**Authors’ Replies to Reviewers’ Comments and Requests**

The authors would like to thank the reviewers for their comments and time in reviewing the manuscript. Their suggestions have helped us improve the quality of the manuscript.

**Note:** Reference, figure, equation, and page numbers used below are from the **revised** manuscript unless stated otherwise. Changes to the manuscript are denoted by red text, while green text highlights material from the original manuscript used to address reviewers’ comments.

**Comments from reviewers:**

**Reviewer #1:**

The manuscript presents a double hybrid formulation for two- and three-dimensional elasticity problems. The approximation spaces lead to a uniformly convergent scheme that is independent of the Poisson ratio. The resulting global system is symmetric and positive definite when applied to compressible solids and is a saddle-point problem for incompressible cases. A single mean pressure per element acts as a Lagrange multiplier to impose the incompressibility constraint.

The manuscript is well-written and clear. It also provides a thorough review of related works, highlighting the connection between the proposed formulation and the earlier works of the authors. Detailed convergence studies are presented in Section 6, with the one in Section 6.1 demonstrating the clear superiority of the proposed formulation over the popular Taylor-Hood formulation. Acceptance of the manuscript is recommended, and I have only minor clarification questions.

**Reply**:

Thank you for your positive feedback on the manuscript. Below, we address your comments.

1) The number of degrees of freedom in the global system depends on the number of element faces in a mesh. Thus, a mesh composed of hexahedron elements results in fewer degrees of freedom than one with tetrahedron elements, given the same number of nodes. This is demonstrated in the last problem solved in the paper. Local mesh refinement of hexahedron meshes generally leads to hanging nodes. Is the proposed formulation suitable for discretization with hanging nodes?

**Reply**:

Yes, it is definitely possible to use non-uniform meshes with hanging nodes. In the PhD thesis of Pablo Carvalho (available at: https://repositorio.unicamp.br/acervo/detalhe/1170781), you can see a very similar approach (similar approximation spaces setup but with a single hybridization on the tangential stresses) working with hanging nodes.

2) Please check the definition of the L2 space given on page 3: Isn't component v\_1required to be in L2?

**Reply**:

Thank you for noticing that. We corrected the definition of the space.

3) Problem (39) is defined across six approximation spaces. Nonetheless, multiple degrees of freedom can be condensed at the element level. While this can be done in parallel, it is necessary to integrate all matrices and vectors listed in (41) before performing the static condensation to ultimately arrive at (47). Please comment on the overhead associated with these operations. How large must a problem be for the solution of the global system of equations to dominate the cost?

**Reply**:

The suggestion of the reviewer is relevant. The computation of the condensed element stiffness has a considerable computational cost, however, it was out of the scope of the present work to tackle that. We definitely plan to explore the numerical efficiency in future works. It is worth to mention that the computation of the element stiffness matrix is highly parallelizable, whereas the decomposition of the global system has at least order neq^2 growth.

4) Figure 11 and related discussion: Do the number of dofs used for the plots include the internal and external dofs, or only the external ones?

**Reply**:

For all the convergence analyses, the number of DOFs considered is, in fact, the size of the global system to be solved. Thus, only the external ones are considered. We added this information to the text to make it more clear for the readers:

“We compare the error as a function of the number of degrees of freedom in the global system (external DOFs)...”

**Reviewer #2:**

This paper deals with an old problem, the calculation of quasi-incompressible elastic structures, which has been the subject of many works, starting with Herrmann's formulation (1965), and which can now be solved efficiently using FE industrial software

In this paper, which is of interest, is the adaptation of a numerical formulation that the authors have already proposed and developed in several papers to such elasticity problems, where, as in all approaches, pressure is considered as an additional unknown.

In the proposed formulation, called semi-hybrid-mixed formulation, only the normal component of the displacement is continuous, while the continuity of the tangent component is satisfied in a mean sense.

In view of the illustrations, the proposed numerical formulation works, but it seems no better than the classical ones.

**Reply**:

Thank you for the feedback on the manuscript and for mentioning the paper by Leonard R. Herrmann (1965). We added the reference to our manuscript in the Introduction section as follows:

“… Examples of these techniques include: using reduced integration with hourglass control [20, 21], average nodal pressure formulations [22, 23], extended variational principles with mean pressure function [24]...”.

Below we address your comments.

1-The title is too technical -it should be simplified

**Reply**:

Thank you for the suggestion. We changed the title to:

A Primal Double-Hybrid FEM for 3D Compressible and Incompressible Elasticity Using H(div)–L² Spaces

2- Formulation and notations

--index mistake in the definition of H1

-(u,v) :not defined

**Reply**:

Thank you for noticing about the definition of the H1 space, it has been corrected. Also, we included the definition of the L2 product of (u,v).

3-Taylor-Hood elements

Such elements are seen in the paper as a reference, which is highly debatable. They approximate the pressure by very regular functions that are not compatible with non-homogeneous structures. Moreover, in solid mechanics, one does not like to have k' > k - 1 where the pressure ( a stress) is approximated by polynomials of degree k' and the displacement by polynomials of degree k. It is worth noting that the authors ' approach is compatible with this remark.

**Reply**:

Thank you for sharing your view on using Taylor-Hood elements as a reference. Although we have the same opinion regarding their capabilities of approximating pressure/stress in non-homogeneous structures, they still remain one of the most used techniques for solving incompressible elasticity/Stokes equations. We believe that the defectiveness of Taylor-Hood when applied to heterogeneous structures is not sufficiently documented in the literature.

With respect to the second remark, we also agree with the reviewer. The basis of our numerical method is a primal approximation with hybridized tangential components. The convergence rate of the displacement will always be larger than the rate of the stresses. The qualities of our scheme that we emphasized are its uniform convergence with respect to compressibility and the exact representation of a divergence-free solution for incompressible materials.

4-Hybrid-mixed formulation

-(18):. To say that what's unusual is that the stress is not symmetrical

**Reply**:

Indeed, the way the traction function space is defined may initially seem a bit confusing. While it is well known that the Cauchy stress tensor is symmetric, there are works in the literature that do not enforce this symmetry explicitly when defining the traction space. For example, see Eq. (2.3) of *Araya R., Harder C., Poza A., & Valentin F. (2024). Multiscale hybrid-mixed methods for the Stokes and Brinkman equations – a priori analysis* (ffhal-04425367v1), in the context of Stokes flows, or the second equation on page 314 of *Harder, C., Madureira, A. L., & Valentin, F. (2016). A hybrid-mixed method for elasticity. ESAIM: Mathematical Modelling and Numerical Analysis, 50(2), 311–336*, in the context of elasticity.

On the other hand, some authors choose to explicitly define the Cauchy stress as belonging to the subspace of symmetric tensors, as seen in Eqs. (2.7) and (2.10) of *Acharya, S. K., & Porwal, K. (2022). Primal hybrid finite element method for the linear elasticity problem. Applied Mathematics and Computation, 435, 127462.* However, we note that the discrete traction space described in Section 4 of that work is, in fact, a subspace of the L^2 space composed of polynomial functions — exactly the same space used in our formulation and in the two previously cited references. Therefore, it is not entirely clear to us whether there is a uniquely correct way to define this space.

Moreover, we emphasize that in our scheme the stress field is not being approximated. For this reason, we chose to maintain generality and define the tractions in the same way as done in the first two references.

5-Semi-hybrid-mixed formulation

-(28) is a particular equilibrium relation, only the resultant. A comment is needed

-It is not clear whether the proposed method is capable of calculating a symmetrical stress per element.

-The interest of the second hybridization of tangential stresses is not clear. A more detailed comment is needed

**Reply**:

- Thank you for the comment on the Eq. (28). We modified the text as follows:

“We then conclude that the integral of the computed normal traction is in equilibrium with the

integral of the body forces b, both in value and moment.”

- The proposed method is based on a primal hybridized formulation. The stresses within each element are computed using the constitutive relation between stress and strain. Therefore, in practical terms, the method is indeed capable of computing a symmetric stress tensor per element. However, these stresses do not belong to the H(div) space of symmetric tensor functions. To achieve this, one would need to add an extra step involving the solution of a local Neumann problem using a mixed formulation with symmetric tensor functions for the stresses, where the imposed tractions are those given in Eq. (28).

- The purpose of the double hybridization is to obtain a positive semi-definite element stiffness matrix with a single constraint related to incompressibility. In the case of compressible elasticity, this constraint equation can be statically condensed, resulting in a positive semi-definite matrix. These arguments are discussed in the last paragraph of Section 5.4, which is highlighted in green in the revised manuscript.

This approach was motivated by numerical issues associated with the matrix structure arising from the semi-hybrid formulation, as detailed in the first two paragraphs of Section 5.2, also highlighted in green in the revised manuscript.

6-Uniform stretch of a non-homogeneous solid

I don't share completely the authors' comments: it's obvious with two elements that the very classical formulations, in particular those implemented in industrial software, give the exact solution (k=1 for the displacement and k=0 for the pressure, per element). This should be true for all formulations.

It's obvious that Taylor-Hood elements don't work because of their abnormal regularity for pressure for non-homogeneous solids.

**Reply**:

We agree with the reviewer. On the other hand, the bad performance of Taylor Hood approximations for heterogeneous materials is rarely emphasized in the literature.

7-Extension

Is it easy to extend the proposed formulation to large-displacement elastic problems?

**Reply**:

We appreciate the reviewer’s question. Indeed, extending the proposed formulation to large-displacement problems is part of our future plans. As the method is based on a primal formulation where stresses are not approximated within an element, we don’t expect there to be any restrictions to extend it to large displacement and/or large deformation problems. We believe it will be a matter of adjusting the kinematics, strain and stress measures. There is an insightful paper that tackles this problem, and it could be very useful for us in the future:

Fu, G., Neunteufel, M., Schöberl, J. and Zdunek, A., 2025. A four-field mixed formulation for incompressible finite elasticity. *Computer Methods in Applied Mechanics and Engineering*, *444*, p.118082.