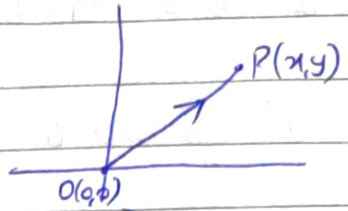
Unit - IV

Vector:→ A quantity that has magnitude as well as direction is called a vector.

Position vector:→ The vector $\overrightarrow{OP}$ having O and P as Initial and terminal points, respectively is called position vector.



$\overrightarrow{OP} = x\hat{i} + y\hat{j}$

Zero Vector:→ A vector whose initial and terminal points coincide is called a zero vector. E.g. $\overrightarrow{AA}$, $\overrightarrow{BB}$ etc. are zero vectors.

Unit vector:→ A vector whose magnitude is unity is called unit vector. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Coinitial vectors:→ Vectors having same initial point are called Coinitial vectors.

Equal vectors:→ Two vectors $\vec{a}$ and $\vec{b}$ are called Equal if they have the same magnitude and direction and written as $\vec{a} = \vec{b}$.

Negative of a vector:→ A vector whose magnitude is the same as that of a given vector but direction is opposite to that of it; is called negative of a vector. E.g. $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$ and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Length of vector :→ Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2+y^2+z^2}$.

→ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then

(1) $\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$.

(2) $\vec{a}-\vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$

(3) $\vec{a}=\vec{b} \iff a_1=b_1,\ a_2=b_2,\ a_3=b_3$.

(4) $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$ where λ is any scalar.

Ex Let $\vec{a}=\hat{i}+2\hat{j}$ and $\vec{b}=2\hat{i}+\hat{j}$. Are the vectors $\vec{a}$ and $\vec{b}$ Equal?

Soln

$|\vec{a}| = \sqrt{1+4} = \sqrt{5}$

$|\vec{b}| = \sqrt{4+1} = \sqrt{5}$

$\Rightarrow |\vec{a}| = |\vec{b}|$

But $\vec{a} \neq \vec{b}$ as their Corresponding Components are not Equal.

Ex Find unit vector in the direction of $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}$.

Soln Unit vector is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$

$\Rightarrow \hat{a} = \dfrac{1}{\sqrt{14}}(2\hat{i}+3\hat{j}+\hat{k}) = \dfrac{2}{\sqrt{14}}\hat{i} + \dfrac{3}{\sqrt{14}}\hat{j} + \dfrac{1}{\sqrt{14}}\hat{k}$.

Ex → Find a vector in the direction of the vector $\vec{a} = \hat{i}-2\hat{j}$ that has magnitude 9 unit.

Soln Unit vector in the direction of $\vec{a}$ is $\hat{a}$

$\Rightarrow \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\hat{i}-2\hat{j}}{\sqrt{1+4}} = \dfrac{\hat{i}}{\sqrt{5}} - \dfrac{2}{\sqrt{5}}\hat{j}$

Now the vector having magnitude $7 = 7\hat{a}$

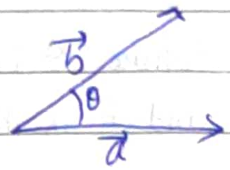
$= 7\left(\dfrac{\hat{i}}{\sqrt{5}} - \dfrac{2}{\sqrt{5}}\hat{j}\right) = \dfrac{7}{\sqrt{5}}\hat{i} - \dfrac{14}{\sqrt{5}}\hat{j}$.

Scalar product or dot product:→ The dot product of two vector $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$$

where θ is the angle b/w $\vec{a}$ and $\vec{b}$.

$0 \le \theta \le \pi$.



→ $\vec{a}\cdot\vec{b}$ is a scalar.

(1) $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular to each other i.e. $\theta = \pi/2$.

(2) If $\theta = 0$ then $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|$

Ex Find angle θ b/w the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$.

Sol^n

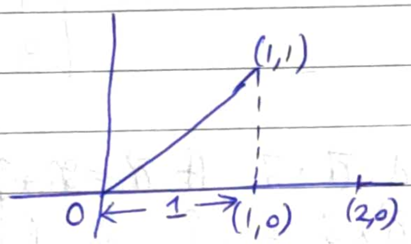
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{1-1-1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{3}\right).$$

→ $\vec{a}\cdot\vec{b}$ actually represents (projection of $\vec{a}$ on $\vec{b}$) × (magnitude of $\vec{b}$.)

E.g. $\vec{a} = \hat{i}+\hat{j}$

$\vec{b} = \hat{i}$



Proj of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{1}{1} = 1.$

If $\vec{b} = 2\hat{i}$

then Proj. of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2}{2} = 1.$

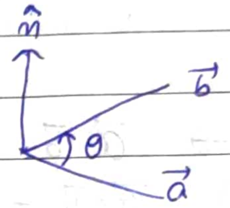
Que Find the Projection of $\vec{a} = 2\hat{i}+3\hat{j}+2\hat{k}$ on $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$.

Sol^n Proj of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{2+6+2}{\sqrt{1+4+1}} = \frac{10}{\sqrt{6}} = \frac{5\sqrt{6}}{3}.$

Vector product or cross product:→ The cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, . \, \hat{n}$$

where θ is the angle b/w $\vec{a}$ and $\vec{b}$ and $0 \le \theta \le \pi$ and $\hat{n}$ is the unit vector $\perp$ to $\vec{a}$ and $\vec{b}$ both.



(1) $\vec{a} \times \vec{b}$ is a vector.

(2) $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel to each other i.e. $\theta = 0$ or $\theta = \pi$.

(3) If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}|$

(4) $\hat{i} \times \hat{i} = 0$, $\hat{j} \times \hat{j} = 0$, $\hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = k$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$.

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Ex Find $\vec{a} \times \vec{b}$ If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Sol^n $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$= \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3) = -17\hat{i} + 13\hat{j} + 7\hat{k}$.

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Vector valued Function:→ A function $f: D \to \mathbb{R}^n, n \geq 1$ is called vector valued function.

E.g. $f: \mathbb{R} \to \mathbb{R}^2$ by

$f(x) = x\hat{i} + x^2\hat{j}$ or

$f(x) = (x, x^2)$

Then f is a vector valued function.

$f: D \to \mathbb{R}^3$ by , $D \subseteq \mathbb{R}$

$f(x) = x\hat{i} + x^2\hat{j} + x^3\hat{k}$. or

$f(x) = (x, x^2, x^3)$ is vector valued function.

Scalar valued function or Scalar point function:→ A function f whose Co-domain is subset of Real no.s or $\mathbb{R}$ itself is called Scalar point function.

E.g. $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = xy^2$ is Scalar point function

$f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = xy^2z^3$ is Scalar valued function.

→ $f(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$

$f: \mathbb{R} \to \mathbb{R}^3$

Domain of f $(D_f) = D_{f_1} \cap D_{f_2} \cap D_{f_3}$.

Limit:→ $\lim\limits_{t \to a} f(t)$ Exist Iff limit of Component functions $f_1(t), f_2(t), f_3(t)$ Exist as $t \to a$.

And If $\lim\limits_{t \to a} f_1(t) = l_1$, $\lim\limits_{t \to a} f_2(t) = l_2$, $\lim\limits_{t \to a} f_3(t) = l_3$

Then $\lim\limits_{t \to a} f(t) = l$ where

$l = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$.

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Continuity:→ A vector function $f(t)$ is continuous at $t=a$ iff the component functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous at $t=a$.

Differentiability:→ A vector function $f(t)$ is differentiable at $t=a$ if the component functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are diff. at $t=a$.

and $f'(t) = f_1'(t)\,\hat{i} + f_2'(t)\,\hat{j} + f_3'(t)\,\hat{k}$.

→ $(f(t)\cdot g(t))' = f(t)\cdot g'(t) + f'(t)\cdot g(t)$

→ $(f(t) \times g(t))' = f(t)\times g'(t) + f'(t)\, g(t)$

where $f(t)$ and $g(t)$ are vector valued functions.

→ $f''(t) = f_1''(t)\,\hat{i} + f_2''(t)\,\hat{j} + f_3''(t)\,\hat{k}$.

→ $(f(t)\cdot u(t))' = f'(t)u(t) + f(t)\,u'(t)$ where $u(t)$ is scalar valued fun.

Que Let $v(t) = (\cos t + t^2)(t\hat{i} + \hat{j} + 2\hat{k})$. Find $v'(t)$

Sol^n

$$v'(t) = (\cos t + t^2)'(ti + j + 2k) + (\cos t + t^2)(t\hat{i} + j + 2k)'$$

$$= (-\sin t + 2t)(ti + j + 2k) + (\cos t + t^2)(i)$$

$$= (-t\sin t + 2t^2)i + (-\sin t + 2t)j + (-2\sin t + 4t)k + (\cos t + t^2)i$$

$$= (3t^2 - t\sin t + \cos t)i + (2t - \sin t)(j + 2k)$$

Que $V(t) = (3ti + 5t^2j + 6k)\cdot(t^2i - 2tj + tk)$

Find $v'(t)$

Sol^n

$$v'(t) = (3ti + 5t^2j + 6k)'\cdot(t^2i - 2tj + tk) + (3ti + 5t^2j + 6k)\cdot(t^2i - 2tj + tk)'$$

$$= (3i + 10tj)\cdot(t^2i - 2tj + tk) + (3ti + 5t^2j + 6k)\cdot(2ti - 2j + k)$$

$$= (3t^2 - 20t^2) + (6t^2 - 10t^2 + 6)$$

$$= 9t^2 - 30t^2 + 6 = -21t^2 + 6.$$

H.W $v(t) = (ti + e^t j - t^2k) \times (t^2i + j + t^3k)$,

Find $v'(t)$.

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Que S.T. $(v(t)\times v'(t))' = v(t)\times v''(t)$

Sol^n L.H.S $(v(t)\times v'(t))' = v'(t)\times(v'(t))' + (v(t))'\times v'(t)$

$= v(t)\times v''(t) + v'(t)\times v'(t)$

$= v(t)\times v''(t)$

$(\because v'(t)\times v'(t) = 0)$

due to $i\times i = 0 = j\times j = k\times k$.

Gradient and directional derivative:→

Let f be any scalar valued function. ie. $f: D\to\mathbb{R}, D\subseteq\mathbb{R}^n, n\geq 1$
Then we know total derivative of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\cdot dy$$

$$\Rightarrow df = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right)$$

Then $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \nabla f$ is called gradient of f

denoted by ∇f or grad(f).
where $\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)f$

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$ is called vector differential operator.

→ • The directional derivative of f in the direction of $\vec{b}$ is defined as $\nabla f\cdot\hat{b} = \nabla f\cdot\frac{\vec{b}}{|\vec{b}|}$

* So the directional der. of f in direction of x-axis
$= \nabla f\cdot\hat{i} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\hat{i} = f_x = \frac{\partial f}{\partial x}$ → Partial der. of f w.r.t. x

* dir. der. of f in the direction of y-axis
$= \nabla f\cdot\hat{j} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\hat{j} = \frac{\partial f}{\partial y}$ → Partial der. of f w.r.t y.

Que $f(x,y) = y^2 - 4xy$. Find ∇f at $(1,2)$

Sol^n

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= -4xy\,\hat{i} + (2y-4x)\hat{j}$$

$$\Rightarrow \nabla f(1,2) = -8\hat{i} + 0\hat{j} = -8\hat{i}$$

Que $f(x,y,z) = x^2y^2 + xy^2 - z^2$.

Find $\nabla f(x,y,z)$ at $(3,1,1)$

Ans $7\hat{i} + 24\hat{j} - 2\hat{k}$.

Que If $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$; $|\vec{V}| = r$. Then find $\text{grad}\left(\frac{1}{r}\right)$

Sol^n

$$|\vec{V}| = r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

$$\text{grad}\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)\right]$$

$$= \frac{-1}{r^2}\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right]$$

$$= \frac{-1}{r^2}\left[\frac{\vec{V}}{|\vec{V}|}\right] = \frac{-1}{r^2}\hat{V} = \frac{-\hat{V}}{r^2}.$$

Properties of ∇f:→

let f and g be two diff scalar valued functions.

Then $\nabla(f+g) = \nabla f + \nabla g$.

$\nabla(c_1 f + c_2 g) = c_1 \nabla f + c_2 \nabla g$. , c_1, c_2 arbitrary constants.

$\nabla(fg) = f\nabla g + g\nabla f$.

$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

Que Find the directional der. of $f(x,y,z) = xy^2 + 4xyz + z^2$ at $(1,2,3)$ in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$.

Sol^n let $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

Dir. der. of f in direction of $\vec{b} = \nabla f \cdot \hat{b}$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{\sqrt{9+16+25}} = \frac{1}{5\sqrt{2}}(3\hat{i} + 4\hat{j} - 5\hat{k})$$

So $D_b(f) = (\nabla f) \cdot \hat{b}$

$$\nabla f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= (y^2 + 4yz)\hat{i} + (2xy + 4xz)\hat{j} + (4xy + 2z)\hat{k}$$

Now $D_b(f) = \frac{1}{5\sqrt{2}}\left[3(y^2 + 4yz) + 4(2xy + 4xz) - 5(4xy + 2z)\right]$

$$= \frac{1}{5\sqrt{2}}(3y^2 + 12yz + 8xy + 16xz - 20xy - 10z)$$

at $(1,2,3)$

$$D_b(f)(1,2,3) = \frac{1}{5\sqrt{2}}\left[12 + 72 + 16 + 48 - 40 - 30\right]$$

$$= \frac{78}{5\sqrt{2}}.$$

Que Find the dir. der. of $(x^2y - xy^2z - xyz)$ in the direction of $\hat{i} - \hat{j} + 2\hat{k}$ at point $(1,-1,0)$.

Ans $\frac{-3}{\sqrt{6}}$ or $-\sqrt{\frac{3}{2}}$

Que $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$. Find dir. der. of f at $(-1,1,2)$ in the direction of $\hat{i} - 2\hat{j} + \hat{k}$.

Ans -3.

Que Find the dir. der of $f(x,y) = x^2 + y^2$ in the direction of $\vec{a} = i + j$ at $(1,1)$.

Sol^n dir. der of f in the direction of $\vec{a} = \nabla f \cdot \hat{a}$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\hat{a} = \frac{i+j}{\sqrt{2}}$$

$$\Rightarrow D_{\vec{a}}(f) = \nabla f \cdot \hat{a}$$
$$= (2x\hat{i} + 2y\hat{j}) \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$$
$$= \frac{2x+2y}{\sqrt{2}} = \sqrt{2}x + \sqrt{2}y.$$

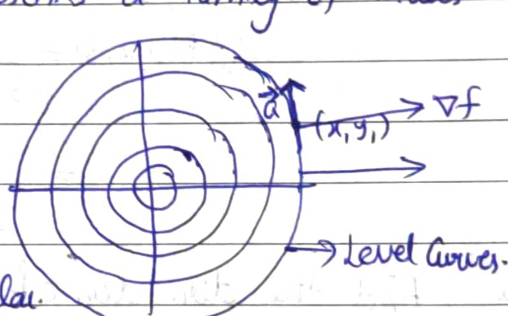
at $(1,1)$, $\sqrt{2}+\sqrt{2} = 2\sqrt{2}$.

Level Surfaces:→ Let $f : \mathbb{R}^3 \to \mathbb{R}$ or $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$ be a scalar valued function.

Then $f(x,y,z) = c$ defines the Equation of a surface and is called a Level Surface of the function.

For different values of c, we obtain different surfaces, no two of which intersect.

Eg. $f(x,y) = x^2 + y^2 = c$ — represents a family of circles.



direction derivative of f in the direction of $\vec{a} = \nabla f \cdot \hat{a} = 0$

($\because f(x,y) = \text{Constant}$)

$\Rightarrow$ ∇f and $\vec{a}$ are Perpendicular.

$\Rightarrow$ ∇f is normal vector to the given surface.

Que Find a ~~unit~~ normal vector to the surface $xy^2 + 2yz = 8$ at the point $(3,-2,1)$.

Solⁿ let $f(x,y,z) = xy^2 + 2yz = 8$. —(1)

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\Rightarrow \nabla f = y^2\hat{i} + (2xy+2z)\hat{j} + (2y)\hat{k}$$

$\nabla f(3,-2,1) = 4\hat{i} - 10\hat{j} - 4\hat{k}$ is a normal vector to (1).

→ $\nabla f(x_1,y_1) \cdot ((x-x_1)\hat{i} + (y-y_1)\hat{j}) = 0$ is the Equⁿ of tangent line to the curve $f(x,y) = c$.

Que Find the normal vector to the Surface $z = \sqrt{x^2+y^2}$ at the pt. $(3,4,5)$.

Sol^n → let $f(x,y,z) = z - \sqrt{x^2+y^2} = 0$ be the Surface.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{-1(2x)}{2\sqrt{x^2+y^2}}\hat{i} - \frac{1(2y)}{2\sqrt{x^2+y^2}}\hat{j} + \hat{k}$$

$$= -\frac{x}{z}\hat{i} - \frac{y}{z}\hat{j} + \hat{k}, \quad (z \neq 0)$$

at $(3,4,5)$; $\nabla f(3,4,5) = -\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} + \hat{k}$.

Que Find the angle b/w the Surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the pt. $(1,1,1)$.

Sol^n → let $f_1(x,y,z) = x\log z - y^2 + 1 = 0$

$f_2(x,y,z) = x^2 y - 2 + z = 0$.

$\nabla f_1(x,y,z) = \log z\,\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$

$\Rightarrow \nabla f_1(1,1,1) = -2\hat{j} + \hat{k}$

$\nabla f_2(x,y,z) = 2xy\hat{i} + x^2\hat{j} + \hat{k}$

$\Rightarrow \nabla f_2(1,1,1) = 2\hat{i} + \hat{j} + \hat{k}$

Thus ~~Cos~~ $\nabla f_1 \cdot \nabla f_2 = |\nabla f_1|\,|\nabla f_2| \cos\theta$

$\Rightarrow \cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1||\nabla f_2|} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

HW Find the unit normal vector to the given Surfaces / Curves at a given pt.

(1) $x^2 + y^2 = 25$ at $(3,4)$

Ans:- $\frac{3\hat{i} + 4\hat{j}}{5}$

(2) $x^2 + 2y^2 + z^2 = 4$ at $(1,1,1)$

Ans:- $\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$

(3) $z^2 = x^2 - y^2$ at $(2,1,\sqrt{3})$

Ans:- $\frac{2\hat{i} - \hat{j} - \sqrt{3}\hat{k}}{\sqrt{8}}$.

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Maximum and Minimum Directional derivatives

We Know that for a given Scalar valued function f, directional der. of f in the direction of $\vec{a}$ is

$$D_{\vec{a}}f = \nabla f \cdot \hat{b}$$
$$= |\nabla f|\,|\hat{b}|\cos\theta$$
$$= |\nabla f| \cdot \cos\theta \qquad (\because |\hat{b}|=1)$$

where θ is angle b/w ∇f and $\hat{b}$. , $0 \le \theta \le \pi$
$$\Rightarrow -1 \le \cos\theta \le 1.$$
$$\Rightarrow \cancel{D_a f} \quad -|\nabla f| \le |\nabla f|\cos\theta \le |\nabla f|$$
$$\Rightarrow -|\nabla f| \le D_a f \le |\nabla f|$$

$\Rightarrow$ Max value of dir. der. is $|\nabla f|$ and it occurs when $\theta = 0$.
& Min value of dir der is $-|\nabla f|$ and it occurs when $\theta = \pi$.

Imp
- $\theta = 0 \Rightarrow$ ∇f and $\hat{b}$ have same direction and ∇f and $\hat{b}$ are Parallel.
- $\theta = \pi \Rightarrow$ ∇f and $\hat{b}$ have opp. directions and Parallel.

Hence dir. der is max in the direction of (∇f) and dir. der is min in the direction of $-(\nabla f)$.

Dir. der. is 0 when $\nabla f \cdot \hat{b} = 0$
$\Rightarrow$ ∇f and $\hat{b}$ are Perpendicular.

Que Find a vector that gives the direction of max. rate of increase and find the max rate.

(i) $e^{2y}\cos x$ at $\left(\frac{\pi}{4}, 0\right)$

$$f(x,y) = e^{2y}\cos x$$
$$\nabla f = -e^{2y}\sin x\,\hat{i} + 2e^{2y}\cos x\,\hat{j}.$$
$$|\nabla f| = \sqrt{(e^{2y}\sin x)^2 + (2e^{2y}\cos x)^2}$$
$$= \sqrt{e^{4y}(\sin^2 x + 4\cos^2 x)}$$
$$\nabla f(\pi/4, 0) = -\frac{1}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} = \frac{-\hat{i} + 2\hat{j}}{\sqrt{2}}$$

$|\nabla f(\frac{\pi}{4}, 0)| = \sqrt{\frac{5}{2}}$

$\therefore$ Max Rate $= \sqrt{\frac{5}{2}}$

and vector that gives the dir. of max rate increase $= \frac{-i+2j}{\sqrt{2}}$.

H.W (1) $3x^2 + y^2 + 2z^2$ at $(0,1,2)$

Ans $2(\hat{j} + 4\hat{k})$ and $2\sqrt{17}$.

(2) $6xyz$ at $(-1,2,1)$

Ans $6(2\hat{i} - \hat{j} - 2\hat{k})$ and 18.

Que Find the vector that gives the dir of min rate of Increase and find the min rate.

(1) $x^3 - xy^2 + y^3$ at $(-2,1)$

Ans $-(11\hat{i} + 7\hat{j})$ and $-\sqrt{170}$

(2) $x^2 - y^2 + z^2$ at $(1,2,1)$

Ans $-2(\hat{i} - 2\hat{j} + \hat{k})$ and $-2\sqrt{6}$.

Que Find the angle b/w the two Surfaces at given point.

(1) $z = x^2 + y^2$; $z = 2x^2 - 3y^2$ at $(2,1,5)$.

Ans $\cos^{-1}\left(\sqrt{21/101}\right)$

(2) $x^2 + y^2 + z^2 = 9$; $z + 3 = x^2 + y^2$ at $(-2,1,2)$

Ans. $\cos^{-1}\left(8/3\sqrt{21}\right)$

Que Find the Eqn of tangent plane to $f(x,y,z) = x^2 - 3y^2 - z^2 = 2$ at $(3,1,2)$.

Soln

Eqn of tangent plane is $\nabla f(3,1,2) \cdot ((x-3)\hat{i} + (y-1)\hat{j} + (z-2)\hat{k}) = 0$

$\Rightarrow (2x\hat{i} - 6y\hat{j} - 2z\hat{k}) \cdot ((x-3)\hat{i} + (y-1)\hat{j} + (z-2)\hat{k}) = 0$

$\Rightarrow (6\hat{i} - 6\hat{j} - 4\hat{k}) \cdot ((x-3)\hat{i} + (y-1)\hat{j} + (z-2)\hat{k}) = 0$

$\Rightarrow 6(x-3) - 6(y-1) - 4(z-2) = 0$

$\Rightarrow 6x - 6y - 4z - 4 = 0$

Que Find the Equ^n of tangent plane to : $z = 16 - x^2 - y^2$ at $(1,3,6)$.

Ans $2x + 6y + z = 26$

Que Find the Equ^n of tangent plane to $xy + yz + zx = -1$ at $(1,-1,2)$.

Ans $x + 3y + 2 = 0$.

→ Divergence of a vector field:→

Let $f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$. be a vector field or vector valued function.

Then $\text{div } f = \nabla \cdot f$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Que Find the divergence of $V = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}$

Solⁿ:→ $\text{div } V = \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(e^{xyz})$

$$= 2xy^2 + 2xz + e^{xyz} \cdot xy.$$

Que P.T. $\text{div}(fv) = f(\text{div } v) + (\text{grad } f) \cdot v$, where f is a scalar function.

where $V = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Solⁿ:→ $\text{div}(fv) = \nabla \cdot (fv)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (fv_1\hat{i} + fv_2\hat{j} + fv_3\hat{k})$$

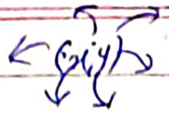
$$= \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= \left(f \cdot \frac{\partial v_1}{\partial x} + v_1 \cdot \frac{\partial f}{\partial x}\right) + \left(f \cdot \frac{\partial v_2}{\partial y} + v_2 \cdot \frac{\partial f}{\partial y}\right) + \left(f \cdot \frac{\partial v_3}{\partial z} + v_3 \cdot \frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \left(v_1 \cdot \frac{\partial f}{\partial x} + v_2 \cdot \frac{\partial f}{\partial y} + v_3 \cdot \frac{\partial f}{\partial z}\right)$$

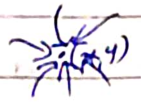
$$= f(\text{div } v) + (\text{grad } f) \cdot v$$

→ $\text{div}(v) > 0 \rightarrow$ Source


diverge

$\text{div}(v) < 0 \rightarrow$ Sink

$\text{div}(v) = 0 \rightarrow$ v is called Solenoidal vector.


Converge

Curl of vector field :→ Let $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{Curl } V = \nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Que $V = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}$

Find Curl V

Sol^n

$$\text{Curl } v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 - z^3 & 2xyz & e^{xyz} \end{vmatrix}$$

$$= i(xze^{xyz} - 2xy) - \hat{j}(yze^{xyz} + 3z^2) + \hat{k}(2yz - 2x^2y)$$

→ Curl of divergence i.e. Curl(div f) = $\nabla \times \nabla\cdot f$ is not defined as $\nabla\cdot f$ is Scalar valued function, But Curl is Evaluated for vector valued function.

→ Curl (grad f) = 0 or $\nabla \times \nabla f = 0$.

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right) = 0.$$

H.W

$\text{Div}(\text{Curl } v) = \nabla\cdot(\nabla \times V) = 0$

~~Grad (div v) =~~ $\cancel{\nabla(\nabla \cdot \vec{V}) = \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}}$ ~~where~~ $\cancel{v = v_1\hat{i} + v_2\hat{j}}$.

Que $\vec{F} = 2x^2\hat{i} + y^2\hat{j} + \alpha z\hat{k}$
at $(1,2,1)$ $\vec{F}$ is solenoidal vector. Find α.

Solⁿ $div\vec{F} = 4x + 2y + \alpha$
at $(1,2,1)$
$div\vec{F} = 0$
$\Rightarrow 4 + 4 + \alpha = 0 \Rightarrow \alpha = -8$

$\rightarrow$ $div(grad f) = \nabla^2 f$ where f = Scalar valued function.

Pf $grad f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$div(grad f) = \nabla\cdot(\nabla f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)f$$
$$= \nabla^2 f. \quad (\nabla^2 - \text{Laplacian operator})$$

Que Compute div(v), Curl(v) and verify that div(Curl v) = 0.

(1) $V = x\hat{i} + 2y\hat{j} + z\hat{k}$
Ans 4, 0 vector.

(2) $V = xy\hat{i} + yz\hat{j} + zx\hat{k}$
Ans:- $x+y+z$; $-(iy + jz + kx)$

(3) $V = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$
Ans:- $2(x+y+z)$; 0 vector.

(4) Show that $V = (2x+3y)\hat{i} + (x-y)\hat{j} - (x+y+z)\hat{k}$.
is Solenoidal vector.

(5) If $V = -(x+y+2)\hat{i} - 2\hat{j} + (x+y)\hat{k}$. Show that $V\cdot Curl\, V = 0$.

→ If Curl(f) = 0 Then f is called Conservative vector field.

Line Integral:→ Let f be a vector valued function. Let C be a simple smooth curve and (x, y, z) be any point on the curve. $r = x\hat{i} + y\hat{j} + z\hat{k}$

Then $$\int_C f \cdot dr = \int_C (f_1\hat{i} + f_2\hat{j} + f_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

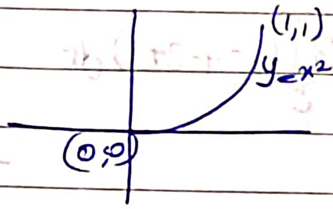
$$= \int_C f_1 dx + f_2 dy + f_3 dz$$

defines a line Integral of f over C.

Que Find the line Integral of $F(x,y) = x^2\hat{i} + y\hat{j}$ along the curve $y = x^2$ from (0,0) to (1,1).

Sol^n

$$\int_C F \cdot dr = \int_C (x^2\hat{i} + y\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$



$$= \int_C (x^2 dx + y\, dy)$$

$$y = x^2$$
$$\Rightarrow dy = 2x\, dx$$

$$\Rightarrow \int_C F \cdot dr = \int_0 x^2 dx + x^2(2x) dx$$

$$= \int_0^1 (x^2 + 2x^3)\, dx$$

$$= \left(\frac{x^3}{3} + \frac{x^4}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Que Find the line Integral of $F = xy\hat{i} + y^2\hat{j} + e^z\hat{k}$ over the curve C whose parametric representation is given by $x = t^2$, $y = 2t$, $z = t$; $0 \le t \le 1$.

Sol^n $$\int_C F \cdot dr = \int_C xy\, dx + y^2 dy + e^z dz$$

$$= \int_0^1 2t^3 (2t\, dt) + 4t^2 (2\, dt) + e^t dt$$

$$= \int_0^1 (4t^4 + 8t^2 + e^t)\,dt$$

$$= \left(\frac{4t^5}{5} + \frac{8}{3}t^3 + e^t\right)_0^1 = \frac{4}{5} + \frac{8}{3} + e - 1$$

$$= \frac{37}{15} + e$$

Que Find $\int_C (x^2+yz)\,dz$ where C is given by $x=t,\ y=t^2,\ z=3t;$ $1 \le t \le 2.$

Sol^n:→

$x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = 3t \Rightarrow dz = 3\,dt$

$$3\int_C (t^2+3t^3)\,dt = 3\int_1^2 (t^2+3t^3)\,dt$$

$$= 3\left[\frac{t^3}{3} + \frac{3t^4}{4}\right]_1^2$$

$$= 3\left[\frac{8}{3} + \frac{48}{4} - \frac{1}{3} - \frac{3}{4}\right]$$

$$= 3\left[\frac{7}{3} + \frac{45}{4}\right] = \frac{163}{4}.$$

Que $\int_C (x+y)\,dx - x^2\,dy + (y+z)\,dz$

where C is $x^2 = 4y,\ z = x,\ 0 \le x \le 2$

Ans $\frac{10}{3}$

Que Evaluate the line Integral $\int_C F\cdot dr$

(i) $F = xi + (\sin y)j + k$; C is given by $x = t^2,\ y = t,\ z = 2t;$ $0 \le t \le 1$

Ans $\frac{7}{2} - \cos(1)$

(2) $F = x^2y\hat{i} - xy^2\hat{j}$; $r(t) = t\hat{i} + t^2\hat{j}$; $0 \le t \le 3$.
Ans:- $-20169/35$

(3) $F = e^x\hat{i} + xe^{xy}\hat{j} + \hat{k}$; $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$; $0 \le t \le 2$.
Ans:- $(2e^8 + 3e^2 + 19)/3$.

Line Integrals Independent of path:→ When a differential equation $f_1dx + f_2dy + f_3dz$ is an Exact differential Equation then line Integral $\int_C F.dr$ is path Independent. and Conversely.

Que $F = x\hat{i} + y\hat{j}$. Find $\int F.dr$ along $y = x^2$ from $(0,0)$ to $(1,1)$.

Soln→
$$\int F.dr = \int (x\hat{i} + y\hat{j}).(dx\hat{i} + dy\hat{j})$$
$$= \int x dx + y dy$$

$x dx + y dy = 0 \rightarrow$ is Exact differential Equn
$$x dx + y dy = d\left(\frac{x^2+y^2}{2}\right)$$
$$\therefore \int F.dr = \int_{(0,0)}^{(1,1)} d\left(\frac{x^2+y^2}{2}\right) = \left(\frac{x^2+y^2}{2}\right)_{(0,0)}^{(1,1)}$$
$$= \frac{2}{2} - 0 = 1.$$

Que→ $\int_P^Q 2xy^2dx + (2x^2y+1)dy$; $P:(-1,2)$; $Q:(2,3)$

Soln→
$M = 2xy^2$, $N = 2x^2y + 1$
$\frac{\partial M}{\partial y} = 4xy$; $\frac{\partial N}{\partial x} = 4xy$

$\int M dx + \int$ terms in N not Cont $x dy = C$

$\Rightarrow \int 2xy^2 dx + \int 1\, dy = C$
$\Rightarrow x^2y^2 + y = C$ $\quad \Rightarrow f(x,y) = x^2y^2 + y$

$$\int_P^Q d(x^2y^2+y) = (x^2y^2+y)_P^Q$$

$$= (x^2y^2+y)_{(-1,2)}^{(2,3)} = (36+3)-(4+2)$$

$$= 33.$$

Que $\int_P^Q (1-\sin x \sin y)\,dx + \int (1+\cos x \cos y)\,dy$; $P:\left(\frac{\pi}{4},\frac{\pi}{4}\right)$; $Q:\left(\frac{\pi}{2},0\right)$

Ans:- $-1/2$

Que $\int_P^Q (y^3+2xy^2)\,dx + (3xy^2+2x^2y+1)\,dy$; $P:(-2,1)$; $Q:(1,2)$

Ans: 11.

Que $\int_P^Q (2xz+y)\,dx + (x+z)\,dy + (x^2+y)\,dz$; $P:(-1,2,3)$; $Q:(2,2,4)$

Que S.T. $\int_C (yz-1)\,dx + (z+xz+z^2)\,dy + (y+xy+2yz)\,dz$ is Independent of the path of Integration from (1,2,2) to (2,3,4). Evaluate the Integral.

Sol^n

$f(x,y,z) = yz-1$; $g(x,y,z) = z+xz+z^2$; $h(x,y,z) = y+xy+2yz$.

$f_y = z$; $f_z = y$

$g_x = z$; $g_z = 1+2z+x$

$h_x = y$; $h_y = 1+2z+x$

$f_y = g_x$; $f_z = h_x$; $g_z = h_y$

$\Rightarrow$ Integral is path Independent.

$\Rightarrow \exists\ \phi(x,y,z)$ s.t.

$d\phi = f\,dx + g\,dy + h\,dz$

But $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$\Rightarrow \frac{\partial \phi}{\partial x} = yz-1;\quad \frac{\partial \phi}{\partial y} = z+xz+z^2;\quad \frac{\partial \phi}{\partial z} = y+xy+2yz.$

$\phi(x,y,z) = xyz - x + h(y,z)$

$\frac{\partial \phi}{\partial y} = xz + \frac{\partial h}{\partial y}$

$\Rightarrow z + \cancel{xz} + z^2 = \cancel{xz} + \frac{\partial h}{\partial y}$

$\Rightarrow z + z^2 = \frac{\partial h}{\partial y}$

$\Rightarrow h(y,z) = zy + z^2y + s(z)$

$\therefore \phi(x,y,z) = xyz - x + yz + yz^2 + s(z)$

$\frac{\partial \phi}{\partial z} = xy + y + 2yz + s'(z)$

$\Rightarrow y + xy + 2yz = \cancel{xy} + \cancel{y} + \cancel{2yz} + s'(z)$

$\Rightarrow s'(z) = 0 \Rightarrow s(z) = c$

$\Rightarrow \phi(x,y,z) = xyz - x + yz + yz^2 + C.$

$\therefore$ The value of Integral is

$$\int_C d\phi(x,y,z) = \int_{(1,2,2)}^{(2,3,4)} d(xyz - x + yz + yz^2)$$

$$= \left(xyz - x + yz + yz^2\right)_{(1,2,2)}^{(2,3,4)} = \underline{67.}$$

Que Find the value of $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ from $P(-1,2)$ to $Q(2,3)$

Ans: $-\sqrt{13}-\sqrt{5}$.

Que Show that $\int_P^Q (3x^2+2xy z)dx + (1+x^2 z)\,dy + x^2y\,dz$; P is path Independent from P(1,1,1) and Q(-2,-3,-4) and Evaluate the Integral.

Ans 34

Que S.T. $\int_P^Q (2xz+y)\,dx + (x+z)\,dy + (x^2+y)\,dz$ is path Independent and Evaluate the Integral where P: (-1, 2,3) and Q: (3,3,4)

Ans: 21

→ Let $F = fi + gj + hk$

If $\int_C F \cdot dr = \int_C f dx + g dy + h dz$

is path Independent then F is called ~~pot~~ Conservative vector field.

Que Find whether $F = (2x + ye^{xy})i + (2y + xe^{xy})j$ is Conservative vector field. If it is, then find the potential function.

Sol^n $F = (2x+ye^{xy})i + (2y+xe^{xy})j$

$$\int F \cdot dr = \int (2x+ye^{xy})dx + (2y+xe^{xy})dy$$

$$\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy}$$

$$\frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$$

⇒ Eq^n is Exact.

$$\int M dx + \int \text{Terms In N not Con. } x\, dy = C$$

$$\Rightarrow \int 2x\,dx + \int ye^{xy}dx + \int 2y\,dy = C$$

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$\Rightarrow x^2 + \frac{ye^{xy}}{xy} + y^2 = c$

$\Rightarrow x^2 + e^{xy} + y^2 = c$

$\therefore \int F \cdot dr = \int d(x^2 + y^2 + e^{xy})$

$= x^2 + y^2 + e^{xy}$

Que $F = \cos(x+y)(i+j) = \cos(x+y)i + \cos(x+y)j$
Is F Conservative vector field? If Yes then find the potential function.
Ans: $\sin(x+y)$

Que $F = yzi + xzj + xyk$
Is F Conservative vector field. If Yes then find the potential fun.

Sol^n $\int F \cdot dr = \int yz\,dx + xz\,dy + xy\,dz$

$f = yz;\ g = xz,\ h = xy$
$f_y = z;\ g_x = z$
$f_z = y;\ h_x = y$
$g_z = x;\ h_y = x$

$\Rightarrow$ Eq^n is Exact
$\Rightarrow$ F is Conservative vector field

$\Rightarrow \exists\ \phi(x,y,z)$ S.t

$\frac{\partial \phi}{\partial x} = yz;\ \frac{\partial \phi}{\partial y} = xz;\ \frac{\partial \phi}{\partial z} = xy$

$\phi(x,y,z) = xyz + h(y,z)$

$\frac{\partial \phi}{\partial y} = xz + \frac{\partial h}{\partial y}$

$\Rightarrow xz = xz + \frac{\partial h}{\partial y}$

$\Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h(y,z) = s(z)$

$\Rightarrow \phi(x,y,z) = xyz + s(z)$

$\frac{\partial \phi}{\partial z} = xy + s'(z)$

$\Rightarrow xy = xy + s'(z)$

$\Rightarrow s'(z) = 0$

$\Rightarrow s(z) = c$

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$\Rightarrow \phi(x,y,z) = xyz + c$

$\therefore \int F \cdot dr = \int d(xyz) = xyz$

Que $F = 2xy\,i + (x^2 + 2yz)\,j + y^2 k$

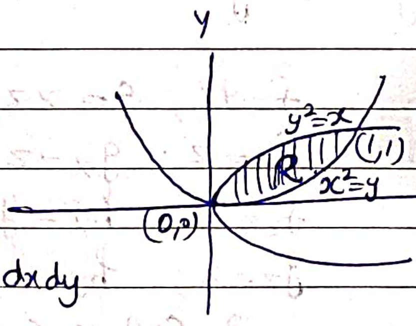
Is F Conservative? If yes then find the potential function.

Ans - $x^2 y + y^2 z$.

★ **Green's Thm:** $\rightarrow$ Let C be a smooth closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ are continuous on R,

then $\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$

Ques Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$ where C is the boundary of the region in the first Quadrant that is bdd by the curves $y^2 = x$ and $x^2 = y$.



Sol^n $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy =$

$$\iint_R \left(\frac{\partial}{\partial x}(y+2x) - \frac{\partial}{\partial y}(x^2+y^2)\right) dx\,dy.$$

$$= \iint_R (2 - 2y)\,dx\,dy.$$

$$\int_0^1 \int_{y^2}^{\sqrt{y}} (2-2y)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_{x^2}^{\sqrt{x}} (2-2y)\,dy\,dx$$

$$= \int_0^1 \left(2x - 2xy\right)_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(2\sqrt{y} - 2y\sqrt{y} - 2y^2 + 2y^3\right) dy$$

$$= \boxed{\frac{11}{30}}$$

Que $\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx - x\,dy$ where C is $\{(x,y) \mid x^2+y^2=1\}$ bounding a region R.

Evaluate the Integral.

Sol^n

$$\int_C y\,dx - x\,dy = \iint_R (-1-1)\,dx\,dy$$

$$= -2\iint_R dx\,dy = -2\cdot \text{area of circle}$$

$$= -2\times\pi(1)^2$$

$$= -2\pi.$$

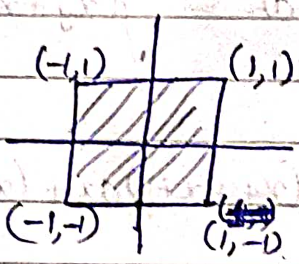
Que $\vec{F} = \sin x\,\hat{i} + e^{x}\hat{j}$

$C = \{(x,y) \mid x=\pm 1, y=\pm 1\} \rightarrow$ Rectangle.

Evaluate the line Integral.

Sol^n

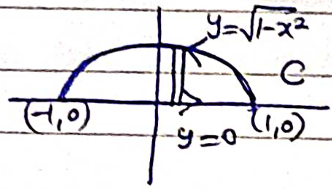
$$\int_C \vec{F}\cdot\vec{dr} = \int_C \sin x\,dx + e^{y}dy$$



$$= \iint_R \frac{\partial}{\partial y}(\sin x) - \frac{\partial}{\partial x}(e^{y})$$

$$= 0$$

Que $\rightarrow$ $\vec{F} = (x^2-xy^2)\hat{i} + y^2\hat{j}$



C is the closed curve bounded by Semi-circle and x-axis. Find the Line Integral.

Sol^n $\rightarrow$

$$\int_C \vec{F}\cdot\vec{dr} = \int_C (x^2-xy^2)\,dx + y^2dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2-xy^2)\right)dx\,dy$$

$$= \iint_R 2xy\,dx\,dy$$

$$= 2\iint_R xy\,dx\,dy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (xy)\,dy\,dx$$

$$= \cancel{2}\int_{-1}^{1} \frac{x(1-x^2)}{\cancel{2}}dx$$

$$= \int_{-1}^{1} x - x^3\,dx$$

$$= 0$$

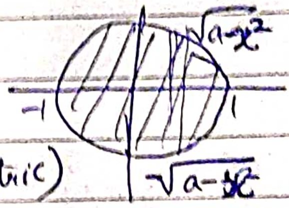
$\int \vec{F}\cdot d\vec{r} =$

Que $\int (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy$ and $C = \{(x,y) \mid x^2+y^2 = a^2\}$ is the Curve bounding a region R. Then Find the Integral.

Sol^n $\int (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy$

$$= \iint_R \big((\cos x \cos y) - \cos x \cos y + x\big)\,dx\,dy$$

$$= \iint_R x\,dx\,dy$$



$$= 0 \qquad (\because \text{ Curve C is Symmetric})$$

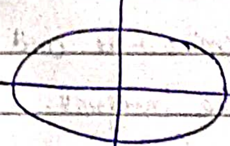
Que $\int \vec{F}\cdot d\vec{r} = \int (xy+x+y)\,dx + (xy+x-y)\,dy$ over the Region $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Sol^n $\int (xy+x+y)\,dx + (xy+x-y)\,dy = \iint_R \big((y+1)-(x+1)\big)\,dx\,dy$

$$= \iint_R (y-x)\,dx\,dy$$

$$= \iint_R y\,dx\,dy - \iint_R x\,dx\,dy$$

$$= 0 \quad \because$$



Que Find $\frac{1}{\pi}\oint_C \left(3y - e^{\cos(x^2)}\right)dx + \left(7x+\sqrt{y^4+11}\right)dy$

over positive oriented circle $x^2+y^2=9$.

Sol^n $\frac{1}{\pi}\oint_C \left(3y - e^{\cos x^2}\right)dx + \left(7x+\sqrt{y^4+11}\right)dy$

$$= \frac{1}{\pi}\iint_R (7-3)\,dx\,dy$$

$$= \frac{4}{\pi}\iint_R dx\,dy = \frac{4}{\pi}\times\pi(3)^2 = \boxed{36}$$

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Que: $\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$; over the Curve $C : x^2+y^2=4$.

$= \oint \frac{x\,dy}{x^2+y^2} - \frac{y}{x^2+y^2}\,dx$ → This is Exact Equation.
But f and g are not Continuous functions at (0,0).

$= \iint$

So we Cannot apply green's Thm.

We'll use parametric Equations
$x = 2\cos t$, $y = 2\sin t$; $0 \le t \le 2\pi$

$$\int_0^{2\pi} \frac{2\cos t\,(2\cos t)\,dt - (2\sin t)(-2\sin t)\,dt}{4}$$

$$= \int_0^{2\pi} \frac{4\cos^2 t + 4\sin^2 t}{4}\,dt = 2\pi.$$

Que $\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}$ over the Curve $C = \{(x,y) \mid (x-9)^2+(y-8)^2=1\}$

⇓ Exact Equation.
f & g are Continuous Everywhere Excep (0,0).
and (0,0) is Excluded from the Region now.

So $\oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = 0$.

Que $\oint_C \frac{x\,dy - y\,dx}{x^2+4y^2}$; $C : \{(x,y) \mid x^2+y^2=4\}$

$$= \iint_R \frac{\partial}{\partial x}\left(\frac{x}{x^2+4y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+4y^2}\right) dx\,dy$$

$$= \iint_R \left[\frac{(x^2+4y^2) - x(2x)}{(x^2+4y^2)^2} + \frac{(x^2+4y^2)(1) - y(8y)}{(x^2+4y^2)^2}\right] dx\,dy$$

$$= \iint_R \left(\frac{4y^2-x^2}{(x^2+4y^2)^2} + \frac{\cancel{4y^2}\; x^2-4y^2}{(x^2+4y^2)^2}\right) dx\,dy$$

⇓

Exact but f & g are not cts at (0,0).

So use $x = 2\cos t,\ y = 2\sin t$

$$\oint_C \frac{x\,dy - y\,dx}{x^2+4y^2} = \int_0^{2\pi} \frac{(2\cos t)(2\cos t)\,dt - (2\sin t)(-2\sin t)\,dt}{4\cos^2 t + 4(4\sin^2 t)}$$

$$= \int_0^{2\pi} \frac{4\,dt}{4\cos^2 t + 16\sin^2 t}$$

$$= \int_0^{2\pi} \frac{dt}{\cos^2 t + 4\sin^2 t}$$

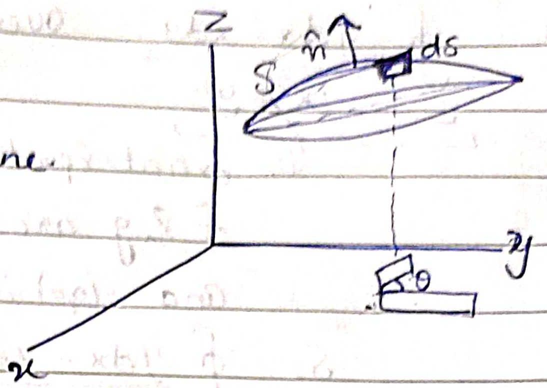
$$= \int_0^{2\pi} \frac{\sec^2 t\,dt}{1+4\tan^2 t}$$

$\tan t = z$

$\sec^2 t\,dt = dz$

$$= 4\int_0^{\pi/2} \frac{\sec^2 t\,dt}{1+4\tan^2 t} = 4\int_0^{\infty} \frac{dz}{1+4z^2} \rightarrow \text{Solve } \boxed{\text{Ans} = \pi}$$

Surface Integral:→



Let S be a Surface in xyz-plane.
and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Let $\hat{n}$ be the normal unit vector to the surface

$$\Rightarrow \hat{n} = \frac{\nabla S}{|\nabla S|}$$

$\vec{F}\cdot\hat{n}$ — will give the component of $\vec{F}$ in the direction of $\hat{n}$
$(\because \vec{F}\cdot\hat{n} = (\text{Proj of } \vec{F} \text{ on } \hat{n})(\text{magnitude of } \hat{n}) = \text{Proj of } \vec{F} \text{ on } \hat{n})$

θ is the angle b/w S and xy-plane.

$\Rightarrow$ θ is angle b/w their normals i.e. $\hat{n}$ and $\hat{k}$

$$\Rightarrow \cos\theta = \frac{\hat{n}\cdot\hat{k}}{|\hat{n}||\hat{k}|} = \hat{n}\cdot\hat{k}$$

$$\therefore ds\cos\theta = dx\,dy$$

$$\Rightarrow ds = \frac{dx\,dy}{\cos\theta}$$

Surface Integral is $\iint \vec{F}.\hat{n}\, ds$

$$= \iint \vec{F}.\hat{n} \frac{dx\,dy}{\cos\theta}$$

Que $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and $S = \{x^2+y^2+z^2=1 - \text{Hemisphere}\}$
Find the Surface Integral over S.

Sol $\hat{n} = \frac{\overrightarrow{\nabla f}}{|\nabla f|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$\vec{F}.\hat{n} = x^2+y^2+z^2$

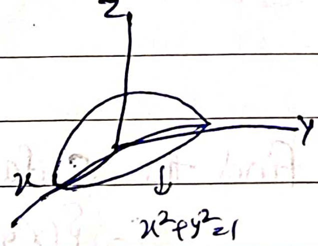
$$S.I. = \iint_S \vec{F}.\hat{n}\, ds = \iint_S (x^2+y^2+z^2). \frac{dx\,dy}{\cos\theta}$$

$$= \iint_S (x^2+y^2+z^2) \cdot \frac{dx\,dy}{\hat{n}.\hat{k}}$$

$$= \iint_S (x^2+y^2+z^2) \frac{dx\,dy}{z}$$

$$= \iint \frac{dx\,dy}{\sqrt{1-x^2-y^2}} \rightarrow \text{on } xy \text{ plane.}$$

Proj of $x^2+y^2+z^2=1$ on xy plane is Circle $x^2+y^2=1$



So $S.I. = \iint \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$ over $x^2+y^2=1$

$x = r\cos\theta,\ y = r\sin\theta$
$x^2+y^2 = r^2$

$$S.I = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{1-r^2}} = \int_0^1 \left(\frac{r\,dr}{\sqrt{1-r^2}}\right) 2\pi$$

Put $\sqrt{1-x^2} = t$
$\frac{-2x}{2\sqrt{1-x^2}} dx = dt$

$$= 2\pi \left(-\sqrt{1-r^2}\right)\Big|_0^1$$

$$= -2\pi(-1) = \boxed{2\pi}$$

Gauss divergence Thm:→

Let S be a closed, Smooth Surface. let $f_1, f_2, f_3, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$

are Continuous function on region 'D'. — where D is a closed and bdd region in 3-D space.

Then $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$

$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Que Find the Surface Integral of $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$ over the Surface $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ — ~~…~~ sphere

Soln

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1+1+1 = 3$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 3\iiint_D dx\,dy\,dz$$

$$= 3\times \text{volume of sphere}$$

$$= 3\times\frac{4}{3}\pi = 4\pi$$

Que Find the Surface Integral of $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$ Over the $S = \{(x,y,z) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}$

Soln

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$= 3\iiint_D dx\,dy\,dz = 3\times \text{volume of ellipsoid.}$$

$$= 3\times\frac{4}{3}\pi abc = 4\pi ac$$

Que $\vec{F}: \mathbb{R}^3 \to \mathbb{R}$ twice differentiable scalar field s.t.
$div(\nabla F) = 6$
let S be a Surface $\{(x,y,z) | x^2+y^2+z^2 = 1\}$
Then Find $\iint_S \nabla F \cdot \hat{n}\, ds$

$$= \iiint_D div(\nabla F)\, dx\,dy\,dz$$

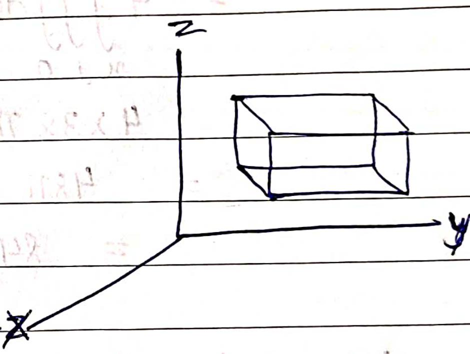
$$= 6 \iiint_D dx\,dy\,dz$$

$$= 6 \times \text{Volume of sphere}$$

$$= 6 \times \frac{4\pi r^3}{3} = 8\pi$$

Que→ Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$. and S be a Surface $\{(x,y,z) | 0 \le x \le 1,$ $0 \le y \le 2,\ 0 \le z \le 3\}$. Then find $\iint_S \vec{F} \cdot \hat{n}\, ds$

Solⁿ $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_D div\vec{F}\, dx\,dy\,dz$



$$= 3 \iiint_D dx\,dy\,dz$$

$$= 3 \times \text{Volume of Cuboid}$$

$$= 3 \times l \times b \times h$$

$$= 3 \times 1 \times 2 \times 3$$

$$= 18$$

Que→ $\vec{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$
$S = \{(x-1)^2 + (y-2)^2 + (z-3)^2 = 1\}$ Find $\iint_S \vec{F} \cdot \hat{n}\, ds$

Solⁿ $\frac{16\pi}{3}$.

Que $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, $S = \{(x,y,z) | x^2 + y^2 = 4;\ z = 0, z = 3\}$
Find $\iint_S \vec{F} \cdot \hat{n}\, ds$.

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Sol^n

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 4 - 4y + 2z$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$= \iiint_D (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_0^3 \int_{-4}^{4} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+2z)\,dy\,dx\,dz$$

$$= \iiint 4\,dy\,dx\,dz - \underbrace{\iiint 4y\,dy\,dx\,dz}_{0} + \iiint 2z\,dy\,dx\,dz$$

$$= 4\iint_{x\,y}\int_0^3 dz\,dy\,dx + \iint_{x\,y}\left[z^2\right]_0^3 dy\,dz$$

$$= 4\times 3\times \pi(2)^2 + 9\times\pi\times(2)^2$$

$$= 48\pi + 36\pi$$

$$= 84\pi \quad \text{Ans}$$

H.W.

Que Let D be the region bdd by the closed surface $x^2+y^2=16$, $z=0$ and $z=4$. Find $\iint_S \vec{F}\cdot\hat{n}\,ds$ when

$$\vec{F} = 3x^2\hat{i} + 6y^2\hat{j} + z\hat{k}$$

Ans 64π.

Que Find $\iint_S \left[\left(\frac{2x}{\pi} + \sin y^2\right)x + \left(e^{z} - \frac{y}{\pi}\right)y + \left(\frac{2z}{\pi} + \sin^2 y\right)z\right] ds$

where $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$.

Sol^n

$$\vec{\nabla F} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla F}}{|\nabla F|} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{F} = \left(\frac{2x}{\pi} + \sin y^2\right)\hat{i} + \left(e^z - \frac{y}{\pi}\right)\hat{j} + \left(\frac{2z}{\pi} + \sin^2 y\right)\hat{k}.$$

$$\operatorname{div}\vec{F} = \frac{2}{\pi} - \frac{1}{\pi} + \frac{2}{\pi} = \frac{3}{\pi}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$= \frac{3}{\pi}\iiint_D dx\,dy\,dz$$

$$= \frac{3}{\pi} \times \frac{4}{3}\pi \times (1)^3 = 4 \text{ Ans}$$

Que $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \dfrac{x\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \dfrac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \dfrac{z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$S = \{1 \le x^2+y^2+z^2 \le 4\}$

Find $\iint_S \vec{F}\cdot\hat{n}\,ds$

Solⁿ $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2}(1) - x\left(\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}(\cancel{2}x)\right)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}\left[(x^2+y^2+z^2) - 3x^2\right]}{(x^2+y^2+z^2)^{3/\cancel{2}}}$$

$$\frac{\partial f_1}{\partial x} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

Sim. $\dfrac{\partial f_2}{\partial y} = \dfrac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$ & $\dfrac{\partial f_3}{\partial z} = \dfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

$$\Rightarrow \operatorname{div}\vec{F} = 0 \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = 0$$

→ $\iint_S \vec{F}\cdot\hat{n}\,ds$ is also called Flux of a vector field F over a Surface S.

→ $S = \{x^2+y^2+z^2=1\}$ → hemisphere.

let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$

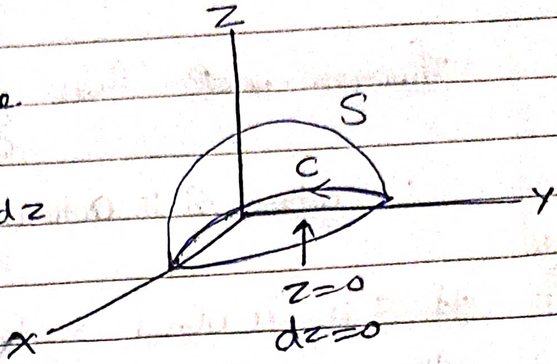
$$\int_C \vec{F}\cdot d\vec{r} = \int_C x\,dx + y\,dy + z\,dz$$

$$= \int_C x\,dx + y\,dy$$

↓ over $x^2+y^2=1$.

⇓

This Can be easily solved now.



→ So we need to Evaluate the line Integral over Circle C, which is part of some Surface S, ie. Evaluation of Line Integral in 3-dimension.

→ let $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$
and S be the upper half of the sphere ie. hemisphere
Then find $\int \vec{F}\cdot d\vec{r}$.

Solⁿ → $\int \vec{F}\cdot d\vec{r} = \int_C (2x-y)dx + 0\,dy$ as $z=0$. in xy plane

where C is $x^2+y^2=1$.

$$= \iint dx\,dy = \pi \quad \text{(Using Green's Thm)}$$

Stokes' Thm :→ If S is an open Surface, $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector field. let c be a closed boundary of Surface 's'. $f_1, f_2, f_3, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are Continuous function. Then $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$

where $\hat{n}$ is outward unit normal vector to S.

Note (1) If S is a Closed Surface Then we Can apply Gauss divergence Thm.

$$\Rightarrow \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iiint_D \operatorname{div}(\nabla\times\vec{F})\,dx\,dy\,dz$$

But $\operatorname{div}(\nabla\times\vec{F}) = \operatorname{div}(\operatorname{Curl}\vec{F}) \equiv 0$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = 0$$

Ex $\vec{F} = 2x\hat{i} + y\hat{j} + z\hat{k}$

and $S = \{x^2+y^2+z^2=1\}$ – upper half of the sphere.

$\Rightarrow$ S is open Surface.

and c be the closed boundary of S. Find $\int_C \vec{F}\cdot d\vec{r}$

Sol^n $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$

$$\nabla\times\vec{F} = \operatorname{Curl}\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & y & z \end{vmatrix} = 0$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = 0$$

Que Find $\oint_C \sin z\, dx - \cos x\, dy + \sin y\, dz$ where

c is boundary of $0 \le x \le \pi$, $0 \le y \le 2$, $z = 4$.

Sol^n (I) $z = 4 \Rightarrow dz = 0$

$$\int_C \sin 4\, dx - \cos x\, dy = \iint_R \frac{\partial(-\cos x)}{\partial x} - \frac{\partial(\sin 4)}{\partial y}\, dx\, dy$$

where R is the region Rectangle bdd by $0 \le x \le \pi$ and $0 \le y \le 2$.

$$= \int_0^2\int_0^\pi \sin x\, dx\, dy$$

$$= \int_0^\pi\int_0^2 \sin x\, dy\, dx$$

$$= \int_0^\pi \sin x\, (y)\Big|_0^2\, dx$$

$$= -2\left[\cos x\right]_0^\pi$$

$$= -2\left[-1-1\right] = 4.$$

(II) $\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$

$$= i\left[\cos y\right] - j\left[-\cos z\right] + \hat{k}\left[\sin x\right]$$

$$= \cos y\,\hat{i} + \cos z\,\hat{j} + \sin x\,\hat{k}$$

$\hat{n} = \hat{k}$ the outward unit normal to the given surface S.

$\Rightarrow \text{Curl } \vec{F} \cdot \hat{n} = \sin x.$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds = \iint_S \sin x\, ds$$

$$= \iint \sin x\, dx\, dy \text{ over } \begin{matrix} 0 \le x \le \pi \\ 0 \le y \le 2 \end{matrix}$$

$$= 4 \text{ (as shown above)}$$

Que $\vec{F} = (z^2 - y)\hat{i} + (x - 2yz)\hat{j} + (2xz - y^2)\hat{k}$
over S - Surface of the sphere
$x^2 + y^2 + z^2 = 9,\ z \geq 0$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C is $x^2 + y^2 = 9,\ z = 0$.

Sol^n→ The unit normal vector to the surface S is

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{1}{3}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y & x - 2yz & 2xz - y^2 \end{vmatrix}$$

$$= \hat{i}(-2y + 2y) - \hat{j}(2z - 2z) + \hat{k}(1+1)$$

$$= 2\hat{k}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

$$= \iint \frac{2}{3} z \cdot \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$= \frac{2}{3} \iint z \cdot \frac{dx\,dy}{(z/3)}$$

$$= 2 \iint_S dx\,dy$$

$= 2 \times$ Area of Circle

$$= 2 \times \pi \times (3)^2$$

$= 18\pi$ Ans

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Que Find $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$

Where S is the Surface of the Sphere $x^2+y^2+z^2=16$, $z>0$.

and C is the Circle $x^2+y^2=16$, $z=0$.

Ans:- 16π