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Bessel Equation:→

$$x^2y'' + xy' + (x^2-n^2)y=0 \quad —(1)$$

is Called Bessel Equation. where n is a real Constant.

Gamma function:→ Generalization of the factorial function to non-Integral values.

Gamma function is defined as

$$\Gamma x = \int_0^\infty t^{x-1} e^{-t} dt.$$

$$\Gamma 1 = \int_0^\infty (t)^0 \cdot e^{-t} dt = \left(-e^{-t}\right)_0^\infty = 1$$

$\Rightarrow \Gamma 1 = 1.$

(1) $\Gamma x+1 = x\Gamma x$

$\because \Gamma x+1 = \int_0^\infty t^x e^{-t} dt$

$$= \left[t^x\left(-e^{-t}\right)\right]_0^\infty - \int_0^\infty x t^{x-1}\left(\frac{e^{-t}}{-1}\right) dt$$

$$= 0 + \int_0^\infty x t^{x-1} e^{-t} dt$$

$$= x\Gamma x$$

$\Rightarrow \Gamma 2 = 1\Gamma 1 = 1$

$\Gamma 3 = \Gamma 2+1 = 2\Gamma 2 = 2\times 1 = \lfloor 2$

$\Gamma 4 = \Gamma 3+1 = 3\Gamma 3 = 3\times 2! = 3!$ and so on.

→ If x is a natural no. then $\Gamma x = \lfloor x-1$.

E.g. $\Gamma 4 = \lfloor 4-1 = 3! = 6.$

→ $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Rightarrow \Gamma 3/2 = \Gamma \frac{1}{2}+1 = \frac{1}{2}\Gamma \frac{1}{2} = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}.$

→ Find $\Gamma\left(\frac{-1}{2}\right)$, $\Gamma\left(\frac{-3}{2}\right)$, $\Gamma\left(\frac{-5}{2}\right)$

We know $\Gamma(x) = \frac{\Gamma(x+1)}{x}$

$$\Rightarrow \Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{-1/2} = -2\,\Gamma\left(\frac{1}{2}\right)$$

$$= -2\sqrt{\pi}$$

IIly $$\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{-3/2} = \frac{\Gamma\left(\frac{-1}{2}\right)}{-3/2} = -2\sqrt{\pi} \times \left(\frac{-2}{3}\right)$$

$$= \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(\frac{-5}{2}\right) = \frac{\Gamma\left(\frac{-5}{2}+1\right)}{-5/2} = \frac{\Gamma\left(\frac{-3}{2}\right)}{-5/2} = \frac{-2}{5}\left(\frac{4}{3}\sqrt{\pi}\right)$$

$$= \frac{-8}{15}\sqrt{\pi}$$

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Bessel's Differential Equation and Bessel function:→

The Equation of the form

$$x^2y'' + xy' + (x^2 - m^2)y = 0 \quad \text{— (1)}$$

is called Bessel's equation, ~~at~~ of order m, where n is a non-negative real number.

$$y'' + \frac{y'}{x} + \frac{x^2 - m^2}{x^2} y = 0.$$

⇒ $x = 0$ is a regular singular point as $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2 - m^2}{x^2}$ not analytic at $x = 0$.

But $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$.

The Solⁿ of (1) about $x = 0$ will be of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n \cdot (n+r)(n+r-1) x^{n+r-2}$$

Substitute y, y', y'' in (1), we get

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - m^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \left[(n+r)(n+r-1) + (n+r) - m^2\right] x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \left[(n+r)^2 - m^2\right] x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

Equating Coeff of x^r and x^{r+1} to zero;

x^r; $r^2 - m^2 = 0 \Rightarrow r = \pm m.$ $\rightarrow$ (Indicial Equation)

x^{r+1}; $C_1[(r+1)^2 - m^2] = 0$

$\Rightarrow C_1(1+2r) = 0 \quad (\because r = \pm m)$

$\Rightarrow C_1 = 0 \quad$ ($\because$ r is positive or non-negative)

$\therefore$ Remaining terms of Summation are

$$\sum_{n=2}^{\infty} C_n ((n+r)^2 - m^2) x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} C_{n+2} ((n+r+2)^2 - m^2) x^{n+r+2} + \sum_{n=0}^{\infty} C_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left([(n+r+2)^2 - m^2] C_{n+2} + C_n \right) x^{n+r+2} = 0$$

Comp. Coeff, we get

$$((n+r+2)^2 - m^2) C_{n+2} = -C_n$$

$$\Rightarrow C_{n+2} = \frac{-C_n}{(n+r+2)^2 - m^2}$$

For $r = m$ $\quad n=0$; $C_2 = \frac{-C_0}{(0+m+2)^2 - m^2} = \frac{-C_0}{(m+2)^2 - m^2}$

$$= \frac{-C_0}{(\cancel{m}+2-\cancel{m})(m+2+m)} = \frac{-C_0}{2^2(1+m)}$$

$C_3 = C_5 = C_7 = \_\_ = 0 \quad (\because C_1 = 0)$

$n=2$; $C_4 = \frac{-C_2}{(2+m+2)^2 - m^2} = \frac{-C_2}{2^3(2+m)} = \frac{+C_0}{2^4 \lfloor 2 \, (1+m)(2+m)}$

Similarly $C_{2n} = \dfrac{(-1)^n \cdot C_0}{2^{2n} \lfloor n \, (1+m)(2+m) \_\_ (n+m)}$; $n = 1, 2, 3 \ldots$

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$\therefore$ The Solⁿ of Bessel's Equation for $r = m$ is

$$y_1(x) = C_0 x^m \left(1 - \frac{x^2}{2^2(1+m)} + \frac{x^4}{2^4 \lfloor 2 (1+m)(2+m)} + \ldots\right)$$

where $C_0 = \dfrac{1}{2^m \Gamma m+1}$

$y_1(x)$ is called Bessel's function of first kind and is denoted by $J_m(x)$.

$$\therefore J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma m+n+1} \left(\frac{x}{2}\right)^{2n+m}.$$

Put $r = -m$ and solve in similar manner, we obtain

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \Gamma n-m+1} \left(\frac{x}{2}\right)^{2n-m}.$$

So the general Solⁿ of (1) is

$$y(x) = A J_m(x) + B J_{-m}(x).$$

$J_m(x)$ and $J_{-m}(x)$ are called Bessel's functions.

Properties:→

(1) $\quad [x^m J_m(x)]' = x^m J_{m-1}(x)$

Pf:→ LHS $\quad x^m J_m(x) = x^m \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!\, \Gamma n+m+1} \left(\dfrac{x}{2}\right)^{2n+m}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+2m}}{n!\, \Gamma n+m+1 \cdot 2^{2n+m}}$$

$$[x^m J_m(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+2m)\, x^{2n+2m-1}}{n!\, \Gamma n+m+1 \;\; 2^{2n+m}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+m)}{n!\, (n+m) \Gamma n+m} \frac{x^{2n+2m-1}}{2^{2n+m}}$$

$$= x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m)} \left(\frac{x}{2}\right)^{2n+m-1}$$

$$= x^m J_{m-1}(x)$$

Cauchy - Euler Equation:→

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots + a_{n-1} x D + a_n) y = Q(x) \quad -(1)$$

where $a_0, a_1 \dots a_n$ are constants.

Put $x = e^z$

$\log x = z$

$\frac{1}{x} dx = dz \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

Let $\frac{d}{dx} = D, \frac{d}{dz} = D_1$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow xD = D_1$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow \cancel{x^2\frac{d^2y}{dx^2}} = x^2D^2 = D_1^2 - D_1 = D_1(D_1-1)$$

Similarly $x^3D^3 = D_1(D_1-1)(D_1-2)$

$\vdots$

$$x^nD^n = D_1(D_1-1)(D_1-2)\dots(D_1-(n-1))$$

So $\left(a_0[D_1(D_1-1)\dots(D_1-(n-1))] + a_1 \_\_\_ + a_{n-2}D_1(D_1-1) + a_{n-1}D_1 + a_n\right)y = Q(z) \quad -(2)$

So (2) becomes a linear diff Eqn with constant Coefficients.

In short; to solve

$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots + a_{n-1} x D + a_n)\, y = Q(x)$ — (1)

(I) Put $x = e^z$ or $z = \log x$

(II) let $D = \frac{d}{dx}$ and $D_1 = \frac{d}{dz}$

Put $xD = D_1$, $x^2D^2 = D_1(D_1-1)$, ..., $x^nD^n = D_1(D_1-1)\dots(D_1-(n-1))$

(III) (1) becomes $f(D_1)\, y = Q(z)$ — (2)

Solve (2) using methods discussed Earlier.

and find Gen Sol^n $y = \phi(z)$

Put $z = \log x$ in the final sol^n.

Que $x^2y'' + xy' - 4y = 0$

Sol^n $(x^2D^2 + xD - 4)\, y = 0$

Put $xD = D_1$, $x^2D^2 = D_1(D_1-1)$

$(D_1(D_1-1) + D_1 - 4)\, y = 0$

$\Rightarrow (D_1^2 - 4)y = 0$

$\Rightarrow$ A.E. is $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

$y(z) = c_1 e^{2z} + c_2 e^{-2z}$

$\Rightarrow y(x) = c_1 x^2 + c_2 x^{-2}$ $(\because z = \log x$ or $x = e^z)$

Que $x^2y'' + y = 3x^2$

Sol^n $(x^2D^2 + 1)\, y = 3x^2$

Put $x = e^z$

and $x^2D^2 = D_1(D_1-1)$

$(D_1(D_1-1) + 1)\, y = 3e^{2z}$

$\Rightarrow (D_1^2 - D_1 + 1)\, y = 3e^{2z}$

A.E $m^2 - m + 1 = 0$

$\Rightarrow m = \left(\frac{1 \pm \sqrt{3}i}{2}\right)$

$y_c(z) = e^{z/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}z\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}z\right)\right]$

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P.I is $\frac{3e^{2z}}{D_1^2 - D_1 + 1}$

$\Rightarrow y_p(z) = \frac{\cancel{3}e^{2z}}{\cancel{3}} = e^{2z}$

$y(z) = y_c(z) + y_p(z)$

Put $z = \log x$

$y(x) = (x)^{1/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x^2.$

Que $x^2y'' + 5xy' + 4y = x\log x.$

Que $(x^4D^3 + 2x^3D^2 - x^2D + x)y = 1.$

Que $x^3y''' + 2xy' - 2y = x^2\log x + 3x.$

Second order Diff Eqn

Linear Combination of functions:→ Let $f_1(x), f_2(x) \ldots f_n(x)$ be n function. Then $c_1 f_1(x) + c_2 f_2(x) + \_\_ + c_n f_n(x)$ where $c_1, c_2 \_\_ c_n \in \mathbb{R}$ is called a linear combination of given functions.

Linearly dependent / Independent functions:→ let $f_1(x), f_2(x) \_\_ f_n(x)$ be n functions. Then If

$$c_1 f_1(x) + c_2 f_2(x) + \_\_ + c_n f_n(x) = 0$$

$$\Rightarrow c_1 = c_2 = \ldots = c_n = 0$$

$$\Rightarrow f_1(x), \_\_ f_n(x) \text{ are linearly Independent.}$$

If $\exists$ some $c_i \neq 0$ and $c_1 f_1(x) + \_\_ + c_n f_n(x) = 0$

Then $f_1(x), \_\_, f_n(x)$ are called linearly dependent.

$\because$ If $c_1 \neq 0$ Then

$$f_1(x) = -\frac{1}{c_1}\left[c_2 f_2(x) + \_\_ + c_n f_n(x)\right]$$

In other words, If any function can be expressed as linear combination of other functions, Then the given functions are linearly dependent.

Ex $f_1(x) = x^2, \; f_2(x) = x^3, \; f_3(x) = 6x^2 - x^3$

Soln let $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$$\Rightarrow c_1 x^2 + c_2 x^3 + c_3(6x^2 - x^3) = 0$$

$$\Rightarrow (c_1 + 6c_3)x^2 + (c_2 - c_3)x^3 = 0$$

$$\Rightarrow c_1 + 6c_3 = 0, \; c_2 = c_3$$

$$\Rightarrow c_1 = -6c_3, \; c_2 = c_3 \qquad \text{If } c_3 = 1 \Rightarrow c_1 = -6, c_2 = 1$$

$$\Rightarrow 6x^2 - x^3 = 6f_1(x) - f_2(x)$$

$$\Rightarrow f_1, f_2, f_3 \text{ are L.D.}$$

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Que S.T. $x^2-1,\ 3x^2,\ 2-5x^2$ are L.D.

Sol^n $c_1(x^2-1)+c_2(3x^2)+c_3(2-5x^2)=0$

$\Rightarrow (c_1+3c_2-5c_3)x^2+(2c_3-c_1)=0$

$\Rightarrow c_1+3c_2-5c_3=0$

$2c_3-c_1=0 \Rightarrow c_1=2c_3$

$\Rightarrow 2c_3-5c_3=-3c_2$

$\Rightarrow c_2=c_3$

If $c_3=1,\ c_1=2,\ c_2=1$

$\Rightarrow$ given functions are L.D.

Que $f(x)=x,\ g(x)=|x|$, on $D=[0,\infty)$

$g(x)=x\ \forall\ x\geq 0$

$\Rightarrow g(x)=1f(x)$

$\Rightarrow f$ & g are L.D on D

If $D=(-\infty,0]$

$g(x)=-x\ \forall\ x<0$

$\Rightarrow g(x)=-f(x)\ \forall\ x\in(-\infty,0)$

$\Rightarrow f$ & g are L.D on D.

If $D=(-\infty,\infty)$ or $\mathbb{R}$

$g(x)=x=f(x)\ \forall\ x\in[0,\infty)$

$g(x)=-x=-f(x)\ \forall\ x\in(-\infty,0)$

$\Rightarrow f$ & g are L.I on D.

If $D=[-1,1]$

$\Rightarrow g(x)$ & $f(x)$ are L.I on D

→ If f and g are L.I. on D, then f & g may be L.I or L.D on $S\subseteq D$. i.e. Subset of D.

→ If f & g are L.I. on D Then f & g will be L.I on $S\supseteq D$ i.e. Superset of D.

Wronskian:→ Let $f_1(x), f_2(x) \ldots f_n(x)$ be n functions.

Then $W(f_1(x), \ldots, f_n(x)) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x)$

(D) Let $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$ —(1)

$a_0(x) \neq 0 \; \forall x$, $a_0(x), a_1(x), a_2(x)$ are continuous fun. $\forall x$. If y_1 & y_2 are Solⁿ of (1) then

$a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \;] \times y_2$

$a_0 y_2'' + a_1 y_2' + a_2 y_1 = 0 \;] \times y_1$

$\Rightarrow \; a_0 (y_2 y_1'' - y_2'' y_1) + a_1 (y_1' y_2 - y_2' y_1) = 0$ —(2)

$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

$= y_1(x) y_2'(x) - y_2(x) y_1'(x)$

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i.e. $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$\Rightarrow W'(y_1, y_2) = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_2' y_1'} - y_2 y_1''$

$= y_1 y_2'' - y_2 y_1''$

So (2) becomes $a_0(-w') + a_1(-w) = 0$

$\Rightarrow a_0(x) w' + a_1(x) w = 0$

$\Rightarrow \frac{dw}{dx} = \frac{-a_1(x)}{a_0(x)} w$

$\Rightarrow \int \frac{dw}{w} = \int \frac{a_1(x)}{a_0(x)} dx$

$\Rightarrow \log w = -\int \frac{a_1(x)}{a_0(x)} dx + \log \alpha$

$\Rightarrow \boxed{w = \alpha e^{-\int \frac{a_1(x)}{a_0(x)} dx}} \rightarrow$ Abel's formula.

★ So wronskian is a solution of first order linear diff equation
$a_0(x) w' + a_1(x) w = 0$.

★ $e^{-\int \frac{a_1(x)}{a_0(x)} dx}$ is always positive.

So If $\alpha = 0$ Then $w = 0$

If $\alpha > 0$ Then $w > 0$

If $\alpha < 0$ Then $w < 0$.

$\Rightarrow w(x) = \alpha \cdot e^{-\int \frac{a_1(x)}{a_0(x)} dx}$ is either throughout zero or nowhere zero.

i.e. $W(x) \equiv 0 \ \forall x$ or $W(x) \neq 0 \ \forall x$.

Thm $\rightarrow$ Let y_1 & y_2 be two solutions of (D). Then $W(y_1, y_2)(x) = 0$ Iff y_1 and y_2 are linearly dependent.

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Solⁿ → Let $W(y_1, y_2)(x) = 0$

$$\Rightarrow \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0 \Rightarrow y_1 y_2' - y_2 y_1' = 0$$

$$\Rightarrow \int \frac{y_2'}{y_2} = \int \frac{y_1'}{y_1}$$

$$\Rightarrow \log y_2 = \log y_1 + \log c$$

$$\Rightarrow y_2 = c y_1$$

$\Rightarrow$ y_1 and y_2 are linearly dependent.

Conversely Let y_1 and y_2 are L.D.

$$\Rightarrow y_1 = c y_2 \text{ or } y_2 = c y_1$$

$$\Rightarrow W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & c y_1(x) \\ y_1'(x) & c y_1'(x) \end{vmatrix} = 0.$$

Hence

y_1 and y_2 are L.I. $\Leftrightarrow$ $W(y_1, y_2) \neq 0, \ \forall x.$

→ Note that above results are not true for general functions.

Ex let $f(x) = x^2$ and $g(x) = x|x|$

then $g(x) = \begin{cases} x^2 & \text{if } x > 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

$\Rightarrow$ g and f are L.I on $(-\infty, \infty)$

But $W(f,g)(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$

$= 2x^2|x| - 2x^2|x| = 0$

$\Rightarrow$ f and g are L.D.

So Contradiction.

$\because \frac{d}{dx} x|x| = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$

$= 2 \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Let $y'' - 2xy' + y = 0$ and $y_1; y_2$ be its two Solutions.

Q:- Let $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 1$, $y_2'(0) = -1$. Find $W(y_1, y_2)(1)$?

Solⁿ:-

$$W(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$$

But $W(y_1, y_2)(x) = \alpha . e^{-\int \frac{a_1(x)}{a_0(x)} dx}$

$$a_0(x) = 1, \quad a_1(x) = -2x$$

$$\Rightarrow W(y_1, y_2)(x) = \alpha e^{-\int -2x\,dx} = \alpha e^{x^2}$$

$$\therefore W(y_1, y_2)(0) = \alpha e^0 = \alpha$$

$$\Rightarrow \boxed{\alpha = -1}$$

$$\Rightarrow W(y_1, y_2)(x) = -e^{x^2}$$

$$\Rightarrow \boxed{W(y_1, y_2)(1) = -e}$$

Que:- Let $y_1(x)$ and $y_2(x)$ be two solⁿ of $(1-x^2)y'' - 2xy' + (\sec x)y = 0$, with $y_1(0) = 1$, $y_1'(0) = 0$, If $W(\frac{1}{2}) = \frac{1}{3}$ then find $y_2'(0)$.

Solⁿ

$$(1-x^2)y'' - 2xy' + (\sec x)y = 0 \quad -(1)$$

$(1-x^2) \neq 0$ iff $x \neq \pm 1$.

$$W(x) = \alpha e^{-\int \frac{-2x}{1-x^2} dx}$$

$$= \alpha e^{\int \frac{2x}{1-x^2} dx} \qquad \text{Put } 1-x^2 = t$$

$$= \alpha e^{-\int dt/t} = \alpha e^{-\log t} \qquad -2x\,dx = dt$$

$$= \frac{\alpha}{t} = \frac{\alpha}{1-x^2}$$

$$W(0) = \frac{\alpha}{1} = \alpha \qquad \text{and } W\left(\tfrac{1}{2}\right) = \frac{\alpha}{1-\frac{1}{4}} = \frac{4\alpha}{3} = \frac{1}{3}$$

$$\Rightarrow \alpha = \frac{1}{4}$$

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \alpha \quad \Rightarrow \quad \begin{vmatrix} 1 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix} = \alpha = \frac{1}{4}$$

$$\Rightarrow y_2'(0) = 1/4.$$

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$$a_0(x)\, y^n + a_1(x)\, y^{n-1} + \_\_ + a_n(x)\, y = Q(x) \quad -(1)$$

where $a_0(x), a_1(x) \_\_, a_n(x)$ & $Q(x)$ are functions of x only.

is called a linear O.D.E of order n.

→ If $Q(x) = 0$ Then Equ^n (1) is called homogeneous linear differential equ^n otherwise it is called non-hom. Equ^n.

→ If $a_i(x)$; $i=1, \_\_ n$ are Constants then (1) is called linear diff. Equ^n with Constant Coefficients.

→ If $\exists\ a_i(x)$ s.t. $a_i(x)$ is a non-Constant function of x Then (1) is called Linear diff Equ^n with variable Coefficients.

E.g. $y'' + 4y' + 3y = x^2 e^x$ — Non-Hom, Second order Linear diff Equ^n with Constant

$x^2 y'' + xy' + (x^2-4)y = 0$ →

Hom, Second order with variable coefficient.

$y'' + xy' = e^x$ → Non-hom, Second order with var. Coeff.

→ (1) Can be written as

$$F(D)\, y = Q(x)$$

Where $F(D) = a_0(x) D^n + a_1(x) D^{n-1} + \_\_ + a_{n-1}(x) D + a_n(x)$.

Here, D is a differential operator. , $D = \frac{d}{dx}$

$$D(f(x)) = \frac{df}{dx} = f'.$$

Eg. $D(x^n) = nx^{n-1}$. ; $D(\sin x) = \cos x$.

If f and g are two diff function Then

$$D(f+g) = D(f) + D(g)$$

$$D(\alpha f) = \alpha D(f).$$

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i.e. D is a linear operator.

So any Second order linear diff. Eqn
Eg $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2$

Can be written as $(D^2+2D+2)y = x^2$.
i.e. $F(D)y = x^2$
Where $F(D) = D^2+2D+2$.

Solution of 2nd order Hom linear diff Eqn with Constant Coeff's

Consider $ay'' + by' + cy = 0$, a, b, c are Constants. — (1)

Or $F(D)y = 0$ where $F(D) = (aD^2 + bD + c)$

So the auxiliary Eqn or Char. Eqn is
$am^2 + bm + c = 0$ (Replace D by m).

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow \text{are char. Roots.}$$

(1) If $b^2-4ac > 0$ Then roots are real and distinct
(2) If $b^2-4ac = 0$, Then roots are real and equal.
(3) If $b^2-4ac < 0$, Then roots are Complex.

(1) If $b^2-4ac > 0$:- $m = m_1$ and $m = m_2$.
Then gen Soln of (1) is $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

$$\left[\begin{array}{ll} \because y' - my = 0; & y' + my = 0 \\ y = e^{mx} + c & y = e^{-mx} + c \end{array} \right]$$

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(2) $b^2-4ac=0$ $\quad m=m_1=m_2$.

Then gen Solⁿ is $y(x) = c_1e^{mx}+c_2xe^{mx}$

(3) If $b^2-4ac<0$ $\quad m=p\pm iq$

gen Solⁿ is $y = e^{px}(c_1\cos qx + c_2\sin qx)$

$\because y = c_1e^{(p+iq)x} + c_2e^{(p-iq)x}$

$= c_1e^{px}.e^{iqx} + c_2e^{px}e^{-iqx}$

$= e^{px}[c_1(\cos qx + i\sin qx) + c_2(\cos qx - i\sin qx)$

$= e^{px}[c_1\cos qx + c_2\sin qx]$

Que $y''-y'-6y=0$

Solⁿ $(D^2-D-6)y=0$

Char Equⁿ is $m^2-m-6=0$

$\Rightarrow m^2-3m+2m-6=0$

$\Rightarrow m(m-3)+2(m-3)=0$

$\Rightarrow m=3,-2$

$y(x) = c_1e^{3x}+c_2e^{-2x}$

Que $4y''-8y'+3y=0$

Solⁿ $(4D^2-8D+3)y=0$

Char equⁿ is $4m^2-8m+3=0$

$\Rightarrow 4m^2-2m-6m+3=0$

$\Rightarrow 2m(2m-1)-3(2m-1)=0$

$\Rightarrow m=1/2, 3/2$

$y(x) = c_1e^{1/2x}+c_2e^{3/2x}$

Que

Que $4y'' + 4y' + y = 0$

Sol^n $(4D^2 + 4D + 1)y = 0$

Char. Equ^n is $4m^2 + 4m + 1 = 0 \Rightarrow (2m+1)^2 = 0$

$\Rightarrow m = -1/2, -1/2$

$\Rightarrow y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2}$

Que $y'' - 4y' - 5y = 0$

Sol^n $(D^2 - 4D - 5)y = 0$

char. Equ^n is $m^2 - 4m - 5 = 0$

$\Rightarrow m^2 + m - 5m - 5 = 0$

$\Rightarrow m(m+1) - 5(m+1) = 0$

$\Rightarrow m = 5, -1$

$y(x) = c_1 e^{5x} + c_2 e^{-x}$

Que $y'' + 2y' + 2y = 0$

Sol^n $(D^2 + 2D + 2)y = 0$

Char Equ^n is $m^2 + 2m + 2 = 0$

$\Rightarrow m = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$= \dfrac{-2 \pm 2i}{2} = -1 \pm i$

$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$

Que $y'' + 4y' + 13y = 0$

Sol^n $(D^2 + 4D + 13)y = 0$

char. Equ^n is $m^2 + 4m + 13 = 0$

$m = \dfrac{-4 \pm \sqrt{16-52}}{2}$

$= -2 \pm 3i$

$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$

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Higher order hom. diff Equ^n with Constant Coeff :→

$$a_0 y^n + a_1 y^{n-1} + \_\_ + a_n y = 0.$$

(1) Real and distinct roots :→ let $m_1, m_2, \_\_ m_n$ be n roots.

Gen Sol^n is $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \_\_ + C_n e^{m_n x_n}$

(2) Real and Repeated roots :→ let $m = m_1 = m_2 = \_\_ = m_n$

then Gen Sol^n is $y(x) = C_1 e^{m_0 x} + C_2 x e^{mx} + C_3 x^2 e^{mx} + \_\_ + C_n x^n e$

(3) Complex Roots :→ $p_1 + iq_1, p_2 + iq_2, \_\_, p_k + iq_k$

then Gen Sol^n is $y(x) = e^{p_1 x}(C_1 \cos q_1 + C_2 \sin q_1 x)$
$+ e^{p_2 x}(C_3 \cos q_2 x + C_4 \sin q_2 x) + \_\_$
$+ e^{p_k x}(C_k \cos q_k x + C_k \sin q_k x)$

Que $y''' - 2y'' - 5y' + 6y = 0$

Sol^n $(D^3 - 2D^2 - 5D + 6)y = 0$

Char. equ^n is $m^3 - 2m^2 - 5m + 6 = 0$

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	$\boxed{0}$

$m^2 - \cancel{6}m - 6 = 0$

$\Rightarrow m = \dfrac{6 \pm \sqrt{36 + 24}}{2}$

$= \dfrac{6 \pm \sqrt{60}}{2}$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$m = 3, -2$

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} + C_3 e^{x}.$$

Que $y''' - y'' - 4y' + 4y = 0.$

Sol^n $(D^3 - D^2 - 4D + 4)y = 0$

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Char. Equ^n is $m^3-m^2-4m+4=0$

$\Rightarrow m^2(m-1)-4(m-1)=0$

$\Rightarrow (m-1)(m^2-4)=0$

$m=1,2,-2$

$y(x)= c_1e^x+c_2e^{2x}+c_3e^{-2x}$.

Qu $y^4-5y^2+4y=0$.

Sol^n $(D^4-5D^2+4)y=0$

Ch. Equ^n is $m^4-5m^2+4=0$

$\Rightarrow (m^2-4)(m^2-1)=0$

$\Rightarrow m=\pm1,\pm2$

$y(x)= c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-2x}$.

Qu $4y^{iv}-12y'''-y''+27y'-18y=0$

Sol^n $(4D^4-12D^3-D^2+27D-18)y=0$.

Char. Equ^n is $4m^4-12m^3-m^2+27m-18=0$

$m=1$,

1	4	-12	-1	27	-18
		4	-8	-9	18
	4	-8	-9	18	0

$4m^3-8m^2-9m+18=0$

$4m^2(m-2)-9(m-2)=0$

$m=2,\ m=\pm3/2$

$y(x)= c_1e^x+c_2e^{2x}+c_3e^{3x/2}+c_4e^{-3/2x}$.

Qu $y'''-3y'-2y=0$

Sol^n $(D^3-3D-2)y=0$

Char Equ^n is $m^3-3m-2=0$

$m=-1$

-1	1	0	-3	-2
		-1	1	2
	1	-1	-2	$\boxed{0}$

$m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$\Rightarrow (m+1)(m-2) = 0$

$\Rightarrow m = -1, -1, 2$

$y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 x e^{-x}$.

$= c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$.

Que $8y''' - 12y'' + 6y' - y = 0$

Solⁿ $(8D^3 - 12D^2 + 6D - 1) y = 0$

Char. Equⁿ $8D^3 - 12D^2 + 6D - 1 = 0$.

$\Rightarrow (2m-1)^3 = 0$

$\Rightarrow m = 1/2 ; 1/2, 1/2$.

$y(x) = (c_1 + c_2 x + c_3 x^2) e^{x/2}$.

HW Que $y''' + 3y'' - 4y = 0$

Que $y^{iv} + 5y'' + 4y = 0$

Solⁿ $(D^4 + 5D^2 + 4) y = 0$

Char. Equⁿ is $\cancel{D^4} + m^4 + 5m^2 + 4 = 0$

$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$

$\Rightarrow m^2(m^2+4) + 1(m^2+4) = 0$

$\Rightarrow m = \pm i, \pm 2i$

$y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$.

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HW **Qu** $y^{iv} + 2y''' + 11y'' + 18y' + 18 = 0$.

Qu $y^{iv} + 32y'' + 256y = 0$

Sol^n $(D^4 + 32D^2 + 256)y = 0$

Char. Equ^n is $m^4 + 32m^2 + 256 = 0$

$\Rightarrow (m^2+16)^2 = 0$

$\Rightarrow m = \pm 4i, \pm 4i$

$y(x) = (C_1 + C_2 x)\cos 4x + (C_3 + C_4 x)\sin 4x$.

Non-Homogeneous linear diff. Equation with Constant Coefficients

$F(D)\,y = Q(x)$ — (1)

If D is a differential operator, then its inverse D^{-1} is an integral operator s.t. $D^{-1} D(f(x)) = f(x)$

i.e. $\frac{1}{D}$ or $D^{-1}(f) = \int f(x)dx$.

from (1); $y = (F(D))^{-1} Q(x) = \frac{Q(x)}{F(D)}$

Case I If $Q(x) = e^{\alpha x}$.

Then Substitute $D = \alpha$. $\therefore$ Solⁿ is $y(x) = \frac{Q(x)}{F(\alpha)}$ Provided $F(\alpha) \neq 0$.

Ex $y'' - 2y' - 3y = 3e^{2x}$

Solⁿ General solution of non-homogeneous diff. Equation

= General solution of hom. diff Eqⁿ + Particular Integral

= Complementary fun + P.I

i.e. = C.F + P.I

$(D^2 - 2D - 3)y = 0$

Aux Eqⁿ is $m^2 - 2m - 3 = 0$

$\Rightarrow (m-3)(m+1) = 0$

$\Rightarrow m = 3, -1$

$\therefore \; y_c(x) = C_1 e^{-x} + C_2 e^{3x}$.

P.I is $y_p(x) = \dfrac{3e^{2x}}{F(D)} = \dfrac{3e^{2x}}{D^2-2D-3}$

Put $D=2$

$= \dfrac{3e^{2x}}{4-4-3} = -e^{2x}$.

$\therefore$ Gen Sol^n is $y(x) = y_c(x) + y_p(x)$

$= C_1 e^{-x} + C_2 e^{3x} - e^{2x}$.

Que $y'' + y' - 6y = 5e^{-3x}$.

Sol^n $(D^2+D-6)y = 5e^{-3x}$

Char. Equ is

$m^2+m-6=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m = 2, -3$

$\therefore \; y_c(x) = C_1 e^{2x} + C_2 e^{-3x}$

$y_p(x) = \dfrac{5e^{-3x}}{D^2+D-6} = \dfrac{5e^{-3x}}{(-3)^2-3-6} = \dfrac{5e^{-3x}}{0}$ — (Case fails)

$= \dfrac{x \cdot 5e^{-3x}}{2D+1} = \dfrac{5xe^{-3x}}{-5} = -xe^{-3x}$

$\therefore$ Gen Sol^n is $y(x) = y_c(x) + y_p(x)$

$\Rightarrow y(x) = C_1 e^{2x} + C_2 e^{-3x} - xe^{-3x}$.

Que $4y'' - 4y' + y = e^{x/2} \Rightarrow (4D^2-4D+1)y = e^{x/2}$

Sol^n Char Equ^n is $4m^2-4m+1=0$

$\Rightarrow (2m-1)^2 = 0 \Rightarrow m = 1/2, 1/2$

$y_c(x) = C_1 e^{x/2} + C_2 x e^{x/2} = (C_1 + C_2 x)\, e^{x/2}$.

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$$y_p(x) = \frac{e^{x/2}}{4D^2-4D+1}$$

$$= \frac{e^{x/2}}{4\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)+1} = \frac{e^{x/2}}{1-2+1=0} \text{ -(Test fails)}$$

$$= \frac{x e^{x/2}}{8D-4} = \frac{x e^{x/2}}{4-4=0}$$

$$= \frac{x^2 e^{x/2}}{8}$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= (c_1 + c_2 x)e^{x/2} + \frac{x^2}{8}e^{x/2}.$$

HW $9y''' + 3y'' - 5y' + y = 42e^x + 64e^{x/3}$.

Sol^n $(9D^3 + 3D^2 - 5D + 1)y = 42e^x + 64e^{x/3}$

A.E is $9m^3 + 3m^2 - 5m + 1 = 0$

$\Rightarrow m = -1$ Satisfies

$$\begin{array}{r|rrrr} -1 & 9 & 3 & -5 & 1 \\ & & -9 & 6 & -1 \\ \hline & 9 & -6 & 1 & \boxed{0} \end{array}$$

$$9m^2 - 6m + 1 = 0$$

$$(3m-1)^2 = 0$$

$$\therefore m = 1/3,\ 1/3,\ -1$$

$$y_c(x) = (c_1 + c_2 x)e^{x/3} + c_3 e^{-x}$$

$$y_p(x) = \frac{42e^x}{9D^3+3D^2-5D+1} + \frac{64e^{x/3}}{9D^3+3D^2-5D+1}$$

$$= \frac{42e^x}{9+3-5+1} + \frac{64e^{x/3}}{9\times\frac{1}{27}+3\times\frac{1}{9}-5\times\frac{1}{3}+1}$$

$\frac{1}{3}+\frac{1}{3}$

$$= \frac{42}{8}e^x + \frac{64}{0} \rightarrow \text{Test fails}$$

$$\frac{64e^{x/3}}{9D^3+3D^2-5D+1} \leftarrow \text{Test fails}$$

$$\frac{64\,x\,e^{x/3}}{27D^2+6D-5} = \frac{64xe^{x/3}}{3\cancel{27}\left(\frac{1}{9}\right)+\cancel{6}2\left(\frac{1}{3}\right)-5}$$

$5-5=0 \rightarrow$ again fails

$$\frac{64x^2e^{x/3}}{54D+6} = \frac{64x^2e^{x/3}}{54\left(\frac{1}{3}\right)+6} = \frac{64}{24}x^2e^{x/3}$$

$$= \frac{8}{3}x^2e^{x/3}$$

$$\therefore y(x) = y_c(x)+y_p(x)$$
$$= (C_1+C_2x)e^{x/3}+C_3e^{-x}+\frac{21}{4}e^{x}+\frac{8}{3}x^2e^{x/3}.$$

Que $16y''+8y'+y=48xe^{-x/4}$.

Ans:- $(C_1x+C_2)e^{-x/4}+\frac{1}{2}x^3e^{-x/4}$.

Case II When $Q(x)=\sin\alpha x$ or $\cos\alpha x$.

Substitute $D^2=-\alpha^2$.

then $y_p(x)=\dfrac{Q(x)}{F(D^2)}=\dfrac{Q(x)}{F(-\alpha^2)}$ Provided $F(-\alpha^2)\neq 0$.

Que $y''+4y=6\cos x$

Sol^n $(D^2+4)y=6\cos x$

A.E. is $m^2+4=0$

$\cancel{\Rightarrow m=0,-4}$ $\Rightarrow m=\pm 2i$

$y_c(x)=C_1\cos 2x+C_2\sin 2x$

$$y_p(x)=\frac{6\cos x}{D^2+4}=\frac{6\cos x}{-1+4}=\frac{6}{3}\cos x=2\cos x$$

$$\therefore y(x)=y_c(x)+y_p(x)$$
$$=C_1\cos 2x+C_2\sin 2x+2\cos x.$$

Qu $2y'' + y' - y = 16\cos 2x$

Sol^n $(2D^2 + D - 1)y = 16\cos 2x$

A.E is $2m^2 + m - 1 = 0$

$\Rightarrow m = \frac{-1 \pm \sqrt{1+8}}{2} = -1, \frac{1}{2}$

$y_C(x) = C_1 e^{-x} + C_2 e^{x/2}$

$y_p(x) = \frac{16\cos 2x}{2D^2 + D - 1} = \frac{16\cos 2x}{-8 + D - 1} = \frac{16\cos 2x}{D - 9} \times \frac{D+9}{D+9}$

$= \frac{(D+9)\, 16\cos 2x}{D^2 - 81}$

$= \frac{(D+9)\, 16\cos 2x}{-85}$

$= \frac{D(16\cos 2x) + 144\cos 2x}{-85}$

$= \frac{-32\sin 2x + 144\cos 2x}{-85}$

$= \frac{-16}{85}(9\cos 2x - 2\sin 2x)$

$\therefore y(x) = y_c(x) + y_p(x)$

$= C_1 e^{-x} + C_2 e^{x/2} - \frac{16}{85}(9\cos 2x - 2\sin 2x)$

HW (1) $y'' - 5y' + 4y = 65\sin 2x$

Ans $y(x) = Ae^{x} + Be^{4x} + \frac{13}{2}\cos 2x.$

(2) $y''' - y'' + 4y' - 4y = \sin 3x$

Ans:- $Ae^{x} + B\cos 2x + C\sin 2x + (3\cos 3x + \sin 3x)/50.$

Qu $y'' + y = 6\sin x$

Sol^n $(D^2 + 1)y = 6\sin x$

A.E is $m^2+1=0$

$\Rightarrow m = \pm i$

$y_c(x) = c_1 \cos x + c_2 \sin x$

$y_p(x) = \dfrac{6\sin x}{D^2+1} = \dfrac{6\sin x}{-1+1=0}$ → Test fails

$\dfrac{x \cdot 6\sin x}{2D} = 3x \dfrac{1}{D}(\sin x)$

$= -3x\cos x$

Que $y'' - 4y' + 13y = 18e^{2x}\sin 3x$

Case III If $Q(x) = e^{\alpha x} h(x)$

Then $y_p(x) = \dfrac{1}{F(D)}\left[e^{\alpha x} \cdot h(x)\right]$

$= e^{\alpha x} \dfrac{1}{F(D+\alpha)} h(x)$

Ex $y'' - 4y' + 13y = 18e^{2x}\sin 3x$

Sol $(D^2 - 4D + 13)y = 18e^{2x}\sin 3x$

A.E. is

$m^2 - 4m + 13 = 0$

$m = \dfrac{4 \pm \sqrt{16-52}}{2}$

$= \dfrac{4 \pm \sqrt{-36}}{2}$

$= \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$y_c(x) = e^{2x}(c_1\cos 3x + c_2 \sin 3x)$

$y_p(x) = \dfrac{18e^{2x}\sin 3x}{(D^2-4D+13)}$

$= 18e^{2x} \dfrac{1}{(D+2)^2 - 4(D+2) + 13} \sin 3x$

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$$= 18e^{2x}\left[\frac{1}{D^2+9}\sin 3x\right]$$

$$= 18e^{2x}\left[\frac{1}{-9+9=0}\sin 3x\right.$$ → Case fails

Put $D^2 = -9$

$$= 18x\frac{e^{2x}}{2D}\sin 3x$$

$$= 9xe^{2x}\int \sin 3x\,dx$$

$$= 9xe^{2x}\left(\frac{-\cos 3x}{3}\right) = -3xe^{2x}\cos 3x$$

$$y(x) = y_c(x) + y_p(x)$$

Que $(D^2+2D+5)y = e^{-x}\cos 2x.$

Solⁿ:- AE is $(m^2+2m+5)=0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y_c(x) = e^{-x}(C_1\cos 2x + C_2\sin 2x)$$

$$y_p(x) = \frac{e^{-x}\cos 2x}{D^2+2D+5}$$

$$= e^{-x}\frac{1}{(D-1)^2+2(D-1)+5}\cos 2x$$

$$= e^{-x}\left[\frac{1}{D^2+4}\cos 2x\right]$$

$$= e^{-x}\left[\frac{1}{-4+4=0}\cos 2x\right]$$ Put $D^2=-4$

→ Case fails

$$\text{So } y_p(x) = \frac{e^{-x}\cdot x}{2D}\cos 2x$$

$$= \frac{e^{-x}x}{2}\left[\int \cos 2x\,dx\right] = \frac{xe^{-x}\sin 2x}{4}$$

$\therefore\ y(x) = y_c(x) + y_p(x)$

$\Rightarrow y(x) = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + \dfrac{x e^{-x} \sin 2x}{4}$

HW (1) $(2D^2 - 7D + 3)y = \sin 2x$.

(2) $(D^2 - 4D + 5)y = 24 e^{2x} \sin x$.

(3) $(D^2 + 4D + 3)y = e^{2x} \cos x$.

Case IV If $f(x) = x^{\alpha}$, $\alpha > 0$ and Integer.

Then $y_p(x) = \dfrac{1}{F(D)} \cdot x^{\alpha}$

$\rightarrow$ Then Expand $[F(D)]^{-1}$ in ascending powers of D.

Ques → $y'' + 16y = 64x^2$

Sol^n → $(D^2 + 16)y = 64x^2$

AE is $m^2 + 16 = 0$

$\Rightarrow m = \pm 4i$

$y_c(x) = C_1 \cos 4x + C_2 \sin 4x$.

$$y_p(x) = \frac{64x^2}{D^2+16}$$

$$= \frac{64}{16}\left[1 + \frac{D^2}{16}\right]^{-1} x^2$$

$$= 4\left[1 - \frac{D^2}{16}\right] x^2$$

$$= 4\left[x^2 - \frac{1}{16}(2)\right] = 4x^2 - \frac{1}{2}.$$

$y(x) = C_1 \cos 4x + C_2 \sin 4x + 4x^2 - \frac{1}{2}$.

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Que $(D^2+25)y = 9x^3$

Soln AE is $m^2+25=0$

$\Rightarrow m = \pm 5i$

$y_c(x) = C_1\cos 5x + C_2\sin 5x.$

$$y_p(x) = \frac{9x^3}{D^2+25} = \frac{1}{25}\left[1+\frac{D^2}{25}\right]^{-1} 9x^3$$

$$= \frac{1}{25}\left[1-\frac{D^2}{25}+\frac{(-1)(-2)}{\lfloor 2}\cdot\left(\frac{D^2}{25}\right)^2 + \dots\right]9x^3$$

$$= \frac{1}{25}\left[9x^3 - \frac{1}{25}(54x)\right]$$

$$= \frac{225x^3-54x}{625}$$

$y(x) = y_c(x) + y_p(x)$

$$= C_1\cos 5x + C_2\sin 5x + \frac{225x^3-54x}{625}.$$

Legendre Equation and Legendre Polynomials

(1) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a real constant.
(In most application n is a positive Integer or whole number)
Equation (1) is Called Legendre differential Equation.

Let $n=0$ in (1)

$(1-x^2)y'' - 2xy' = 0 \quad (2)$ → We can observe that $y =$ Constant is solⁿ of (2).

So we can choose $y(x) = 1$.

let $n=1$ in (1)

$(1-x^2)y'' - 2xy' + 2y = 0 \quad (3)$ → We note that $y = x$ is solⁿ of (3)

So we can choose $y(x) = x$.

let $n=2$ in (1)

$(1-x^2)y'' - 2xy' + 6y = 0 \quad (4)$ → we note that $y = x^2$ is one of the solⁿ of (4). So we Select $y(x) = x^2$.

and so on.

⟹ The Solⁿ of (1) Corresponding to different values of n are polynomials.

→ <u>To Find the Solⁿ of (1) about $x=0$.</u>

Let the Solⁿ of (1) about $x=0$ is of the form

$$y(x) = \sum_{r=0}^{\infty} C_r x^{n-r}$$

$$y'(x) = \sum_{r=1}^{\infty} (n-r) C_r x^{n-r-1}$$

$$y''(x) = \sum_{r=2}^{\infty} (n-r)(n-r+1) C_r x^{n-r-2}$$

Substitute $y(x)$, $y'(x)$ and $y''(x)$ in (1), we find the solution.

of the form $y(x) = C_1 y_1 + C_2 y_2$.

where $y_1 = a_0 \left(x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} + \ldots \right)$

↳ Called legendre polynomial of first kind, and denoted by $\boxed{P_n(x)}$.

$y_2 = b \left(x^{-n-1} + \frac{n(n+1)}{2(2n+3)} x^{-n-3} + \frac{n(n+1)(n+2)(n+3)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots \right)$

↳ Called legendre Solution of Second Kind, and denoted by $\boxed{Q_n(x)}$

The value of a in y_1 is

$$a = \frac{1.3.5\ldots(2n-1)}{n}$$

The general term of $P_n(x)$ will be

$$\frac{1.3.5\ldots(2n-1)}{n}\left[\frac{(-1)^r n(n-1)(n-2)\ldots(n-2r+1)\, x^{n-2r}}{(2.4.6\ldots 2r)(2n-1)(2n-3)\ldots(2n-2r+1)}\right]$$

$$= \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!}$$

$$\Rightarrow \boxed{P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!}} \longrightarrow \text{Legendre Polynomial.}$$

$n=0;\ P_0(x) = 1$

$n=1;\ P_1(x) = \frac{2!\, x^1}{2^1 0!\, 1!\, 1!} = x$

$n=2;\ P_2(x) = \sum_{r=0}^{[1]} \frac{(-1)^r (4-2r)!\, x^{2-2r}}{2^2 r!\,(2-r)!\,(2-2r)!}$

$$= \frac{4!\,x^2}{4\ 2!\ 2!} + \frac{(-1)\ 2!\ x^0}{2^2\,1!\ 1!\ 0!}$$

$$= \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \sum_{r=0}^{[3/2]} \frac{(-1)^r\,(2n-2r)!\ x^{n-2r}}{2^n\ r!\ (n-r)!\ (n-2r)!}$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

Similarly $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Que→ Express $P(x) = 3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$ as a polynomial in x, where $P_n(x)$ is legenedre polynomial of order n.

Solⁿ→ $P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2-1)$; $P_3(x) = \frac{1}{2}(5x^3-3x)$

$$P(x) = 3\left[\frac{1}{2}(5x^3-3x)\right] + \cancel{2}\left[\frac{1}{\cancel{2}}(3x^2-1)\right] + 4(x) + 5(1)$$

$$= \frac{15}{2}x^3 - \frac{9}{2}x + 3x^2 - 1 + 4x + 5$$

$$= \frac{15}{2}x^3 + 3x^2 - \frac{1}{2}x + 4.$$

$$= \frac{1}{2}\left[15x^3 + 6x^2 - x + 8\right].$$

Ques→ Express $f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$ in terms of legendre polynomials.

Solⁿ→ $1 = P_0(x)$; $x = P_1(x)$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow \frac{2P_2(x)+1}{3} = x^2 \Rightarrow x^2 = \frac{1}{3}(2P_2 + P_0(x))$$

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$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow 2P_3 = 5x^3 - 3P_1$$

$$\Rightarrow \frac{2P_3 + 3P_1}{5} = x^3$$

$$\Rightarrow \boxed{x^3 = \frac{1}{5}(2P_3 + 3P_1)}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$\Rightarrow 8P_4(x) = 35x^4 - 30\left(\frac{1}{3}(2P_2 + P_0)\right) + 3P_0$$

$$\Rightarrow 8P_4 = 35x^4 - 10(2P_2 + P_0) + 3P_0$$

$$\Rightarrow 8P_4 + 20P_2 + 7P_0 = 35x^4$$

$$\Rightarrow \boxed{x^4 = \frac{1}{35}(8P_4 + 20P_2 + 7P_0)}$$

$$f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$$

$$= \frac{1}{35}(8P_4 + 20P_2 + 7P_0) + 2\left(\frac{1}{5}(2P_3 + 3P_1)\right) - 6\left(\frac{1}{3}(2P_2 + P_0)\right) + 5P_1 - 3P_0$$

$$= \frac{8}{35}P_4 + \frac{20}{35}P_2 + \frac{7}{35}P_0 + \frac{4}{5}P_3 + \frac{6}{5}P_1 - 4P_2 - 2P_0 + 5P_1 - 3P_0$$

$$= \frac{1}{35}(8P_4 + 28P_3 - 120P_2 + 217P_1 - 168P_0).$$

Que Express $x^3 + x + 1$ in Legendre polynomials.

Sol $1 = P_0$; $x = P_1$; $x^2 = \frac{1}{3}(2P_2 + P_0)$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow 2P_3 = 5x^3 - 3P_1 \quad \Rightarrow \frac{1}{5}(2P_3 + 3P_1) = x^3$$

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$$x^3+x+1 = \frac{1}{5}(2P_3+3P_1)+P_1+P_0$$

$$= \frac{2}{5}P_3+\frac{3}{5}P_1+P_1+P_0$$

$$= \frac{2}{5}P_3+\frac{8}{5}P_1+P_0$$

$$= \frac{1}{5}(2P_3+8P_1+5P_0).$$

H.W

Que (1) $3x^2+5x-6$
(2) $4x^3+3x^2+2x-6$
(3) $5x^4+3x^3-6x^2-2x+3$
$\rightarrow$ Express In Legendre polynomials.

Que (1) $6P_3(x)-2P_1(x)+P_0(x)$
(2) $4P_3(x)+6P_2(x)-3P_1(x)-2P_0(x)$
(3) $8P_4(x)+2P_2(x)+P_0(x)$
(4) $5P_4(x)+10P_3(x)+2P_2(x)+P_1(x)$
$\rightarrow$ Express In terms of polynomials of x.

Que:→ $(1-x^2)y''-2xy'+6y=0 \longrightarrow (1)$

If Sol^n of (1) is $y(x)$ then find

$$\int_{-1}^{1} y(x)\,(x+x^2)\,dx$$

Sol^n

(1) is legendre polynomial with $n=2$

So Sol^n will be $P_2(x)=\frac{1}{2}(3x^2-1)=y(x)$

$$\frac{1}{2}\int_{-1}^{1}(3x^2-1)(x+x^2)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3+3x^4-x-x^2)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}(3x^4-x^2)\,dx$$

$$= \frac{2}{2}\int_{0}^{1}(3x^4-x^2)\,dx = \frac{3}{5}-\frac{1}{3}=\boxed{\frac{4}{15}}$$

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Rodrigue's Formula:→

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}$$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2^1} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2}(2x) = x$$

$$P_2(x) = \frac{1}{2^2 \lfloor 2} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\left[2(x^2-1)(2x)\right]'$$

$$= \frac{1}{8}(4x^3-4x)'$$

$$= \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 \lfloor 3} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{8 \lfloor 3}\left[\frac{d^2}{dx^2}\left(3(x^2-1)^2(2x)\right)\right]$$

$$= \frac{1}{8}\left[\frac{d^2}{dx^2}(x^2-1)^2(x)\right]$$

$$= \frac{1}{8}\left[\frac{d}{dx}\left[(x^2-1)^2 + x(2(x^2-1))(2x)\right]\right]$$

$$= \frac{1}{8}\left[\frac{d}{dx}\left((x^2-1)^2 + 4x^2(x^2-1)\right)\right]$$

$$= \frac{1}{8}\left[2(x^2-1)(2x) + 16x^3 - 8x\right]$$

$$= \frac{1}{8}\left[4x^3 - 4x + 16x^3 - 8x\right] = \frac{20x^3 - 12x}{8}$$

$$= \frac{5x^3 - 3x}{2} = \frac{1}{2}(5x^3 - 3x)$$

Recurrence Relations for Legendre Polynomials:-

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

E.g. $P_0(x) = 1$, $P_1(x) = x$

Find $P_2(x)$, $P_3(x)$, $P_4(x)$.

$n=1$; $2P_2(x) = 3xP_1(x) - P_0(x)$

$= 3x(x) - 1 = \frac{3x^2-1}{2}$

$\Rightarrow P_2(x) = \frac{3x^2-1}{2}$

$n=2$; $3P_3(x) = 5xP_2(x) - 2P_1(x)$

$= 5x\left(\frac{3x^2-1}{2}\right) - 2x$

$= \frac{15x^3-5x}{2} - 2x = \cancel{9x^3-x} \; \frac{15x^3-9x}{2}$

$\Rightarrow P_3(x) = \frac{1}{2}(5x^3-3x)$

$n=3$; $4P_4(x) = 7xP_3(x) - 3P_2(x)$

$= 7x\left(\frac{5x^3-3x}{2}\right) - 3\left(\frac{3x^2-1}{2}\right)$

$= \frac{35x^4-21x^2}{2} - \left(\frac{9x^2-3}{2}\right)$

$= \frac{35x^4-30x^2+3}{2}$

$\Rightarrow P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Que Using the Recurrence Relations

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

Evaluate $P_2(1.5)$ and $P_3(2.1)$.

Solⁿ: $n=1$; $2P_2(x) = 3xP_1(x) - P_0(x)$

$2P_2(1.5) = 3(1.5)P_1(1.5) - P_0(1.5) = 3(1.5)(1.5) - 1$

$= 5.75$

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$\Rightarrow P_2(1.5) = 2.875.$

Similarly $n=2$

$3P_3(x) = 5x P_2(x) - 2P_1(x)$

$3P_3(2.1) = 5(2.1)P_2(2.1) - 2P_1(2.1)$

$P_1(2.1) = 2.1$

$P_2(2.1) = \frac{1}{2}(3(2.1)^2 - 1) = 6.115$

$\Rightarrow \boxed{P_3(2.1) = 20.0025.}$

Orthogonality property of Legendre polynomials:→

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} 0 & \text{If } m \neq n \\ \frac{2}{2n+1} & \text{If } m = n. \end{cases}$$

Que If $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{3}$ then n Equals

Sol^n $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} = \frac{2}{3}$

~~$\Rightarrow 2 = 30n + 15 \Rightarrow n = \frac{-13}{30}$~~

$\Rightarrow 2n+1 = 3 \Rightarrow \boxed{n = 1}$

Que Let $(1-x^2)y'' - 2xy' + n(n+1)y = 0$; let $y_n(x)$ be its solution

If $\int_{-1}^{1} (y_n^2 + y_{n+1}^2)\,dx = \frac{16}{15}$ then find n.

Sol^n We Know that $\int_{-1}^{1} y_n^2(x)\,dx = \frac{2}{2n+1}$

$\Rightarrow \int_{-1}^{1} y_{n+1}^2\,dx = \frac{2}{2(n+1)+1} = \frac{2}{2n+3}$

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$$\therefore \int_{-1}^{1} (y_n^2 + y_{n+1}^2)\,dx = \frac{2}{2n+1} + \frac{2}{2n+3} = \frac{16}{15}$$

$$\Rightarrow \frac{2n+1+2n+3}{(2n+1)(2n+3)} = \frac{8}{15}$$

$$\Rightarrow \frac{4n+4}{(2n+1)(2n+3)} = \frac{8}{15}$$

$$\Rightarrow (n+1)15 = 2(2n+1)(2n+3)$$

$$\Rightarrow 15n+15 = 8n^2+16n+6$$

$$\Rightarrow 8n^2+n-9=0$$

$$\Rightarrow 8n^2+9n-8n-9=0$$

$$\Rightarrow 8n(n-1)+9(n-1)=0$$

$$\Rightarrow (n-1)(8n+9)=0$$

$$\Rightarrow \boxed{n=1}$$

(1) $P_n(1) = 1$;

(2) $P_n(-1) = (-1)^n$

(3) $P_n(-x) = (-1)^n P_n(x)$

(4) $\int_{-1}^{1} P_n(x)\,dx = 0 \quad \forall\, n \geq 1.$

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Power Series Solution about an ordinary point:→

let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad -(1)$

~~The Power~~ Let $x = x_0$ is an ordinary point of (1).

Then the power series solution of (1) about $x = x_0$ is of the form $y(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \ldots$

where $c_0, c_1, c_2 \ldots$ are constants.

Ex Find the power series solution about $x=0$ of the differential Equation $y' - 2y = 0$

Sol^n The given diff. Equ^n is $y' - 2y = 0 \quad -(1)$

Let the power series solution of (1) about $x=0$ is of the form $y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$

$\Rightarrow y'(x) = c_1 + 2c_2x + 3c_3x^2 + \ldots$

Substitute $y(x)$ and $y'(x)$ in (1), we get

$(c_1 + 2c_2x + 3c_3x^2 + \ldots) - 2(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots) = 0$

$\Rightarrow (c_1 - 2c_0) + 2(c_2 - c_1)x + (3c_3 - 2c_2)x^2 + \ldots + [(m+1)c_m - 2c_m]x^m + \ldots = 0$

Compare coeff of various powers of x, we get.

$c_1 - 2c_0 = 0 \Rightarrow c_1 = 2c_0$

$2c_2 - 2c_1 = 0 \Rightarrow c_2 = c_1 = 2c_0$

$3c_3 - 2c_2 = 0 \Rightarrow 3c_3 = 2c_2$

$\Rightarrow c_3 = \frac{2c_2}{3} = \frac{4c_0}{3} \ldots$

$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$

$= c_0 + 2c_0x + 2c_0x^2 + \frac{4}{3}c_0x^3 + \ldots$

$= c_0\left[1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots\right]$

$= c_0\left[1 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots\right] = c_0e^{2x}$

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Que Find the power Series Solution of $y''-y=0$ about $x=0$.

Sol^n $\Rightarrow$ let the power Series Sol^n about $x=0$ is

$$y(x) = \sum_{m=0}^{\infty} c_m x^m$$

$$y'(x) = \sum_{m=1}^{\infty} m\, c_m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2}.$$

So $\quad y''-y=0$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} - \sum_{m=0}^{\infty} c_m x^m = 0$$

$$\Rightarrow \underbrace{\sum_{t=0}^{\infty} (t+2)(t+1)\, c_{t+2}\, x^t}_{\text{Put } m-2=t} - \sum_{m=0}^{\infty} c_m x^m = 0$$

~~Or~~ $\cancel{\sum_{m=0}}$

$$\text{Or} \quad \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{m=0}^{\infty} c_m x^m = 0$$

Comparing Coeff of like powers of x, we get

$$(m+2)(m+1) c_{m+2} - c_m = 0$$

$$\Rightarrow (m+2)(m+1) c_{m+2} = c_m$$

$$\Rightarrow c_{m+2} = \frac{c_m}{(m+1)(m+2)}$$

$$\Rightarrow c_2 = \frac{c_0}{(1)(2)} = \frac{1}{2} c_0 = \frac{1}{\lfloor 2} c_0$$

$$c_3 = \frac{c_1}{(2)(3)} = \frac{1}{6} c_1 = \frac{1}{\lfloor 3} c_1$$

$$c_4 = \frac{c_2}{(3)(4)} = \frac{1}{12} \times \frac{1}{2} c_0 = \frac{1}{\lfloor 4} c_0$$

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$$C_5 = \frac{C_3}{(4)(5)} = \frac{1}{\lfloor 3 \cdot 4 \cdot 5} C_1 = \frac{1}{\lfloor 5} C_1$$

and so on.

$$\therefore y(x) = C_0 + C_1 x + \frac{1}{\lfloor 2} C_0 x^2 + \frac{1}{\lfloor 3} C_1 x^3 + \frac{1}{\lfloor 4} C_0 x^4 + \frac{1}{\lfloor 5} C_1 x^5 + \cdots$$

$$= C_0\left(1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots\right) + C_1\left(x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots\right)$$

where C_0 and C_1 are arbitrary constants.

Que Find the power series solⁿ of $(1-x^2)y'' + 2xy' + y = 0$. about $x=0$. — (1)

Solⁿ⇒ let $y(x) = \sum_{m=0}^{\infty} C_m x^m$

$$y'(x) = \sum_{m=1}^{\infty} m C_m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$$

Put $y(x)$, $y'(x)$ and $y''(x)$ in (1), we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} + 2x\left(\sum_{m=1}^{\infty} m C_m x^{m-1}\right) + \sum_{m=0}^{\infty} C_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) C_m x^m + 2\sum_{m=1}^{\infty} m C_m x^m + \sum_{m=0}^{\infty} C_m x^m = 0.$$

$\Rightarrow$ For $\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$ Put $m-2=t$, $m=t+2$

$$\sum_{t=0}^{\infty} (t+2)(t+1) C_{t+2} x^t$$

or $\sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2} x^m$ → we can replace t by m – just notation

$$\therefore \sum_{m=0}^{\infty}(m+2)(m+1)C_{m+2}x^m - \sum_{m=2}^{\infty}m(m-1)C_m x^m + 2\sum_{m=1}^{\infty}mC_m x^m + \sum_{m=0}^{\infty}C_m x^m = 0$$

$$\Rightarrow 2C_2 + 6C_3 x + \sum_{m=2}^{\infty}(m+2)(m+1)C_{m+2}x^m - \sum_{m=2}^{\infty}m(m-1)C_m x^m + 2C_1 x + 2\sum_{m=2}^{\infty}mC_m x^m + C_0 + C_1 x + \sum_{m=2}^{\infty}C_m x^m = 0$$

$$\Rightarrow 2C_2 + 6C_3 x + 2C_1 x + C_0 + C_1 x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)C_{m+2} - m(m-1)C_m + 2mC_m + C_m\right]x^m = 0$$

$$\Rightarrow 2C_2 + C_0 = 0 \Rightarrow C_2 = \frac{-C_0}{2}$$

$$6C_3 + 3C_1 = 0 \Rightarrow C_3 = \frac{-1}{2}C_1$$

$$(m+2)(m+1)C_{m+2} - m(m-1)C_m + 2mC_m + C_m = 0$$

$$\Rightarrow (m+2)(m+1)C_{m+2} - (m^2 - 3m - 1)C_m = 0$$

$$\Rightarrow C_{m+2} = \frac{(m^2-3m-1)C_m}{(m+1)(m+2)} \; ; \; m$$

$$\Rightarrow C_2 = \frac{-C_0}{2}, \quad C_3 = \frac{-1}{2}C_1, \quad C_4 = \frac{-3}{12}C_2 = \frac{1}{8}C_0 \ldots$$

$$\therefore y(x) = C_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 \ldots\right) + C_1\left(x - \frac{x^3}{2} + \ldots\right)$$

Series Solutions of Differential Equations:→

Analytic function:→ A function $f(x)$ is called analytic at point x_0 if it is differentiable at x_0 and at every point in the neighbourhood of x_0.

Eg. e^x, $\sin x$, $\cos x$, polynomial functions are analytic everywhere.

A rational function of the form $\frac{f(x)}{g(x)}$ is analytic at all x Except at those values of x for which $g(x)=0$.

Eg. $\frac{x}{x^2-3x+2}$ is analytic everywhere Except at x s.t. $x^2-3x+2=0$
ie $x=1$ and $x=2$.

Ordinary point:→ Let $y''+P(x)y'+Q(x)y=0$ — (1)

A point $x=x_0$ is called an ordinary point of (1) if $P(x)$ and $Q(x)$ are analytic functions at $x=x_0$.

If $x=x_0$ is not an ordinary point then it is called singular point of (1).

⇒ Singular points are of two types:-

(1) Regular Singular point (2) Irregular Singular point.

Regular Singular point:→ A Singular point $x=x_0$ is called regular singular pt of (1) if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x=x_0$.

A Singular point which is not regular is called Irregular Singular point of (1).

Que↦ $2x^2 y'' + 7x(x+1)y' - 3y = 0 \quad -(1)$

$\Rightarrow y'' + \frac{7x(x+1)}{2x^2} y' - \frac{3y}{2x^2} = 0$

or $y'' + \frac{7(x+1)}{2x} y' - \frac{3}{2x^2} y = 0$

$P(x)$ and $Q(x)$ are analytic Everywhere Except at $x=0$.

$\Rightarrow x=0$ is a Singular point of (1)

$(x-0)\frac{7(x+1)}{2x} = \frac{7(x+1)}{2}$ is analytic at $x=0$

and $(x-0)^2\left(\frac{-3}{2x^2}\right) = \frac{-3}{2}$ is analytic at $x=0$

$\Rightarrow x=0$ is a regular Singular pt. of (1).

Que S.T. $x=0$ is an ordinary point of $(x^2-1)y'' + xy' - y = 0$ but $x=1$ is a regular Singular point.

Solⁿ↦ $(x^2-1)y'' + xy' - y = 0$

$\Rightarrow y'' + \frac{x}{x^2-1} y' - \frac{y}{x^2-1} = 0$

$P(x) = \frac{x}{x^2-1}$ and $Q(x) = \frac{-1}{x^2-1}$ are analytic Everywhere Except at $x = \pm 1$

$\Rightarrow x=0$ is an ordinary pt and $x=1$ is a Singular pt.

Now $(x-1)p(x) = (x-1)\frac{x}{x^2-1} = \frac{x}{x+1}$

$(x-1)^2 Q(x) = (x-1)^2\left(\frac{-1}{(x-1)(x+1)}\right) = \frac{-(x-1)}{(x+1)}$

are analytic at $x=1 \Rightarrow x=1$ is a regular Singular pt

Que $x^2(x+1)^2y'' + (x^2-1)y' + 2y = 0$ — (1)

$$\Rightarrow xy'' + \frac{(x-1)(x+1)}{x^2(x+1)^2}y' + \frac{2y}{x^2(x+1)^2} = 0$$

$$\Rightarrow y'' + \frac{(x-1)}{x^2(x+1)}y' + \frac{2y}{x^2(x+1)^2} = 0$$

$$P(x) = \frac{x-1}{x^2(x+1)} \;;\; Q(x) = \frac{2}{x^2(x+1)^2}$$

are analytic Everywhere Except at $x=0$ and $x=-1$

$\Rightarrow$ $x=0$ and $x=-1$ are Singular pts of (1)

$$x\,P(x) = \frac{x(x-1)}{x^2(x+1)} = \frac{x-1}{x(x+1)} \rightarrow \text{not analytic at } x=0$$

$$\text{and } x^2Q(x) = x^2\left(\frac{2}{x^2(x+1)^2}\right) = \frac{2}{(x+1)^2} \rightarrow \text{analytic at } x=0$$

$\Rightarrow$ $x=0$ is an irregular Singular pts.

$$(x+1)P(x) = \frac{(x+1)(x-1)}{x^2(x+1)} = \frac{x-1}{x^2}$$

$$(x+1)^2Q(x) = \cancel{(x+1)^2}\cdot\frac{2}{x^2\cancel{(x+1)^2}} = \frac{2}{x^2}$$

are analytic at $x=-1$

$\Rightarrow$ $x=-1$ is a regular Singular point.

H.W **Que** Determine the Singularities of the following Equations:

(1) $x(x-1)^3y'' + 2(x-1)^3y' + 3y = 0$ at $x=0$ and $x=1$.

(2) $3xy'' + 2x(x-1)y' + 5y = 0$

(3) $y'' + (1-x)y' + (1-x)^2y = 0$.

(4) $x^3y'' + y\sin x = 0$

Unit-II

Dependent | Independent variables :→ The variable whose value is assigned is called Independent variable and the variable whose value is obtained w.r.t. assigned value is called dependent variable.

If $f: A \to B$ be any function

then $\forall\, x \in A$, $\exists$ unique $y \in B$ s.t.

$$\underbrace{y}_{\text{dependent variable}} = f(\underbrace{x}_{\text{Independent variable}})$$

Differential Equation :→ An equation containing dependent variables, Independent variables and the derivatives of dependent variables w.r.t Independent variables.

If $y = f(x)$ | If $y = f(x,t)$

Then $\frac{dy}{dx} + \sin x = y$ | $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 2\sin t$

$\frac{dy}{dx} = y - x$ | $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = x + t$

Ordinary Diff. Equⁿ (ODE) | Partial Diff. Equⁿ (PDE)

Order of ODE :→ The highest order derivative in the differential equation is called the order of differential equation.

Degree of ODE :→ The highest power of the highest derivative in the differential Equⁿ provided all the derivatives are in natural power.

Eg. $y' + \sqrt{y} = \sin x$ order - 1, degree - 1

$y'' + \sqrt{y'} = x$ order - 2

$\Rightarrow y'' - x = -\sqrt{y'}$

$\Rightarrow (y'' - x)^2 = (-\sqrt{y'})^2$

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$\Rightarrow (y'')^2 + x^2 - 2xy'' = y'$

$\Rightarrow$ Order $= 2$, degree $= 2$

$$- \begin{cases} y'' + e^{y'} = x & \text{order} - 2 \\ y'' + \left(1 + y' + \frac{(y')^2}{\lfloor 2} + \ldots\right) = x \end{cases}$$

$\downarrow$ These two are not same as only finitely many terms should be there in an equation.

$\Rightarrow$ degree of this ODE is not deffined.

$y' + |y| = x$ order $= 1$, degree $= 1$

$|y'| + y = x$ Order $= 1$, degree not defined.
($\because$ Power of y' is not natural power)

Ques $(y'')^{3/2} = (y'')^{2/3}$ Find degree and order.

Sol^n $\Rightarrow (y'')^9 = (y'')^4$ order $= 2$, degree $= 9$

If we do $\dfrac{(y'')^9}{(y'')^4} = 1$

$\Rightarrow (y'')^5 = 1 \Rightarrow$ degree $= 5$ $\rightarrow$ This is wrong.

Linear and Non-linear ODE:$\rightarrow$

The differential equation $f(x, y, y', y'', \ldots, y^{(n)}) = 0$ is called linear ODE if

(1) all the derivatives and dependent variables are of degree 1.

(2) There does not exist any ~~product~~ term which contains product of dependent variables and derivatives or two derivatives.

E.g. $y \cdot y'$ or $y \cdot y''$ or $y' \cdot y''$ should not be present.

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Ques Which of following is/are linear Diff. Equ^n

(a) $y'' + \sin y = x$

(b) $y'' + y \cdot y' = x$

✓(c) $y'' + xy' = \sin x$

(d) $y'' + xy = y' \cdot y$

→ An ODE is linear diff equ^n $\Rightarrow$ deg of this diff Equ^n = 1

But Converse need not to be true.

i.e. If deg of any diff Equ^n = 1 $\not\Rightarrow$ It is linear diff Equ^n.

E.g. $y \cdot y' = x$. Order = 1, degree = 1

↳ But it is Non-Linear.

→ Solution of diff Equ^n $f(x, y, y', y'', \ldots, y^{(n)}) = 0$

The function $\phi(x)$ is called solution of $f = 0$ defined on domain D If

(1) $\phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)$ Exists $\forall\, x \in D$

(2) ϕ satisfies the diff. Equ^n $f = 0$

i.e. $f(x, \phi, \phi', \phi'', \ldots, \phi^{(n)}) = 0$.

First order first degree ODE:→

Any of these two forms.

$$\left[\begin{array}{l} \dfrac{dy}{dx} = f(x,y) \quad -(1) \\[2ex] M(x,y)\,dx + N(x,y)\,dy = 0 \quad -(2) \end{array}\right]$$

Formation of differential Eqn →

Ex $y = A\cos 2x + B\sin 2x$

$y' = -A(\sin 2x)(2) + B\cos 2x(2)$

$= -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x$

$= -4y$

$\Rightarrow y'' + 4y = 0.$ → Second order, Linear Diff Eqn.

Ex $y = cx + \frac{1}{c}$; $c \neq 0$

$y' = c$

$\Rightarrow y = y'(x) + \frac{1}{y'}$

$\Rightarrow yy' = x(y')^2 + 1$

$\Rightarrow x(y')^2 - yy' + 1 = 0$ — First order, Second degree, Non-Linear.

H.W (1) $y = \frac{a}{x^2} + bx$; a, b arbitrary Constants

(2) $y = ce^{qx}$; q: fixed constant

(3) $y = C\cos(Pt - a)$: P: fixed Constant.

Find order and degree of following ODE's.

(4) $[1 + (y')^2]^{1/2} = x^2 + y$

(5) $y' = \sin y$

(6) $(1 + y')^{1/2} = y''$.

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Separable ODEs:→

$$y' = f(x,y)$$

If $f(x,y) = g(x)$ – function of one variable x alone

then $\int \frac{dy}{dx} = \int g(x) + c$

$$\Rightarrow y = F(x) + c$$

If $f(x,y) = g(x) \cdot h(y)$ – Separable form

Then $\frac{dy}{dx} = g(x)h(y)$

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$$

$$\Rightarrow A(y) = B(x) + c.$$

Que $y' - 2y + a = 0$, a is fixed Constant. Find gen sol^n

Sol^n $\frac{dy}{dx} = 2y - a \Rightarrow \frac{dy}{2y-a} = dx$, $y \neq \frac{a}{2}$ — (1)

For $y = \frac{a}{2}$; $\frac{dy}{dx} = 0$ and the given diff Eq^n is satisfied

$\Rightarrow y = \frac{a}{2}$ is a solution.

But It is not a general solution as this doesnot Contain any arbitrary Constant.

So Integrating (1), we get

$$\int \frac{dy}{2y-a} = \int dx + c$$

$$\Rightarrow \frac{1}{2}\log|2y-a| = x + c$$

$$\Rightarrow \log|2y-a| = 2x + 2c$$

$$\Rightarrow |2y-a| = e^{2x} \cdot e^{2c},$$

$$\Rightarrow 2y - a = k \cdot e^{2x} \text{ where } k = \pm e^{2c}.$$

or $y = Ce^{2x} + \frac{a}{2}$; where $c = \frac{k}{2}$.

Que $y' + xy = x$

Sol^n $\frac{dy}{dx} = x - xy$

$\Rightarrow \frac{dy}{dx} = x(1-y)$

$\Rightarrow \int \frac{dy}{1-y} = \int x\,dx$; $\cancel{y \neq 1}$

$\Rightarrow -\log|1-y| = x + c$

$\Rightarrow \log|1-y| = -x - c$

$\Rightarrow |1-y| = e^{-x-c}$

$\Rightarrow |1-y| = e^{-x} \cdot e^{-c}$

$\Rightarrow |1-y| = ke^{-x}$, $\cancel{k = e^{-c}}$

$\Rightarrow 1-y = ke^{-x}$ $(k = \pm e^{-c})$

$\Rightarrow$

~~Que y log y dy/dx =~~

Que $x\frac{dy}{dx} = y\log y$

$\int \frac{dy}{y\log y} = \int \frac{dx}{x} + c$

Put $\log y = t$

$\frac{1}{y}dy = dt$

$\Rightarrow \int \frac{dt}{t} = \int \frac{dx}{x} + c$

$\Rightarrow \log t = \log x + \log c$

$\Rightarrow \underline{t = cx}$.

Que Using $xy = v$, reduce $xy' = e^{-xy} - y$ into Separable

Sol^n

$$xy = v$$
$$xy' + y = \frac{dv}{dx}$$

$$\Rightarrow xy' = e^{-xy} - y$$
$$\Rightarrow xy' + y = e^{-xy}$$
$$\Rightarrow \frac{dv}{dx} = e^{-v}$$

$$\Rightarrow \int \frac{dv}{e^{-v}} = \int dx$$

$$\Rightarrow e^{v} = x + c$$
$$\Rightarrow e^{xy} = x + c \quad \text{or} \quad xy = \log|x+c|$$

HW (1) $xy' = y + x^2 \tan\left(\frac{y}{x}\right)$; $\frac{y}{x} = t$

(2) $(xy' - y) \cos\left(\frac{y}{x}\right) + x = 0$; $\frac{y}{x} = t$.

→ Equations Reducible to Separable form:→

$$\frac{dy}{dx} = f(ax + by + c)$$

Put $ax + by + c = z$

$$a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$$\Rightarrow \frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

$$\Rightarrow \frac{dz}{dx} = a + bf(z)$$

$$\Rightarrow \int \frac{dz}{a + bf(z)} = \int dx + c$$

Ex

$$y' = (2y-y^2)^{1/2}$$

$$= (y(2-y))^{1/2}$$

$$y' = \sqrt{y}\sqrt{2-y}$$

$$\Rightarrow dy$$

Ex

$$y' = (4x+y)^2$$

Put $4x+y = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} - 4 = t^2 \Rightarrow \frac{dt}{dx} = t^2+4$$

$$\Rightarrow \int \frac{dt}{t^2+4} = \int dx + c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x+c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y}{2}\right) = x+c$$

$$\Rightarrow \tan^{-1}\left(\frac{4x+y}{2}\right) = 2(c+x)$$

H.W. Ex

$$\frac{dy}{dx} = (2x-y+1)^2$$

Que

$$y' = \sqrt{2y-y^2}$$

Put $y-1 = t$

$$\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} = \sqrt{2(t+1) - (t+1)^2}$$

$$= \sqrt{2t + 2 - t^2 - 1 - 2t}$$

$$= \sqrt{1-t^2}$$

$$\int \frac{dt}{\sqrt{1-t^2}} = \int dx + c$$

$$\sin^{-1} t = x + c$$

$$\sin^{-1}(y-1) = x + c$$

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Homogeneous function:⇒

A function $f(x,y)$ is said to be homogeneous function of deg. n if by substituting $x=\lambda x$, $y=\lambda y$ Produces

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Here n Can be an Integer or any real number.

E.g. $f(x,y) = x^2+y^2$

$f(\lambda x,\lambda y) = \lambda^2x^2+\lambda^2y^2 = \lambda^2(x^2+y^2) = \lambda^2 f(x,y)$

$\Rightarrow$ homogeneous fun of deg 2.

E.g. $f(x,y) = \dfrac{x^2+y^2+xy}{x^2-y^2}$, $x \neq y$

$f(\lambda x, \lambda y) = \dfrac{\lambda^2x^2+\lambda^2y^2+\lambda^2xy}{\lambda^2x^2-\lambda^2y^2} = \lambda^0 f(x,y)$

$\Rightarrow$ hom. function of degree 0.

E.g. $f(x,y) = x^3 \log\left[\dfrac{\sqrt{x+y}}{\sqrt{x-y}}\right]$

$f(\lambda x,\lambda y) = \lambda^3x^3\left[\log\left(\dfrac{\sqrt{\lambda x+\lambda y}}{\sqrt{\lambda x-\lambda y}}\right)\right] = \lambda^3 x^3 \log\left[\sqrt{\dfrac{x+y}{x-y}}\right]$

$= \lambda^3 f(x,y)$

$\Rightarrow$ hom. function of degree 3.

Homogeneous first order differential equation:⇒

$y' = f(x,y)$ is called a hom. equation if $f(x,y)$ is a hom. function of degree 0.

E.g. $y' = \dfrac{x^3+y^3+x^2y}{x^3+y^3}$ is a homogeneous equation.

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Method to solve hom. Equation:→

Put $y = vx$ or $x = vy$
and Reduce the given ODE to a separable form.

Ex $(x^2+4y^2+xy)\,dx - x^2\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+4y^2+xy}{x^2}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2+4x^2v^2+x^2v}{x^2}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = 1+4v^2+\cancel{v}$$

$$\Rightarrow \int \frac{dv}{1+4v^2} = \int \frac{dx}{x} + c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(2v) = \log|x| + \log c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{2y}{x}\right) = \log|cx|$$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{x}\right) = 2\log|cx|$$

$$\Rightarrow 2y = x\tan[2\log|cx|].$$

Ex $xy' = xe^{-y/x} + y$

$$y' = e^{-y/x} + \frac{y}{x}$$

Put $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \cancel{v} + x\frac{dv}{dx} = e^{-v} + \cancel{v} \qquad \Rightarrow \int \frac{dv}{e^{-v}} = \int \frac{dx}{x} + \log c$$

$\Rightarrow e^v = \log x + \log c$

$\Rightarrow e^v = \log|cx|$

$\Rightarrow e^{y/x} = \log|cx|$

<u>Ex</u> $(x+y)(xy'-y)y = x^3.$

$\Rightarrow (x^2y' - xy + xyy' - y^2)\,y = x^3$

$\Rightarrow x^2yy' - xy^2 + xy^2y' - y^3 = x^3$

$\Rightarrow y'(x^2y + xy^2) = x^3 + y^3 + xy^2$

$$\Rightarrow y' = \frac{x^3+y^3+xy^2}{x^2y+xy^2}$$

Put $y = vx$

$$\text{So } v + x\frac{dv}{dx} = \frac{x^3+v^3x^3+v^2x^3}{vx^3+v^2x^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^3+v^2}{v+v^2} - v$$

$$= \frac{1+\cancel{v^3}+\cancel{v^2}-\cancel{v^2}-\cancel{v^3}}{v+v^2}$$

$$= \frac{1}{v(v+1)}$$

$$\Rightarrow \int (v+v^2)\,dv = \int \frac{dx}{x} + \log|c|$$

$$\Rightarrow \left(\frac{v^2}{2} + \frac{v^3}{3}\right) = \log|cx|$$

$$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 = \log|cx|$$

$$\Rightarrow 3xy^2 + 2y^3 = 6x^3\log|cx| \quad \text{Ans.}$$

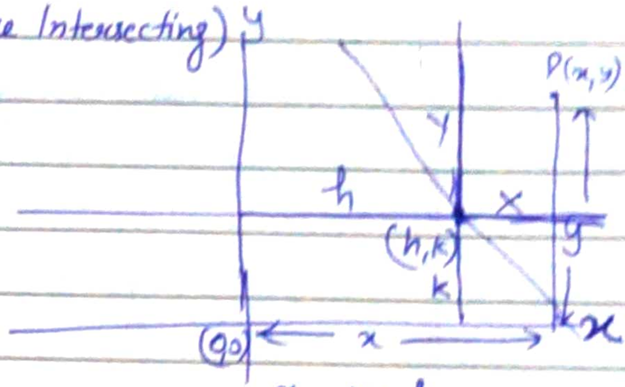
<u>HW</u> (1) $xy' = y + x\sec\left(\frac{y}{x}\right)$

(2) $(2xy + x^2)\,y' = 3y^2 + 2xy.$

Equⁿ Reducible to hom. Equation:→

(1) $$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

If $\frac{a}{a'} \neq \frac{b}{b'}$ (⇒ lines are Intersecting)



$x = X + h$

$y = Y + k$

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

$$= \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

Find h, k s.t. $ah+bk+c=0$ & $a'h+b'k+c'=0$ ($\because$ (h, k) is the Intersecting points of these two lines)

$$\therefore \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} \rightarrow \text{Reduced Into hom. form.}$$

(2) If $\frac{a}{a'} = \frac{b}{b'} = \lambda$ (say) (When lines are Parallel)

So $$\frac{dy}{dx} = \frac{(a'x+b'y)\lambda+c}{a'x+b'y+c'}$$

Put $a'x+b'y = u$

$$a' + b'\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \frac{1}{b'}\left(\frac{du}{dx} - a'\right) = \frac{u\lambda+c}{u+c'} \rightarrow \text{Reduced Into Separable form.}$$

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Que $\frac{dy}{dx} = \frac{x+3y-2}{2x+6y+1}$

$\frac{1}{2} = \frac{3}{6}$

$\therefore$ Put $x+3y = V$

$1+3\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{1}{3}\left(\frac{dv}{dx}-1\right) = \frac{V-2}{2V+1}$

$\Rightarrow \frac{dv}{dx} = \frac{3V-6}{2V+1} + 1$

$= \frac{3V-6+2V+1}{2V+1} = \frac{5V-5}{2V+1}$

$\Rightarrow \int \frac{2V+1}{5(V-1)} dv = \int dx + c$

$\Rightarrow \frac{2}{5}\int \frac{Vdv}{V-1} + \frac{1}{5}\int \frac{dv}{V-1} = x + c$

$\Rightarrow \frac{2}{5}V + \frac{2}{5}\log|V-1| = x+c$

$\Rightarrow 2(x+3y) + 2\log|x+3y-1| = 5x+c$

$\Rightarrow 6y + 2\log|x+3y-1| = 3x+c$

$\Rightarrow$

Que $\frac{dy}{dx} = \frac{y+3}{x+y+2}$

Put $x = X+h$, $y = Y+k$

$\frac{dY}{dX} = \frac{Y+k+3}{X+h+Y+k+2}$

So $k+3=0$, $h+k+2=0$

$K=-3$; $h=1$

$\frac{dY}{dX} = \frac{Y}{X+Y}$

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Put $Y = Vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$V + x\frac{dv}{dx} = \frac{vx}{\cancel{x} + vx}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v} - v$$

$$= \frac{\cancel{v} - \cancel{v} - v^2}{1+v}$$

$$= \frac{-v^2}{1+v}$$

$$\Rightarrow \int \frac{1+v}{v^2} dv = -\int \frac{dx}{x} + \log c$$

$$\Rightarrow \int \frac{1}{v^2} dv + \int \frac{1}{v} dv = -\log|x| + \log c$$

$$\Rightarrow \frac{v^{-2+1}}{-2+1} + \log|v| + \log|x| = \log c$$

$$\Rightarrow -\frac{1}{v} + \log|vx| = \log c$$

$$\Rightarrow \frac{-x}{y} + \log|y| = \log c$$

$$\Rightarrow -\frac{x-1}{y+3} + \log(y+3) = \log c$$

H.W. (1) $(2x+y-1)\,dy + (4x+2y-3)\,dx = 0$

(2) $(y-x+1)\,dy - (y-x+2)\,dx = 0$

(3) $x^2 dy - xy\,dx + y^2 e^{x^2/y^2} dy = 0.$

Exact Differential Equation:-

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a differentiable function.

Then $df(x,y) = 0$ is called an exact differential equation. And its general solution is $f(x,y) = c$ where c is an arbitrary constant.

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy \quad -(1)$$

Let a first order first degree diff Equ^n as

$$M(x,y)dx + N(x,y)dy = 0 \quad -(2)$$

Comparing (1) & (2), we get

$$M(x,y) = \frac{\partial f}{\partial x} \; ; \; N(x,y) = \frac{\partial f}{\partial y}.$$

We assume that M and N have continuous partial derivatives exist

$$\text{So } \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Now If second order partial derivatives are continuous then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence an Equ^n $M(x,y)dx + N(x,y)dy$ is an exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\Rightarrow$ There exist a function $f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = M(x,y) \quad -(A) \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x,y) \quad -(B)$$

$f(x,y)$ can be determined from (A) & (B).

Integrate (A) w.r.t. x, we get

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$\Rightarrow$ ~~Put~~ $f(x,y) = K(x,y) + g(y)$ where $K(x,y) = \int M(x,y)\,dx$.

Put $f(x,y)$ in (B), we get

$$\frac{\partial f}{\partial y} = \frac{\partial K}{\partial y} + g'(y) = N(x,y)$$

$$\Rightarrow \quad g'(y) = N(x,y) - \frac{\partial K}{\partial y}$$

$$\Rightarrow \quad g(y) = \int\left(N(x,y) - \frac{\partial K}{\partial y}\right)dy + C$$

So $\boxed{f(x,y) = K(x,y) + \int\left(N(x,y) - \frac{\partial K}{\partial y}\right)dy + C}$

Que $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$.

Soln $M(x,y) = 3x^2 + 2e^y$; $N(x,y) = 2xe^y + 3y^2$

$$\frac{\partial M}{\partial y} = 2e^y \; ; \; \frac{\partial N}{\partial x} = 2e^y$$

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the given Eqn is Exact.

$\Rightarrow$ There Exist $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

or $\frac{\partial f}{\partial x} = 3x^2 + 2e^y$ —(A) ; $\frac{\partial f}{\partial y} = 2xe^y + 3y^2$ —(B)

from (A); $f(x,y) = \int(3x^2 + 2e^y)\,dx + g(y)$

$$f(x,y) = x^3 + 2xe^y + g(y)$$

Put in (B); $\frac{\partial f}{\partial y} = \cancel{2xe^y} + g'(y) = \cancel{2xe^y} + 3y^2$

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$$g'(y) = 3y^2$$

$$\Rightarrow \int g(y)dy = \int 3y^2 dy + c$$

$$\Rightarrow g(y) = y^3 + c$$

$$\therefore f(x,y) = x^3 + 2xe^y + y^3 + c.$$

Que $e^x(\cos y\, dx - \sin y\, dy) = 0. \quad -(1)$

Sol^n

$$e^x \cos y\, dx - e^x \sin y\, dy = 0$$

$$M = e^x \cos y; \quad N = -e^x \sin y.$$

$$\frac{\partial M}{\partial y} = -e^x \sin y; \quad \frac{\partial N}{\partial x} = -e^x \sin y$$

$\Rightarrow$ (1) is Exact Diff. Eqn.

$$\Rightarrow \exists f(x,y) \text{ s.t. } \frac{\partial f}{\partial x} = e^x \cos y \ -(A) \quad \text{and} \quad \frac{\partial f}{\partial y} = -e^x \sin y \ -(B)$$

from (A); $f(x,y) = \int e^x \cos y\, dx + g(y)$

$$\Rightarrow f(x,y) = e^x \cos y + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -e^x \sin y + g'(y)$$

Put in (B), we get

$$-e^x \cancel{\sin y} + g'(y) = -e^x \cancel{\sin y}$$

$$\Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c$$

$$\boxed{\therefore f(x,y) = e^x \cos y + c} = c_1$$

$$\Rightarrow \boxed{e^x \cos y = c_2}$$

Que $(y+x^3)dx + (ax+by^3)dy = 0$

Sol^n → $M = y+x^3$; $N = ax+by^3$

$\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = a$

⇒ The Equ^n is Exact for $a=1$ no matters what b we choose

∴ $(y+x^3)dx + (x+by^3)dy = 0$ is Exact

⇒ $\exists f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = y+x^3 \quad -(A) \; ; \quad \frac{\partial f}{\partial y} = x+by^3. \quad -(B)$$

$$f(x,y) = \int (y+x^3)dx + g(y)$$

$$\Rightarrow f(x,y) = xy + \frac{x^4}{4} + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = x + g'(y)$$

Put in (B), we get $\cancel{x} + g'(y) = \cancel{x} + by^3$

$$\Rightarrow g'(y) = by^3$$

$$\Rightarrow g(y) = \int by^3 dy + c$$

$$\Rightarrow g(y) = \frac{by^4}{4} + c$$

$$\therefore f(x,y) = xy + \frac{x^4}{4} + \frac{by^4}{4} + c.$$

Direct formula: →

$Mdx + Ndy = 0$ is Exact diff Equ^n if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The Sol^n $f(x,y) = c$ is given by

$$\int_{y \text{ Constant}} M dx + \int (\text{terms in } N \text{ not containing } x)\, dy = c.$$

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Ex $(y+x^3)dx + (x+by^3)dy = 0$.

Sol^n $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Eq^n is Exact

Solution is given by

$$\int_{y\,Const} M\,dx + \int(\text{terms in } N \text{ not Containing } x)\,dy = C$$

$$\int_{y\text{-Constant}} (y+x^3)\,dx + \int_{x\text{-Const}} (by^3)\,dy = C$$

$$\Rightarrow xy + \frac{x^4}{4} + \frac{by^4}{4} = C.$$

Que $e^x \cos y\,dx - e^x \sin y\,dy = 0$

Sol^n Sol^n is given by

$$\int_{y\,const} M\,dx + \int(\text{terms in } N \text{ not Containing } x)\,dy = C.$$

$$\Rightarrow \int_{y\text{-Const}} (e^x \cos y)\,dy = C$$

$$\Rightarrow \boxed{e^x \cos y = C}$$

H.W (1) $(1+e^x)dx + y\,dy = 0$

(2) $(x e^{xy} + 2y)\,dy + y e^{xy}\,dx = 0$

(3) $2xy\,dx + (x^2+1)\,dy = 0$

(4) $(1+x^2)\,dy + 2xy\,dx = 0$

(5) $x\,dx + y\,dy = 2y\,(x^2+y^2)\,dy$.

Under what Conditions, the following diff Equ^n is Exact.

(6) $xy^3\,dx + ax^2y^2\,dy = 0$

(7) $(ax+y)dx + (Kx+by)dy = 0$.

Integrating factor:→

If $M(x,y)dx + N(x,y)dy = 0$ —(1) is a non-exact equation then sometimes Equⁿ (1) can be made exact by multiplying it with some function $\phi(x,y)$. This $\phi(x,y)$ is called an Integrating factor (I.F.).

i.e. If $M(x,y)dx + N(x,y)dy = 0$ is non-exact and if we multiply it with $\phi(x,y)$ s.t.

$$\phi(x,y)\,M(x,y)dx + \phi(x,y)\,N(x,y)dy = 0$$

is an exact diff Equⁿ Then $\phi(x,y)$ is called an Integrating factor (I.F.)

→ Integrating factor for a first order first degree differential Equation may or may not exist.

→ Consider the first order first degree differential Equⁿ

$$x\,dy - y\,dx = 0. \quad —(1)$$ is non-Exact

Choose $\phi(x,y) = \frac{1}{x^2}$ s.t.

$$\frac{1}{x^2}(x\,dy - y\,dx) = 0 \Rightarrow \frac{1}{x}dy - \frac{y}{x^2}dx = 0$$ is an exact diff Equⁿ.

$\because M = \frac{-y}{x^2}$, $N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2} \text{ and } \frac{\partial N}{\partial x} = \frac{-1}{x^2} \Rightarrow \text{Equ}^n \text{ is Exact}$$

$\therefore \phi(x,y) = \frac{1}{x^2}$ is an I.F for Equⁿ (1) $\forall x \neq 0$.

Choose $\phi(x,y) = \frac{1}{y^2}$ s.t. $\frac{1}{y^2}(x\,dy - y\,dx) = 0 \Rightarrow \frac{x}{y^2}dy - \frac{1}{y}dx = 0$

$M = \frac{-1}{y}$, $N = \frac{x}{y^2}$

$\partial M/\partial y = 1/y^2$ $\quad \partial N/\partial x = 1/y^2 \Rightarrow$ Equⁿ is Exact

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$\Rightarrow \phi(x,y) = \frac{1}{y^2}$ is an I.F. of Equ^n(1) $\forall\ y \neq 0$.

Choose $\phi(x,y) = \frac{1}{xy}$ St.

$$\frac{1}{xy}(x\,dy - y\,dx) = 0 \Rightarrow \frac{1}{y}dy - \frac{1}{x}dx = 0$$

$$M = \frac{-1}{x},\ N = \frac{1}{y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0 \Rightarrow \text{Equ}^n \text{ is Exact}$$

$\Rightarrow \phi(x,y) = \frac{1}{xy}$ is an I.F. of Equ^n (1) $\forall\ x \neq 0, y \neq 0$.

Choose $\phi(x,y) = \frac{1}{x^2+y^2}$ St.

$$\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 0$$

$$M = \frac{-y}{x^2+y^2};\ N = \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\Rightarrow$ Equ^n is Exact

So $\phi(x,y) = \frac{1}{x^2+y^2}$ is an I.F. of Equ^n(1) $\forall\ x \neq 0, y \neq 0$.

$\Rightarrow$ Let $Mdx + Ndy = 0$ —(1) is a non-exact differential Equation. Then ϕ is an I.F of (1) Such that ϕ is function of v only where v is function of x and y.

$$\Rightarrow \phi(v)\,[Mdx + Ndy] = 0 \quad \text{is Exact diff Eqn}$$

$$\text{or } (\phi(v)M)\,dx + (\phi(v)N)\,dy = 0 \quad \text{is Exact Eqn}$$

$$\Rightarrow \frac{\partial[\phi(v)M]}{\partial y} = \frac{\partial}{\partial x}[\phi(v)N]$$

$$\Rightarrow \frac{\partial}{\partial y}(\phi(v))M + \phi(v)\frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(\phi(v))N + \phi(v)\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{d\phi}{dv}\cdot\frac{\partial v}{\partial y}\cdot M + \phi(v)\frac{\partial M}{\partial y} = \frac{d\phi}{dv}\cdot\frac{\partial v}{\partial x}\cdot N + \phi(v)\cdot\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{d\phi}{dv}\left[\frac{\partial v}{\partial y}\cdot M - \frac{\partial v}{\partial x}\cdot N\right] + \phi(v)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = 0$$

$$\Rightarrow \phi'(v)\,[M\cdot v_y - Nv_x] = \phi(v)\,[N_x - M_y]$$

$$\Rightarrow \frac{\phi'(v)}{\phi(v)} = \frac{M_y - N_x}{Nv_x - Mv_y}$$

$$\Rightarrow \frac{M_y - N_x}{Nv_x - Mv_y} = \boxed{\frac{\phi'(v)}{\phi(v)} = f(v)}$$

$$\Rightarrow \int \frac{\phi'(v)}{\phi(v)}\,dv = \int f(v)\,dv$$

$$\Rightarrow \log\phi(v) = \int f(v)\,dv + c$$

$$\Rightarrow \phi(v) = \alpha\cdot e^{\int f(v)dv} \qquad \boxed{\alpha = e^c}$$

$$\text{or } \boxed{\phi(v) = e^{\int f(v)dv}} \text{ is an I.F. of (1)}$$

Hence we have Shown that For $Mdx + Ndy = 0$ —(1) — a non Exact diff Equation, Then there Exist an Integrating factor ϕ which is function of v only Iff $\dfrac{M_y - N_x}{Nv_x - Mv_y}$ is fun of v only and $\phi(v) = e^{\int f(v)dv}$ is an I.F of (1) where $f(v) = \dfrac{M_y - N_x}{Nv_x - Mv_y}$.

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Ex Solve $(5x^3+12x^2+6y^2)\,dx + 6xy\,dy = 0$ —(1)

Sol^n $M = 5x^3+12x^2+6y^2$; $N = 6xy$.

$\frac{\partial M}{\partial y} = 12y$; $\frac{\partial N}{\partial x} = 6xy$

$\Rightarrow$ Equ^n (1) is not Exact

So let $V = \cancel{\#}x$

$V_x = 1,\ V_y = 0$

Then $\frac{M_y - N_x}{NV_x - MV_y} = \frac{12y-6xy}{6xy(1)-0} = \frac{6y}{6xy} = \frac{1}{x}$ is fun of x only

$\Rightarrow \phi(x) = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$ is an I.F of (1)

$\therefore\ x(5x^3+12x^2+6y^2)\,dx + 6x^2y\,dy = 0$

$\frac{\partial M}{\partial y} = \cancel{6}\ 12xy$; $\frac{\partial N}{\partial x} = 12xy$

$\Rightarrow$ Equ^n becomes Exact.

So Sol^n is $\int_{y\text{-const}} M\,dx + \int(\text{Terms in } N \text{ not Containing } x)\,dy = c$

$\Rightarrow \int (5x^4+12x^3+6xy^2)\,dx + 0 = c$

$\Rightarrow \frac{5x^5}{5} + \frac{12x^4}{4} + 6y^2\left(\frac{x^2}{2}\right) = c$

$\Rightarrow \boxed{x^5 + 3x^4 + 3x^2y^2 = c}$

Ex $\rightarrow$ $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$ —(1)

Sol^n $M = 3x^2y^3e^y + y^3 + y^2$; $N = x^3y^3e^y - xy$

$\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$

$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$

$\Rightarrow$ Equ^n (1) is not Exact.

Let $v = y$

$v_x = 0, \quad v_y = 1$

So $\dfrac{M_y - N_x}{Nv_x - Mv_y} = \dfrac{9x^2y^2e^y + 3y^2 + 2y + y}{0 - (3x^2y^3e^y + y^3 + y^2)(1)}$

$= \dfrac{9x^2y^2e^y + 3y^2 + 3y}{-(3x^2y^3e^y + y^3 + y^2)}$

$= \dfrac{3\,\cancel{(3x^2y^2e^y + y^2 + y)}}{-y\,\cancel{(3x^2y^2e^y + y^2 + y)}} = -\dfrac{3}{y} = f(y)$

$\therefore \quad \phi(y) = e^{\int -3/y\,dy} = e^{-3\log y} = y^{-3} = \dfrac{1}{y^3}$ is an I.F. of (1)

So $\dfrac{1}{y^3}\left[3x^2y^3e^y + y^3 + y^2\right]dx + \dfrac{1}{y^3}\left[x^3y^3e^y - xy\right]dy = 0$

$\Rightarrow \left(3x^2e^y + 1 + \dfrac{1}{y}\right)dx + \left(x^3e^y - \dfrac{x}{y^2}\right)dy = 0$

$\dfrac{\partial M}{\partial y} = 3x^2e^y + x^2\left[\dfrac{y^3(0) - 1(3y^2)}{y^6}\right]$

$= 3x^2e^y - \dfrac{3x^2}{y^4}$

$\dfrac{\partial N}{\partial x} = 3x^2e^y - 1$

$\dfrac{\partial M}{\partial y} = 3x^2e^y - \dfrac{1}{y^2}$

$\dfrac{\partial N}{\partial x} = 3x^2e^y - \dfrac{1}{y^2}$ $\Rightarrow$ Equ^n Reduces to Exact diff Equ^n

Sol^n is $\displaystyle\int_{y\text{-const}} M\,dx + \int(\text{terms in N not containing } x)\,dy = c$

$\Rightarrow \displaystyle\int\left(3x^2e^y + 1 + \frac{1}{y}\right)dx + \int 0\,dy = c$

$\Rightarrow \boxed{x^3e^y + x + \dfrac{x}{y} = c}$.

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Que $\left[y+x f(x^2+y^2)\right]dx + \left[y f(x^2+y^2) - x\right]dy = 0$ —(1)

$M = y + x f(x^2+y^2)$; $N = y f(x^2+y^2) - x$

$$\frac{\partial M}{\partial y} = 1 + x f'(x^2+y^2)(2y)$$

$$= 1 + 2xy f'(x^2+y^2)$$

$$\frac{\partial N}{\partial x} = y f'(x^2+y^2)(2x) - 1 = 2xy f'(x^2+y^2) - 1.$$

$\Rightarrow$ Eqn (1) is not Exact.

$$M_y - N_x = 1 + \cancel{2xy f'(x^2+y^2)} - \cancel{2xy f'(x^2+y^2)} + 1$$
$$= 2$$

S.T. $\frac{1}{x^2+y^2}$ is an I.F.

$$\frac{1}{x^2+y^2}\left[y + x f(x^2+y^2)\right]dx + \frac{1}{x^2+y^2}\left[y f(x^2+y^2) - x\right]dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)\left[1 + 2xy f'(x^2+y^2)\right] - \left[y + x f(x^2+y^2)\right]\left[2y\right]}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2 + 2xy(x^2+y^2) f'(x^2+y^2) - 2y^2 - 2xy f(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2 + 2xy(x^2+y^2) f'(x^2+y^2) - 2xy f(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)\left[y f'(x^2+y^2)(2x) - 1\right] - (y f(x^2+y^2) - x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2xy(x^2+y^2) f'(x^2+y^2) - x^2 - y^2 - 2xy f(x^2+y^2) + 2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2 + 2xy(x^2+y^2) f'(x^2+y^2) - 2xy f(x^2+y^2)}{(x^2+y^2)^2}$$

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Que $(x+y+1)\,dx + (2x+2y+1)\,dy = 0$

Sol^n $\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = 2$

Not Exact diff Equ^n.

let $V = x+y$

$V_x = 1$, $V_y = +1$

$$\frac{M_y - N_x}{NV_x - MV_y} = \frac{1-2}{(2x+2y+1)-(x+y+1)}$$

$$= \frac{-1}{x+y}$$

$$\phi(v) = e^{\int \frac{-1}{x+y} d(x+y)} = e^{-\log|x+y|} = \frac{1}{x+y}$$

$$\frac{1}{x+y}\left[(x+y+1)\,dx\right] + \left(\frac{2x+2y-1}{x+y}\right) dy = 0$$

$M = \frac{x+y+1}{x+y}$; $N = \frac{2(x+y)+1}{x+y}$

$M = 1 + \frac{1}{x+y}$; $N = 2 + \frac{1}{x+y}$

$\frac{\partial M}{\partial y} = \frac{-1}{(x+y)^2}$; $\frac{\partial N}{\partial x} = \frac{-1}{(x+y)^2}$

Sol^n is $\int\limits_{y\ \text{const}} \left(1 + \frac{1}{x+y}\right) dx + \int 2\,dy = c$

$\Rightarrow$

$x + \log|x+y| + 2y = c$

$\Rightarrow$ $\boxed{(x+2y) + \log|x+y| = c.}$

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H.W (1) Solve $(x^2+y^2+x)dx + xy\,dy = 0$ Ans:- $4x^3+3x^4+6x^2y^2 = c$.
$I.F. = x$

(2) $(2xy^4e^y + 2xy^3+y)dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$
$I.F. = 1/y^4$; Sol^n is $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

(3) $(x^4+y^4)\,dx - xy^3\,dy = 0$
Sol^n is $\boxed{y^4 = 4x^4 \log x + cx^4}$

(4) $y^2dx + (x^2-xy-y^2)\,dy = 0$
Sol^n is $\boxed{(x-y)\,y^2 = c(x+y)}$

(5) $y(2xy+1)dx + x(1+2xy-x^3y^3)dy = 0$
Sol^n is $\boxed{y = Ce^{-(3xy+1)/3x^3y^3}}$

(6) $(x^2+y^2)\,dx - 2xy\,dy = 0$.
Sol^n is $\boxed{x^2-y^2 = Cx}$

→ If $M(x,y)\,dx + N(x,y)dy = 0$ is a non-exact differential equation. The functions $M(x,y)$ and $N(x,y)$ are such that they are homogeneous function of degree n and $Mx+Ny \neq 0$ Then $\frac{1}{Mx+Ny}$ is an Integrating factor.

If $Mx+Ny = 0 \Rightarrow \frac{M}{N} = \frac{-y}{x}$

So $M\,dx + N\,dy = 0$

$\Rightarrow \frac{M}{N}dx + dy = 0$

$\Rightarrow \frac{-y}{x}dx + dy = 0$

$\Rightarrow -y\,dx + x\,dy = 0$ —(1)

So $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$ are I.F of (1).

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Ex:→ $(2xy+x^2)\frac{dy}{dx} = 3y^2+2xy$ —(1)

Soln ⇒ $(2xy+x^2)dy - (3y^2+2xy)dx = 0$

$M = -(3y^2+2xy)$; $N = 2xy+x^2$

$\frac{\partial M}{\partial y} = -6y-2x$; $\frac{\partial N}{\partial x} = 2y+2x$

⇒ (1) is not Exact.

M and N are hom. functions of degree 2.

∴ $\frac{1}{Mx+Ny}$ will be an I.F.

i.e. $\frac{1}{(2xy+x^2)y+(-3y^2-2xy)x} = \frac{1}{2xy^2+x^2y-3xy^2-2x^2y}$

$= \frac{1}{-x^2y-xy^2} = \frac{-1}{xy(x+y)}$

Using this I.F (1) becomes

$\frac{1}{xy(x+y)}(3y^2+2xy)dx - \frac{1}{xy(x+y)}(2xy+x^2)dy = 0$

⇒ $\frac{3y+2x}{x(x+y)}dx - \frac{(2y+x)}{y(x+y)}dy = 0.$

Now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (check)

Soln is $\int \frac{3y+2x}{x(x+y)}dx + \int \frac{-1}{y}dy = c$

⇒ $\int \frac{3y\,dx}{x(x+y)} + 2\int \frac{dx}{x+y} \overset{-\log|y|}{=} c$

⇒ $\int \frac{3y\,dx}{x(x+y)} + 2\log|x+y| - \log y = \log c$

$\int \frac{3y\,dx}{x(x+y)} = \frac{A}{x} + \frac{B}{x+y}$

$3y = A(x+y)+Bx$

$x=0$: $3y = Ay ⇒ A=3$

$x=-y$; $3y = B(-y)$

$B=-3$

∴ $\int \frac{3y\,dx}{x(x+y)} = \int \frac{3}{x}dx - \int \frac{3}{x+y}dx$

$= 3\log|x| - 3\log|x+y|$

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$\therefore f(x,y) = 3\log|x| - 3\log|x+y| + 2\log|x+y| - \log y = \log c$

$\Rightarrow \log x^3 - \log(x+y) - \log y = \log c$

$\Rightarrow \log \dfrac{x^3}{y(x+y)} = \log c$

$\Rightarrow \dfrac{x^3}{y(x+y)} = c \Rightarrow \boxed{x^3 = cy(x+y)}$

Ques→ $(2x+y)\,dy - (x+2y)\,dx = 0$ —(1)

Solⁿ $M = -(x+2y)$; $N = 2x+y$

$\dfrac{\partial M}{\partial y} = -2$; $\dfrac{\partial N}{\partial x} = 2 \Rightarrow$ (1) is not Exact.

I.F is $\dfrac{1}{Mx+Ny}$ ($\because$ M & N are hom fun of deg. 1)

$= \dfrac{1}{-(x+2y)x+(2x+y)y} = \dfrac{1}{-x^2 \cancel{-2xy} + \cancel{2xy} + y^2} = \dfrac{1}{y^2-x^2}$

$\dfrac{1}{y^2-x^2}(2x+y)\,dy - \dfrac{1}{y^2-x^2}(x+2y)\,dx = 0$

$M = \dfrac{-1}{y^2-x^2}(x+2y)$; $N = \dfrac{1}{y^2-x^2}(2x+y)$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (check)

$\int \dfrac{x+2y}{x^2-y^2}\,dx + \int 0\,dy = c$

$\Rightarrow \int \dfrac{x+y+y}{(x-y)(x+y)}\,dx = c$

$\Rightarrow \int \dfrac{1}{x-y}\,dx + \int \dfrac{y}{x^2-y^2}\,dx = c$

$\Rightarrow \log|x-y| + \dfrac{y}{2y}\log\left|\dfrac{x-y}{x+y}\right| = \log c$

$\Rightarrow \log|x-y| + \dfrac{1}{2}\log\left|\dfrac{x-y}{x+y}\right| = \log c$

$\Rightarrow \log|x-y| + \log\left|\dfrac{x-y}{x+y}\right|^{1/2} = \log c$

$\Rightarrow \log\left((x-y)\dfrac{(x-y)^{1/2}}{(x+y)^{1/2}}\right) = \log c$

$\Rightarrow (x-y)^{3/2} = c\sqrt{x+y}$

$\Rightarrow \boxed{(x-y)^3 = c(x+y)}$

H.W.

(1) $3x^2y^4 dx + 4x^3y^3 dy = 0$ — Exact

(2) $3y\,dx + 2x\,dy = 0$ $\quad I.F. = \sqrt{x}$

(3) $xy\,dx - (x^2+y^2)dy = 0$ $\quad I.F. = 1/y^3$.

~~(4)~~

Linear Differential Equation:→

$$\frac{dy}{dx} + P(x)y = r(x) \quad —(1)$$

$$\Rightarrow dy + (P(x)y - r(x))dx = 0 \quad —$$

$$M = P(x)y - r(x)\ ;\ N = 1$$

$$\frac{\partial M}{\partial y} = P(x),\ \frac{\partial N}{\partial x} = 0$$

$$\cancel{IF}\ \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{NV_x - MV_y}$$

Chose $V = x$
$V_x = 1$
$V_y = 0$

$$\therefore \frac{M_y - N_x}{N} = \frac{P(x)}{1} = P(x) \quad \text{— fun of } x \text{ only.}$$

$$I.F. = e^{\int P(x)dx}.$$

Multiply Eqn (1) with the I.F., we get

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x).y.e^{\int P(x)dx} = e^{\int P(x)dx}.r(x)$$

$$\Rightarrow \frac{d}{dx}\left[y\,e^{\int P(x)dx}\right] = e^{\int P(x)dx}.r(x)$$

Integrating both sides w.r.t x, we get

$$\boxed{y.\,e^{\int P(x)dx} = \int r(x).\,e^{\int P(x)dx}\,dx + c}$$

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Que $x\frac{dy}{dx} = 2y + x^4 + 6x^2 + 2x, \quad x \neq 0.$

Sol^n $\frac{dy}{dx} - \frac{2y}{x} = x^3 + 6x + 2$

$P(x) = \frac{-2}{x}, \quad Q(x) = x^3 + 6x + 2$

$I.F. = e^{\int -2/x\, dx} = e^{-2\log|x|} = \frac{1}{x^2}$

$y \cdot \frac{1}{x^2} = \int \frac{(x^3 + 6x + 2)}{x^2} dx + c$

$= \int \left(x + \frac{6}{x} + \frac{2}{x^2}\right) dx + c$

$\Rightarrow \frac{y}{x^2} = \frac{x^2}{2} + 6\log|x| + 2\left(\frac{x^{-1}}{-1}\right) + c$

$\Rightarrow y = \frac{x^4}{4} + 6x^2 \log|x| - 2x + cx^2$

Que $(x-a)\frac{dy}{dx} + 3y = 12(x-a)^3$

Sol^n $\frac{dy}{dx} + \left(\frac{3}{x-a}\right)y = 12(x-a)^2$

$I.F. = e^{\int \frac{3}{x-a} dx} = e^{3\log|x-a|} = (x-a)^3$

$y \cdot (x-a)^3 = \int (12)(x-a)^2 \cdot (x-a)^3\, dx + c$

$= 12\frac{(x-a)^6}{6} + c$

$= 2(x-a)^6 + c$

$\Rightarrow y = 2(x-a)^3 + c(x-a)^{-3}.$

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<u>Que</u> $\cot 3x \frac{dy}{dx} - 3y = \cos 3x + \sin 3x \; ; \; 0 < x < \frac{\pi}{2}$

<u>Sol^n</u>
$$\frac{dy}{dx} - \frac{3}{\tan 3x} y = \sin 3x + \frac{\sin 3x}{\cos 3x}$$

$$I.F. = e^{\int -3/\tan 3x \, dx} = e^{-3\int \cot 3x \, dx}$$
$$= e^{\log|\cos 3x|} = \cos 3x.$$

$$y \cdot \cos 3x = \int \left(\sin 3x + \frac{\sin^2 3x}{\cos 3x} \right) \cos 3x \, dx + C$$
$$= \frac{1}{2} \int \sin 6x + (1 - \cos 6x) \, dx + c$$
$$= \frac{1}{2} \left[-\frac{\cos 6x}{6} + x - \frac{\sin 6x}{6} \right] + c$$

$$\Rightarrow \; y \cos 3x = \frac{-1}{12} \cos 6x + \frac{x}{2} - \frac{1}{12} \sin 6x + c$$

<u>H.W</u>
(1) $(1+x^2) y' + 2xy = x \sin x$ Ans:- $y = (c + \sin x - x \cos x) / (1+x^2)$
(2) $y' + 3y = e^{2x} + 6$ Ans:- $y = \frac{1}{5} e^{2x} + 2 + c e^{-3x}$
(3) $xy' + (1+2x) y = 1 + x e^{-2x}$
Ans:- $2x e^{2x} y = x^2 + e^{2x} + c$

<u>I.F. of Separable Equation:</u>→ let $f(x) g(y) dx + F(x) G(y) dy = 0$ —(1) is a non-Exact differential equation.

Then $\frac{1}{g(y) F(x)}$ is an I.F. of (1)

$$\therefore \; \frac{1}{g(y) F(x)} \left[f(x) g(y) dx + F(x) G(y) dy \right] = 0$$

$$\Rightarrow \; \frac{f(x)}{F(x)} dx + \frac{G(y)}{g(y)} dy = 0$$

$\Rightarrow \; \frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ ⇒ Equ^n becomes Exact diff Equ^n.

Determination of $M(x,y)$ and $N(x,y)$ Such that Equation is Exact:→

(1) $(x^3 + xy^2)\,dx + N(x,y)\,dy = 0 \quad —(1)$ is Exact. Find $N(x,y)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\because (1) \text{ is Exact})$$

$$M = x^3 + xy^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2xy$$

$$\therefore \frac{\partial N}{\partial x} = 2xy$$

$$\Rightarrow \int \frac{\partial N}{\partial x}\,dx = \int 2xy\,dx + g(y)$$

$$\Rightarrow N = x^2y + g(y)$$

(2) $(x^{-2}y^{-2} + xy^{-3})\,dx + N(x,y)\,dy = 0$ is Exact. Find $N(x,y)$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = x^{-2}y^{-2} + xy^{-3}$$

$$\frac{\partial M}{\partial y} = x^{-2}(-2y^{-3}) + x(-3)y^{-4}$$

$$= \frac{-2}{x^2y^3} - \frac{3x}{\cancel{x}y^4}$$

$$\therefore \frac{\partial N}{\partial x} = \frac{-2}{x^2y^3} - \frac{3x}{\cancel{x}y^4}$$

$$\text{So } N = \int \frac{-2}{x^2y^3}\,dx - \int \frac{3x\,dx}{\cancel{x}y^4} + g(y)$$

$$\Rightarrow N = \frac{-2}{y^3}\left(\frac{x^{-2+1}}{-2+1}\right) - \frac{3x^2}{2y^4} + g(y)$$

$$\Rightarrow N = \frac{2}{xy^3} - \frac{3x^2}{2y^4} + g(y)$$

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(3) $M(x,y)\,dx + (2x^2y^3 + x^4y)\,dy = 0$. —(1) Find $M(x,y)$

Solⁿ As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ($\because$ (1) is Exact)

$N = 2x^2y^3 + x^4y$

$\frac{\partial N}{\partial x} = 4xy^3 + 4x^3y$

$\therefore \frac{\partial M}{\partial y} = 4xy^3 + 4x^3y$.

$\Rightarrow M = \int (4xy^3 + 4x^3y)\,dy + g(x)$

$\Rightarrow M = xy^4 + 2x^3y^2 + g(x)$

(4) $M(x,y)\,dx + (2ye^x + y^2e^{3x})\,dy = 0$; Find $M(x,y)$.
—(1)

Solⁿ:→ as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ($\because$ (1) is Exact)

$\frac{\partial N}{\partial x} = 2ye^x + 3y^2e^{3x}$.

$\therefore \frac{\partial M}{\partial y} = 2ye^x + 3y^2e^{3x}$

$M = \int (2ye^x + 3y^2e^{3x})\,dy + g(x)$

$\Rightarrow M = y^2e^x + y^3e^{3x} + g(x)$.

Variation of Perameters:→

$$y'' + P(x)\,y' + q(x)\,y = R(x) \quad -(1)$$

Let y_1 and y_2 be two L.I. solutions of $y''+P(x)y'+q(x)y=0$ $-(2)$

Then Complementary fun. of (1) is $y_c(x) = a y_1 + b y_2$.

Let us assume that $y(x) = A(x)\,y_1 + B(x)\,y_2$ is a general solution of (1). $-(3)$

$\because$ y_1 and y_2 are sol^n of (2)

So $y_1'' + P(x)y_1' + q(x)y_1 = 0 \quad -(4)$

and $y_2'' + P(x)y_2' + q(x)y_2 = 0 \quad -(5)$

From (3) $y = Ay_1 + By_2$ where A, B are fun of x.

$\Rightarrow y' = Ay_1' + A'y_1 + By_2' + B'y_2$

Let $A'y_1 + B'y_2 = 0 \quad -(6)$

$\Rightarrow y' = Ay_1' + By_2'$

$\Rightarrow y'' = A'y_1' + Ay_1'' + B'y_2' + By_2''$

Put the values of y, y', y'' in (1), we get

$$(A'y_1' + Ay_1'' + B'y_2' + By_2'') + P(x)\,(Ay_1' + By_2') + q(x)\,(Ay_1 + By_2) = R(x)$$

$$\Rightarrow A(y_1'' + P(x)y_1' + q(x)y_1) + B(y_2'' + P(x)y_2' + q(x)y_2) + A'y_1' + B'y_2' = R(x)$$

Using (4), (5), we get

$$A'y_1' + B'y_2' = R \quad -(7)$$

From (6) and (7) find A' and B'.

$$A'y_1 + B'y_2 = 0$$

$$A'y_1' + B'y_2' - R = 0$$

$$\begin{array}{cccccc} A' & B' & 1 & & & \\ y_2 & 0 & y_1 & y_2 \\ y_2' & -R & y_1' & y_2' \end{array}$$

$$\frac{A'}{-y_2 R} = \frac{B'}{+Ry_1} = \frac{1}{y_1y_2' - y_1'y_2}$$

$$\Rightarrow A' = \frac{-Ry_2}{W(y_1,y_2)} \quad \text{and} \quad B' = \frac{+Ry_1}{W(y_1,y_2)}$$

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$$\Rightarrow A(x) = \int \frac{-R y_2}{W(y_1, y_2)} dx + c_1 \quad ; \quad B(x) = \int \frac{R y_1}{W(y_1, y_2)} dx + c_2$$

So From (3) $y(x) = A(x) y_1 + B(x) y_2$

$$\Rightarrow y(x) = -\int \frac{R y_2 \, dx}{W(y_1, y_2)} \cdot y_1 + c_1 y_1 + \int \frac{R y_1}{W(y_1, y_2)} dx \cdot y_2 + c_2 y_2$$

$$\Rightarrow y(x) = \underbrace{c_1 y_1 + c_2 y_2}_{C.F.} + \underbrace{y_1 \int \frac{-R y_2}{W(y_1, y_2)} dx + \int \frac{R y_1}{W(y_1, y_2)} dx \cdot y_2}_{P.I.}$$

In Short:→ For $y'' + P(x) y' + q(x) y = R(x)$ —(1)

Find its C.F. i.e. $y(x) = c_1 u + c_2 v$.

Let Gen Sol^n of (1) is $y(x) = Au + Bv$

where A, B are fun of x.

where $A = \int \frac{-vR}{W} dx + c_1$; $B = \int \frac{uR}{W} dx + c_2$.

&

$$W = W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v.$$

Que:→ $y'' - y = e^x$ —(1)

Sol^n $(D^2 - 1) y = e^x$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c(x) = c_1 e^x + c_2 e^{-x}$

let Gen Sol^n of (1) is

$y(x) = A e^x + B e^{-x}$

$A = -\int \frac{vR}{W} dx + c_1$ and $B = \int \frac{uR}{W} dx + c_2$

where $u = e^x$, $v = e^{-x}$, $R = e^x$

and $W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x e^{-x}$

$= -2$

So $A = \cancel{-}\int \frac{e^{-x} \cdot e^x}{\cancel{-}2} dx = \frac{1}{2} x + c_1$

$B = \int \frac{e^x \cdot e^x}{-2} dx + c_2 = \frac{-1}{4} e^{2x} + c_2.$

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So $y(x) = \left(\frac{1}{2}x + c_1\right)e^{x} + \left(-\frac{1}{4}e^{2x} + c_2\right)e^{-x}$

$= \frac{1}{2}xe^{x} + c_1e^{x} - \frac{1}{4}e^{x} + c_2e^{-x}$

$= \cancel{c_1e^{x}} \left(c_1 - \frac{1}{4}\right)e^{x} + c_2e^{-x} + \frac{1}{2}xe^{x}$

$= \underbrace{c_3e^{x} + c_2e^{-x}}_{C.F.} + \underbrace{\frac{1}{2}xe^{x}}_{P.I.}$

Que $y'' + y = \sec x$ — (1)

Solⁿ $(D^2+1)y = \sec x$

AE $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c(x) = c_1 \cos x + c_2 \sin x$

let Gen Solⁿ of (1) is $y(x) = A\cos x + B\sin x$.

where $A = \int \frac{-VR}{W}dx + c_1$ & $B = \int \frac{UR}{W}dx + c_2$.

~~$A = \int \sin x \sec x\, dx$~~

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore A = \int -\sin x \cdot \sec x\, dx + c_1$

$= -\int \tan x\, dx + c_1$

$= \log|\cos x| + c_1$

$B = \int \cos x \cdot \sec x\, dx + c_2 = x + c_2$

$\therefore y(x) = (\log|\cos x| + c_1)\cos x + (x + c_2)\sin x$

$= (c_1\cos x + c_2\sin x) + x\sin x + \log|\cos x| \cdot \cos x$

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Qu: $y'' + a^2 y = \operatorname{cosec} ax$ —(1)

Sol^n: $(D^2 + a^2)y = \operatorname{cosec} ax$

A.E: $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

$y_c(x) = C_1 \cos ax + C_2 \sin ax$

let Gen Sol^n of (1) is

$y(x) = A \cos ax + B \sin ax$

when $A = -\int \frac{vR}{W} dx + C_1$ & $B = \int \frac{uR}{W} dx + C_2$.

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$A = -\int \frac{\sin ax \cdot \operatorname{cosec} ax}{a} dx + C_1$$

$$\boxed{A = \frac{-1}{a} x + C_1}$$

$$B = \int \frac{\cos ax \cdot \operatorname{cosec} ax}{a} dx + C_2$$

$$\boxed{B = \frac{1}{a^2} \log|\sin ax| + C_2}$$

$$\therefore \; y(x) = \left(\frac{-x}{a} + C_1\right)\cos ax + \left(\frac{1}{a^2}\log|\sin ax| + C_2\right)\sin ax$$

$$y(x) = C_1 \cos ax + C_2 \sin ax - \frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \log|\sin ax|.$$

H.W (1) $y'' + 4y = 4\tan 2x$

(2) $y'' + 4y = \cot 2x$

(3) $y'' - 2y' + y = x e^x \log x, \; x > 0.$

(4) $y'' + y = \frac{1}{1 + \sin x}$