcal{T}_h^K$ the spaces $\mathbf{V}_h(K) \cap H_0^1(K)^d$ and $Q_h(K) \cap L_0^2(K)$ are compatible (see [17, Theorem 4.1] and [18, Theorem 3.1]). Since $\mathbf{V}_h(K) \cap H_0^1(K)^d \subset \mathbf{V}_h(K)$, the spaces $Q_h(K) \cap L_0^2(K)$ and $\mathbf{V}_h(K)$ are also inf-sup stable. As for the pressure $q_h = 1_K \in Q_h(K)$, we select $\boldsymbol{v}_h$ in the lowest-order Raviart-Thomas space $RT_0(K) \subset \mathbf{V}_h(K)$ and then (5.20) holds.

Next, we assume $\boldsymbol{\theta} = \mathbf{0}$, e.g., $\mathscr{N}_{a,h}^{\perp}(K) = [\mathbf{V}_h(K) \cap L_0^2(K)^d] \times Q_h(K)$. Given $q_h \in Q_h(K)$, take $\boldsymbol{v}_h \in \mathbf{V}_h(K)$ from the previous case, and define $\boldsymbol{w}_h := \boldsymbol{v}_h - \frac{1}{|K|} \int_K \boldsymbol{v}_h \, \mathrm{d}\boldsymbol{x} \in \mathbf{V}_h(K) \cap L_0^2(K)^d$. Note that $\|\boldsymbol{w}_h\|_{1,K} \leq C\, \|\boldsymbol{v}_h\|_{1,K}$, and then (5.20) holds as

$$(\nabla \cdot \boldsymbol{w}_h, q_h)_K = (\nabla \cdot \boldsymbol{v}_h, q_h)_K \geq C\, \|q_h\|_{0,K} \|\boldsymbol{v}_h\|_{1,K} \geq C\, \|q_h\|_{0,K} \|\boldsymbol{w}_h\|_{1,K}.$$

When $\mathcal{T}_h^K$ is a mesh of one element, the condition (5.20) follows noting that the image of the Raviart-Thomas space $RT_k(K) \subset \mathbf{V}_h(K)$ by the divergence operator coincides with $\mathbb{P}_{k-1}(K)$. Therefore, we recover the condition (H1) from the combination of (5.19) and (5.20) using [65, Theorem 3]. Finally, classical arguments based on the best approximation property of the Galerkin method and interpolation results applied to each $K \in \mathscr{P}_{\mathcal{H}}$ result in (5.18). ☐

The two-level MHM method (5.1) with the stable second-level solver (3.1) is well-posed and convergent. This is establish in the next theorem.

Theorem 5.3. *Assume that $\mathcal{T}_h^K$ satisfies Assumption (M). Then, the method (5.1) with the local problems (3.1) defined using (2.10)-(2.11) is well-posed with $k - \ell \geq d$ and $\ell \geq 0$ if $\boldsymbol{\Lambda}_H^{\ell}$ is discontinuous and $\ell \geq 1$ otherwise. Moreover, if $(\boldsymbol{u}, p) \in H^{m+1}(\mathscr{P}_{\mathcal{H}})^d \times H^m(\mathscr{P}_{\mathcal{H}})$, with $1 \leq m \leq \ell + 1$, and $(\nu \nabla \boldsymbol{u} - p\mathbb{I}) \in \boldsymbol{H}(\mathrm{div}; \Omega)$ then there exist C such that*

$$\begin{aligned} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{H,h}\|_{\boldsymbol{\Lambda}} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^{H,h}\|_{\mathbf{V}} &\leq C\, (H^m + h^m) \left(\|\boldsymbol{u}\|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}} \right), \\ \|\boldsymbol{u} - \boldsymbol{u}_{H,h}\|_{\mathbf{V}} + \|p - p_{H,h}\|_Q &\leq C\, (H^m + h^m) \left(\|\boldsymbol{u}\|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}} \right), \end{aligned} \tag{5.21}$$

where $(\boldsymbol{\lambda}_{H,h}, \boldsymbol{u}_0^{H,h})$ solves (5.1) with (2.10)-(2.11), and $(\boldsymbol{u}_{H,h}, p_{H,h})$ is given in (5.2).

Proof. First, observe that (H3) holds by the arguments used in the proof of Theorem 5.1 and (H1) follows from Lemma 5.8. It remains to show condition (H2), which is a consequence of the local inf-sup condition of the generalized Taylor-Hood element presented in the proof of Lemma 5.8, and the Fortin mapping proposed in Lemma 5.4. Specifically, note that given $(0,q) \in \mathscr{N}_{b,h}$, there exists $\boldsymbol{v} \in H_0^1(\Omega)^d$ such that

$$\nabla \cdot \boldsymbol{v} = q \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{v}\|_{1,\Omega} \leq C\,\|q\|_Q. \tag{5.22}$$

Then, we adapt the technique of [44, Lemma 3.5], and define the following "corrector" mapping $\mathscr{M}_h : H_0^1(\Omega)^d \to \mathbf{V}_h \cap H_0^1(\mathscr{P}_{\mathcal{H}})^d$ as

$$\int_K r_h \nabla \cdot \mathscr{M}_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} = \int_K r_h \nabla \cdot (\boldsymbol{v}_K - \Pi_h(\boldsymbol{v})|_K)\,\mathrm{d}\boldsymbol{x} \quad \text{for all } r_h \in Q_h(K) \cap L_0^2(K), \tag{5.23}$$

where $H_0^1(\mathscr{P}_{\mathcal{H}})^d$ denotes the space of functions $\boldsymbol{v}_K \in H_0^1(K)^d$, for all $K \in \mathscr{P}_{\mathcal{H}}$. Observe that $\mathscr{M}_h(\boldsymbol{v})|_K \in \mathbf{V}_h(K) \cap H_0^1(K)^d$ exists under the mesh assumption (M) which allows the use of [17, Theorem 4.1] and [18, Theorem 3.1], and then

$$\|\mathscr{M}_h(\boldsymbol{v})\|_{\mathbf{V}} \leq C\,\|\boldsymbol{v}\|_{\mathbf{V}}. \tag{5.24}$$

Next, using $k \geq d + \ell \geq 2$ by assumption, we set $\Theta_h : H_0^1(\Omega)^d \to \mathbf{V}_h$ as

$$\Theta_h(\boldsymbol{v}) := \Pi_h(\boldsymbol{v}) + \mathscr{M}_h(\boldsymbol{v}),$$

and then from (5.11), (5.24) and (5.22)

$$\|\Theta_h(\boldsymbol{v})\|_{\mathbf{V}} \leq \|\Pi_h(\boldsymbol{v})\|_{\mathbf{V}} + \|\mathscr{M}_h(\boldsymbol{v})\|_{\mathbf{V}} \leq C\,\|\boldsymbol{v}\|_{\mathbf{V}} \leq C\|q\|_Q.$$

Notice from the properties of operator $\Pi_h(\cdot)$ in (5.10) and $\mathscr{M}_h(\boldsymbol{v})|_{\partial K} = 0$ for all $K \in \mathscr{P}_{\mathcal{H}}$, that for all $\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_H^\ell$ it holds

$$(\boldsymbol{\mu}_h, \Theta_h(\boldsymbol{v}))_{\partial\mathscr{P}} = (\boldsymbol{\mu}_h, \Pi_h(\boldsymbol{v}))_{\partial\mathscr{P}} = 0 \;\Rightarrow\; (\Theta_h(\boldsymbol{v}), 0) \in \mathscr{N}_{b,h}.$$

In addition, for all $q \in Q_h$, $q = \widetilde{q} + q^0$ with $\widetilde{q} \in L_0^2(\mathscr{P}_{\mathcal{H}})$ and $q^0 \in \mathbb{P}_0(\mathscr{P}_{\mathcal{H}})$,

$$\begin{aligned}
\int_K q_K \nabla \cdot \Theta_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} &= \int_K \widetilde{q}_K \nabla \cdot \Theta_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} + \int_K q_K^0 \nabla \cdot \Theta_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} \\
&= \int_K \widetilde{q}_K \nabla \cdot \boldsymbol{v}_K \,\mathrm{d}\boldsymbol{x} + \int_K q_K^0 \nabla \cdot \Pi_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} + \int_K q_K^0 \nabla \cdot \mathscr{M}_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} \\
&= \int_K q_K \nabla \cdot \boldsymbol{v}_K \,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where we used the definition of operador $\mathscr{M}_h(\cdot)$ in (5.23), Lemma 5.4 and $\int_K q_K^0 \nabla\cdot\mathscr{M}_h(\boldsymbol{v})|_K \,\mathrm{d}\boldsymbol{x} = 0$ since $\mathscr{M}_h(\boldsymbol{v})|_K \in H_0^1(K)^d$. We conclude that, given $(0,q) \in \mathscr{N}_{b,h}$, there exists $(\widetilde{\boldsymbol{v}},0) := (\Theta_h(\boldsymbol{v}),0) \in \mathscr{N}_{b,h}$ such that

$$(q, \nabla\cdot\widetilde{\boldsymbol{v}})_\Omega \;=\; (q, \nabla\cdot\boldsymbol{v})_\Omega \;=\; \|q\|_Q^2 \quad \text{and} \quad \|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}} \le C\|q\|_Q. \tag{5.25}$$

Let $(\boldsymbol{v},q) \in \mathscr{N}_{b,h}$ and define $\boldsymbol{w} := \boldsymbol{v} - \varepsilon\,\widetilde{\boldsymbol{v}}$. We observe $(\boldsymbol{w},q) \in \mathscr{N}_{b,h}$ and then there exists positive constants C_1 and C_2 independent of mesh parameters, such that

$$\begin{aligned}
a_h(\boldsymbol{v},q;\,\boldsymbol{w},q) &= a_h(\boldsymbol{v},q;\,\boldsymbol{v},q) - \varepsilon\, a_h(\boldsymbol{v},q;\,\widetilde{\boldsymbol{v}},0)\\
&\ge C_1\,\|\boldsymbol{v}\|_{\mathbf{V}}^2 - \varepsilon\left[\nu(\nabla\boldsymbol{v},\nabla\widetilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}} + (\boldsymbol{\theta}\,\boldsymbol{v},\widetilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}} - (q,\nabla\cdot\widetilde{\boldsymbol{v}})_{\mathscr{P}_{\mathcal{H}}}\right]\\
&\ge C_1\,\|\boldsymbol{v}\|_{\mathbf{V}}^2 - \varepsilon\, C_2\|\boldsymbol{v}\|_{\mathbf{V}}\|\widetilde{\boldsymbol{v}}\|_{\mathbf{V}} + \varepsilon\|q\|_Q^2\\
&\ge C_1\,\|\boldsymbol{v}\|_{\mathbf{V}}^2 - \varepsilon\, C_2 C\|\boldsymbol{v}\|_{\mathbf{V}}\|q\|_Q + \varepsilon\,\|q\|_Q^2\\
&\ge \left(C_1 - \frac{C_2 C\,\varepsilon}{2\,\gamma}\right)\|\boldsymbol{v}\|_{\mathbf{V}}^2 + \varepsilon\left(1 - \frac{C_2 C\gamma}{2}\right)\|q\|_Q^2\\
&\ge\; C_3\|\boldsymbol{v},q\|_{\mathbf{V}\times Q}^2,
\end{aligned}$$

where we chose the positive constants γ and ε such that C_3 is a positive constant independent of mesh parameters. In addition, we have

$$\|\boldsymbol{w},q\|_{\mathbf{V}\times Q} \le \|\boldsymbol{v},q\|_{\mathbf{V}\times Q} + \varepsilon\,\|\widetilde{\boldsymbol{v}},0\|_{\mathbf{V}\times Q} \le \|\boldsymbol{v},q\|_{\mathbf{V}\times Q} + C\,\varepsilon\,\|0,q\|_{\mathbf{V}\times Q} \le C\,\|\boldsymbol{v},q\|_{\mathbf{V}\times Q},$$

and the well-posedness follows from Theorem 3.1. The error estimates (5.21) follow from Theorem 3.2, Lemma 4.1 and estimate (5.18) using $\min\{\ell+1,k\} = \ell+1$. ☐

Remark 5.9 (The one-element sub-mesh case)**.** *We note that if $\mathcal{T}_h^K$ is a mesh of an element ($\mathcal{H} = H = h$), then the MHM method with the stable Taylor-Hood element at the second level coincides with the non-conforming Galerkin method with the Crouzeix–Raviart element for polynomial degrees $k = 1, 2, 3$. In fact, note that the MHM method can be recast as find $(\boldsymbol{u}_h, p_h) \in \mathscr{N}_{b,h}$*

$$a_h(\boldsymbol{u}_h,p_h;\boldsymbol{v}_h,q_h) \;=\; (\boldsymbol{f},\boldsymbol{v}_h)_\Omega \quad \textit{for all } (\boldsymbol{v}_h,q_h) \in \mathscr{N}_{b,h},$$

where the product space with weakly continuous polynomial velocity and pressure with zero mean value $\mathscr{N}_{b,h}$ is given in (3.3)*. Then, we recognize* [35] *for $\ell = 0$ and $k = 1$ ($d = 2, 3$),* [42] *for $\ell = 1$ and $k = 2$ ($d = 2$) and* [34] *for $\ell = 2$ and $k = 3$ ($d = 2$). Consequently, the two-level stable MHM method using one-element sub-meshes is well-posed and satisfies*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{V}} + \|p - p_h\|_Q \le C\,h^m\left(\|\boldsymbol{u}\|_{m+1,\mathscr{P}_{\mathcal{H}}} + \|p\|_{m,\mathscr{P}_{\mathcal{H}}}\right), \tag{5.26}$$

with $1 \leq m \leq \ell + 1$. *The pair of spaces* $\mathbf{V}_h \times Q_h$ *with polynomial degree* $k \geq \ell + d$ *and* $\ell \geq 2$ *in 2D and* $\ell \geq 0$ *in 3D are inf-sup stable as it includes the (stable) Scott-Vogelius element (c.f.* [63, 66]*). Then, the two-level MHM method is also well-posed in those cases and satisfies* (5.26). □

Remark 5.10 (Stable or stabilized MHM: Pros and cons)**.** *Stabilized and stable MHM methods share equivalent convergence rates and local conservation properties (see Remark 5.7). However, we note that the mesh constraint given in* (5.7) *is unnecessary in the case of the stable version of the MHM method. Other difference is that in the particular case of the one-element sub-mesh, the two-level stable MHM method is point-wisely divergence-free since* $\nabla \cdot \boldsymbol{u}_{H,h}|_K \in \mathbb{P}_{k-1}(K)$ *and*

$$\int_K \nabla \cdot \boldsymbol{u}_{H,h}\, q_k \; d\boldsymbol{x} = 0 \quad \rightarrow \quad \nabla \cdot \boldsymbol{u}_{H,h} = 0 \quad \textit{for all } q_k \in \mathbb{P}_{k-1}(K) \textit{ and } K \in \mathscr{P}_{\mathcal{H}}.$$

In favor of the two-level stabilized MHM method, one has that it allows local polynomial interpolation of equal order, which easy computational implementations. It is also tailored to handle singularly perturbed reactive flows at the local level, which can avoid local refinements to capture boundary layers. □

6. Numerical benchmarks

In this section, we assess the two-level stabilized MHM method (5.1) using USFEM as a local solver (Section 5.1), performing convergence tests and verifying that this method is robust when simulating fluid flows in highly heterogeneous porous media. Details of the underlying algorithm and its implementation can be found in [7, Algorithm 1].

6.1. **Convergence studies.** We first focus on the Stokes model ($\boldsymbol{\theta} = \mathbf{0}$) with $\nu = 1$. The domain Ω is $]0,1[\times]0,1[$, the function $\boldsymbol{f}$ is chosen such that the exact solution is given by

$$\begin{aligned} u_1(x,y) &= -256\, x^2(x-1)^2 y(y-1)(2y-1), \\ u_2(x,y) &= -u_1(y,x), \\ p(x,y) &= 150\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right). \end{aligned}$$

The discontinuous version of $\boldsymbol{\Lambda}_H^\ell$ was validated in [7], and then we assume hereafter that $\boldsymbol{\Lambda}_H^\ell$ is the space of piecewise continuous polynomial functions. We set $\ell = 1, 2$ and first validate the method using USFEM with the elements $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$ and $\mathbb{P}_4(K)^2 \times \mathbb{P}_4(K)$ in one-element sub-meshes (i.e., $\mathcal{H} = H = h$), respectively. We observe convergence rates concerning H in Figure 1 that perfectly agree with the theoretical estimates of Theorem 5.2.

Using the broken norm in the $\boldsymbol{H}(\mathrm{div};\mathscr{P}_{\mathcal{H}})$ space, we also verify in Figure 1 the convergence of the two-level stress variable $\boldsymbol{\sigma}_{H,h} := \nu\nabla_{\mathcal{H}}\boldsymbol{u}_{H,h} - p_{H,h}\mathbb{I}$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{H,h}\|^2_{\boldsymbol{H}(\mathrm{div};\mathscr{P}_{\mathcal{H}})} := \sum_{K\in\mathscr{P}_{\mathcal{H}}} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{H,h}\|^2_{\boldsymbol{H}(\mathrm{div};K)},$$

which is not covered in Theorem 5.2. Owing to the regularity of the exact solution, we also observe that the error in the $L^2(\Omega)$ norm for the velocity converges of order $O(H^{\ell+2})$ as expected.

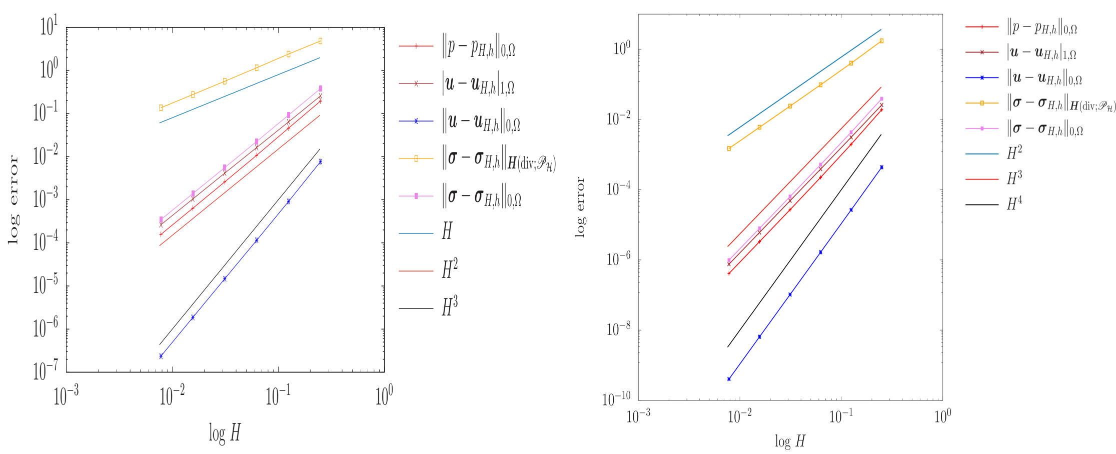

FIGURE 1. Convergence curves for $\boldsymbol{\Lambda}^1_H$ with $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$ (left) and $\boldsymbol{\Lambda}^2_H$ with $\mathbb{P}_4(K)^2 \times \mathbb{P}_4(K)$ (right). In both case we used one element to solve the second level local problems (here $\nu = 1$ and $\boldsymbol{\theta} = \boldsymbol{0}$).

Next, we maintain $\mathcal{H}$ fixed and refine the face partitions ($H \to 0$). Since the exact solution is regular, we expect to achieve super-convergence with an extra $O(H^{1/2})$ rate, as pointed out in Remark 4.3. This is, indeed, obtained as shown in Figure 2 (right) using the $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$ element which validates the theory. We note a decrease in the number of degrees of freedom needed (see Figure 2 (left)) to achieve a given error threshold when we compared it with the strategy to refine the first-level mesh ($\mathcal{H} \to 0$).

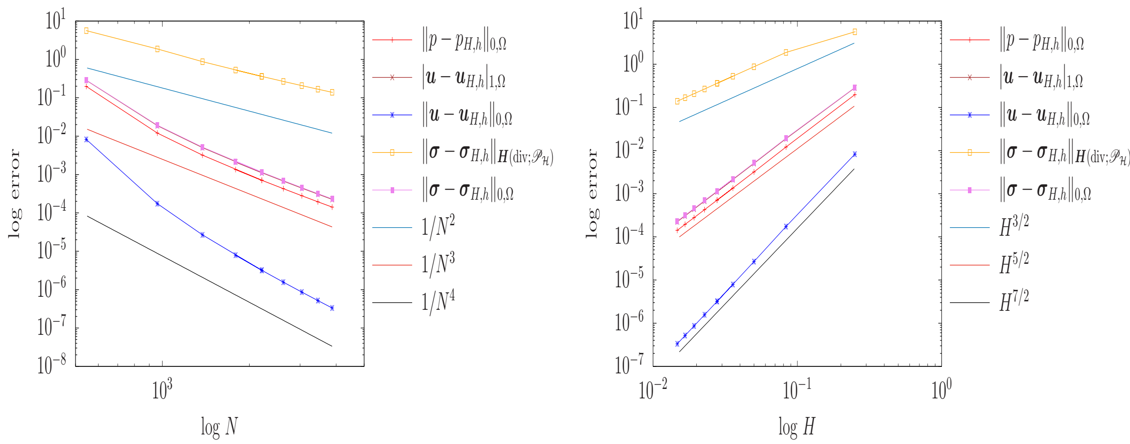

Figure 2. Convergence curves for $\mathbf{\Lambda}_H^1$ with $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$ in terms of `dof` (left) and the diameter of the face meshes H (right). In all cases we fixed to 64 the number of elements of the first level mesh (here $\nu = 1$ and $\boldsymbol{\theta} = \mathbf{0}$).

Table 1 shows convergence when using continuous or piecewise discontinuous space $\mathbf{\Lambda}_H^1$. To do so, we keep the number of mesh elements constant at the first level (64 elements) and refine the mesh only at the skeleton. Note that using the piecewise continuous space $\mathbf{\Lambda}_H^1$, we obtain better error results for a given number of degrees of freedom than the discontinuous case. However, the discontinuous option can still be attractive, especially when the physical coefficients are discontinuous or exhibit multiscale behavior.

	$\|p - p_{H,h}\|_{0,\Omega}$		$\|\boldsymbol{u} - \boldsymbol{u}_{H,h}\|_{0,\Omega}$		$\|\boldsymbol{u} - \boldsymbol{u}_{H,h}\|_{1,\Omega}$	
`dof`	$\mathbf{\Lambda}_{H,h}$ (cont)	$\mathbf{\Lambda}_{H,h}$,(disc)	$\mathbf{\Lambda}_{H,h}$ (cont)	$\mathbf{\Lambda}_{H,h}$,(disc)	$\mathbf{\Lambda}_{H,h}$ (cont)	$\mathbf{\Lambda}_{H,h}$,(disc)
545	0.1978e+00	0.1978e+00	0.8244e-02	0.8244e-02	0.2857e+00	0.2857e+00
961	0.1206e-01	0.3500e-01	0.3500e-01	0.6158e-03	0.1879e-01	0.4665e-01
1377	0.3197e-02	0.1275e-01	0.2673e-04	0.1435e-03	0.5017e-02	0.1667e-01
1793	0.1348e-02	0.6178e-02	0.7905e-05	0.5169e-04	0.2122e-02	0.8093e-02
2209	0.7108e-03	0.3535e-02	0.3203e-05	0.2343e-04	0.1120e-02	0.4614e-02
2625	0.4271e-03	0.2236e-02	0.1563e-05	0.1230e-04	0.6739e-03	0.2917e-02
3041	0.2797e-03	0.1519e-02	0.8617e-06	0.7135e-05	0.4418e-03	0.1980e-02
3457	0.1948e-03	0.1086e-02	0.5181e-06	0.4453e-05	0.3079e-03	0.1415e-02
3873	0.1421e-03	0.8087e-03	0.3322e-06	0.2940e-05	0.2246e-03	0.1053e-02

Table 1. History of convergence of the error in terms of the number of degrees of freedom in the skeleton, for $\mathbf{\Lambda}_H^1$ continuous or discontinuous.

Next, we are interested in the exact solution of the Brinkman problem ($\boldsymbol{\theta} = \mathbb{I}$) using the same analytical solution above, just changing the definition of the exact pressure by

$p(x, y) = (x - y)^6 - 1/28$. We do not show the convergence in relation to H and $\mathcal{H}$ because this is close to that obtained for the Stokes problem. Instead, we address in Figure 3 the behavior of the error in relation to the diffusion coefficient ν when it tends to zero.

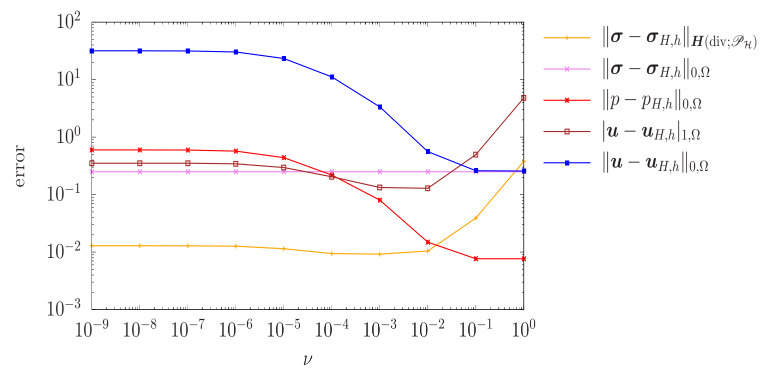

FIGURE 3. Error sensitivity in relation to different values of ν. Here we use $\boldsymbol{\Lambda}_H^1$ with $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$, with a coarse mesh of 64 elements.

We note that the error remains limited for a wide range of ν values. This indicates that the constant in the error estimates in Theorem 5.2 depends only slightly in terms of physical coefficients, which verifies the robustness of the two-level stabilized MHM method. The analysis of such property, which appears closely related to the accuracy of second-level USFEM in dealing with vanishing coefficient problems, deserves further theoretical investigation.

6.2. **A highly heterogeneous case.** This numerical test illustrates the capacity of the MHM method to simulate a fluid flow with a highly heterogeneous porous media on top of a coarse mesh. Indeed, such a domain represents a quite realistic prototype of a reservoir. In this context, we adopt the following version of the Brinkman model

$$-\mu\,\Delta\boldsymbol{u} + \boldsymbol{\theta}\,\boldsymbol{u} + \nabla p = 0 \quad \text{in } \Omega, \qquad \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{6.1}$$

with $\boldsymbol{\theta} := \mu^* \mathcal{K}^{-1}$, where $\mathcal{K}$ is the permeability tensor, μ is the fluid viscosity, and μ^* is the so-called *effective* viscosity of the fluid. Generally, the value of μ^* depends on the properties of the porous media. For example, if there are large variations in the material properties, μ^* might not be considered a homogeneous coefficient. Nevertheless, it is often assumed that such an effective viscosity is homogeneous and that $\mu = \mu^*$. Here we adopt a heterogenous isotropic permeability coefficient $\mathcal{K}$, which is obtained from layer 36 of the 85 layers in SPE10 project [31] (second dataset). The domain $\Omega :=]0, 1200[\times]0, 2200[$, $\mu = \mu^* = 0.3$. The permeability tensor $\mathcal{K}$ and the boundary conditions are depicted in Figure 4.

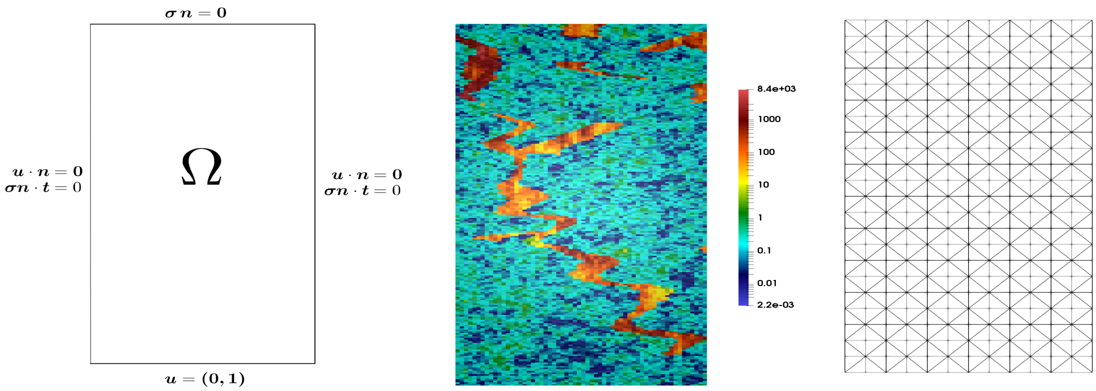

FIGURE 4. The statement of the problem (left), the permeability field $\mathcal{K}$ in a logarithmic scale (center), and the coarse mesh with 528 triangle elements used for the MHM computations (right).

A reference solution is calculated by solving the problem using USFEM on a mesh with $1,081,344$ elements ($1,625,283$ degrees of freedom) with $\mathbb{P}_1(K)^2 \times \mathbb{P}_1(K)$ element. We perform the calculations using the MHM method with the discontinuous space $\mathbf{\Lambda}_1^{10}$ on a coarse structured mesh with 528 elements, as shown in Figure 4 (right). The number of degrees of freedom is $33,040$. The coarse partition faces are not aligned with changing coefficients, and multiple scales still persist within the elements. The submeshes contain one hundred triangles with $\mathbb{P}_3(K)^2 \times \mathbb{P}_3(K)$ interpolation. We note a good agreement between the reference and the MHM solution through the pressure isolines and $|\boldsymbol{u}_{H,h}|$, and the pressure profiles (see Figures 5–7) .

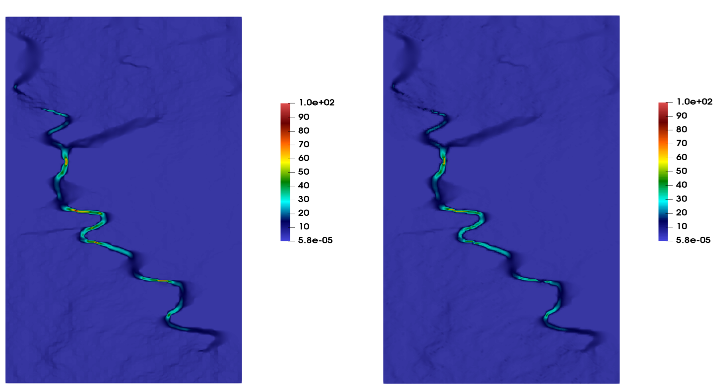

FIGURE 5. Isolines of the velocity magnitude. The reference solution (left) and the MHM solution (right).

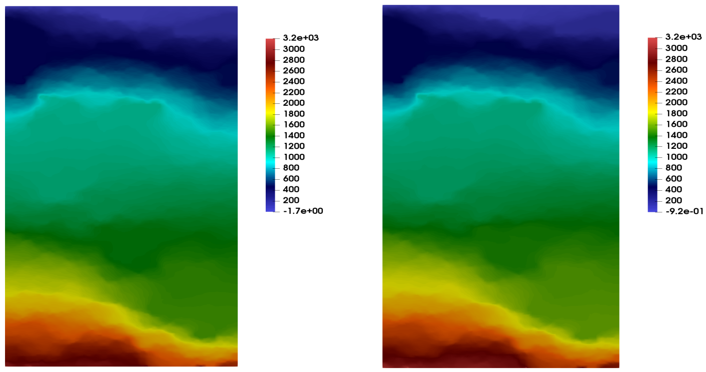

FIGURE 6. Pressure isolines. Comparison between the reference solution (left) and the MHM solution (right).

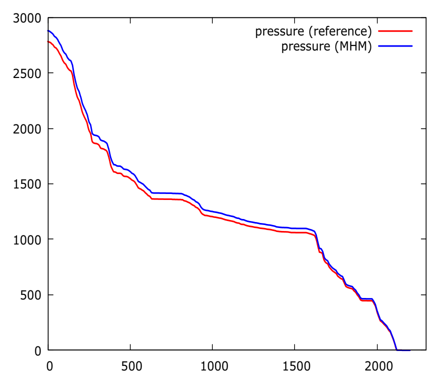

FIGURE 7. Pressure profiles of pressure at $x = 600$. Reference versus MHM solutions.

7. Conclusion

Hybridization was used to characterize exact velocity and pressure variables in terms of local and global problems, producing new face-based multiscale numerical methods for the Stokes/Brinkman model. The Neumann local problems respond to multiscale basis functions that incorporate physical and geometric aspects of the coarse mesh, while the global formulation responds to the degrees of freedom. The present work is also the first to prove that, under conditions of local regularity, the two-level MHM method originally given in [7] is well-posed and achieves better approximation results. Such results are previously established in an abstract way, starting at the continuous level and assuming generic second-level solvers that satisfy some local properties. Then, by particularizing the second-level solver via a stabilized and stable finite element method and (dis)continuous polynomial interpolation for the flow variable (e.g., Lagrange multiplier), we demonstrate that the underlying MHM methods are high-order super-convergent for pressure and velocity variables and release local mass conservative numerical velocity fields and numerical flux in local equilibrium with external forces. These properties are achieved without post-processing. In the case of a sub-mesh composed of only one element, we establish relationships between the MHM method and classical non-conforming methods in the literature. The numerical tests validated the theoretical results and fully complement and support the extensive numerical validation first proposed in [7]. The robustness of the proposed MHM methods with respect to physical coefficients is verified numerically, but its precise demonstration is left for future work.

Appendix A. Equivalence of norms in broken spaces

Consider a set of Hilbert spaces Z_i with inner products $(\cdot,\cdot)_{Z_i}$, $1 \leq i \leq N$ and denote by F_i a linear bounded functional acting on Z_i. Given the Hilbert space $Z := \Pi_{i=1}^N Z_i$ with inner product $(\cdot,\cdot)_Z := \sum_{i=1}^N (\cdot,\cdot)_{Z_i}$, consider linear bounded functional

$$F(\boldsymbol{w}) := \sum_{i=1}^N F_i(w_i), \quad \text{for all } \boldsymbol{w} = (w_1,\ldots,w_N) \in Z. \tag{A.1}$$

We take the usual norms of these functionals:

$$\|F\|_{Z'} = \sup_{\boldsymbol{w}\in Z} \frac{F(\boldsymbol{w})}{\|\boldsymbol{w}\|_Z} \quad \text{and} \quad \|F_i\|_{Z_i'} = \sup_{w_i\in Z_i} \frac{F_i(w_i)}{\|w_i\|_{Z_i}}.$$

Lemma A.1.

$$\|F\|_{Z'}^2 = \sum_{i=1}^N \|F_i\|_{Z_i'}^2.$$

Proof. The Riesz Representation Theorem guarantees existence of $\boldsymbol{z}_F \in Z$ and $z_{F_i} \in Z_i$ such that

$$F(\boldsymbol{w}) = (\boldsymbol{z}_F, \boldsymbol{w})_Z, \quad \|\boldsymbol{z}_F\|_Z = \|F\|_{Z'},$$
$$F_i(w_i) = (z_{F_i}, w_i)_{Z_i}, \quad \|z_{F_i}\|_{Z_i} = \|F_i\|_{Z_i'}.$$

Use z_F^i to denote the components of $\boldsymbol{z}_F$. Then, since Z is a product space on one hand, and by definition eq. (A.1) on the other, it holds that for all $\boldsymbol{w} = (w_1,\ldots,w_N) \in Z$,

$$\sum_{i=1}^N (z_{F_i}, w_i)_{Z_i} = \sum_{i=1}^N (z_F^i, w_i)_{Z_i}.$$

We easily see that $z_{F_i} = z_F^i$ for $1 \leq i \leq N$. Therefore,

$$\|F\|_{Z'}^2 = \sum_{i=1}^N \|z_F^i\|_{Z_i}^2 = \sum_{i=1}^N \|z_{F_i}\|_{Z_i}^2 = \sum_{i=1}^N \|F_i\|_{Z_i'}^2.$$

□

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