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Primal hybrid finite element method for the linear elasticity problem

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ABSTRACT

In this manuscript, we study a primal hybrid finite element method for two dimensional linear elasticity problem. We derive a priori error estimates for both primal and hybrid variables. The rate of convergence of the method is independent of the Lamé parameters, which illustrates the robustness of the method. Numerical experiments are presented to validate the theoretical findings.

1. Introduction

Primal hybrid finite element method is based on an extended variational principle introduced by Pian and Tong [18]. Many numerical techniques such as mixed techniques, (cf. [4,21]) mortar element techniques (cf. [5,12]) falls into the category of primal hybrid method, where different discretizations can be coupled with the help of Lagrange multipliers at interelement boundaries. These Lagrange multipliers are physical quantities related to the physics of the problem. For instance, for the Poisson model it is the flux (cf. [20]) and for the biharmonic model there are two associated Lagrange multipliers, one is bending moments and another is Kirchhoff shear force (cf. [19]). For the linear elastic model problem, Lagrange multiplier represents the normal stress. The best part of primal hybrid method method is that it approximate both the primal and hybrid variable simultaneously which would be cumbersome (results in loss of accuracy) using standard finite element methods. In this method, the inter-element continuity requirement is withdrawn at the expense of Lagrange multipliers which makes this method more flexible in the sense that, we can allow discontinuities across inter-element boundaries to reduce the computational cost, particularly in case of hp-adaptivity for more complex problems. Patch test (cf. [15,19,20]) build a bridge between primal hybrid method and nonconforming methods, which project it as a generalized nonconforming method. That is, the primal variable in the discrete primal hybrid form coincides with the solution of corresponding nonconforming form in the sense that both the solutions meets at the Gauss–Legendre quadrature points. In practice, one can extract the flux from the nonconforming solution using primal hybrid form more accurately, namely by flux recovery technique cf. [9].

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Primal hybrid finite element method is firstly used by Raviart and Thomas [20] to solve second order elliptic problems, which is then extended for the case of fourth order elliptic problems by Quarteroni [19]. Park and Milner studied this method for the nonlinear second order elliptic problems (cf. [16,17]) and article [8] discusses an application to optimal shape problem. For the convergence analysis of this method for second order parabolic problems we refer to Acharya and Patel [1], Acharya and Porwal [2], Shokeen et al. [22], Swann [23] for the fourth order parabolic problems. We refer to Boffi et al. [6], Devloo et al. [13] for a general discussion on hybrid methods. Existing literature on the numerical analysis of this method is very limited and certainly many aspects (such as convergence on uniform and adaptive meshes, recovery of fluxes) of this method are yet to be explored for various linear and nonlinear problems.

Linear elasticity problem, a simplified version of nonlinear elasticity problem, is important in studying the deformation of solid objects with respect to given load. This model has many applications in structural analysis and engineering design. To the best of the knowledge of authors, there is no work available till now on primal hybrid method for linear elasticity problems. In this article, an attempt is made to analyze the convergence behavior of the primal hybrid finite element method for the pure displacement linear elasticity boundary value problem in two dimension with a homogeneous isotropic elastic material. Considering the fact that various finite element approximations exhibit the locking phenomenon for the linear elasticity problem, in the sense that the convergence rates deteriorates as the Poisson ratio ν approaches 0.5, it becomes important to have the robust numerical approximation. In this article, we derive a priori error estimates for both primal and hybrid variables. Therein, the rate convergence of the method is independent of the Lamé parameters, which shows the robustness of the method.

The plan of this manuscript is as follows. In the next section, we introduce the problem setting, some important notions and preliminary results. The primal hybrid method is introduced in Section 3. Followed by that, in Section 4, we discuss the discrete problem and related discrete spaces. The convergence analysis in presented in Section 5. Finally, the numerical tests, validating the theoretical results, are discussed in Section 6 followed by conclusions in Section 7.

2. Linear elasticity problem and its functional setting

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and convex polygonal Lipschitz domain with boundary Γ, representing the area occupied by an elastic body. Consider the linear elasticity model problem

$$-\sum_{j=1}^{2} \frac{\partial}{\partial x_j}(\sigma_{ij}(\bar{u})) = f_i \text{ in } \Omega,\ \ i = 1,2,\ \ \bar{u} = \bar{g} \text{ on } \Gamma, \tag{2.1}$$

where the symmetric stress tensor $\sigma_{ij}(\bar{u}) = \lambda\left(\sum_{k=1}^{2}\varepsilon_{kk}(\bar{u})\right)\delta_{ij} + 2\mu\varepsilon_{ij}(\bar{u})$ with $\bar{u} = (u_1, u_2)$ the displacement vector, the strain tensor $\varepsilon_{ij}(\bar{u}) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$, δ_{ij} is the Kronecker delta function, $\lambda \geq 0$ and $\mu > 0$ are the Lamé coefficients, $\bar{f} = (f_1, f_2) \in (L^2(\Omega))^2$ is the volume force on Ω, $\bar{g} = (g_1, g_2) \in (H^{1/2}(\Gamma))^2$.

We can rewrite the problem (2.1) as

$$-\sum_{j=1}^{2}\left(\mu\frac{\partial^2 u_i}{\partial^2 x_j} + (\mu+\lambda)\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right) = f_i \text{ in } \Omega,\ \ i = 1,2,\ \ \bar{u} = \bar{g} \text{ on } \Gamma. \tag{2.2}$$

Note that, although the Neumann boundary conditions are highly relevant to this type of problem, we consider Dirichlet boundary condition which suffice our purpose. In subsequent discussion, $H^m(\Omega)$, $m \in \mathbb{R}_+$ denotes the standard Sobolev space over the domain Ω, with norm $||v||_{H^m(\Omega)}$ and seminorm $|v|_{H^m(\Omega)}$ cf. [3,14]. For a vector valued function $\bar{v} = (v_1, v_2) \in (H^m(\Omega))^2$ we define the product norm $||\cdot||_{H^m(\Omega)^2}$ and seminorm $|\cdot|_{H^m(\Omega)^2}$ as

$$||\bar{v}||^2_{H^m(\Omega)^2} = \sum_{i=1}^{2}||v_i||^2_{H^m(\Omega)} \text{ and } |\bar{v}|^2_{H^m(\Omega)^2} = \sum_{i=1}^{2}|v_i|^2_{H^m(\Omega)}.$$

Let $H^{m-1/2}(\Gamma)$ be the space of first order trace ϕ of the function $v \in H^m(\Omega)$ on Γ equipped with the norm

$$||\phi||_{m-1/2,\Gamma} = \inf_{v\in H^m(\Omega), v|_\Gamma = \phi}||v||_{m,\Omega}.$$

Further, $H^1_0(\Omega)$ denotes the space of functions in $H^1(\Omega)$ whose trace vanishes on the boundary Γ. Denote by $H^{-1/2}(\Gamma)$, the dual of $H^{1/2}(\Gamma)$ and equip it with the norm $||\cdot||_{-1/2,\Gamma}$ defined by

$$||\cdot||_{-1/2,\Gamma} = \sup_{\phi\in H^{1/2}(\Gamma),\ \phi\neq 0}\frac{\langle\cdot,\phi\rangle_{1/2,\Gamma}}{||\phi||_{1/2,\Gamma}},$$

where $\langle\cdot,\cdot\rangle_{1/2,\Gamma}$ denotes the duality paring between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Let

$$G = \left\{\bar{v} = (v_1, v_2) \in (H^1(\Omega))^2 : \bar{v} - \bar{u}_D \in (H^1_0(\Omega))^2\right\},$$

where $\bar{u}_D = (u_{D1}, u_{D2}) \in (H^1(\Omega))^2$ is a function such that $\bar{u}_D = \bar{g}$ on Γ.

The variational form of (2.2) is to find $\bar{u} \in G$ satisfying

$$\mu \int_\Omega \nabla\bar{u} : \nabla\bar{v}\, dx + (\mu+\lambda)\int_\Omega div(\bar{u})div(\bar{v})\, dx = \bar{f}(\bar{v}), \tag{2.3}$$

where for any $\tilde{\eta}, \tilde{\tau} \in \mathbb{R}^{2\times 2}$, $\tilde{\eta} : \tilde{\tau} = \sum_{i,j=1}^{2} \eta_{ij}\tau_{ij}$, $\bar{f}(\bar{v}) = \int_\Omega \bar{f}\cdot\bar{v}dx$. The well-posedness of this problem can be ensured by using the Lax–Milgram lemma [14].

We state the following two lemmas, for their proof we refer to [7].

Lemma 2.1. *For $\bar{v} \in (H^2(\Omega))^2 \cap (H^1_0(\Omega))^2$, there exists a positive constant C_Ω (depending only on Ω) and $\bar{w} \in (H^2(\Omega))^2 \cap (H^1_0(\Omega))^2$ such that $div(\bar{w}) = div(\bar{v})$ and*

$$||\bar{w}||_{H^2(\Omega)^2} \le C_\Omega ||div(\bar{v})||_{H^1(\Omega)^2}. \tag{2.4}$$

Let $\{a_i\}_{i=1}^n$ to be the collection of all vertices of Ω and $\Gamma = \cup_{i=1}^n \overline{\Gamma}_i$, where Γ_i are the open line segments joining a_i and a_{i+1}, for $1 \le i \le n$. We denote τ_i to be the positively oriented unit tangent towards Γ_i and ν_i be the unit outward normal along Γ_i.

Lemma 2.2. *Let $\bar{g}|_{\Gamma_i} \in (H^{3/2}(\Gamma_i))^2$ and $\bar{g}|_{\Gamma_i}(a_{i+1}) = \bar{g}|_{\Gamma_{i+1}}(a_{i+1})$ for $1 \le i \le n$. Then the problem (2.2) has a unique solution $\bar{u} \in (H^2(\Omega))^2$. If $\sum_{i=1}^{n}\int_{E\Gamma_i} \bar{g}|_{\Gamma_i}\cdot\nu_i d\gamma = 0$ and*

$$\frac{\partial}{\partial\tau_i}\bar{g}|_{\Gamma_i}\cdot\nu_{i+1} = \frac{\partial}{\partial\tau_{i+1}}\bar{g}|_{\Gamma_{i+1}}\cdot\nu_i$$

at a_i for $1 \le i \le n$, then there exists a positive constant C_Ω such that

$$||\bar{u}||_{H^2(\Omega)^2} + \lambda||div(\bar{u})||_{H^1(\Omega)} \le C_\Omega\left(||\bar{f}||_{L^2(\Omega)^2} + \sum_{i=1}^{n}||\bar{g}|_{\Gamma_i}||_{H^{3/2}(\Gamma_i)}\right). \tag{2.5}$$

Next, we introduce some notions to be used in setting up the primal hybrid formulation of the problem under consideration. Let $\mathcal{T}_h$ be a non-overlapping partition of the set $\overline{\Omega}$ with shape regular triangles K ($\max_{K\in\mathcal{T}_h}\frac{h_K}{\rho_K} \le \delta$, where h_K is the diameter of the triangle and ρ_K is the diameter of the inscribed circle in K cf. [10]) with $h = \max_{K\in\mathcal{T}_h} h_K$ such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h}\overline{K}$$

and for two different K_1 and K_2 in $\mathcal{T}_h$, $\overline{K}_1\cap\overline{K}_2$ is either empty, or an edge or a vertex common to both the triangles. For any $K \in \mathcal{T}_h$, we denote by ∂K the boundary of K and $\nu = (\nu_1, \nu_2)$ to be the unit outward normal along the ∂K.

Let $\mathcal{E}^h$ be the set of all edges of $\mathcal{T}_h$, $\mathcal{E}^h(K)$ be the set of all edges of an element $K \in \mathcal{T}_h$ and $\mathcal{E}^h_i$, $\mathcal{E}^h_b$ denote the set of all interior and boundary edges of $\mathcal{T}_h$, respectively. Let for each $E \in \mathcal{E}^h$, $\nu_E = (\nu_E^1, \nu_E^2)$ be the outward unit normal and $\tau_E = (\tau_E^1, \tau_E^2)$ be the tangent direction to E. We define the jump of a scalar valued function g on any $E \in \mathcal{E}^h$ which is such that $E = \partial K_1 \cap \partial K_2$, as $[\![g]\!]_E = (g|_{K_1})|_E - (g|_{K_2})|_E$.

Define the broken Sobolev space X as

$$X = \left\{\bar{v} = (v_1, v_2) \in (L^2(\Omega))^2 : \forall K \in \mathcal{T}_h, \bar{v}|_K \in (H^1(K))^2\right\}$$

provided with a norm

$$||\bar{v}||_X = \left(\sum_{K\in\mathcal{T}_h} |||\bar{v}|_K|||^2_{1,K}\right)^{1/2}, \tag{2.6}$$

where $|||\bar{v}|_K|||^2_{1,K} = \sum_{i=1}^{2}\left(|v_i|_K|^2_{H^1(K)} + h_K^2||v_i|_K||^2_{L^2(K)}\right)$.

We define the space Q as

$$Q = \left\{\mathbf{q} = (q_{ij})_{2\times 2} : q_{ij} \in L^2(\Omega)\ \forall\ i, j = 1, 2,\ div(q) \in (L^2(\Omega))^2, \text{and } q_{ij} = q_{ji}\right\} \tag{2.7}$$

[1] equipped with the norm

$$||\mathbf{q}||_Q = \left(\sum_{i,j} ||q_{ij}||^2_{L^2(\Omega)} + ||q_{ij/j}||^2_{L^2(\Omega)}\right)^{1/2}. \tag{2.8}$$

Given any $\mathbf{q} \in Q$, its normal components are defined as $(\sum_{j=1}^2 q_{1j}\nu_j, \sum_{j=1}^2 q_{2j}\nu_j) \in (H^{-1/2}(\Gamma))^2$, where (ν_1, ν_2) is the unit outward normal along Γ and the following Greens formula holds: for $\bar{v} \in (H^1(\Omega))^2$ and $\mathbf{q} \in Q$,

$$\int_\Omega \left(\sum_{i,j} \frac{\partial v_i}{\partial x_j} q_{ij} + \sum_{i,j} v_i \frac{\partial q_{ij}}{\partial x_j}\right) dx = \int_\Gamma \sum_{i,j} v_i q_{ij} \nu_j d\gamma. \tag{2.9}$$

Next, we define the space

$$M = \Big\{ \bar{\chi} = (\chi_1, \chi_2) \in \prod_{K \in \mathcal{T}_h} (H^{-1/2}(\partial K))^2 : \text{there exists } \mathbf{q} \in Q$$
$$\text{such that, } \left(\sum_{j=1}^2 q_{1j}\nu_j, \sum_{j=1}^2 q_{2j}\nu_j\right) = (\chi_1, \chi_2) \text{ on } \partial K, \forall K \in \mathcal{T}_h \Big\} \tag{2.10}$$

equipped with the norm

$$||\bar{\chi}||_M = \inf_{\left\{\mathbf{q}\in Q \,:\, \left(\sum_{j=1}^2 q_{1j}\nu_j, \sum_{j=1}^2 q_{2j}\nu_j\right) = (\chi_1, \chi_2) \text{ on } \partial K, \forall K \in \mathcal{T}_h\right\}} ||\mathbf{q}||_Q. \tag{2.11}$$

Let $b(\cdot,\cdot) : X \times M \to \mathbb{R}$ be a bilinear form defined by,

$$b(\bar{v}, \bar{\chi}) = -\sum_{K \in \mathcal{T}_h} \sum_{i=1}^2 \langle \chi_i, v_i \rangle_{1/2, \partial K}, \tag{2.12}$$

$$= -\sum_{i=1}^2 \sum_{E \in \mathcal{E}^h} \int_E \chi_i [\![v_i]\!]_E \; d\gamma. \tag{2.13}$$

In the following lemma, we characterize the space $(H_0^1(\Omega))^2$, which will help in proving the existence and uniqueness of the primal hybrid solution.

Proposition 2.3. $(H_0^1(\Omega))^2 = \{\bar{v} \in X : b(\bar{v}, \bar{\chi}) = 0, \;\forall \bar{\chi} \in M\}$.

Proof. Consider $\bar{v} \in (H_0^1(\Omega))^2$, by the definition, v_1 and v_2 vanishes on $E \in \mathcal{E}_b^h$ and the jumps $[\![v_1]\!]_E$ and $[\![v_2]\!]_E$ vanishes on the $E \in \mathcal{E}_i^h$. Hence the bilinear form $b(\bar{v}, \bar{\chi})$ vanishes for all $\bar{\chi} \in M$. Conversely, let $\bar{v} \in X$ be such that $b(\bar{v}, \bar{\chi}) = 0, \;\forall \bar{\chi} \in M$. To prove $\bar{v} \in (H_0^1(\Omega))^2$, consider $\bar{\chi} = (\chi_1, 0)$ with

$$\chi_1 = \begin{cases} \varphi \text{ on } E \in \mathcal{E}^h, \;\; \varphi \in H_0^1(E) \\ 0 \text{ otherwise.} \end{cases}$$

Then $b(\bar{v}, \bar{\chi}) = -\sum_{E \in \mathcal{E}^h} \int_E \chi_1 [\![v_1]\!]_E d\gamma = -\int_E [\![v_1]\!]_E \varphi d\gamma = 0$ for all $\varphi \in H_0^1(E)$, $E \in \mathcal{E}^h$. Which implies $[\![v_1]\!]_E = 0$ a.e. for all $E \in \mathcal{E}^h$. Similarly by taking $\bar{\chi} = (0, \chi_2)$ with

$$\chi_2 = \begin{cases} \varphi \text{ on } E \in \mathcal{E}^h, \;\; \varphi \in H_0^1(E) \\ 0 \text{ otherwise,} \end{cases}$$

we get,

$$\begin{aligned} 0 = b(\bar{v}, \bar{\chi}) &= -\sum_{E \in \mathcal{E}^h} \int_E \chi_2 [\![v_2]\!]_E d\gamma \\ &= -\int_E [\![v_2]\!]_E \varphi d\gamma \quad \forall \; \varphi \in H_0^1(E), \end{aligned}$$

which implies $[\![v_2]\!]_E = 0$ a.e. for any $E \in \mathcal{E}^h$ and hence the desired result follows. □

[1] Here, we use the notation $\cdot_{/j} := \frac{\partial}{\partial x_j}$.

3. Primal hybrid formulation

We define the primal hybrid formulation of the problem (2.1) as follows. Find $(\bar{u}, \bar{\kappa}) \in X \times M$ such that

$$a(\bar{u}, \bar{v})+b(\bar{v}, \bar{\kappa}) = \bar{f}(\bar{v}) \ \forall \ \bar{v} \in X, \tag{3.1}$$

$$b(\bar{u}, \bar{\chi}) = \langle \bar{\chi}, \bar{u}_D \rangle \ \forall \bar{\chi} \in M, \tag{3.2}$$

where

$$a(\bar{u}, \bar{v}) = \sum_{K \in \mathcal{T}_h} \mu \int_K \nabla \bar{u} : \nabla \bar{v} \ dx + (\mu + \lambda) \int_K div(\bar{u}) div(\bar{v}) dx, \ \ \langle \bar{\chi}, \bar{u}_D \rangle = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \chi_i, u_{Di} \rangle_{1/2, \partial K}.$$

Next, we discuss some important results motivated from Raviart and Thomas [20] which will be helpful in establishing the well-posedness of the above primal hybrid formulation.

Theorem 3.1. *A continuous linear functional $L(\cdot)$ defined on the space X vanishes on $(H_0^1(\Omega))^2$ if and only if there exist a unique $\bar{\chi} \in M$ such that*

$$L(\bar{v}) = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \chi_i, v_i \rangle_{1/2, \partial K} \quad \forall \bar{v} \in X.$$

Proof. Firstly, we observe that for any continuous linear functional $L_K(\cdot)$ on $(H^1(K))^2$, $K \in \mathcal{T}_h$, there exists $q_{ij} \in L^2(K)$ $i, j = 1, 2$ and $q_i \in L^2(K), i = 1, 2$ such that

$$L_K(\bar{v}) = \int_K (\sum_{i,j} q_{ij} \frac{\partial v_i}{\partial x_j} + \sum_i q_i v_i) \ dx.$$

Thus for any continuous linear functional $L(\cdot)$ on X, there exists $q_{ij} \in L^2(\Omega)$ $i, j = 1, 2$ and $q_i \in L^2(\Omega), i = 1, 2$ such that

$$L(\bar{v}) = \sum_{K \in \mathcal{T}_h} \int_K (\sum_{i,j} q_{ij} \frac{\partial v_i}{\partial x_j} + \sum_i q_i v_i) \ dx.$$

Let $L(\cdot)$ vanishes on $(H_0^1(\Omega))^2$ that is

$$\int_\Omega (\sum_{i,j} q_{ij} \frac{\partial v_i}{\partial x_j} + \sum_i q_i v_i) \ dx = 0 \quad \forall \ \bar{v} \in (H_0^1(\Omega))^2,$$

which yields $(q_1, q_2) = div(\mathbf{q})$ in the sense of distribution, where $\mathbf{q} = (q_{ij})_{2\times 2}$. Observe that since $div(\mathbf{q}) \in (L^2(\Omega))^2$, $\mathbf{q} \in Q$ and

$$\begin{aligned} L(\bar{v}) &= \sum_{K \in \mathcal{T}_h} \int_K (\sum_{i,j} q_{ij} \frac{\partial v_i}{\partial x_j} + \sum_{i,j} \frac{\partial q_{ij}}{\partial x_j} v_i) \ dx \\ &= \sum_{K \in \mathcal{T}_h} \sum_{i,j} \langle q_{ij} \nu_j, v_i \rangle_{1/2, \partial K}, \end{aligned}$$

where the last equation resulted from the integration by parts. Thus we have existence of $\bar{\chi} = (\sum_{j=1}^{2} q_{1j}\nu_j, \sum_{j=1}^{2} q_{2j}\nu_j) \in M$ such that

$$L(\bar{v}) = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \chi_i, v_i \rangle_{1/2, \partial K} \quad \forall \bar{v} \in X.$$

Note that, here $\mathbf{q} \in Q$ is not unique. We proceed to prove the uniqueness of $\bar{\chi} \in M$.

Let $\sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \chi_i, v_i \rangle_{1/2, \partial K} = 0 \quad \forall \bar{v} \in X$ which gives for all $K \in \mathcal{T}_h$, $\sum_{i=1}^{2} \langle \chi_i, v_i \rangle_{1/2, \partial K} = 0, \ \forall \ \bar{v} \in (H^1(K))^2$. Thus $\bar{\chi} = \bar{0}$ from the surjectivity of the first order trace map.

Conversely, suppose that there exists $\bar{\chi} \in M$ such that

$$L(\bar{v}) = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \chi_i, v_i \rangle_{1/2, \partial K} \quad \forall \bar{v} \in X,$$

therein, it is evident that $L(\cdot)$ vanishes on $(H_0^1(\Omega))^2$. $\square$

Theorem 3.2. *The continuous problem (3.1) and (3.2) has a unique solution* $(\bar{u}, \bar{\kappa}) \in X \times M$. *Further* $\bar{u} \in G$ *is the solution of* (2.3) *and*

$$\bar{\kappa} = (\kappa_1, \kappa_2) = \mu \frac{\partial \bar{u}}{\partial \nu} + (\mu + \lambda) div(\bar{u})\nu \quad \text{on} \quad \partial K \ \forall \ K \in \mathcal{T}_h. \tag{3.3}$$

Proof. Let $(\bar{u}, \bar{\kappa}) \in X \times M$ be the solution of (3.1) and (3.2). From Proposition 2.3, we have $\bar{u} \in G$ and for each $\bar{v} \in (H_0^1(\Omega))^2$,

$$a(\bar{u}, \bar{v}) = \mu \int_\Omega \nabla \bar{u} : \nabla \bar{v} \ dx + (\mu + \lambda) \int_\Omega div(\bar{u}) div(\bar{v}) \ dx = \int_\Omega \bar{f} \cdot \bar{v} \ dx.$$

Hence $\bar{u}$ is the solution of (2.3).

Conversely, suppose that $F(\bar{v}) = \int_\Omega \bar{f} \cdot \bar{v} \ dx - a(\bar{u}, \bar{v})$ be a linear functional on X which vanishes on $(H_0^1(\Omega))^2$ where $\bar{u} \in G$ is the solution of (2.3). From Theorem 3.1, we have that the existence of unique $\bar{\kappa} \in M$ such that

$$b(\bar{v}, \bar{\kappa}) = F(\bar{v}) = \int_\Omega \bar{f} \cdot \bar{v} \ dx - a(\bar{u}, \bar{v}),$$

which shows that $(\bar{u}, \bar{\kappa}) \in X \times M$ is the solution of (3.1) and (3.2). Further in view of (2.6), using Green's formula in each $K \in \mathcal{T}_h$, we have for any $\bar{v} \in X$

$$\begin{aligned} b(\bar{v}, \bar{\kappa}) &= \int_\Omega \bar{f} \cdot \bar{v} \ dx - a(\bar{u}, \bar{v}) \\ &= -\mu \int_\Omega \Delta \bar{u} \cdot \bar{v} dx - (\mu + \lambda) \int_\Omega \nabla(\nabla \cdot \bar{u}) \cdot \bar{v} dx - a(\bar{u}, \bar{v}) \\ &= -\sum_{K \in \mathcal{T}_h} \langle \mu \frac{\partial \bar{u}}{\partial \nu} + (\mu + \lambda) div(\bar{u})\nu, \bar{v} \rangle_{1/2, \partial K} \\ &= -\sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \langle \mu \frac{\partial u_i}{\partial \nu} + (\mu + \lambda) div(\bar{u})\nu_i, v_i \rangle_{1/2, \partial K}. \end{aligned} \tag{3.4}$$

The relation (3.3) then can be observed by choosing $\bar{v} = (\phi, 0)$ and $\bar{v} = (0, \phi)$, respectively for any $\phi \in H^1(K)$, $K \in \mathcal{T}_h$ in (3.4). □

4. Finite element spaces and discrete problem

In this section, we define the finite element spaces to discretize the problem (3.1) and (3.2). Here, we assume each triangle $K \in \mathcal{T}_h$ is an image of a unit right angled triangle $\hat{K}$ through an invertible affine map F_K (cf. [10]) defined as: for $x \in K$, $F_K(x) = B_K x + b_K$, where B_K is an invertible 2×2 matrix and b_K is a vector in $\mathbb{R}^2$. Let $P_k(\hat{K})$ be the space of all polynomials over $\hat{K}$ of degree $\leq k$ in x_1 and x_2 variables. Let $\hat{P}(\hat{K})$ be the finite dimensional subspace of $H^1(\hat{K})$ defined as

$$P_1(\hat{K}) \subset \hat{P}(\hat{K}) \subset H^1(\hat{K}).$$

Now we define the finite dimensional subspace of $(H^1(K))^2$

$$P_K = \left\{ \bar{v}_h \in (H^1(K))^2 : \bar{v}_h = \hat{\bar{v}}_h \circ F_K^{-1}, \ \hat{\bar{v}}_h \in (\hat{P}(\hat{K}))^2 \right\}$$

and the finite dimensional space X_h of X as

$$X_h = \left\{ \bar{v}_h \in (L^2(\Omega))^2 : \bar{v}_h|_K \in P_K, \ \forall K \in \mathcal{T}_h \right\}.$$

To enforce the continuity of functions in X_h along interelement boundaries, we define the discrete Lagrange multiplier space as follows.

Let $\hat{S}_m$ be the space of all functions defined over $\partial \hat{K}$ whose restriction to any edge $\hat{E}$ of $\partial \hat{K}$ are polynomials of degree $\leq m$. We define a finite dimensional subspace of $L^2(\partial \hat{K})$ by $\hat{S}$ and defined as $\hat{S}_0 \subseteq \hat{S}$. Next we define

$$S_{\partial K} = \left\{ \bar{\chi}_h \in (L^2(\partial K))^2 : \bar{\chi}_h = \hat{\bar{\chi}}_h \circ F_K^{-1}, \ \hat{\bar{\chi}}_h \in (\hat{S})^2 \right\}.$$

The Lagrange multiplier space is defined as

$$M_h = \left\{ \bar{\chi}_h \in \prod_{K \in \mathcal{T}_h} S_{\partial K} : \bar{\chi}_h|_{\partial K_1} + \bar{\chi}_h|_{\partial K_2} = \bar{0} \text{ on } K_1 \cap K_2 \text{ for adjacent pair of triangles } K_1 \text{and } K_2 \right\}.$$

The discrete form of the problem (3.1) and (3.2) is to find $(\bar{u}_h, \bar{\kappa}_h) \in X_h \times M_h$ such that

$$a(\bar{u}_h, \bar{v}_h) + b(\bar{v}_h, \bar{\kappa}_h) = \bar{f}(\bar{v}_h) \qquad \forall \bar{v}_h \in X_h, \tag{4.1}$$

$$b(\bar{u}_h, \bar{\chi}_h) = \langle \bar{\chi}_h, \bar{u}_{Dh} \rangle \quad \forall \bar{\chi}_h \in M_h, \tag{4.2}$$

where $\bar{u}_{Dh}$ is a suitable interpolation of $\bar{u}_D$.

A generalized nonconforming space V_h is defined by

$$V_h = \{\bar{v}_h \in X_h : b(\bar{v}_h, \bar{\chi}_h) = 0, \ \forall \bar{\chi}_h \in M_h\}. \tag{4.3}$$

The generalized nonconforming problem for the primal variable is to seek $\bar{u}_h = \bar{w}_h + \bar{u}_{Dh}$ such that $\bar{w}_h \in V_h$ satisfies

$$a(\bar{u}_h, \bar{v}_h) = \bar{f}(\bar{v}_h) - a(\bar{u}_{Dh}, \bar{v}_h), \ \forall \bar{v}_h \in V_h. \tag{4.4}$$

Theorem 4.1. *The problem (4.4) has a unique solution $\bar{u}_h \in V_h$.*

Proof of this theorem follows using the Lax–Milgram lemma [14] and the following lemma.

Lemma 4.2. $||\bar{v}_h||_h = a(\bar{v}_h, \bar{v}_h)^{1/2}$ *defines a norm on V_h.*

Proof. We have,

$$a(\bar{v}_h, \bar{v}_h) = \sum_{K\in\mathcal{T}_h} \Big(\mu \int_K \nabla\bar{v}_h : \nabla\bar{v}_h dx + (\mu+\lambda) \int_K (div(\bar{v}_h))^2 dx\Big).$$

Now, $a(\bar{v}_h, \bar{v}_h) = 0$ with $\bar{v}_h = (v_{h1}, v_{h2})$ implies that we have $\frac{\partial v_{h1}}{\partial x_1} = \frac{\partial v_{h1}}{\partial x_2} = \frac{\partial v_{h2}}{\partial x_1} = \frac{\partial v_{h2}}{\partial x_2} = 0$ on each element $K \in \mathcal{T}_h$. Thus, v_{h1} and v_{h2} are constants over each $K \in \mathcal{T}_h$. To show $\bar{v}_h$ is a constant vector over Ω, consider two adjacent triangles K_1 and K_2 with $E = \partial K_1 \cap \partial K_2$ and let $\bar{v}_h = (c_1, c_2)$ in K_1, $\bar{v}_h = (c_3, c_4)$ in K_2. Set $\bar{\chi}_h = (\chi_{h1}, 0)$ with

$$\chi_{h1}|_{\partial K_1} = \begin{cases} 1 \text{ on } E \\ 0 \text{ on } \partial K_1/E \end{cases}$$

and

$$\chi_{h1}|_{\partial K_2} = \begin{cases} -1 \text{ on } E \\ \ \ 0 \ \text{ on } \partial K_2/E. \end{cases}$$

Now using the definition of V_h, we get, $0 = b(\bar{v}_h, \bar{\chi}_h) = (c_1 - c_3) \int_E d\gamma$ which implies $c_1 = c_3$. Likewise, by taking $\bar{\chi}_h = (0, \chi_{h2})$ with

$$\chi_{h2}|_{\partial K_1} = \begin{cases} 1 \text{ on } E \\ 0 \text{ on } \partial K_1/E \end{cases}$$

and

$$\chi_{h2}|_{\partial K_2} = \begin{cases} -1 \text{ on } E \\ \ \ 0 \ \text{ on } \partial K_2/E. \end{cases}$$

we get $0 = b(\bar{v}_h, \bar{\chi}_h) = (c_2 - c_4) \int_E d\gamma$ which implies $c_2 = c_4$ and thus $\bar{v}_h$ is a constant vector over Ω.

Let $\bar{v}_h = (c, d)$ be the constant vector over Ω and for any edge $E_b \subset \Gamma$, set $\bar{\chi}_h^1 = (\chi_1, 0)$ and $\bar{\chi}_h^2 = (0, \chi_1)$ with

$$\chi_1|_{E_b} = \begin{cases} 1 \text{ on } E_b \\ 0 \text{ elsewhere.} \end{cases}$$

Now using the definition of V_h, $0 = b(\bar{v}_h, \bar{\chi}_h^1) = c\int_E d\gamma$ and $0 = b(\bar{v}_h, \bar{\chi}_h^2) = d\int_E d\gamma$, which implies $c = d = 0$. This proves the desired claim. □

For ease of further analysis, we consider $\bar{g} = \bar{0}$ in the problem (2.2). Estimates in case of nonhomogeneous $\bar{g}$ satisfying the hypothesis of the Lemma 2.2, can be derived similarly.

Theorem 4.3. *The problem (4.1) and (4.2) has a unique solution $(\bar{u}_h, \bar{\kappa}_h) \in X_h \times M_h$ iff the following compatibility condition holds.*

$$\{\bar{\chi}_h \in M_h : b(\bar{v}_h, \bar{\chi}_h) = 0 \ \forall \bar{v}_h \in X_h\} = \{\bar{0}\}. \tag{4.5}$$

Proof. The problem (4.1) and (4.2) is posed over finite dimensional space therefore it suffices to prove either existence or uniqueness of $(\bar{u}_h, \bar{\kappa}_h) \in X_h \times M_h$. In this case, it is easy to establish the uniqueness. To observe this, assume $\bar{f} = \bar{0}$. Then, from (4.1) we find

$$a(\bar{u}_h, \bar{v}_h) + b(\bar{v}_h, \bar{\kappa}_h) = 0 \ \forall \bar{v}_h \in X_h.$$

In particular, by taking $v_h = u_h$ in the last equation and using (4.2) thereafter, we obtain that $\bar{u}_h = \bar{0}$. Now, from primal hybrid form, we get $b(\bar{v}_h, \bar{\kappa}_h) = 0 \ \forall \bar{v}_h \in X_h$. Hence, we observe that the compatibility condition (4.5) holds iff $\bar{\kappa}_h = \bar{0}$. □

Lemma 4.4. *Assume*

$$\left\{ \hat{\bar{\chi}}_h \in (\hat{S})^2 : \int_{\partial\hat{K}} \hat{\bar{v}}_h \cdot \hat{\bar{\chi}}_h d\gamma = 0 \ \forall \hat{\bar{v}}_h \in (\hat{P})^2 \right\} = \{\bar{0}\}. \tag{4.6}$$

Then, the compatibility condition (4.5) holds.

Proof. Using (4.6) and the definitions of $S_{\partial K}$ and P_K, we obtain

$$\left\{ \bar{\chi}_h \in S_{\partial K} : \int_{\partial K} \bar{v}_h \cdot \bar{\chi}_h d\gamma = 0 \ \forall \bar{v}_h \in P_K \right\} = \{\bar{0}\}.$$

for all $K \in \mathcal{T}_h$. Hence the statement of the lemma follows. □

The following lemma shows that the above primal hybrid finite element space satisfies condition (4.6).

Lemma 4.5. *Let* $\hat{\bar{\chi}}_h = (\hat{\chi}_{h1}, \hat{\chi}_{h2}) \in \hat{S}_0 \times \hat{S}_0$ *satisfies*

$$\int_{\partial\hat{K}} \hat{\chi}_{hi} \hat{v}_{hi} d\gamma = 0 \quad \forall v_{hi} \in P_1(\hat{K})|_{\partial\hat{K}}. \tag{4.7}$$

Then, for $i = 1, 2,\ \chi_{hi} = 0$.

Proof. We refer to Lemma 4 of Raviart and Thomas [20] for the proof of this result. □

5. A priori error analysis

This section is devoted to the convergence analysis of the proposed primal hybrid finite element method. Below we discuss few important lemmas.

Lemma 5.1 (see Crouzeix and Raviart [11]). *For* $\chi \in H^1(K)$ *and* $v \in H^1(K)$, *there exists a positive constant C such that for any* $K \in \mathcal{T}_h$ *and* $E \in \mathcal{E}_h(K)$, *we have*

$$\int_E (\chi - \pi_E \chi) v d\gamma \le C \frac{h_K}{\rho_K} |\chi|_{H^1(K)} |v|_{H^1(K)} \tag{5.1}$$

where $\pi_E \chi$ *is the orthogonal projection from* $L^2(E)$ *over* $P_1(E)$.

Lemma 5.2. *For* $\bar{\phi} \in (H^2(\Omega))^2$, *let* $\bar{\psi} = (\psi_1, \psi_2) = \mu \frac{\partial\bar{\phi}}{\partial\nu} + (\mu + \lambda) div(\bar{\phi})\nu \quad \text{on} \quad \partial K \ \forall\ K \in \mathcal{T}_h$. *Then*

$$\inf_{\bar{\chi}_h \in M_h} \sup_{\bar{v} \in X} \frac{b(\bar{v}, \bar{\psi} - \bar{\chi}_h)}{||\bar{v}||_h} \le Ch|\bar{\phi}|_{H^2(\Omega)^2}. \tag{5.2}$$

Proof. We define $\bar{\chi}_h \in M_h$ as follows. Let for $K \in \mathcal{T}_h$ and $E \subset \partial K$, $\bar{\chi}_h = \pi_E(\mu \frac{\partial\bar{\phi}}{\partial\nu} + (\mu + \lambda) div(\bar{\phi})\nu)$ on E. Now using Lemma 5.1, for $\bar{v} \in H^1(K)^2$

$$\begin{aligned} \int_E (\bar{\psi} - \bar{\chi}_h) \cdot \bar{v} d\gamma &= \int_E \left((\mu \frac{\partial\bar{\phi}}{\partial\nu} + (\mu + \lambda) div(\bar{\phi})\nu) - \pi_E(\mu \frac{\partial\bar{\phi}}{\partial\nu} + (\mu + \lambda) div(\bar{\phi})\nu) \right) \cdot \bar{v} d\gamma \\ &\le Ch_K |\bar{\phi}|_{H^2(K)^2} ||\bar{v}||_{H^1(K)^2}. \end{aligned}$$

Now for $\bar{v} \in X$

$$b(\bar{v}, \bar{\psi} - \bar{\chi}_h) = \sum_{E \in \mathcal{E}_h} \int_E (\bar{\psi} - \bar{\chi}_h) \cdot \bar{v} d\gamma \le Ch|\bar{\phi}|_{H^2(\Omega)^2} ||\bar{v}||_h.$$

Hence the desired result of the lemma follows. □

Let the interpolation operator $\bar{\mathcal{I}}_h : (H^2(\Omega))^2 \cap (H_0^1(\Omega))^2 \to V_h$ be defined as

$$\bar{\mathcal{I}}_h \bar{\phi}(m_E) = \frac{1}{|E|} \int_E \bar{\phi} d\gamma, \tag{5.3}$$

where m_E is the mid point of edge E.

$$div(\bar{\mathcal{I}}_h \bar{\phi})|_K = \frac{1}{|K|} \int_K div(\bar{\phi}) dx, \ \ \forall K \in \mathcal{T}_h. \tag{5.4}$$

We have the following approximation properties of $\bar{\mathcal{I}}_h$ [7].

Lemma 5.3. *For any vector valued function* $\bar{\phi} \in (H^2(\Omega))^2$, *it holds that*

$$||\bar{\phi} - \bar{\mathcal{I}}_h \bar{\phi}||_{L^2(\Omega)^2} + h||\nabla(\bar{\phi} - \bar{\mathcal{I}}_h \bar{\phi})||_{L^2(\Omega)^2} \le C_\Omega h |\bar{\phi}|_{H^2(\Omega)^2}. \tag{5.5}$$

We define the orthogonal projection $\bar{\pi}_h$ as:

$$a(\bar{u}-\bar{\pi}_h\bar{u},\bar{v}_h)=0\ \forall\bar{v}_h\in V_h \tag{5.6}$$

which will be required in the subsequent error analysis. The following theorem is motivated from the error analysis presented in Brenner and Sung [7] and Raviart and Thomas [20].

Theorem 5.4. *Let $\bar{u}$ and $\bar{u}_h$ be solutions of the problems (2.2) and (4.4) respectively with $\bar{u}\in(H^2(\Omega))^2\cap(H_0^1(\Omega))^2$. Then there exists a positive constant C_Ω independent of h, λ and μ such that*

$$||\bar{u}-\bar{u}_h||_h\le C_\Omega h||\bar{f}||_{L^2(\Omega)^2}. \tag{5.7}$$

Proof. Using the orthogonal projection (5.6) as in Raviart and Thomas [20], we may write

$$||\bar{u}-\bar{u}_h||_h^2=||\bar{u}-\bar{\pi}_h\bar{u}||_h^2+||\bar{\pi}_h\bar{u}-\bar{u}||_h^2. \tag{5.8}$$

Notice that

$$||\bar{\pi}_h\bar{u}-\bar{u}_h||_h=\sup_{\bar{v}_h\in V_h}\frac{a(\bar{\pi}_hu-\bar{u}_h,\bar{v}_h)}{||\bar{v}_h||_h}=\sup_{\bar{v}_h\in V_h}\frac{a(\bar{u}-\bar{u}_h,\bar{v}_h)}{||\bar{v}_h||_h}. \tag{5.9}$$

Now from (3.1) and (4.4), we get

$$a(\bar{u}-\bar{u}_h,\bar{v}_h)+b(\bar{\kappa}-\bar{\chi}_h,\bar{v}_h)=0.$$

Therefore, from (5.9) we obtain

$$||\bar{\pi}_h\bar{u}-\bar{u}_h||_h=\inf_{\bar{\chi}_h\in M_h}\sup_{\bar{v}_h\in V_h}\frac{-b(\bar{\kappa}-\bar{\chi}_h,\bar{v}_h)}{||\bar{v}_h||_h}. \tag{5.10}$$

A use of Lemma 5.2 by taking $\bar{\phi}=\bar{u}$ therein yields

$$\inf_{\bar{\chi}_h\in M_h}\sup_{\bar{v}_h\in X_h}\frac{b(\bar{v}_h,\bar{\kappa}-\bar{\chi}_h)}{||\bar{v}_h||_h}\le Ch|\bar{u}|_{H^2(\Omega)^2}. \tag{5.11}$$

Now, using the definition of $\bar{\pi}_h\bar{u}$, we get

$$||\bar{u}-\bar{\pi}_h\bar{u}||_h=\inf_{\bar{v}_h\in V_h}||\bar{u}-\bar{v}_h||_h. \tag{5.12}$$

From Lemma 2.1, there exists $\bar{w}\in(H^2(\Omega))^2\cap(H_0^1(\Omega))^2$ such that $div\ \bar{w}=div\ \bar{u}$ and

$$||\bar{w}||_{H^2(\Omega)^2}\le C_\Omega||\bar{f}||_{L^2(\Omega)^2}. \tag{5.13}$$

Further, using Lemma 2.2 and (5.13), we obtain

$$||\bar{w}||_{H^2(\Omega)^2}\le\frac{C_\Omega}{1+\lambda}||\bar{f}||_{L^2(\Omega)^2}. \tag{5.14}$$

A use of (5.14) and Lemma 5.3 yields

$$\begin{aligned}\inf_{\bar{v}_h\in V_h}||\bar{u}-\bar{v}_h||_h&\le||\bar{u}-\bar{\mathcal{I}}_h\bar{u}||_h\\&=\Big(\mu||\nabla(\bar{u}-\bar{\mathcal{I}}_h\bar{u})||^2_{L^2(\Omega)^2}+(\mu+\lambda)||div(\bar{w}-\bar{\mathcal{I}}_h\bar{w})||^2_{L^2(\Omega)}\Big)^{1/2}\\&\le C_\Omega h||\bar{f}||_{L^2(\Omega)^2},\end{aligned} \tag{5.15}$$

therein, we have used $div(\bar{\mathcal{I}}_h\bar{w})=div(\bar{\mathcal{I}}_h\bar{u})$ in obtaining the last estimate, which follows from (5.4).

Thus, the result of the theorem follows from (5.15) and (5.11). □

Theorem 5.5. *Let $\bar{u}$ and $\bar{u}_h$ satisfies the hypothesis of Theorem 5.4. Then, there exists a positive constant C_Ω independent of h, λ and μ such that*

$$||\bar{u}-\bar{u}_h||_{L^2(\Omega)^2}\le C_\Omega h^2||\bar{f}||_{L^2(\Omega)^2}. \tag{5.16}$$

Proof. We refer to Brenner and Sung [7] for the proof of this theorem. □

We use the following operator norm on M generated by the bilinear form $b(\cdot,\cdot)$ which is more convenient to estimate the Lagrange multiplier.

$$||\bar{\chi}||_M=\sup_{\bar{v}\in X}\frac{b(\bar{v},\bar{\chi})}{||\bar{v}||_X}. \tag{5.17}$$

Lemma 5.6 (Discrete inf-sup condition). *Assume the condition (4.6) holds. Then, there exists inf-sup constant $\alpha > 0$ such that*

$$\sup_{\bar{v}_h \in X_h} \frac{b(\bar{v}_h, \bar{\chi}_h)}{||\bar{v}_h||_X} \geq \alpha ||\bar{\chi}_h||_M \quad \forall \bar{\chi}_h \in M_h. \tag{5.18}$$

Proof. For $K \in \mathcal{T}_h$ and $(\bar{v}, \bar{\chi}_h) \in P_K \times M_h$ let

$$\int_{\partial K} \bar{\chi}_h \cdot \bar{v}\, d\gamma = \int_{\partial \hat{K}} \hat{\bar{\chi}}_h \cdot \hat{\bar{v}}\, d\gamma, \tag{5.19}$$

where $\hat{\bar{v}} = (\hat{v}_1, \hat{v}_2) = (v_1 \circ F_K, v_2 \circ F_K) = \bar{v} \circ F_K$ and $\hat{\bar{\chi}}_h = \bar{\chi}_h \circ F_K$. Note that $\hat{\bar{v}} \in (\hat{P})^2$ and from the definition of $\hat{S}$, $\hat{\bar{\chi}}_h \in (\hat{S})^2$. A use of (4.6) yields

$$\int_{\partial \hat{K}} \hat{\bar{\chi}}_h \cdot \hat{\bar{v}}\, d\gamma \geq \hat{\alpha} ||\hat{\bar{\chi}}_h||_{\partial \hat{K}} ||\hat{\bar{v}}||_{H^1(\hat{K})^2} \tag{5.20}$$

where

$$||\bar{\chi}_h||_{\partial K} = \sup_{\bar{v} \in H^1(K)^2} \frac{\int_{\partial K} \bar{\chi}_h \cdot \bar{v} d\gamma}{||\bar{v}||_{H^1(K)^2}}. \tag{5.21}$$

Note that

$$||\bar{v}||_{H^1(K)^2} \leq h_{\hat{K}} \rho_K^{-1} |det(B_K)|^{1/2} ||\hat{\bar{v}}||_{H^1(\hat{K})^2}. \tag{5.22}$$

From (5.19) to (5.21), we get

$$||\bar{\chi}_h||_{\partial K} = \sup_{\bar{\omega} \in H^1(K)^2} \frac{\int_{\partial \hat{K}} \hat{\bar{\chi}}_h \cdot \hat{\bar{\omega}} d\gamma}{||\bar{\omega}||_{H^1(K)^2}}, \tag{5.23}$$

where $\hat{\bar{\omega}} = \bar{\omega} \circ F_K$. Since

$$||\hat{\bar{\omega}}||_{H^1(\hat{K})^2} \leq h_K \rho_{\hat{K}}^{-1} |det(B_K)|^{-1/2} ||\bar{\omega}||_{H^1(K)^2}, \tag{5.24}$$

therefore, from (5.23), we get

$$||\bar{\chi}_h||_{\partial K} \leq h_K \rho_{\hat{K}}^{-1} |det(B_K)|^{-1/2} ||\hat{\bar{\chi}}_h||_{\partial \hat{K}}. \tag{5.25}$$

Using (5.20), (5.22) and (5.25), it follows that

$$\int_{\partial K} \bar{\chi}_h \cdot \bar{v} d\gamma \geq \alpha ||\bar{\chi}_h||_{\partial K} ||\bar{v}||_{H^1(K)^2}, \tag{5.26}$$

where $\alpha = \frac{\hat{\alpha}}{\sigma} \frac{\rho_{\hat{K}}}{h_{\hat{K}}}$. Normalizing $\bar{v}$ such that $||\bar{v}||_{H^1(K)^2} = ||\bar{\omega}||_{H^1(K)^2}$, there exist a function $\bar{v} \in (P_K)^2$ such that

$$-\int_{\partial K} \bar{\chi}_h \cdot \bar{v} d\gamma \geq -\alpha \int_{\partial K} \bar{\chi}_h \cdot \bar{\omega} d\gamma. \tag{5.27}$$

Now, for $\bar{\omega} \in X$, there exists $\bar{v}_h \in X_h$ such that

$$\frac{b(\bar{v}_h, \bar{\chi}_h)}{||\bar{v}_h||_X} \geq \alpha \frac{b(\bar{\omega}, \bar{\chi}_h)}{||\bar{\omega}||_X}.$$

Hence,

$$\sup_{\bar{v}_h \in X_h} \frac{b(\bar{v}_h, \bar{\chi}_h)}{||\bar{v}_h||_X} \geq \alpha \sup_{\bar{\omega} \in X} \frac{b(\bar{\omega}, \bar{\chi}_h)}{||\bar{\omega}||_X} = \alpha ||\bar{\chi}_h||_M.$$

□

Theorem 5.7. *There exists a positive constant C_Ω independent of h, λ and μ such that*

$$||\bar{\kappa} - \bar{\kappa}_h||_M \leq C_\Omega h ||\bar{f}||_{L^2(\Omega)^2}. \tag{5.28}$$

Proof. Using (3.1) and (4.1), we get

$$b(\bar{\kappa}_h - \bar{\chi}_h, \bar{v}_h) = a(\bar{u} - \bar{u}_h, \bar{v}_h) + b(\bar{\kappa} - \bar{\chi}_h, \bar{v}_h). \tag{5.29}$$

For $\bar{\chi}_h \in M_h$, from Lemma 5.6, we may choose a function $\bar{v}_h \in X_h$ such that

$$b(\bar{\kappa}_h - \bar{\chi}_h, \bar{v}_h) \geq \alpha ||\bar{v}_h||_X ||\bar{\kappa}_h - \bar{\chi}_h||_M.$$

Since $||\bar{v}_h||_h \le ||\bar{v}_h||_X$, from (5.29) we get

$$\alpha||\bar{\kappa}_h - \bar{\chi}_h||_M \le ||\bar{u} - \bar{u}_h||_h + ||\bar{\kappa} - \bar{\chi}_h||_M$$

and

$$||\bar{\kappa} - \bar{\kappa}_h||_M \le \frac{1}{\alpha}||\bar{u} - \bar{u}_h||_h + \left(1 + \frac{1}{\alpha}\right) \inf_{\bar{\chi}_h \in M_h} ||\bar{\kappa} - \bar{\chi}_h||_M. \tag{5.30}$$

Considering $\bar{\phi} = \bar{u}$ and $\bar{\psi} = \bar{\kappa}$ in Lemma 5.2, we get

$$\inf_{\bar{\chi}_h \in M_h} \sup_{\bar{v} \in X} \frac{b(\bar{v}, \bar{\kappa} - \bar{\chi}_h)}{||\bar{v}||_h} \le Ch|\bar{u}|_{H^2(\Omega)^2}. \tag{5.31}$$

Noting that $||\bar{v}||_h \le ||\bar{v}||_X$, a use of (5.17) yields

$$\inf_{\bar{\chi}_h \in M_h} ||\bar{\kappa} - \bar{\chi}_h||_M \le C_\Omega h|\bar{u}|_{H^2(\Omega)^2}. \tag{5.32}$$

The result of theorem then follows from (5.30), (5.32) and Theorem 5.4. □

Remark. Notice that in Lemma 5.2, the constant C has the dependence on the Lamé parameters as it comprises of the factor $\max\{\mu, \mu + \lambda\}$. Hence, in Theorems 5.4 and 5.7, if $\lambda \to \infty$, the order of the estimates do not deteriorate as we can find C_Ω sufficiently large to dominate $\max\{\mu, \mu + \lambda\}/(1 + \lambda)$.

6. Numerical examples

This section is devoted to the numerical illustration of theoretical findings. We first describe the algebraic formulation of the discrete problem (4.1) and (4.2). We use the Lagrangian linear basis for X_h, with $N = \dim(X_h)$ and let $\{\bar{\psi}_i, i = 1 \ldots .N_M\}$ denote the basis for M_h with $N_M = \dim(M_h)$, essentially $N = 6 \times$ number of elements in $\mathcal{T}_h$ and $N_M = 2 \times$ number of edges in $\mathcal{T}_h$. Note that, the basis of M_h comprises of the vectors from the set $\{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. We denote by $\mathbb{A} \in \mathbb{R}^{N \times N}$ the matrix representation of the bilinear form $a(\cdot, \cdot)$ and $\mathbb{B} \in \mathbb{R}^{N_M \times N}$ denotes the matrix representation of the bilinear form $b(\cdot, \cdot)$ defined in (2.12). Further, let $\mathbf{F} \in \mathbb{R}^N$ be the vector representation of the right hand side in (4.1), $\mathbf{U} \in \mathbb{R}^N$ and $\Lambda \in \mathbb{R}^{N_M}$ denote respectively the vector representations of the unknown solution $\bar{u}_h$ and Lagrange multiplier $\bar{\kappa}_h$.

The matrix formulation of (4.1) and (4.2) is then stated as: to find $(\mathbf{U}, \Lambda) \in \mathbb{R}^N \times \mathbb{R}^{N_M}$ satisfying

$$\mathbb{A}\ \mathbf{U} + \mathbb{B}'\ \Lambda = \mathbf{F}, \tag{6.1}$$

$$\mathbb{B}\ \mathbf{U} = \mathbf{0}. \tag{6.2}$$

In case of the non-homogeneous Dirichlet data $\bar{u} = \bar{g}$ on Γ, the right hand side vector in (6.2) will change to $\mathbb{L} \in \mathbb{R}^{N_M}$ which is the vector representation of $\langle \bar{\chi}_h, \bar{u}_{Dh} \rangle,\ \bar{\chi}_h \in M_h$.

Below, we report numerical results for two test examples. In both the examples the Lamé parameter can be computed as

$$\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 - 2\nu)(1 + \nu)},$$

where Young modulus E and Poisson's ratio ν which are prescribed in examples. Taking into consideration that primal variable of the saddle point problem coincides with the solution of the corresponding non-conforming problem at the Gauss-Legendre points, we also report the convergence behavior of the non-conforming counterpart for the primal variable, therein the discrete space V_h is taken to be the Crouzeix-Raviart nonconforming finite element space [11].

Example 1. Let $\Omega = (0, 1) \times (0, 1)$, $E = 1$ and $\nu = 0.45$. The true solution is given by

$$\bar{u} = (x^2 - y^2, x^2 + y^2).$$

The source function $\bar{f}$ can be computed using (2.2) knowing the true solution $\bar{u}$. The convergence behavior of error $\bar{u} - \bar{u}_h$ in $(L^2)^2$ and $(H^1)^2$ norms for primal-hybrid method as well as non-conforming method are shown in Tables 1 and 2, respectively. We observe linear convergence in the energy norm and quadratic convergence in the $(L^2)^2$ norm which matches with our theoretical results. Table 1 also depicts the linear convergence of the Lagrange multiplier error $\|\bar{\kappa} - \bar{\kappa}_h\|_M$ with respect to the mesh size. The plots of the exact and discrete solutions are reported in Figs. 1 and 2. Fig. 1 shows the plot of first components of the exact and discrete solutions while Fig. 2 shows the second components of the exact and discrete solutions.

Next, numerical example illustrates the locking-free convergence in various norms for the method under consideration.

Table 1
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of primal-hybrid method for Example 1.

h	$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u}-\bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)	$\|\bar{\kappa}-\bar{\kappa}_h\|_M$	Order (h)
1/2	2.2114 1e−1	–	1.8067 1e+0	–	3.5269 1e+0	–
1/4	6.5555 1e−2	1.7542	1.0128 1e+0	0.8350	1.8065 1e+0	0.9653
1/8	1.7493 1e−2	1.9059	5.2944 1e−1	0.9358	9.1689 1e−1	0.9783
1/16	4.4606 1e−3	1.9715	2.6890 1e−1	0.9774	4.6222 1e−1	0.9882
1/32	1.1203 1e−3	1.9934	1.3513 1e−1	0.9927	2.3209 1e−1	0.9939

Table 2
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of non-conforming method for Example 1.

h	$\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u}-\bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)
1/2	3.0522 1e−1	–	1.5298 1e+0	–
1/4	9.0352 1e−2	2.1196	8.4549 1e−1	1.0325
1/8	2.4102 1e−2	2.0923	4.3919 1e−1	1.0371
1/16	6.1455 1e−3	2.0642	2.2250 1e−1	1.0272
1/32	1.5434 1e−3	2.0393	1.1172 1e−1	1.0169

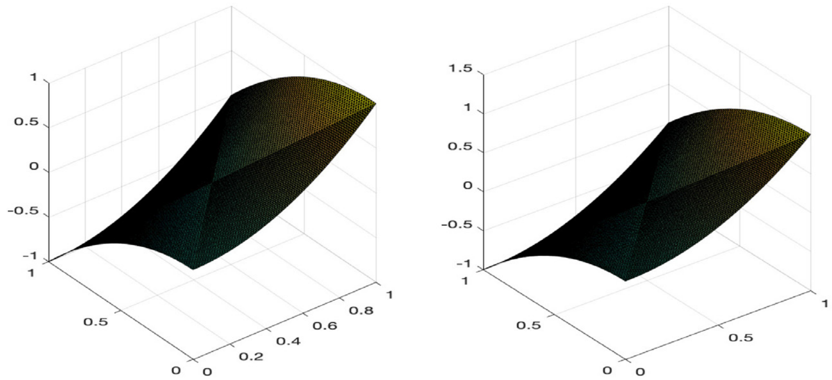

Fig. 1. First components of the exact displacement vector (left) and the discrete displacement vector (right) at $h = \frac{1}{64}$ for Example 1.

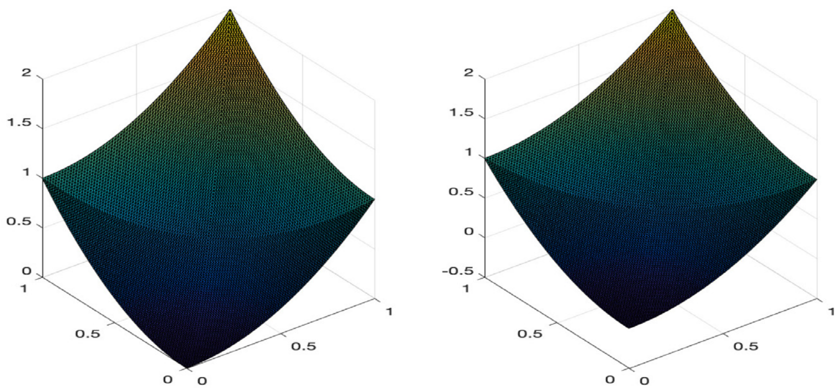

Fig. 2. Second components of the exact displacement vector (left) and the discrete displacement vector (right) at $h = \frac{1}{64}$ for Example 1.

Example 2. Let $\Omega = (0,1) \times (0,1)$ and the exact solution is $\bar{u} = (u_1, u_2)$ with

$$u_1 = \sin(2\pi y)\big(\cos(2\pi x) - 1\big) + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y)$$

and

$$u_2 = \sin(2\pi x)\big(\cos(2\pi y) - 1\big) + \frac{1}{1+\lambda}\sin(\pi x)\sin(\pi y).$$

The source function $\bar{f}$ can be computed using (2.2) and Dirichlet boundary data is chosen to match the exact solution. The Young modulus is taken to be $E = 2.5$.

Table 3
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of primal-hybrid method for Example 2 for $\nu = 0.4$.

h	$\nu = 0.4$					
	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)	$\|\bar{\kappa} - \bar{\kappa}_h\|_M$	Order (h)
1/2	5.2262 1e−1	–	5.1241 1e+0	–	5.9918 1e+0	–
1/4	1.1420 1e−1	2.1942	2.2791 1e+0	1.1688	3.0878 1e+0	0.9564
1/8	3.0760 1e−2	1.8925	1.1794 1e+0	0.9505	1.5562 1e+0	0.9886
1/16	7.8827 1e−3	1.9643	5.9579 1e−1	0.9851	7.8111 1e−1	0.9944
1/32	1.9845 1e−3	1.9899	2.9873 1e−1	0.9960	3.9112 1e−1	0.9979

Table 4
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of non-conforming method for Example 2 for $\nu = 0.4$.

h	$\nu = 0.4$			
	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)
1/2	5.7347 1e−1	–	5.7683 1e+0	–
1/4	1.4799 1e−1	2.3586	2.9877 1e+0	1.1455
1/8	4.1786 1e−2	2.0022	1.5531 1e+0	1.0359
1/16	1.0844 1e−2	2.0376	7.8551 1e−1	1.0297
1/32	2.7390 1e−3	2.0310	3.9396 1e−1	1.0185

Table 5
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of primal-hybrid method for Example 2 for $\nu = 0.499999$.

h	$\nu = 0.499999$					
	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)	$\|\bar{\kappa} - \bar{\kappa}_h\|_M$	Order (h)
1/2	5.0344 1e−1	–	5.0987 1e+0	–	5.5428 1e+0	–
1/4	1.1293 1e−1	2.1564	2.2701 1e+0	1.1674	2.8631 1e+0	0.9530
1/8	3.1118 1e−2	1.8596	1.1753 1e+0	0.9497	1.4584 1e+0	0.9732
1/16	8.0525 1e−3	1.9502	5.9389 1e−1	0.9848	7.2577 1e−1	1.0068
1/32	2.0337 1e−3	1.9853	2.9780 1e−1	0.9958	3.6093 1e−1	1.0078

Table 6
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of non-conforming method for Example 2 for $\nu = 0.499999$.

h	$\nu = 0.499999$			
	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)
1/2	5.6119 1e−1	–	5.4080 1e+0	–
1/4	1.4696 1e−1	2.3331	2.8357 1e+0	1.1241
1/8	4.2362 1e−2	1.9695	1.4779 1e+0	1.0318
1/16	1.1097 1e−2	2.0234	7.4841 1e−1	1.0278
1/32	2.8114 1e−3	2.0265	3.7551 1e−1	1.0179

Tables 3 –8 illustrate the numerical performance of primal-hybrid method as well as non-conforming method for three values of Poisson's ratio $\nu = 0.4$, $\nu = 0.499999$ and $\nu = 0.49999999$, confirming the quadratic convergence of error in $(L^2)^2$ norm and linear convergence in H^1 norm. The convergence behavior of the Lagrange multiplier error $\|\bar{\kappa} - \bar{\kappa}_h\|_M$ is also reported in Tables 3, 5 and 7 for Poisson's ratio $\nu = 0.4$, $\nu = 0.499999$ and $\nu = 0.49999999$, and we clearly observe the linear convergence.

Locking free convergence phenomenon can be observed from the numerical results reported in Tables 5–8 corresponding to $\nu = 0.499999$ and $\nu = 0.49999999$ ($\lambda \to \infty$), respectively, ensuring the robustness of the method with respect to Lamé parameter λ.

7. Concluding remarks

In this manuscript, we considered the primal hybrid formulation of the linear elasticity problem and its corresponding nonconforming formulation. The existence and uniqueness of the solution to the primal hybrid formulation has been established. We further derive a priori error estimates of underlying primal hybrid method for both the variables. Our numerical experiments illustrates that the rate of convergence of the method for both the variables is optimal and is independent of the Lamé parameters, which guarantees the robustness of the method.

Table 7
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of primal-hybrid method for Example 2 for $\nu = 0.49999999$.

	$\nu = 0.49999999$					
h	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)	$\|\bar{\kappa} - \bar{\kappa}_h\|_M$	Order (h)
1/2	5.0344 1e−1	–	5.0987 1e+0	–	5.5428 1e+0	–
1/4	1.1293 1e−1	2.1564	2.2701 1e+0	1.1674	2.8631 1e+0	0.9530
1/8	3.1118 1e−2	1.8596	1.1753 1e+0	0.9497	1.4584 1e+0	0.9732
1/16	8.0525 1e−3	1.9502	5.9389 1e−1	0.9848	7.2577 1e−1	1.0068
1/32	2.0337 1e−3	1.9854	2.9780 1e−1	0.9958	3.6093 1e−1	1.0078

Table 8
Order of convergence in $(L^2)^2$ and $(H^1)^2$ norms of non-conforming method for Example 2 for $\nu = 0.49999999$.

	$\nu = 0.49999999$			
h	$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)^2}$	Order (h)	$\|\bar{u} - \bar{u}_h\|_{H^1(\Omega)^2}$	Order (h)
1/2	5.6119 1e−1	–	5.4080 1e+0	–
1/4	1.4696 1e−1	2.331	2.8357 1e+0	1.1241
1/8	4.2362 1e−2	1.9695	1.4779 1e+0	1.0318
1/16	1.1097 1e−2	2.0234	7.4841 1e−1	1.0278
1/32	2.8114 1e−3	2.0265	3.7551 1e−1	1.0179

Data availability

No data was used for the research described in the article.

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