Eigenvalue Problems:→

Let A be $n \times n$ square matrix. A may be singular or non-singular. Consider the hom. system of equations

$$AX = \lambda X \Rightarrow (A - \lambda I)X = 0 \quad -(1)$$

(1) always has a unique or trivial solution. The values of λ for which (1) has a non-trivial solution, are called the eigenvalues or characteristic values of A. The corresponding non-trivial solution X are called eigen vectors or characteristic vectors of A.

- The problem of determining the eigenvalues and eigenvectors of a square matrix A is called an Eigenvalue problem.

- If the hom. system (1) has a non-trivial solution, then the Rank of $(A - \lambda I) < n$

$\Rightarrow (A - \lambda I)$ is a singular matrix

$\Rightarrow \det(A - \lambda I) = 0.$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding the determinant gives the polynomial of degree n in λ; $P_n(\lambda) = |A - \lambda I| = 0 \quad -(2)$

is called characteristic equation of matrix A.

The roots of (2) are called the eigenvalues.

- Sum of eigenvalues = Trace(A)
- Product of eigenvalues = $|A|$
- If any one of the eigenvalue is zero then $|A| = 0 \Rightarrow$ A is Singular matrix.
- If A is singular $\Rightarrow |A| = 0 \Rightarrow$ one of the eigenvalues is zero.

– If A is diagonal or upper triangular or lower triangular matrix, then the diagonal elements of A are the eigenvalues of A.

– Let λ be an eigenvalues of A and X be the corresponding eigen vector. Then

(1) αA has eigenvalue $\alpha\lambda$ and the corr. eigenvector is X.

$$AX = \lambda X \Rightarrow \alpha AX = \alpha\lambda X$$
$$\Rightarrow (\alpha A)X = (\alpha\lambda)X$$

(2) A^m has eigenvalue λ^m and the corr. eigenvector is X for any positive Integer m.

$$AX = \lambda X$$
$$A(AX) = A(\lambda X) = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$$
$$\Rightarrow A^2 X = \lambda^2 X$$

Similarly $A^m X = \lambda^m X$.

(3) A^{-1} has the eigenvalue $(1/\lambda)$ and the corresponding eigen vector is X; Provided A^{-1} Exists.

$$\because AX = \lambda X$$
$$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$$
$$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$$
$$\Rightarrow IX = \lambda(A^{-1}X)$$
$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X \quad \text{or} \quad A^{-1}X = \left(\frac{1}{\lambda}\right)X.$$

(4) A and A^T have the same eigenvalues.

(5) For a real matrix A, If $\alpha + i\beta$ is an eigen value then $\alpha - i\beta$ is also an eigen value of A.

Que Find the eigenvalues and the corresponding eigenvectors of the following matrices.

(1) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $\qquad (A - \lambda I)X = 0$ — System

Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\Rightarrow \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\Rightarrow \lambda = -2, 5. \quad \text{— Two eigen values.}$$

Corresponding to eigen value $\lambda = -2$

$$(A + 2I)X = 0$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x + 4y = 0$$

$$\Rightarrow x = \frac{-4y}{3}$$

Let $y = 1 \Rightarrow x = -4/3$

$\therefore$ or $y = 3, \Rightarrow x = -4.$

So Eigen Vector is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$.

$\underline{\lambda = 5}$

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x + 4y = 0$$

$$\Rightarrow x = y \qquad \text{let } x = y = 1$$

So Eigen vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Que:→ $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$(A-\lambda I)X = 0 \rightarrow$ System of hom. equation

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 + 1 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = 1 \pm i$

Corr. to $\lambda = 1+i$

$(A-\lambda I)X = 0$

$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - (1+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -ix + y = 0 \quad \Rightarrow x = +\frac{1}{i}y = (-iy)$

and $-x - iy = 0$

$y = 1, \quad x = -i$

So Eigen vector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Corr. to $\lambda = 1-i$

$(A-\lambda I)X = 0$

$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$ix + y = 0 \Rightarrow y = -ix$

$x = 1; \quad y = -i$

So eigenvector is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ or $\begin{bmatrix} i \\ 1 \end{bmatrix}$

→ Eigenvectors Corresponding to distinct eigen values are linearly Independent.

Que (1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ (3) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (4) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Cayley – Hamilton Thm: → Every matrix 'A' satisfies its own characterstic equation.

— Verify Cayley Hamilton Thm for $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Also obtain A^{-1} and A^3.

Sol^n $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

The char. equation of A is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$$

$A^2 = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$; $A^3 = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$

$-A^3 + 3A^2 - A + 3I = 0$

So $A^3 = 3A^2 - A + 3I$

$$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$$

and $A^{-1} = \frac{1}{3}[A^2 - 3A + I]$

$$= \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

→ Eigen values of A are the roots of
$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$
$\Rightarrow \lambda = 3, i, -i$
⇒ Eigen values of A^2 are λ^2
$\Rightarrow 9, i^2, (-i)^2$
$= 9, -1, -1.$

— Spectral radius of a matrix A is $\rho(A) =$
Largest eigenvalue in magnitude.
$= \max\limits_i |\lambda_i|$

Eg. Spectral radius of A used in above Question is 3.

→ Algebraic multiplicity of an eigen value is the no. of times the eigen value repeats itself.

Eg. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ → upper triangular matrix
$\Rightarrow \lambda = 1, 1, 1$ (diagonal Entries)

$|A - \lambda I| = 0$ — Char. Equation
Char. Equ. is $(1-\lambda)^3 = 0$
$\lambda = 1$ repeats 3 times
$\Rightarrow$ A.M. of $\lambda = 1$ is 3.

To find the eigen vector corresponding to $\lambda = 1$,

$(A-I)X=0$

$$\left(\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)X=0$$

$$\Rightarrow \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=0$$

$\Rightarrow$ $y=0$

$z=0$ and x arbitrary

$\Rightarrow$ $(1,0,0)$ If $x=1$ will be an Eigen vector.

$\Rightarrow$ Corresponding to $\lambda=1$, we have only one L.I. Eigen vector.

(2) $A=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$ $\rightarrow$ Again upper Triangular matrix

$\Rightarrow \lambda=1,1,1$

$\Rightarrow$ A.M. of $\lambda=1$ is 3.

To find Eigen Vector Corr. to $\lambda=1$

$(A-I)X=0$

$$\left[\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow$ $y=0$, x and z are arbitrary

Choose $x=0$, $z=1$

$(0,0,1)$

Choose $z=0$; $x=1$

$(1,0,0)$

$\Rightarrow$ $(0,0,1)$ & $(1,0,0)$ $\rightarrow$ Two L.I. Eigen vectors Corr. to $\lambda=1$.

Evergreen
Page No.
Date : / /201

(3) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ diagonal Matrix

$\Rightarrow \lambda = 1, 1, 1$

A.M. of $\lambda = 1$ is 3.

To find Eigen vector Corr. to $\lambda = 1$;

$(A - I) X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ x, y, z all are arbitrary.

$\Rightarrow x = 1, y = 0, z = 0$

$x = 0, y = 1, z = 0$

$x = 0, y = 0, z = 1$

$\Rightarrow (1,0,0)\ (0,1,0)\ (0,0,1) \rightarrow 3$ L.I. Eigen vectors.

Hence, If A.M. of any λ is > 1 i.e. λ is repeating more than once, then how many L.I. vectors are there corresponding to that particular λ?

Ans $n - \text{Rank}(A - \lambda I)$

In (1), $\text{Rank}(A - \lambda I) = 2$

$\Rightarrow 3 - 2 = 1$ L.I. Eigen vector

In (2); $\text{Rank}(A - \lambda I) = 1$

$\Rightarrow 3 - 1 = 2$ L.I. Eigen vector

In (3); $\text{Rank}(A - \lambda I) = 0$

$\Rightarrow 3 - 0 = 3$ L.I. Eigen vectors.

$\rightarrow$ The no. of L.I. Eigen vectors corresponding to λ ~~are~~ is called the geometric multiplicity (G.M.) of λ.

Diagonalizable Matrix: → A Square matrix of order n is diagonalizable iff it has n L.I. Eigen vectors.

→ If $A_{n\times n}$ matrix and has all the n eigen values as distinct
i.e. $\lambda_1, \lambda_2 -- \lambda_n$ are n eigen values of A and $\lambda_i \neq \lambda_j \ \forall i,j$
⇒ A has n L.I. Eigen vector Corresponding to each λ
⇒ A is diagonalizable matrix.

Thus If A has distinct Eigenvalues ⇒ A is diagonalizable.
But Converse need not to be true.
i.e. If A is diagonalizable then A Can have repeated eigenvalues as well.
E.g. In (3) above, $\lambda = 1, 1, 1 \rightarrow$ A.M. is 3
But We have 3 L.I. Eigen vectors Corr. to $\lambda = 1$
⇒ A is diagonalizable

→ Every diagonal matrix is diagonalizable matrix, but Converse need not to be true.
i.e. A diagonalizable matrix need not to be a diagonal matrix.

E.g. $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ → Not diagonal matrix.

But $|A - \lambda I| = 0$
$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$
$\Rightarrow \lambda = 1, 2, 3$
⇒ All eigen values are distinct
⇒ A is diagonalizable.

→ A is diagonalizable if A is similar to a diagonal matrix.
⇒ ∃ an Invertible matrix P s.t. $P^{-1}AP = D$
or $A = PDP^{-1}$

Note that this P matrix can be obtained by using the L.I. Eigenvectors corr. to eigenvalues of A.

Ex The eigenvectors of 3x3 matrix A corr. to eigenvalues 1, 1, 3 are $(1,0,-1)^T$, $(0,1,-1)^T$, $(1,1,0)^T$ respectively. Find A.

Sol^n $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \rightarrow$ Matrix Consisting of eigenvectors.

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow$ Matrix Consisting of Eigenvalues.

We find $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A = PDP^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Que Find the matrix A if

(1) Eigenvalues are 2, 2, 4 and Eigenvectors are $(-2,1,0)^T$, $(-1,0,1)^T$, $(1,0,1)^T$

Ans $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(2) Eigenvalues are 1, −1, 2 and Eigenvectors are $(1,1,0)^T$; $(1,0,1)^T$; $(3,1,1)^T$.

Ans:- $\begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$

(3) Eigenvalues are 1, 1, 1 and Eigenvectors are $(-1,1,1)^T$, $(1,-1,1)^T$, $(1,1,-1)^T$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Symmetric matrix:→ If $A = A^T$

i.e. $a_{ij} = a_{ji} \ \forall i, j$

– All eigenvalues of A and A^T are same.

– The eigenvalues of a Symmetric matrix are real.

Skew Symmetric matrix:→ If $A = -A^T$ or $A^T = -A$.

i.e. $a_{ij} = -a_{ji} \ \forall i, j$

i.e. $a_{ii} = -a_{ii} \ \forall i$

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow a_{ii} = 0$

$\Rightarrow$ all the diagonal entries are zero.

→ The eigenvalues of a Skew symmetric matrix are Zero or Pure Imaginary.

i.e. Skew Symmetric matrix Cannot have real eigenvalues.

Orthogonal Matrix:→ A real matrix A is orthogonal if

$A^T A = I$

E.g. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix

→ The eigen values of an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs.
i.e. If λ is an eigenvalue of an orthogonal matrix A
Then $|\lambda| = 1$.
$\Rightarrow \lambda$ Could be 1 or -1 or $i, -i$

→ If A is an orthogonal matrix then $|A| = \pm 1$.
[$\because |\lambda| = 1 \ \forall \ \lambda$ Eigenvalues of A.
and product of eigenvalues $= |A|$]

Homogeneous System of Linear Equations :→

$$a_{11}x_1 + a_{12}x_2 + \_\_ + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \_\_ + a_{2n}x_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots \qquad — (1)$$
$$a_{m1}x_1 + a_{m2}x_2 + \_\_ + a_{mn}x_n = 0.$$

$\Rightarrow \quad AX = 0$

where $A = \begin{bmatrix} a_{11} & a_{12} & — & a_{1n} \\ a_{21} & a_{22} & — & a_{2n} \\ | & | & & | \\ a_{m1} & a_{m2} & — & a_{mn} \end{bmatrix}_{m\times n}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

→ System (1) is always Consistent since $X=0$ is always a Solution, Known as Trivial solution.

⇒ homogeneous system Can have Either
(a) Unique Solution i.e. Trivial solution
(b) Infinite no. of solutions.

→ System (1) has a unique solution if Rank(A) = no. of unknowns (n).

→ If Rank(A) < no. of unknowns (n), then system (1) has Infinite no. of solutions.

→ As Rank (A) ≤ min {m,n}
If $m < n$, Then System (1) always possesses a non-trivial Solution ⇒ Infinite no. of solutions.

Ex Solve
$$2x + y = 0$$
$$x - y = 0$$
$$3x + 2y = 0.$$

Sol^n
$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$AX = 0$

Evergreen
Page No.
Date : / /201

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1/2$
$R_3 \leftarrow R_3 - 3R_1/2$

$$= \begin{bmatrix} 2 & 1 \\ 0 & -3/2 \\ 0 & 1/2 \end{bmatrix} \quad R_3 \leftarrow R_3 + \frac{R_2}{3} = \begin{bmatrix} 2 & 1 \\ 0 & -3/2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank(A) = 2 = no. of unknowns
$\Rightarrow$ System has only a trivial solution.

Que Solve
$$x + y - z + w = 0$$
$$2x + 3y + z + 4w = 0$$
$$3x + 2y - 6z + w = 0$$

Solⁿ

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As A is 3X4 matrix. $\Rightarrow m < n$
$\Rightarrow$ System has Infinite no. of solutions.

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank(A) = 2 < no. of unknowns.
$$x + y - z + w = 0$$

$$y + 3z + 2w = 0$$
$$\Rightarrow y = -3z - 2w$$
and $x = -y + z - w$
$$\Rightarrow x = 3z + 2w + z - w = 4z + w.$$
z and w are arbitrary.

Que Find the Solution of the following hom. System $Ax = 0$ where A is given by

(1) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$ Ans $[-\alpha, \alpha, \alpha]$; α arbitrary

(2) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 6 & 12 \end{bmatrix}$ Ans:- $[-2\alpha/3, 7\alpha/3, -8\alpha/3; \alpha]$; α arbitrary.

(3) $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$ Ans:- $[0, 0, 0]$

(4) $\begin{bmatrix} 1 & 1 & +2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}$ Ans:- $[-15\alpha, 13\alpha, \alpha]$; α arbitrary.

Gauss Jordan Method:→ Let A be a non-singular matrix of order n.

Form the augmented matrix $[A|I]$

Using elementary row operations, we obtain

$$[A|I] \longrightarrow [I|B]$$

Thus $B = A^{-1}$

Que:→ Using Gauss Jordan method, find the inverse of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Sol^n Write the augmented matrix $[A|I]$.

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Using elementary row operations, Convert $[A|I]$ into $[I/B]$.

$R_1 \leftarrow -R_1$ Then

$R_2 \leftarrow R_2 - 3R_1$

$R_3 \leftarrow R_3 + R_1$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2/2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

$R_3 \leftarrow (R_3)/5$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_1 \leftarrow R_1 - \frac{3}{2} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{7}{2} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$$\therefore A^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

Que Find the Inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solⁿ

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

~~$R_3 \leftarrow R_3 + R_2$~~

$R_2 \leftarrow R_2 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$R_3 \leftarrow (-R_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \text{ Ans}$$

Que $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Ans: $\begin{bmatrix} 3 & -5 & 6 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$

Linear Independence of vectors:→
Let V be a vector space, $V_1, V_2 \ldots V_n$ be n vectors of V. Then $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n$ is called a Linear Combination of vectors where $\alpha_1, \alpha_2 \ldots \alpha_n \in F(= \mathbb{R}$ or $\mathbb{C})$

→ The Set of vectors $\{V_1, V_2, \ldots V_n\}$ is said to be linearly dependent if there Exist scalars $\alpha_1, \alpha_2 \ldots \alpha_n$, not all zero such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n = 0.$$

→ The Set of Vectors $\{V_1, V_2, \ldots V_n\}$ is said to be Linearly Independent if $\alpha_1 V_1 + \ldots + \alpha_n V_n = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

In other words, the Set of vectors $\{V_1 \ldots V_n\}$ is Linearly dependent iff atleast one element/vector Can be written as a Linear Combination of the other vectors.

i.e. Let $\alpha_i \neq 0$

Then $V_i = -\frac{1}{\alpha_i}\left[\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_{i-1} V_{i-1} + \alpha_{i+1} V_{i+1} + \ldots + \alpha_n V_n\right].$

Ex Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,0,1) \in \mathbb{R}^3$.
S.T. $\{V_1, V_2, V_3\}$ is Linearly Independent.

Sol^n Let $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0.$

$\Rightarrow \alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,0,1) = 0$

$\Rightarrow (\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = 0 = (0,0,0)$

$\Rightarrow \alpha_1 = 0$, $-\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = 0$

$-\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_3 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow \{V_1, V_2, V_3\}$ is L.I.

Ex Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,2,1)$, $V_4 (1,0,3) \in \mathbb{R}^3$
Then S.T. $\{V_1, V_2, V_3, V_4\}$ is L.D.

Sol^n Consider the Equation $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$\Rightarrow \alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,2,1) + \alpha_4(1,0,3) = 0$

$\Rightarrow (\alpha_1 + \alpha_4, -\alpha_1 + \alpha_2 + 2\alpha_3, -\alpha_2 + \alpha_3 + 3\alpha_4) = 0$

$\Rightarrow \alpha_1 + \alpha_4 = 0 \Rightarrow \alpha_1 = -\alpha_4$

$-\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_2 + 2\alpha_3 = -\alpha_4$

$-\alpha_2 + \alpha_3 + 3\alpha_4 = 0 \Rightarrow \alpha_2 - \alpha_3 = 3\alpha_4$

$3\alpha_3 = -4\alpha_4$

$\alpha_3 = \frac{-4}{3}\alpha_4$

$\Rightarrow \alpha_2 = \alpha_3 + 3\alpha_4$

$= \left(\frac{-4}{3} + 3\right)\alpha_4 = 5/3\,\alpha_4$

$\therefore \alpha_1 = -\alpha_4, \ \alpha_2 = \frac{5}{3}\alpha_4, \ \alpha_3 = \frac{-4}{3}\alpha_4.$

If $\alpha_4 = 1, \ \alpha_1 = -1, \ \alpha_2 = \frac{5}{3}, \ \alpha_3 = \frac{-4}{3}$

So not all scalars are zero

$\Rightarrow \{v_1, v_2, v_3, v_4\}$ are L.D.

→ Singleton Set is L.I. Set

$\because$ let $A = \{a\}$ be a singleton set, $A \neq \phi$, $a \neq 0$

let $\alpha a = 0$; α is a scalar

Either $\alpha = 0$, or $a = 0$

$a \neq 0 \Rightarrow \alpha = 0$

$\Rightarrow A = \{a\}$ is Singleton Set

→ Every Subset of L.I. Set is L.I.

i.e. Let $S = \{v_1, v_2 \_\_ v_n\}$ be a L.I. Set

Then S.T. $S_1 = \{v_1, v_2 \_\_ v_k\}, k < n$ is also L.I.

Sol^n let $\alpha_1 v_1 + \_\_ + \alpha_k v_k = 0$

$\Rightarrow \alpha_1 v_1 + \_\_ + \alpha_k v_k + 0.v_{k+1} + \_\_ + 0 v_n = 0$

And $\{v_1 \_\_ v_n\}$ is L.I.

$\Rightarrow \alpha_1 = 0 = \alpha_2 = \_\_ = \alpha_k = 0$

$\Rightarrow S_1$ is also L.I. Set.

Evergreen
Page No.
Date: / /201

→ ϕ is subset of every set.
⇒ Empty set is Linearly Independent.

→ Superset of Every Linearly dependent set is Linearly dependent.
i.e. Let $S = \{v_1, \ldots v_k\}$ be a LD Set
Then S.T. $S' = \{v_1, \ldots v_n\}$ $n > k$ is also L.D. Set.

Pf. If $S' = \{v_1 \ldots v_n\}$, $n > k$ is Linearly Independent
⇒ Every Subset of S' is also Linearly Independent
⇒ S is also L.I.
— Contradiction.
⇒ S' is L.D. Set.

Que Examine whether the following set of vectors is L.I. or L.D.
(1) (1,2,3,4), (2,0,1,−2), (3,2,4,2)
(2) (1,1,0,1), (1,1,1,1), (−1,1,1,1), (1,0,0,1)

Elementary Row and Column operations:→
(1) Interchange of any two rows ($R_i \sim R_j$)
(2) Multiplication/division of any row by a non-zero scalar (αR_i)
(3) Adding/Subtracting a scalar multiple of any row to another row ($R_i \leftarrow R_i + \alpha R_j$)

Echelon form of a matrix:→ An $m \times n$ matrix is called a row Echelon matrix if the no. of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries is obtained.
↳ i.e. (1) If ith row contains all zeros, then it is true for all subsequent ~~zeros~~ rows.
(2) If a column contains a non-zero Entry of any row then every subsequent entry in this column is zero.
(3) Rows containing all zeros occur only after all non-zero rows.

for Ex $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are the matrices in row Echelon form.

Similary we can define the Column echelon form of a matrix.

Rank of a matrix :→ The number of Linearly Independent rows ~~or~~ ~~Columns~~ of a matrix gives the rank of the matrix.

→ If A is $m \times n$ matrix then rank of $A \leq \min\{m, n\}$.

→ In other words, the no. of non-zero rows in the row echelon form of a matrix gives the rank of the matrix.

Ex Reduce the following matrices to row Echelon form and find their ranks.

(i) $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + 2R_1$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$ $R_3 \leftarrow R_3 + 2R_2 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

⇒ # of non-zero rows in row Echelon form is 2

⇒ Rank of $A = 2$.

H.W (1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ (2) $A = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$

Evergreen
Page No.
Date: / /201

$\Rightarrow$ Examine whether the following set of vectors is L.I. or not:

(1) $(1,2,3,4)$; $(2,0,1,-2)$; $(3,2,4,2)$

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ # of non-zero rows in $A = 2$

$\Rightarrow$ Rank $(A) = 2$.

$\Rightarrow$ given set of vectors are linearly dependent.

<u>Quadratic forms</u> :→ Let $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary vector. A real Quadratic form is an homogeneous expression of the form

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j \quad -(1)$$

in which the total power in each term is 2.

We can write (1) as

$$Q = a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots a_{1n} x_1 x_n$$
$$+ a_{21} x_2 x_1 + a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n$$
$$\vdots$$
$$+ a_{n1} x_n x_1 + a_{n2} x_n x_2 + \cdots + a_{nn} x_n^2$$
$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \cdots + (x_1 x_n)(a_{1n} + a_{n1})$$
$$+ a_{22} x_2^2 + (a_{23} + a_{32}) x_2 x_3 + \cdots + (a_{2n} + a_{n2}) x_2 x_n$$
$$+ \cdots + a_{nn} x_n^2$$
$$= X^T A X \quad \text{(By def}^n \text{ of Matrix multiplication)}$$

↳ (2)

Now set $b_{ij} = \dfrac{(a_{ij} + a_{ji})}{2}$

⇒ $B = (b_{ij})$ Matrix is Symmetric matrix ($\because b_{ij} = b_{ji}$)

So (2) can be written as

$$Q = X^T B X$$

where B is a Symmetric matrix and $b_{ij} = \dfrac{a_{ij} + a_{ji}}{2}$.

<u>Que</u> Obtain the Symmetric matrix B for the Quadratic form

(1) $Q = 2x_1^2 + 3x_1 x_2 + x_2^2$

$B = (b_{ij})$ and $b_{ij} = \dfrac{a_{ij} + a_{ji}}{2}$

$a_{11} = 2$; $a_{12} + a_{21} = 3$; $a_{22} = 1$

$b_{12} = \dfrac{3}{2} = b_{21}$

Hence $B = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$

Que $Q = x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$

$$B = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$\left[\because a_{11} = 1,\ a_{22} = -5,\ a_{33} = 4 \right.$
$a_{12} + a_{21} = 2;\ a_{13} + a_{31} = -4;\ a_{23} + a_{32} = 6.$

Conversely If a symmetric matrix is given then we can find the Corresponding Quadratic form.

Ex $A = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$

then Quadratic form is $Q = X^T A X$ where $X \in R^2$

$$Q = (x_1\ x_2) \begin{pmatrix} 2 & 3/2 \\ 3/2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + \frac{3x_2}{2} & \frac{3}{2}x_1 + x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 2x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_1x_2 + x_2^2$$

$$Q = 2x_1^2 + 3x_1x_2 + x_2^2$$

H.W Que Find the Sym. matrix A for the Quadratic forms

(1) $Q = x_1^2 - 2x_1x_2 + 4x_2x_3 - x_2^2 + x_3^2$

(2) $Q = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 8x_2x_3 + x_2^2.$

Ans (1) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 4 \\ -2 & 4 & 0 \end{bmatrix}$

and Find their Corresponding Quadratic forms (For practice)

Que Examine whether the following set of vectors is L.I. or L.D. Also find the rank of the matrix associated.

(1) $\{(2, 3, 6, -3, 4);\ (4, 2, 12, -3, 6);\ (4, 10, 12, -9, 10)\}$ Ans:- L.D, Rank 2

(2) $\{(3, 2, 4)\ (1, 0, 2)\ (1, -1, -1)\}$ $\longrightarrow$ Ans L.I.; Rank 3

(3) $\{(1, 2, 3, 1)\ (2, 1, -1, 1),\ (4, 5, 5, 3),\ (5, 4, 1, 3)\}$ Ans. L.D., Rank 2

(4) $\{(2, 2, 0, 2);\ (4, 1, 4, 1);\ (3, 0, 4, 0)\}$ Ans L.D. , Rank 2

Gauss Elimination method :→

System of Linear Equations :→

Let $a_{11}x_1 + a_{12}x_2 + \_\_\_ + a_{1n}x_n = b_1$

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \_\_\_ + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \_\_\_ + a_{2n}x_n = b_2 \\ \quad | \qquad\quad | \qquad\qquad\quad | \qquad\quad | \\ a_{m1}x_1 + a_{m2}x_2 \_\_\_ + a_{mn}x_n = b_m \end{array} \quad —(1)$$

So $AX = b$

where $A = \begin{bmatrix} a_{11} & a_{12} & \_\_ & a_{1n} \\ a_{21} & a_{22} & - & a_{2n} \\ | & | & & | \\ a_{m1} & a_{m2} & - & a_{mn} \end{bmatrix}_{m\times n}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_m \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

(1) is the System of m equations and n unknowns.

System (1) is said to be Consistent if it has atleast one solution and Inconsistent, if it has no solution.

There are three Possibilities :–

(1) The System (1) has unique Solution.
(2) System (1) has no solution.
(3) system (1) has Infinite no. of solutions.

Result 1 The non-hom. System of Equation $Ax=b$ has a solution iff $\text{Rank}(A) = \text{Rank}(A|b)$ where $A|b$ is the augmented matrix.

⇒ If $\text{Rank}(A) \neq \text{Rank}(A|b) \Rightarrow$ (1) has no Solution.

→ If $\text{Rank}(A) = \text{Rank}(A|b) = n$ (no. of unknowns) then (1) has unique solution.

→ If $\text{Rank}(A) = \text{Rank}(A|b) \leq r < n$ then (1) has Infinite no. of solutions.

Gauss Elimination method for non-homogeneous system:→

(1) Solve
$$2x+y-z=4$$
$$x-y+2z=-2$$
$$-x+2y-z=2$$

Sol^n→
$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$
$$AX = b$$

Write the augmented matrix $[A|b]$ and reduce it to row echelon form by applying elementary row operations.

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1/2$
$R_3 \leftarrow R_3 + R_1/2$

$$= \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{5}{3}R_2$

$$= \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right]$$

Hence Rank (A) = Rank $(A|b) = 3$

⇒ System has a unique solution.

And solution is

$$\frac{8}{3}z = -8/3 \Rightarrow z = -1$$

$$-\frac{3}{2}y + \frac{5}{2}z = -4 \Rightarrow y = 1$$

$2x + y - z = 4 \Rightarrow x = 1$

So $(x = 1, y = 1, z = -1)$ is the solution.

(2) $2x + z = 3$

$x - y + z = 1$

$4x - 2y + 3z = 3$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right]$

$R_2 \leftarrow R_2 - R_1/2$

$R_3 \leftarrow R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$\Rightarrow$ Rank$(A) = 2$ & Rank$(A|b) = 3$

$\Rightarrow$ System has no solution.

(3) $x - y + z = 1$

$2x + y - z = 2$

$5x - 2y + 2z = 5$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

The augmented form of the matrix is

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & +1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 - 5R_1$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ Rank $(A) =$ Rank $(A|b) = 2 < 3$

$\Rightarrow$ System has Infinite no. of solutions.

$3y - 3z = 0$

$\Rightarrow y = z$

$x - y + z = 1$

$\Rightarrow x = 1$

So $x = 1, y = z$ and z can be chosen arbitrarily.

Que Solve $4x - 3y - 9z + 6w = 0$

$2x + 3y + 3z + 6w = 6$

$4x - 21y - 39z - 6w = -24$

Ans :- Infinite Sol^n

Que Solve $x + 2y - 2z = 1$

$2x - 3y + z = 0$

$5x + y - 5z = 1$

$3x + 14y - 12z = 5$

Ans:- $x = 1, y = 1, z = 1$ Unique Sol^n.

Que
$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

Ans $x=1, y=2, z=2$

Que
$$\begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Ans $x=-1, y=-1/2, z=3/4$

Que
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Ans $[2-\alpha, 1, \alpha, 1]$; α arbitrary.

Que
$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\left[\frac{-1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$

Unit 3

Vector Space :→ Let $V \neq \phi$ set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a vector.

Let $+ : V \times V \rightarrow V$ a binary operation, called vector addition

$\cdot : F \times V \rightarrow V$ an operation, called scalar multiplication

Then $(V, +, \cdot)$ is called a vector space if $\forall\, a, b, c \in V$, $\alpha, \beta \in F$ the following properties are satisfied.

(1) $a + b \in V$

(2) $a + b = b + a$

(3) $(a+b)+c = a+(b+c)$

(4) $a + 0 = a = 0 + a$ where 0 is the zero element in V

(5) $a + (-a) = 0$

(6) $\alpha \cdot a \in V$

(7) $(\alpha+\beta)\cdot a = \alpha \cdot a + \beta \cdot a$

(8) $(\alpha \cdot \beta)\cdot a = \alpha \cdot (\beta \cdot a)$

(9) $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b$

(10) $1 \cdot a = a$ where 1 is unity element in F.

Note that F is the set of real numbers or set of complex numbers, called field of scalars.

→ $V = \{0\}$ is called a trivial vector space.

Ex (1) $V = \{$Set of real numbers$\}$ is a vector space with usual addition and scalar multiplication over $F = \mathbb{R}$

(2) $V = \{$ Set of all continuous functions i.e. $f : [a, b] \rightarrow \mathbb{R} \}$

V is a vector space over $\mathbb{R}$.

(3) $V = \{$ Set of polynomials of degree $\leq n\}$ is a vector space over $F = \mathbb{R}$.

(4) $V = \{$ Set of all $m \times n$ matrices$\}$
V is a vector space over $F = \mathbb{R}$.
Here $0 =$ Null matrix.

(5) let $V = \{$ Set of all polynomials of degree $n\}$.
Then V is not a vector space.
$\because$ let $p_n = a_0 + a_1 x + \_\_ + a_{n-1} x^{n-1} + x^n \in V$
$q_n = b_0 + b_1 x + \_\_ + b_{n-1} x^{n-1} - x^n \in V$
$p_n + q_n = (a_0 + b_0) + (a_1 + b_1)x + \_\_ + (a_{n-1} + b_{n-1}) x^{n-1}$
is a polynomial of deg $(n-1)$
$\Rightarrow p_n + q_n \notin V$.

(6) $V = \{$ Set of all polynomials of degree $n\}$
for $a, b \in V$, $a + b = ab$
and usual scalar multiplication.
Then V is not a vector space.
$\because$ let $p_n = a_0 + a_1 x + \_\_ + a_n x^n \in V$
$q_n = b_0 + b_1 x + \_\_ + b_n x^n \in V$
Then $p_n + q_n = p_n \cdot q_n$ is a polynomial of degree $2n \Rightarrow p_n + q_n \notin V$
$\Rightarrow$ V is not a vector space.

(7) $V = \{\mathbb{R}^2$ i.e. Set of all ordered pairs (x, y), $x, y \in \mathbb{R}$
Vector addition is defined as
$$a + b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$
$$\alpha \cdot a = \alpha(x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3}\right)$$
Check whether V is a vector space or not over $F = \mathbb{R}$.

Solⁿ:→ $a+b \neq b+a$

$\because$ let $a=(x_1,y_1)$, $b=(x_2,y_2)$

Then $a+b = (x_1,y_1)+(x_2,y_2) = (2x_1-3x_2, y_1-y_2)$

$b+a = (x_2,y_2)+(x_1,y_1) = (2x_2-3x_1, y_2-y_1)$

Similarly $(a+b)+c \neq a+(b+c)$

Also $1.(x_1,y_1) = (x_1,y_1) = \left(\frac{x_1}{3},\frac{y_1}{3}\right) \neq (x_1,y_1)$

$\therefore \Rightarrow 1.a \neq a$ where 1 is the unity Element in $\mathbb{R}$.

$\Rightarrow$ V is not a vector space over $\mathbb{R}$

H.W Que: let $V=\mathbb{R}^2$ i.e. set of all ordered pairs.

let $a=(x_1,y_1)$, $b=(x_2,y_2) \in V$

$a+b = (x_1,y_1)+(x_2,y_2) = (x_1x_2, y_1y_2)$

and $\alpha a = \alpha(x_1,y_1) = (\alpha x_1, \alpha y_1)$

Then show that V is not a V.S over $\mathbb{R}$.

→ Subspace:→ let $(V,+,\cdot)$ be a vector space over field F. let W be a non-Empty subset of V. i.e. $W \subseteq V$. Then W is called a subspace of V if W is a vector space under the vector addition '+' and scalar multiplication '·'.

Two step test to check the Subspace:→

let $(V,+,\cdot)$ be a vector space over F

and $W \neq \phi$, $W \subseteq V$

Then W is subspace of V if

$a+b \in W \ \forall \ a,b \in W$

and $\alpha a \in W \ \forall \ \alpha \in F, a \in W$.

One step Test:→ let $(V,+,\cdot)$ be a V.S. over F

and $W \neq \phi$, $W \subseteq V$

Then W is subspace of V if

$\alpha a + \beta b \in W \ \forall \ \alpha,\beta \in F, a,b \in W$.

Ex Let $V = \{(x_1, x_2 \_ x_n) \in \mathbb{R}^n\}$; $F = \mathbb{R}$
V is a V.S. with usual addition & scalar multiplication.

(1) let $W = \{(x_1, \_ x_n) \mid x_1 = 0\} \subseteq V$
Then W is Subspace of V
$\because$ let $a = (0, x_2 \_ x_n)$; $b = (0, y_2, \_ y_n) \in W$
Then $a + b = (0, 2x_2, \_, 2x_n) \in W$
And $\alpha \in F$; $\alpha a \in W$.

(2) let $W = \{(x_1, \_ x_n) \mid x_1 \geq 0\}$
Then W is not a subspace of V.
$\because$ let $\alpha = -1$ let $a = (x_1, \_ x_n) \in W$
Then $\alpha a = -a = -(x_1, \_ x_n)$
$= (-x_1, -x_2, \_ -x_n) \notin W$.

(3) let $W = \{(x_1, \_ x_n) \mid x_2 = x_1 + 1\}$
Then W is not a Subspace of V
$\because$ let $a = (x_1, x_2, \_ x_n)$; $b = (y_1, y_2 \_ y_n) \in W$
$\Rightarrow x_2 = x_1 + 1$; $y_2 = y_1 + 1$.
Then $a + b = (x_1 + y_1, x_2 + y_2, \_, x_n + y_n)$
And $x_2 + y_2 = (x_1 + 1) + (y_1 + 1)$
$= x_1 + y_1 + 2 \neq (x_1 + y_1) + 1$
$\Rightarrow$ W is not a Subspace.

Ex let $V = \{ \overset{p(x) =}{a_0 + a_1 x + \_ + a_m x^m}$ – polynomials of deg. $\leq m\}$
is a V.S. with usual addition and scalar multiplication
$F = \mathbb{R}$

Then
(1) let $W = \{$polynomials of deg $\leq m \mid p(0) = 0\}$
Then W is a Subspace of W
$\because$ let $p(x), q(x) \in W \Rightarrow p(0) = 0 = q(0)$
$p(x) + q(x) \in W$ as $p(0) + q(0) = 0$
And $\alpha p(x) \in W$ as $\alpha p(0) = 0$, $\alpha \in F$.

(2) $W = \{ P(x) \in V \mid P(0) = 1 \}$
is not a Subspace
$\because$ let $P(x), q(x) \in W$
$\Rightarrow P(0) = 1 = q(0)$
Then $P(x) + q(x) \notin W$ $(\because P(0) + q(0) = 2 \neq 1)$

(3) $W = \{ P(x) \in V \mid$ Coefficients are Positive $\}$
Then W is not a Subspace
$\because$ let $P(x) = a_0 + a_1 x + \_\_ + a_m x^m \in W$, $a_0, a_1 \_\_ a_m > 0$.
let $\alpha = -1 \in F$
Then $\alpha P(x) = -a_0 - a_1 x - \_\_ - a_m x^m \notin W$

HW $V = \{$ Set of all $n \times n$ real Square matrices $\}$ is a V.S. with usual matrix addition & scalar multiplication. $F = \mathbb{R}$
(1) $W = \{$ Set of all Symmetric matrices $\}$
(2) $W = \{$ Set of all upper Triangular matrices $\}$
(3) $W = \{$ Set of all $n \times n$ matrices having real positive Elements $\}$.

Check that W is a Subspace of V or not.