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Set :→ A well defined Collection of objects.
Eg. $A = \{a, e, i, o, u\}$
$A = \{$Set of all prime no.s$\}$.

Empty Set :→ A Set having no Element ie. $A = \phi$
↳ Null / void / Empty

Singleton Set :→ A Set Consisting of only one Element
E.g. $A = \{a\}$, $A = \{i\}$.

Finite Set :→ If it is Either void Set or elements Can be Counted by natural no.s 1, 2, 3 --- And this Counting ends at Certain natural no. n.
Eg. $A = \{1, 2, -- n\}$
$A = \{1, 2, 3 - - -\}$ – $\mathbb{N}$ – Infinite Set

Cardinality of Set / order of Set :→ If A is finite then no. of Elements in A is called order / Cardinality of A.
denoted by $|A|$ or $n(A)$.
→ For Infinite Set,

Subset :→ Let A and B be two Sets. If $\forall\, a \in A,\ a \in B$
Then $A \subseteq B$.

R

→ Every Set is a Subset of itself
$\because \forall\, a \in A,\ a \in A \Rightarrow A \subseteq A$.
→ Empty set is a Subset of Every set.
$\because \phi \subseteq A$ as ϕ has no Element.
} Improper Subsets.

→ A Subset B of A is called Proper Subset If $B \neq \phi$, $B \neq A$.

→ **Power Set :→** let A be a set.

Then collection of all subsets of A is called Power set of A.

$$P(A) = \{ B \mid B \subset A \}.$$

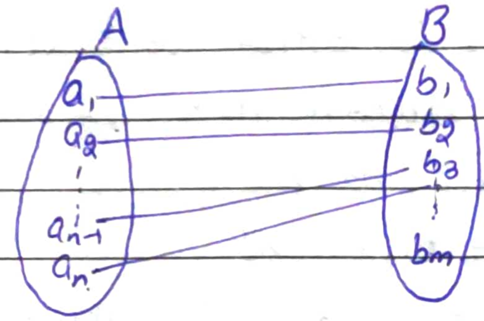
→ **# of Elements In $P(A)$:→** If A is finite Set

let $|A| = n$

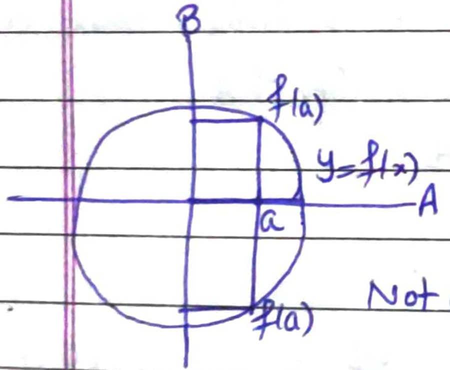
$$\boxed{\cancel{{}^2C_0 + {}^2C_1}} \quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = (1+1)^n = 2^n$$

($\because$ no. of Subsets of A having r elements $= {}^nC_r$)

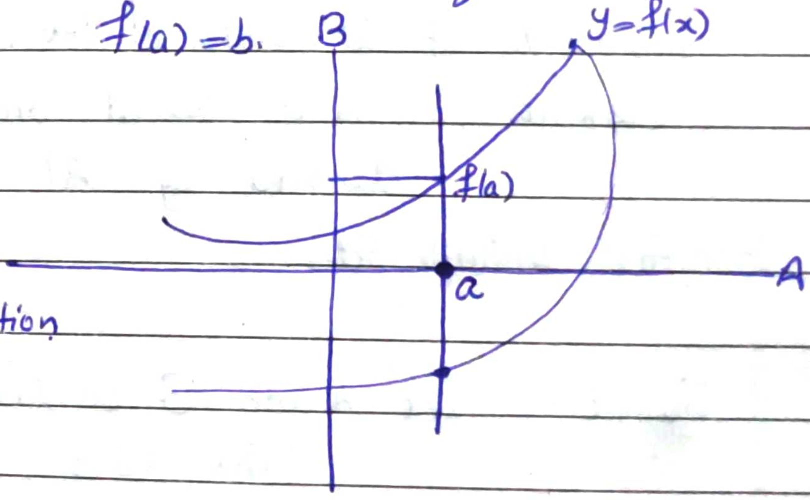
Function :→



let $A, B, \neq \phi$ sets. If $\forall\, a \in A$, $\exists$ unique $b \in B$ s.t: $f(a) = b$.



Not a function



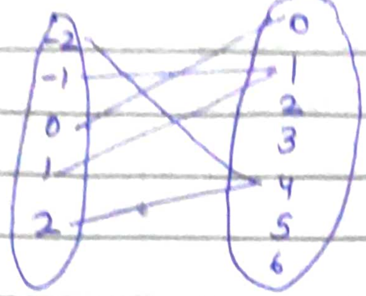
→ For every line passing through x-axis, $\parallel$ to y-axis, It cuts the curve $y = f(x)$ exactly once.

→ A – Domain of f

B – Co-Domain of f.

Ex $f: A \to B$; $A = \{-2, -1, 0, 1, 2\}$

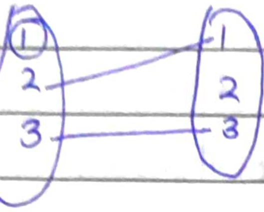
$f(x) = x^2$ $B = \{0, 1, 2, 3, 4, 5, 6\}$.



function

Ex $f: A \to B$

$f(x) = 2x - 3$; $A = \{1, 2, 3\}$, $B = \{1, 2, 3\}$



Not a function.

→ # of functions from $A \to B$ where $|A| = n$, $|B| = m$.

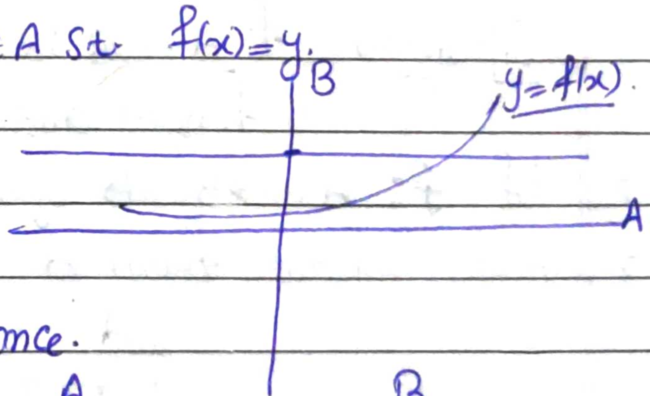
$= m^n$. $\because \underbrace{m \times m \times \dots \times m}_{n \text{ times}}$

→ $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$

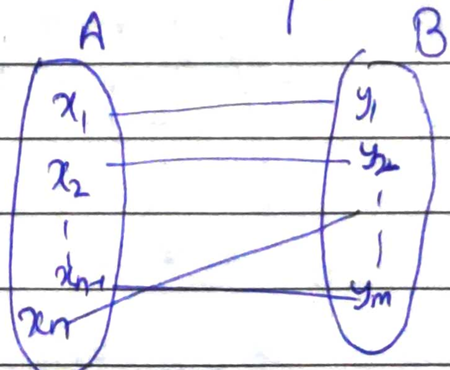
\# of fun. from $A \to B$ s.t. $f(2) = c$, $f(x) \neq b \; \forall x \in$

→ Onto functions :→ $f: A \to B$

$\forall y \in B, \exists x \in A$ s.t. $f(x) = y$.



Geometrically - Every line passing through y-axis ∥ to x-axis Cuts the Curve $y = f(x)$ atleast once.



› If $|B| > |A|$ Then ∄ onto fun. from $A \to B$.

$f: A \to B$ onto $\Rightarrow$

$|A| \geq |B|$.

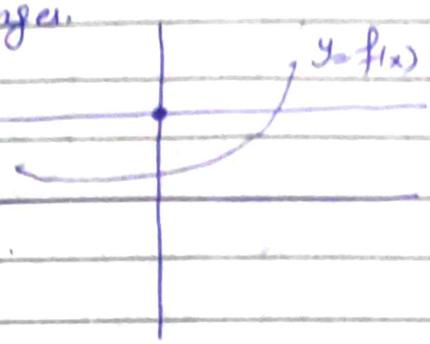
› If $|A| \geq B \Rightarrow \exists f: A \to B$ onto fun.

→ **1-1 function** :→ $f: A \to B$

$\forall\ x_1 \neq x_2,\quad f(x_1) \neq f(x_2)$

or If $f(x_1) = f(x_2)$ Then $x_1 = x_2$.

i.e. Distinct Elements have distinct Images.



Graphically $\forall$ line passing through y-axis, $\parallel$ to x-axis cuts the curve $y = f(x)$ atmost once.

→ $f: A \to B$ is 1-1 $\Rightarrow |A| \leq |B|$

→ If $|A| \leq |B|$ Then $\exists\ f: A \to B$ s.t. f is 1-1.

→ If $|A| > |B|$ Then $\nexists\ f: A \to B$ s.t. f is 1-1.

→ f, 1-1, g 1-1 $\Rightarrow f \circ g$ 1-1

→ f, g onto $\Rightarrow f \circ g$ onto.

→ f, g 1-1, onto $\Rightarrow f \circ g$ 1-1, onto.

→ $f \circ g$ 1-1 onto $\Rightarrow g$ 1-1 & f onto.

→ $f \circ g$ 1-1 $\Rightarrow g$ is 1-1

→ $f \circ g$ is onto $\Rightarrow f$ is onto.

$f: A \to B,\ g: B \to C$

$g \circ f: A \to C$

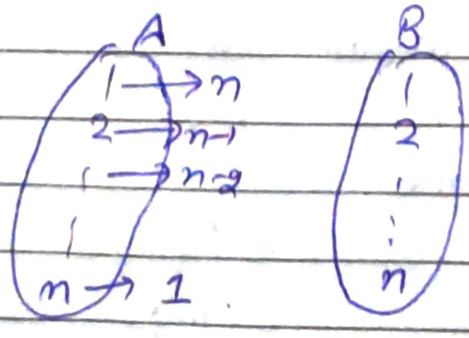
$R(f) \subseteq D(g)$

$f \circ g$ does not Exist.

Bijection :→ $f: A \to B$ is called Bijection If f is 1-1 and onto.

→ If $f: A \to B$ is bijection $\Leftrightarrow |A| = |B|$

→ # of bijection from $A \to B$ is $n!$



$= \lfloor n$ choices.

Real function:→ If the domain and Co-domain of a function are Subsets of $\mathbb{R}$ (set of real no.s)

Range of function $f: A \to B$ is a fun.

$R(f) = \{f(x) \mid x \in A\}$.

$R(f) \subset B$

If $R(f) = B$ Then f is onto.

Domain of Real function:→ Dom of f is the set of all real no.s x for which f(x) is real no.

Ex find dom of fun. $f(x) = \dfrac{1}{\sqrt{9-x^2}}$

$9 - x^2 > 0 \Rightarrow x^2 - 9 < 0$

$\Rightarrow x^2 < 9$

$\Rightarrow x \in (-3, 3)$

Even function → If $f(-x) = f(x) \ \forall x$

odd function → If $f(-x) = -f(x) \ \forall x$

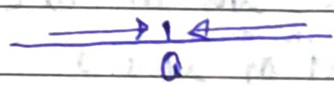
Ex $f(x) = x^2$ Even fun

$f(x) = x^3$ odd fun.

Ex S.T. $f(x) = \log(x + \sqrt{x^2+1})$ is an odd fun.

Limits:→

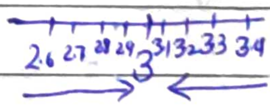
$f(x)$



$x \to a$ from left then $f(x)$

$x \to a$ from right then $f(x)$

$\lim_{x \to 3^-} f(x) = LHL$



$\lim_{x \to 3^+} f(x) = RHL$

If LHL = RHL

Then limit Exists.

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$\lim_{x\to a} f(x)$

LHL Put $x = a-h$, then $\lim_{h\to 0} f(a-h)$

RHL $\lim_{x\to a^+} f(x)$
Put $x = a+h$
then $\lim_{h\to 0} f(a+h)$

Ex
$$f(x) = \begin{cases} \frac{x-|x|}{x} & ; x \neq 0 \\ 2 & ; x = 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{-h-|-h|}{-h} = \lim_{h\to 0} \frac{-h-h}{-h}$$
$$= +2$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{h-|h|}{h} = 0$$

$\lim_{x\to 0} f(x)$ doesn not Exist

Neighbourhood of a pt:→

$a-\delta \quad a \quad a+\delta$

Set of real no.s lying b/w $(a-\delta, a+\delta)$ is called nbd of a of radius δ, denoted by $N_\delta(a)$.

Deleted nbd of a:→ $N_\delta(a) - \{a\}$

So $(a-\delta, a)$ left nbd of a
$(a, a+\delta)$ right nbd of a.

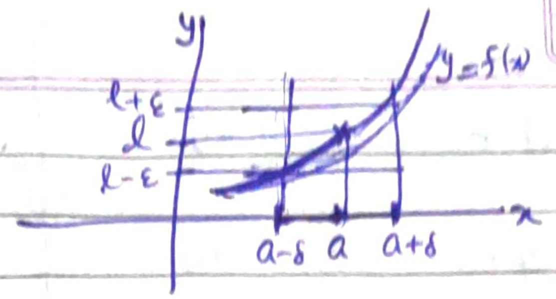
x lies in $(a-\delta, a+\delta) \Rightarrow a-\delta < x < a+\delta \Rightarrow |x-a| < \delta$.

Def^n of limit:→ $\lim_{x\to a} f(x) = l$

$$|f(x) - l| < \varepsilon \text{ if } |x-a| < \delta$$

if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t.
$$|f(x) - l| < \varepsilon \text{ ~~if~~ when } |x-a| < \delta.$$

So l is the lt of $f(x)$ as x tends to a
if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t.
$|f(x) - l| < \varepsilon$ when $|x-a| < \delta$.



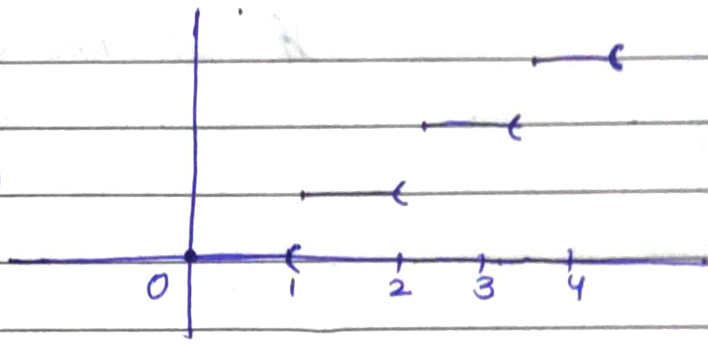
→ $f(x)$ is cts at a if $\lim\limits_{x\to a^-} f(x) = f(a) = \lim\limits_{x\to a^+} f(x)$

→ $f(x)$ is cts at $x=a$ If $f(a)$ is defined

$\lim\limits_{x\to a} f(x)$ Exist

and $\lim\limits_{x\to a} f(x) = f(a)$

Ex $f(x) = [x]$



$$\lim_{x\to n^-} f(x) = \lim_{h\to 0} f(n-h)$$
$$= \lim_{h\to 0} [n-h]$$
$$= n-1$$

$$\lim_{x\to n^+} f(x) = \lim_{h\to 0} f(n+h) = \lim_{h\to 0} [n+h] = n.$$

$\lim\limits_{x\to n^+} f(x) \neq \lim\limits_{x\to n^-} f(x) \Rightarrow \lim$ does not Exist.

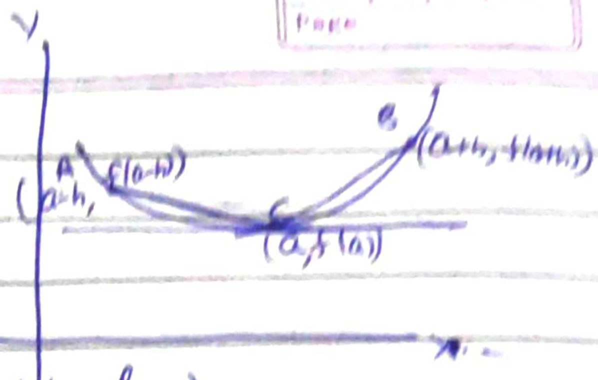
→ Differentiability at a point → $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$

f is diff. at a Iff $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ Exist finitely.

or $\lim\limits_{x\to a^-} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a}$

or $\underbrace{\lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h}}_{LHD} = \underbrace{\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}}_{RHD}$

Slope of $AC = \frac{f(a-h)-f(a)}{-h}$



Slope of $BC = \frac{f(a+h)-f(a)}{h}$

$$\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} (\text{slope of } AC)$$

LHD $= \lim_{A\to C} (\text{Slope of } AC)$

$=$ Slope of tangent line at C.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \text{slope of tangent line at C.}$$

RHD

If LHD $=$ RHD $\iff$ There is ! tangent line at C.

$\iff$ no sharp Edge.

→ $f: \mathbb{R}^2 \to \mathbb{R}$ is a function

If $\forall (x,y) \in \mathbb{R}^2$, $\exists$ unique $z \in \mathbb{R}$ s.t.

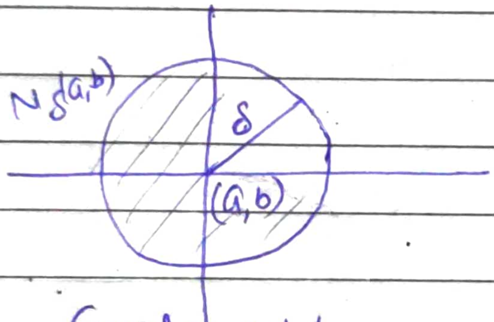
$$z = f(x,y)$$

→ <u>Neighbourhood of (a,b):→</u>

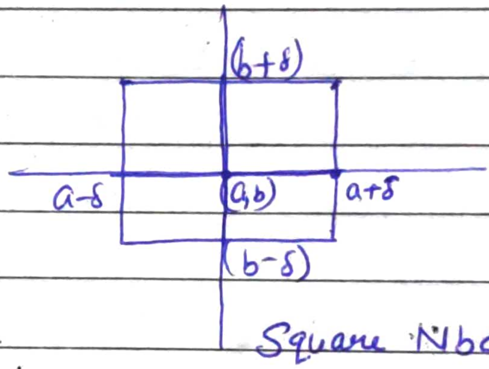
$$N_\delta(a,b) = \{(x,y) \mid \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$$

or

$$N_\delta(a,b) = \{(x,y) \mid |x-a| < \delta,\ |y-b| < \delta\}$$



Circular Nbd



Square Nbd.

→ <u>Limit of a function:→</u>

If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a function

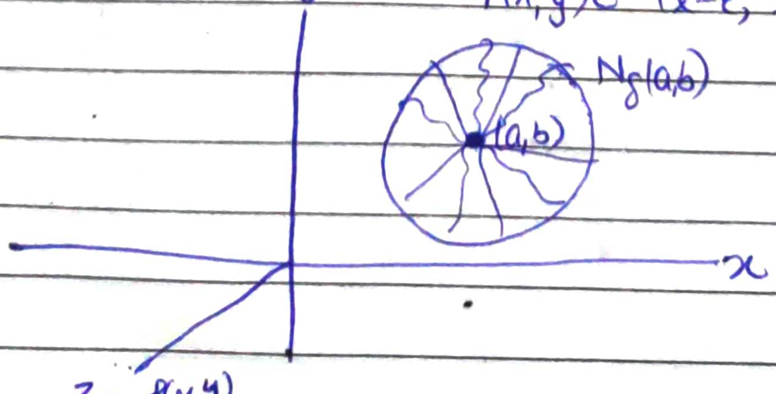
Then $l \in \mathbb{R}$ is called limit of f

as $(x,y) \to (a,b)$ If

$\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t.

~~f(x~~ If $(x,y) \in N_\delta(a,b) - \{(a,b)\}$ Then

$f(x,y) \in (l-\varepsilon, l+\varepsilon)$



Ex $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0 \end{cases}$ $f : \mathbb{R}^2 \to \mathbb{R}$

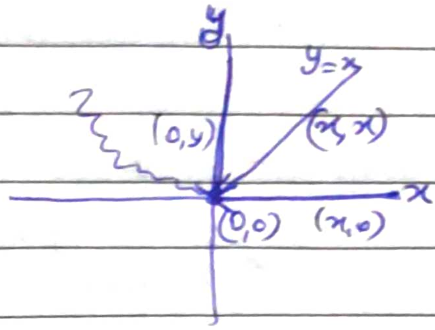
Then at $(0,0)$ $\lim f$ Exist or not.

Sol^n along x-axis

$$\lim_{(x,0)\to(0,0)} f(x,y) = 1$$

along y-axis

$$\lim_{(0,y)\to(0,0)} f(x,y) = -1$$



along line $y = x$

$$\lim_{(x,x)\to(0,0)} f(x,y) = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \text{ doesnot Exist.}$$

($\because$ as we are approaching towards different values in different directions)

Ex→ $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} ; & x-y \neq 0 \text{ or } x \neq y \\ 0 ; & x = y \end{cases}$ Then $\displaystyle \lim_{(x,y)\to(0,0)} f(x,y) = ?$

along x-axis ; $\displaystyle \lim_{(x,0)\to(0,0)} f(x,y) = 0$

along y-axis ; $\displaystyle \lim_{(0,y)\to(0,0)} f(x,y) = 0$

along $y = mx$ line ; $\displaystyle \lim_{(x,mx)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{x^3(1+m^3)}{x(1-m)} = 0$

Put $x = r\cos\theta$; $y = r\sin\theta$; $r = \sqrt{x^2+y^2}$

$(x,y) \to (0,0)$
$\Rightarrow r \to 0$
and θ is any angle.

$$f(x,y) = f(r,\theta) = \frac{r^3(\cos^3\theta + \sin^3\theta)}{r(\cos\theta - \sin\theta)}$$

$$= \frac{r^2(\cos^3\theta + \sin^3\theta)}{(\cos\theta - \sin\theta)}$$

$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1$
$\Rightarrow \theta = \pi/4$

$\Rightarrow$ for all those curves whose slope is 1 at $(0,0)$, $\lim_{(x,y)\to(0,0)} f(x,y)$ may not exist.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}; & x^2+y^2 \neq 0 \\ 0; & x^2+y^2 = 0 \end{cases}$; $\lim_{(x,y)\to(0,0)} f(x,y) = ?$

$x = r\cos\theta, \quad y = r\sin\theta$

$f(r,\theta) = \dfrac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta\sin\theta$

different values for different θ's.

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity $\rightarrow$ If $f: D \rightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^2$
Then f is called continuous function at $P = (a,b) \in D$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}; & x^2+y^2 \neq 0 \\ 0; & x^2+y^2 = 0 \end{cases}$ at $P(0,0) \in \mathbb{R}^2$

$\because \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\Rightarrow f(x,y)$ is not cts at $(0,0)$

$f: \mathbb{R}^2 \to \mathbb{R}$

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Ex:→ $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right); & xy \neq 0 \\ 0 \quad ; & xy = 0 \end{cases}$

Then at $(0,0)$, f is cts or not?

Sol^n

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \left(\because \sin\tfrac{1}{x}, \sin\tfrac{1}{y} \text{ bdd}\right)$$
$$= f(0,0)$$

Ex:→ $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}; & x^2+y^2 \neq 0 \\ 0 \quad ; & x^2+y^2 = 0 \end{cases}$ at $(0,0)$

Sol^n

$x = r\cos\theta$; $y = r\sin\theta$

$(x,y) \to 0 \Rightarrow r \to 0$ and θ is any angle

$$f(r,\theta) = \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = r(\cos^3\theta - \sin^3\theta)$$

$\to 0$ as $r \to 0$

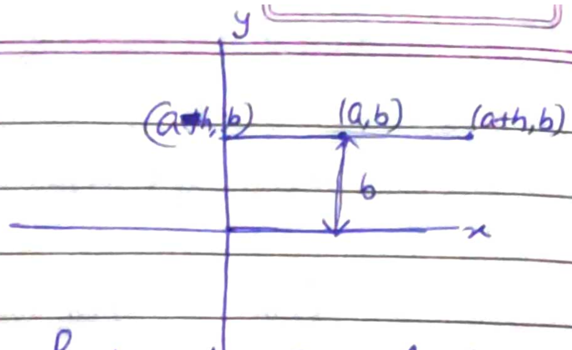
and $(\cos^3\theta - \sin^3\theta)$ is bdd

and $f(0,0) = 0$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Partial Derivatives :→

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$



$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

f_x and f_y are called first order Partial derivatives of f w.r.t. x and y respectively.

Ex $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & ; x^2+y^2=0 \end{cases}$. $P(0,0)$ be any point.

find $f_x(0,0)$ and $f_y(0,0)$.

Solⁿ $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$

But $\lim_{(x,y)\to(0,0)} f(x,y)$ does not Exist

$\Rightarrow$ $f(x,y)$ is not continuous at $(0,0)$.

Ex $f: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right) & ; x\neq 0, y\neq 0 \\ x\sin\left(\frac{1}{x}\right) & ; x\neq 0, y=0 \\ y\sin\left(\frac{1}{y}\right) & ; x=0, y\neq 0 \\ 0 & ; x=0, y=0. \end{cases}$

Find $f_x(0,0)$ & $f_y(0,0)$

Solⁿ $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h\sin\frac{1}{h}}{h}$

$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \to$ doesnot Exist.

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{k\sin\left(\frac{1}{k}\right)}{k} \to$ doesnot Exist

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But $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

$\Rightarrow f(x,y)$ is continuous fun. at $(0,0)$

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$,
S.t. f has both first order Partial derivatives Exist at Point $(a,b) \not\Rightarrow f$ is cts at point (a,b)

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$
S.t. f is continuous at Point $(a,b) \not\Rightarrow f$ has both first order partial derivatives exist at point (a,b).

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$
S.t. $f_x(a,b)$ and $f_y(a,b)$ Exist and are bdd in the neighbour-hood of $(a,b) \Rightarrow f$ is cts function at point (a,b).

H.W (1) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} 1 \; ; & (x,y) \neq (0,0) \\ 0 \; ; & (x,y) = (0,0) \end{cases}$$

Find $f_x(0,0)$ & $f_y(0,0)$.

(2) $$f(x,y) = \begin{cases} \sin\left(\dfrac{xy}{x^2+y^2}\right) \; ; & \text{If } (x,y) \neq (0,0) \\ 0 \; ; & \text{If } (x,y) = (0,0) \end{cases}$$

Find $f_x(0,0)$ and $f_y(0,0)$. Is f Continuous at $(0,0)$.

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Que $f(x,y) = \sqrt{|xy|}$

Find $f_x(0,0)$, $f_y(0,0)$ and Cont. at $(0,0)$

Solⁿ $f(0,0) = 0$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$f_y(0,0) = \lim_{h\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \sqrt{|r^2 \cos\theta \sin\theta|}$

$= \lim_{r\to 0} |r| \sqrt{|\cos\theta \sin\theta|} \to 0 = f(0,0)$

$\Rightarrow f$ is cts at $(0,0)$

Que $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & ; x^2+y^2 = 0 \end{cases}$

is cts at $(0,0)$; $f_x(0,0)$, $f_y(0,0)$?

$f_{xy}(a,b) = \lim_{k\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$

$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k}$

$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$

$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$

$z = f(x,y)$

$\frac{\partial z}{\partial x} = f_x$; $\frac{\partial z}{\partial y} = f_y$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$; $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$

Que

$$f(x,y) = \begin{cases} \dfrac{x^3y}{x^2+y^2} & ; \ x^2+y^2 \neq 0 \\ 0 & ; \ x^2+y^2 = 0 \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \frac{\frac{h^3k}{h^2+k^2}}{h} = \lim_{h\to 0} \frac{h^2k}{h^2+k^2} = 0$$

$$f_{yx}(0,0) = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{h^3k}{(h^2+k^2)k}$$

$$= \lim_{k\to 0} \frac{h^3}{h^2+k^2} = h$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

→ $f(x,y) = \log(xy+2y^2-2x)$

Find $f_x(2,3)$ & $f_y(2,3)$

$$f_y = \frac{1}{xy+2y^2-2x}(x+4y)$$

$$f_x = \frac{1}{xy+2y^2-2x}(y-2)$$

$$f_y(2,3) = \frac{14}{6+18-4} = \frac{14}{20} = 7/10$$

$$f_x(2,3) = \frac{1}{6+18-4}(1) = \frac{1}{20}$$

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Differentiability:→

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

Then f is diff. at point $(a,b) \in D$ If

$$f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$$

where A, B are real constants and $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$.

i.e. $$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - (Ah+Bk)}{\sqrt{h^2+k^2}} = 0.$$

→ If $h=0$ Then

$$f(a,b+k) - f(a,b) = Bk + |k|\,\phi(0,k)$$

$$\Rightarrow \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = B + \lim_{k\to 0} \phi(0,k)$$

$$\Rightarrow B = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = f_y(a,b)$$

Similarly, If $k=0$ Then

$$f(a+h, b) - f(a,b) = Ah + |h|\,\phi(h,0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = A$$

$$\Rightarrow \boxed{A = f_x(a,b)}$$

and $\boxed{B = f_y(a,b)}$

→ If f is differentiable at point $(a,b) \Rightarrow f$ is Continuous at pt (a,b).

→ If f is diff. at point $(a,b) \Rightarrow f_x(a,b)$ and $f_y(a,b)$ Exist.

i.e. Differentiability at $(a,b) \Rightarrow$ Partial derivatives at (a,b).

But Converse need not to be true.

→ If $f_x(a,b)$ and $f_y(a,b)$ Exist and both f_x and f_y are Continuous at $(a,b) \Rightarrow f$ is diff at (a,b).

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Que $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right) & ; x\neq 0, y\neq 0 \\ x\sin\left(\frac{1}{x}\right) & ; x\neq 0, y=0 \\ y\sin\left(\frac{1}{y}\right) & ; x=0, y\neq 0 \\ 0 & ; x=0, y=0. \end{cases}$

Sol^n f is continuous at (0,0)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\sin\left(\frac{1}{h}\right)}{\cancel{h}} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not Exist

$\Rightarrow$ f is not differentiable at (a,b).

Que $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0). \end{cases}$

Sol^n

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta)$$

$$= \lim_{r\to 0} r(\cos\theta\sin\theta) \to 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$\Rightarrow$ f is cts at (0,0).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

$$f(h,k) - f(0,0) = A\cdot h + B\cdot k + \sqrt{h^2+k^2}\cdot\phi(h,k)$$

$$\Rightarrow \frac{hk}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{hk}{h^2+k^2} \quad ; \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$ does not Exist

$\Rightarrow f$ is not diff. at $(0,0)$.

Ex $f(x,y) = |x| + |y|$; $\forall (x,y) \in \mathbb{R}^2$.

Check at point $(0,0)$.

Sol^n $f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} |r|\,(|\cos\theta| + |\sin\theta|)$$

$$= 0 = f(0,0)$$

$\Rightarrow f$ is cts at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

limit does not Exist

$\Rightarrow f$ is not diff at $(0,0)$.

Ex $f(x,y) = \sqrt{|xy|}$; $\forall (x,y) \in \mathbb{R}^2$

Sol^n f is Cont. at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = 0$$

$$f(h,k) - f(0,0) = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$$

$h = r\cos\theta,\ k = r\sin\theta$

$$\lim_{r\to 0} \phi(r,\theta) = \frac{\cancel{|r|}\sqrt{|\cos\theta\sin\theta|}}{\cancel{|r|}}$$

$\hookrightarrow$ Limit does not Exist.

$\Rightarrow f$ is not diff. at $(0,0)$.

H.W

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0. \end{cases}$$

S.T. $f(x,y)$ is cont. at $(0,0)$ but not diff. at $(0,0)$.

Que

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) ; & x \neq 0, y \neq 0 \\ x^2 \sin\left(\frac{1}{x}\right) ; & x \neq 0, y = 0 \\ y^2 \sin\left(\frac{1}{y}\right) ; & x = 0, y \neq 0 \\ 0 ; & x = 0, y = 0. \end{cases}$$

Find $f_x(0,0)$, $f_y(0,0)$ and check If f is diff at $(0,0)$ or not.

→ **Exact differential / Total differential** ⇒

~~Let $z = f(x,$~~

Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

$z = f(x,y)$.

Let dx and dy represent changes in x and y respectively. If the partial derivatives f_x and f_y Exist Then the total differential of z is

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

Or

$$\boxed{dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy}$$

Ex

Let $z = x^4 e^{3y}$

Find dz

Soln

$$\frac{\partial z}{\partial x} = 4x^3 e^{3y}$$

$$\frac{\partial z}{\partial y} = 3x^4 e^{3y}$$

$$dz = (4x^3 e^{3y})dx + (3x^4 e^{3y})dy$$

$$\boxed{dz = x^3 e^{3y}(4dx + 3x\,dy)}$$

Maxima / Minima at a point:→

Def^n:→ Positive definite Matrix:→ Matrix A is called Positive definite If all the principal Minors of A are Positive.

i.e. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

then $M_{11} = a > 0$

$M_{22} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc > 0$

Negative definite Matrix:→ Matrix A is called negative definite If all the principal Minors are alternatively negative and positive starting with the negative sign.

i.e. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $M_{11} = a < 0$

$M_{22} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc > 0.$

So $(-1)^i M_{ii} > 0.$

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

→ Then $(a,b) \in D$ is called point of local Maxima If $\exists\ \delta > 0$ s.t.

$$f(x,y) \leq f(a,b) \quad \forall\ (x,y) \in N_\delta(a,b)$$

→ $(a,b) \in D$ is called point of local Minima If $\exists\ \delta > 0$ s.t.

$$f(x,y) \geq f(a,b) \quad \forall\ (x,y) \in N_\delta(a,b).$$

→ Maximum or Minimum value of function at point (a,b) is called an Extreme value.

→ **Critical Points:→** Let f has first order partial derivatives at point (a,b). Then $(a,b) \in D$ is called a critical point If $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

→ **Saddle points:→** If $(a,b) \in D$ is critical point but it is neither point of Minima nor point of Maxima then (a,b) is called saddle point.

Methodology:→

If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a function s.t. f_x, f_y, f_{xx}, f_{yy}, f_{xy}; f_{yx} exist and continuous at $(a,b) \in D$.

($f_{xy} = f_{yx}$ In this case)

Then (a,b) is point of Extrema

$\Rightarrow f_x(a,b) = 0$ and $f_y(a,b) = 0$.

→ $$J_{(a,b)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(a,b)}$$

→ If J matrix is Positive definite Then (a,b) is Called Point of Minima.

→ If J matrix is negative definite Then (a,b) is called Point of maxima.

→ If $|J_{(a,b)}| < 0$ Then (a,b) is Called Saddle point. ie. neither point of minima nor maxima.

→ If $|J_{(a,b)}| = 0$ Then may or may not have Extrema at point (a,b).

Que $f(x,y) = x^3 + y^2 - 12x - 6y + 40$.

$f_x = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$(2,3), (-2,3)$

↓ ↓ Saddle point

Que $f(x,y) = x^2 + xy + y^2 - 3x - 6y + 11$

$f_x = 0$

$f_y = 0$

→ Let A function $f(x,y)$ be defined in region $\mathbb{R}$.
The minimum and maximum values attained by a function over the entire region including the boundary are called the absolute (global) minimum and absolute (global) maximum values respectively.

Q $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3 = 0$

$\Rightarrow x = \pm 1$

$f_y = 3y^2 - 12 = 0$

$\Rightarrow y = \pm 2$

$(1,2)\ (1,-2)\ (-1,2)\ (-1,-2)$

at $\underline{(1,2)}$

$f_{xx} = 6x = 6$

$f_{xy} = 0 = f_{yx}$

$f_{yy} = 6y = 12$

$$J = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$$

$6 > 0, \quad |J| > 0$

$\Rightarrow (1,2)$ is point of minima.

at $(1,-2)$

$f_{xx} = 6$

$f_{yy} = -12$

$$J = \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix}$$

$6 > 0, \quad \underline{|J| = -72 < 0}$

$\hookrightarrow$ Saddle point

at $(-1,2)$

$f_{xx} = -6, \quad f_{yy} = 12$

$$J = \begin{bmatrix} -6 & 0 \\ 0 & +2 \end{bmatrix}$$

$-6 < 0, \quad |J| = \underline{-72 < 0}$

Saddle pt.

at $(1,-2)$

$f_{xx} = -6, \; f_{yy} = -12$

$$J = \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix}$$

$-6 < 0, \; |J| > 0 \Rightarrow J$ is negative definite

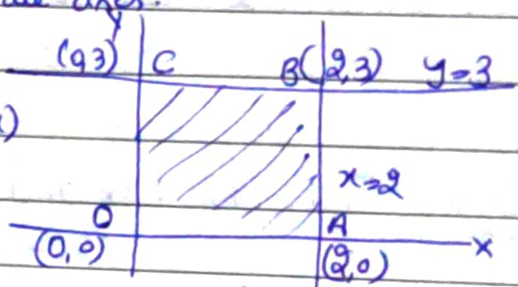
$\Rightarrow$ $(-1,-2)$ is pt of minima.

<u>Que</u>:→ Find the absolute max and min values of

$$f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the I^{st} Quadrant bounded by the lines $x=2$, $y=3$ and the co-ordinate axes.

<u>Solⁿ</u>:→ on the line OA, $y=0$



$\Rightarrow f(x,y) = f(x,0) = 4x^2 - 8x + 4 = g(x)$

$g'(x) = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

$g''(x) = 8 > 0$

$\Rightarrow$ fun. $g(x)$ has minimum at $x=1$

and $g(1) = 0$.

$\therefore$ at $f(0,0) = 4 = g(0)$; $f(2,0) = g(2) = 16 - 16 + 4 = 4$.

On the line OC, $x=0$

$f(x,y) = f(0,y) = 9y^2 - 12y + 4 = h(y)$

$h'(y) = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = 2/3$

$h''(y) = 18 > 0$

$\Rightarrow h(y)$ has min at $y = 2/3$

and $h(2/3) = 4 - 8 + 4 = 0$

$f(0,0) = h(0) = 4$ and $f(0,3) = h(3) = 81 - 36 + 4 = 49$.

on the line AB, $x=2$

$f(x,y) = f(2,y) = 9y^2 - 12y + 4 = h(y)$ → Same as above.

$f(2,3) = f(0,3) = 49$.

on the line BC, $y=3$, $f(x,3) = 4x^2 - 8x + 49$.

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Lagrange's Method of Multipliers:→

Suppose we have to find the Critical / Stationary points of a function $f(x,y,z)=0$ Subject to the Constraint

$g(x,y,z)=0$. — (1)

then The Critical points of the given function f are given by determining the Critical points of F, where

$F = f + \lambda g$

or $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$.

Here λ is Called Lagrange Multiplier.

The Critical points of F are determined by

$F_x = 0$, $F_y = 0$ and $F_z = 0$ — (2)

Ex:→ Find the Max and Min of x^2+y^2 s.t. the Constraint

$3x^2+4xy+6y^2-140=0$

Sol^n:→ $f(x,y) = x^2+y^2$

S.t. $3x^2+4xy+6y^2-140=0$ — (1)

Let $F(x,y) = x^2+y^2+\lambda(3x^2+6y^2+4xy-140)$.

Where λ is Lagrange's Multiplier.

$$\frac{\partial F}{\partial x} = 2x + \lambda(6x+4y) = 0$$

$$\frac{\partial F}{\partial y} = 2y + \lambda(12y+4x) = 0$$

$\Rightarrow$ $(1+3\lambda)x + 2\lambda y = 0$ — (2)

$2\lambda x + (1+6\lambda)y = 0$. — (3)

$\because$ x and y are non-zero,

$$\begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix} = 0 \Rightarrow (1+3\lambda)(1+6\lambda) - 4\lambda^2 = 0$$

$\Rightarrow 14\lambda^2 + 9\lambda + 1 = 0$

$\Rightarrow \boxed{\lambda = -\frac{1}{2}, -\frac{1}{7}}$

$\lambda = \frac{-1}{2} \Rightarrow \left(1-\frac{3}{2}\right)x + 2\left(\frac{-1}{2}\right)y = 0$

$\Rightarrow -\frac{1}{2}x - y = 0 \Rightarrow x = -2y$

From (1); $3x^2 + 4xy + 6y^2 = 140$

$\Rightarrow 3(4y^2) + 4y(-2y) + 6y^2 = 140$

$\Rightarrow 12y^2 - 8y^2 + 6y^2 = 140$

$\Rightarrow 10y^2 = 140 \Rightarrow y^2 = 14$

So $x = -2y \Rightarrow x^2 = 4y^2$

$\Rightarrow x^2 = 14 \times 4 = 56$

So $f(x,y) = x^2 + y^2 = 56 + 14 = 70.$

$\lambda = \frac{-1}{7} \Rightarrow y = 2x$

From (1); $3x^2 + 4x(2x) + 6(4x^2) = 140$

$\Rightarrow 35x^2 = 140 \Rightarrow x^2 = 4$

$\cancel{\Rightarrow x = \pm 2}$

$y = 2x \Rightarrow y^2 = 16$

So $f(x,y) = x^2 + y^2 = 4 + 16 = 20.$

$\Rightarrow$ Min. value of $f(x,y) = 20$ and Max value is 70.

Ex $f(x,y,z) = x^2+y^2+z^2$ s.t $lx + my + nz = p$.

Soln:→ Consider $F(x,y,z) = x^2+y^2+z^2+\lambda(lx+my+nz-p)$

$F_x = 2x + l\lambda = 0$

$\Rightarrow x = -\frac{l\lambda}{2}$

$F_y = 2y + m\lambda = 0 \Rightarrow y = \frac{-m\lambda}{2}$

$F_z = 2z + n\lambda = 0 \Rightarrow z = \frac{-n\lambda}{2}$

$lx + my + nz = p \Rightarrow -\frac{l^2\lambda}{2} - \frac{m^2\lambda}{2} - \frac{n^2\lambda}{2} = p$

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$\Rightarrow -\lambda(l^2+m^2+n^2) = 2p$

$\Rightarrow \boxed{\lambda = \dfrac{-2p}{l^2+m^2+n^2}}$

$\Rightarrow x = \dfrac{lp}{l^2+m^2+n^2}$; $y = \dfrac{mp}{l^2+m^2+n^2}$; $z = \dfrac{np}{l^2+m^2+n^2}$ — (1)

is a critical points.

$F_{xx} = 2$, $F_{yy} = 2$, $F_{zz} = 2$

$F_{xy} = 0$, $F_{yz} = 0$, $F_{zx} = 0$

$$\Rightarrow J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$M_{11} = 2 > 0$

$M_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$M_{33} = |J| = 8 > 0$

$\Rightarrow$ J is positive definite matrix ~~at~~

$\Rightarrow$ (1) is point of Minima of $f(x,y,z)$

and Min value is

$$\frac{l^2p^2 + m^2p^2 + n^2p^2}{(l^2+m^2+n^2)^2} = \frac{p^2}{l^2+m^2+n^2}.$$

Ex Find the Minimum value of $x^2+y^2+z^2$ s.t. $x+y+z = 3a$

Ans: $3a^2$

Ex Find the Min. value of $x^2+y^2+z^2$ s.t. $yz+zx+xy = 3a^2$

Ans: $3a^2$

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Ex Find the Max and Min value of $x^2+y^2+z^2$

S.t. the Constraints $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $z=x+y$.

Sol^n $F(x,y,z) = x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(z-x-y)$

$F_x = 2x + \frac{\lambda_1 x}{2} - \lambda_2 = 0$ — (1)

$F_y = 2y + \frac{2}{5}y\lambda_1 - \lambda_2 = 0$ — (2)

$F_z = 2z + \frac{2z}{25}\lambda_1 + \lambda_2 = 0$ — (3)

Mul (1) by x, (2) by y and (3) by z and adding, we get

$2(x^2+y^2+z^2) + \lambda_1\left(\frac{x^2}{2}+\frac{2}{5}y^2+\frac{2z^2}{25}\right)+\lambda_2(z-x-y) = 0$

$\Rightarrow 2(x^2+y^2+z^2) + 2\lambda_1(1) + \lambda_2(0) = 0$

$\Rightarrow \lambda_1 = -(x^2+y^2+z^2) = -u^2$

Put λ_1 in (1), (2) and (3), we get

$2x - \frac{x}{2}(x^2+y^2+z^2) - \lambda_2 = 0$

~~$\Rightarrow \lambda_2 = 2x - \frac{u^2x}{2} = \frac{4x-u^2x}{2} = \left(\frac{4-u^2}{2}\right)x$~~

$\Rightarrow x\left(2-\frac{1}{2}u^2\right) = \lambda_2$

$\Rightarrow x = \frac{2\lambda_2}{4-u^2}$

$2y + \frac{2}{5}y(-u^2) = \lambda_2$

$\Rightarrow y\left(2-\frac{2}{5}u^2\right) = \lambda_2 \Rightarrow y = \frac{5\lambda_2}{10-2u^2}$

and $2z + \frac{2z(-x^2)}{25} + \lambda_2 = 0$

$\Rightarrow z\left(2 - \frac{2}{25}x^2\right) = -\lambda_2$

$\Rightarrow z = \frac{48\lambda_2}{10-2x^2}$ $\Rightarrow z = \frac{-25\lambda_2}{50-2x^2}$

Now $z = x+y$

$\Rightarrow \frac{-25\lambda_2}{50-2x^2} = \frac{2\lambda_2}{4-x^2} + \frac{5\lambda_2}{10-2x^2}$

$\Rightarrow \frac{-10\lambda_2}{10-2x^2} = \frac{2\lambda_2}{4-x^2}$

$\Rightarrow -5(4-x^2) = 10-2x^2$

$\Rightarrow -20+5x^2 = 10-2x^2$

$\Rightarrow 7x^2 = 30 \Rightarrow x^2 = \frac{30}{7}$

$\Rightarrow \frac{-25}{50-2x^2} = \frac{2}{4-x^2} + \frac{5}{10-2x^2}$; $(\because \lambda_2 \neq 0)$

$= \frac{2(10-2x^2)+5(4-x^2)}{(4-x^2)(10-2x^2)}$

$= \frac{20-4x^2+20-5x^2}{(4-x^2)(10-2x^2)}$

$\Rightarrow \frac{-25}{50-2x^2} = \frac{40-9x^2}{(4-x^2)(10-2x^2)}$

$\Rightarrow -25(4-x^2)(10-2x^2) = (40-9x^2)(50-2x^2)$

$\Rightarrow -25(40-18x^2+2x^4) = 2000-530x^2+18x^4$

$\Rightarrow -1000+450x^2-50x^4 = 2000-530x^2+18x^4$

$\Rightarrow 68x^4 - 980x^2 + 3000 = 0$

$\Rightarrow$

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18
200
25
450

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~~Que~~ 1

→ A f

Que Find the absolute Max and Min values of
$$f(x,y) = 3x^2 + y^2 - x \text{ over the region } 2x^2 + y^2 \le 1.$$

Sol^n $f_x = 6x - 1 = 0 \Rightarrow x = 1/6$

$f_y = 2y = 0 \Rightarrow y = 0.$

point is $\left(\frac{1}{6}, 0\right) \rightarrow$

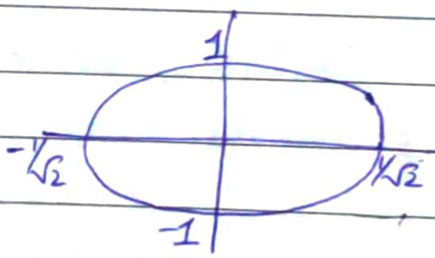
$f_{xx} = 6,\quad f_{yy} = 2,\quad f_{xy} = f_{yx} = 0$

$$J = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \text{positive definite}$$

$\Rightarrow \left(\frac{1}{6}, 0\right)$ is pt of local min

and $f\left(\frac{1}{6}, 0\right) = \frac{3}{36} - \frac{1}{6} = \frac{3-6}{36} = \frac{-3}{36} = \frac{-1}{12}$

on the boundary $y^2 = 1 - 2x^2$



$f(x,y) = 3x^2 + 1 - 2x^2 - x$
$= x^2 - x + 1 = g(x)$

$g'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = 1/2$

$g''(x) = 2 > 0$

$\Rightarrow$ $x = 1/2$ is pt of Minima

at $x = 1/2$, $y = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \left(1/2, \pm\frac{1}{\sqrt{2}}\right)$ are pts of Min. and

$f\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = 3\left(\frac{1}{4}\right) + \frac{1}{2} - \frac{1}{2} = \frac{3}{4}$

At the vertices $f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{3}{2} - \frac{1}{\sqrt{2}} = \frac{3-\sqrt{2}}{2}$.

$f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{3+\sqrt{2}}{2}$, $f(0,1) = 1 = f(0,-1)$

$\Rightarrow$ abs. Min at $\left(\frac{1}{6}, 0\right)$ is $\frac{-1}{12}$

And abs. Max at $\left(\frac{-1}{\sqrt{2}}, 0\right)$ is $\frac{3+\sqrt{2}}{2}$

→ **Derivatives of Composite functions:⇒**

Let $z = f(x, y)$.

Let x and y are functions of some Independent variable 't'. i.e. $x = \phi(t)$ and $y = \psi(t)$

Then $z = f(\phi(t), \psi(t))$ is a composite function of the Independent variable t.

Then $\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$ – Total Differential of f W.r.t 't'.

Similarly, let x and y are functions of two Independent variables u and v

i.e. $x = \phi(u, v)$ and $y = \psi(u, v)$

Then $z = f(\phi(u,v), \psi(u,v))$ is a Composite function of two Independent variables u and v.

So $$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \text{— (A)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \quad \text{— (B)}$$

The rules (A) and (B) are called the chain rules.

Ex $f(x,y) = x\cos y + e^x \sin y$

$x = t^2+1 \; ; \; y = t^3+t$

Find $\frac{df}{dt}$ at $t=0$

Sol^n $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

$\Rightarrow \frac{df}{dt} = (\cos y + e^x \sin y)(2t) + (-x\sin y + e^x \cos y)(3t^2+1)$

at $t=0$, $x=1$, $y=0$

$\Rightarrow \frac{df}{dt} = (1+0)\,2(0) + (e(1))(1)$

$= e$

Ex $f(x,y,z) = x^3 + xz^2 + y^3 + xyz$

$x = e^t \; ; \; y = \cos t \; ; \; z = t^3$ at $t=0$ find $\frac{df}{dt}$

$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$

$\Rightarrow \frac{df}{dt} = (3x^2+z^2+yz)(e^t) + (3y^2+xz)(-\sin t) + (2xz+xy)(3t^2)$

at $t=0$, $x=1$, $y=1$, $z=0$

$\Rightarrow \frac{df}{dt} = 3 + 3(-0) + 1(0)$

$= 3$

Ex $z = f(x,y)$; $x = e^{2u} + e^{-2v}$; $y = e^{-2u} + e^{2v}$

S.T. $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$

Sol^n $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$

$= \frac{\partial f}{\partial x}(2e^{2u}) + \frac{\partial f}{\partial y}(-2e^{-2u})$

$= 2\frac{\partial f}{\partial x}e^{2u} - 2\frac{\partial f}{\partial y}\cdot e^{-2u}$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}\left(-2e^{-2v}\right) + \frac{\partial f}{\partial y}\left(2e^{2v}\right)$$

$$= -2e^{-2v}\cdot\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y} + 2e^{-2v}\frac{\partial f}{\partial x} - 2e^{2v}\frac{\partial f}{\partial y}$$

$$= 2\left(e^{2u} + e^{-2v}\right)\frac{\partial f}{\partial x} - 2\left(e^{-2u} + e^{2v}\right)\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

H.W (1) If $z = \log(x^2+y)$

$x = e^{u+v^2}$; $y = u+v^2$

Then S.T. $2v\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$

(2) If $w = \sqrt{x^2+y^2+z^2}$

$x = u\cos v$; $y = u\sin v$; $z = uv$

Then $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

(3) If $w = f(x,y)$

$x = \sqrt{u^2+v^2}$; $y = \cot^{-1}\left(\frac{v}{u}\right)$

Then $\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \frac{1}{u^2+v^2}\left[(u^2+v^2)\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$

Derivative of Implicit functions:→

The function $f(x,y)=0$ defines Implicit a function $y=\phi(x)$ of one Independent variable x.

Then $df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)} \text{ provided } f_y(x,y) \neq 0$$

Ex $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Find $\frac{dy}{dx}$

Sol^n

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \; ; \; \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{x/a^2}{y/b^2}$$

$$= \frac{-b^2x}{a^2y}, \; y \neq 0.$$

Ex $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = 0$ Find $\frac{dy}{dx}$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(2x-y)}{2y+x} = \frac{y-2x}{2y+x}; \; 2y+x \neq 0 \; ; \; y \neq -x/2$$

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HW (1) $x^y + y^x = \alpha$, α is any Constant, $x>0, y>0$

Find $\frac{dy}{dx}$

Sol^n $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$

$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left(\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$$

(2) Find $\frac{dy}{dx}$ when $\cot^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$, $x>0, y>0$.

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HW Absolute Minima/Maxima

(*) Find the relative and absolute minimum and maximum values for the following functions in the given region R:–

(1) $x^2+y^2-x-y+1$; R: rectangular region ; $0\le x\le 2,\ 0\le y\le 2$.

(2) x^2-y^2-2y ; R: $x^2+y^2\le 1$.

(3) xy ; R: $x^2+y^2\le 1$.

Que Prove that $f(x,y)=x^2-2xy+y^2+x^4+y^4$ has a minima at the origin.

Que Show that $f(x,y)=x^2-3xy^2+2y^4$ has neither a minimum nor maximum value at the origin.

Sol^n

$f(x,y)=x^2+y^2-x-y+1$

$f_x=2x-1=0 \Rightarrow x=1/2$

$f_y=2y-1=0 \Rightarrow y=1/2$

$f_{xx}=2,\ f_{yy}=2,\ f_{xy}=0=f_{yx}$

$J=\begin{bmatrix}2&0\\0&2\end{bmatrix}\rightarrow$ Pos. def

$\Rightarrow\left(\frac{1}{2},\frac{1}{2}\right)$ is pt of local Min.

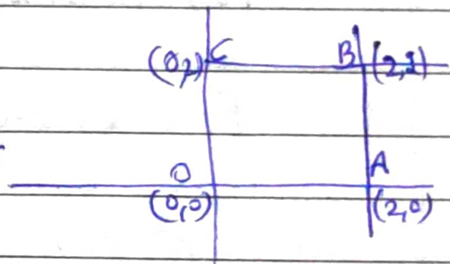
$f\left(\frac12,\frac12\right)=\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}-\frac{1}{2}+1=\frac{1}{2}$

$f(0,0)=1$

$f(2,0)=3$

$f(0,2)=3$

$f(2,2)=5$



along OA, $y=0$

$f(x,y)=g(x)=x^2-x+1$

$g'(x)=2x-1=0\Rightarrow x=1/2$

$g''(x)>0$

$\left(\frac12,0\right)\rightarrow$ Min. $f\left(\frac12,0\right)=\frac14-\frac12+$

$=\frac{1-2+4}{4}=$

along OC, $x=0$

$f(x,y)=f(0,y)=y^2-y+1=h(y)$

$h'(y)=2y-1=0\Rightarrow y=1/2$

$h''(y)>0$

$(0,1/2)$ – Min

$f\left(0,\frac12\right)=3/4$

along AB, $x=2$

$f(2,y)=y^2-y+3$

$f(2,1/2)=1/4-1/2+3=\frac{1-2+6}{4}=5/$

along BC, $y=3$

$f(x,3)=x^2-x+7$

$f(1/2,3)=\frac14-\frac12+7=\frac{1-2+28}{4}=27/4$

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Que xy on $x+2y=2$, $x\geq 0$, $y\geq 0$.

$F = xy + \lambda(x+2y-2)$

$F_x = y+\lambda = 0 \quad \Rightarrow y = -\lambda$

$F_y = x + 2\lambda = 0$

$\Rightarrow x = -2\lambda$

So $-2\lambda - 2\lambda = 2$

$-4\lambda = 2 \Rightarrow \lambda = -1/2$

$x = 1,\ y = 1/2$

Max value at $\left(1, \frac{1}{2}\right)$ is $1/2$

Min value $= 0$.

Que $x+2y$ on the circle $x^2+y^2=1$

$F = x+2y + \lambda(x^2+y^2-1)$

$F_x = 1+2\lambda x = 0 \quad \Rightarrow x = \frac{-1}{2\lambda}$

$F_y = 2+2\lambda y = 0 \Rightarrow y = -1/\lambda$

$x^2+y^2 = 1$

$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 1 \Rightarrow \frac{5}{4} = \lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{5}}{2}$

$\lambda = \frac{\sqrt{5}}{2}$, $x = \frac{-1}{2\times\sqrt{5}/2} = \frac{-1}{\sqrt{5}}$

$y = \frac{-2}{\sqrt{5}}$

So $x+2y = -\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = \frac{-5}{\sqrt{5}} = -\sqrt{5}$

$\lambda = -\frac{\sqrt{5}}{2}$; $x = \frac{1}{\sqrt{5}}$

$y = 2/\sqrt{5}$

$x+2y = \sqrt{5}$

Que Find the Smallest and the largest value of $2x-y$ on the Curve $x-\sin y=0$; $0\le y\le 2\pi$

Sol^n

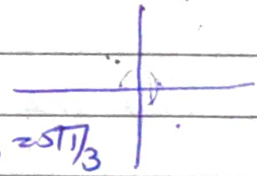
$F=(2x-y)+\lambda(x-\sin y)$

$F_x = 2+\lambda = 0 \Rightarrow \lambda=-2$

$F_y = -1-\cos y\,\lambda = 0 \Rightarrow -1=\cos y(-2)$

$\Rightarrow \frac{1}{2}=\cos y$

$\Rightarrow y=\pi/3,\ 2\pi-\pi/3 = 5\pi/3$



$x-\sin y=0 \Rightarrow x=\sin\pi/3$

$= x=\sqrt{3}/2$

So $\left(\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$ $\quad 2x-y = 2\frac{\sqrt{3}}{2}-\frac{\pi}{3} = \sqrt{3}-\frac{\pi}{3}$

$= \left(\frac{3\sqrt{3}-\pi}{3}\right)$

$x=\sin\frac{5\pi}{3} = \sin\left(2\pi-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$\left(\frac{-\sqrt{3}}{2},\frac{5\pi}{3}\right)$; $2\left(\frac{-\sqrt{3}}{2}\right)-\frac{5\pi}{3} = \frac{-3\sqrt{3}-5\pi}{3} = -\frac{(3\sqrt{3}+5\pi)}{3}$

Que Find the Extreme values of x^2+y^2 when $x^4+y^4=1$

Sol^n

$F= x^2+y^2+\lambda(x^4+y^4-1)$

$F_x = 2x+4\lambda x^3=0$

$F_y = 2y+4\lambda y^3=0$

$\Rightarrow 1+2\lambda x^2=0$

$\Rightarrow x^2=-1/2\lambda$

$1+2\lambda y^2=0 \Rightarrow y^2=-1/2\lambda$

So $x^4+y^4=1 \Rightarrow \frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$

$\Rightarrow \frac{1}{2\lambda^2}=1 \Rightarrow \boxed{\lambda^2=1/2}$

$\Rightarrow \lambda=\pm\frac{1}{\sqrt{2}}$

$\lambda=\frac{1}{\sqrt{2}}$; $x^2=\frac{-1}{2\times\frac{1}{\sqrt{2}}}=\frac{-1}{\sqrt{2}}$

$y^2=\frac{-1}{\sqrt{2}}$

$x^2+y^2=-\sqrt{2}$

$\lambda=\frac{-1}{\sqrt{2}}$; $x^2=\frac{1}{\sqrt{2}}$, $y^2=\frac{1}{\sqrt{2}}$

$x^2+y^2=\sqrt{2}$

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HW (1) Find the Extreme values of xyz when

$x+y+z = a, a>0$ Ans: $\frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

(2) Find the Extreme values of $x^3+8y^3+64z^3$ when $xyz=1$

Ans:- 24 at $\left(2, 1, \frac{1}{2}\right)$

Qu Find the Extreme values of $f(x,y,z) = 2x+3y+z$ s.t. $x^2+y^2=5$ and $x+z=1$.

Solⁿ $F(x,y,z) = 2x+3y+z + \lambda_1(x^2+y^2-5) + \lambda_2(x+z-1)$

$F_x = 2 + 2\lambda_1 x + \lambda_2 = 0$

$F_y = 3 + 2\lambda_1 y = 0$

$F_z = 1 + \lambda_2 = 0$

$\Rightarrow \lambda_2 = -1$

$\Rightarrow 2 + 2\lambda_1 x - 1 = 0 \Rightarrow 2\lambda_1 x + 1 = 0$

$\Rightarrow x = \frac{-1}{2\lambda_1}$

and $y = -3/2\lambda_1$

$x^2+y^2=5 \Rightarrow \frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$

$\Rightarrow 10 = 20\lambda_1^2 \Rightarrow \lambda_1^2 = 1/2$

$\Rightarrow \lambda_1 = \pm\frac{1}{\sqrt{2}}$

$\lambda_1 = \frac{1}{\sqrt{2}}$, $x = \frac{-1}{\sqrt{2}}$, $y = \frac{-3}{\sqrt{2}}$, $x+z=1 \Rightarrow z = 1+\frac{1}{\sqrt{2}}$

$f(x,y,z) = 2x+3y+z = 2\left(\frac{-1}{\sqrt{2}}\right) + 3\left(\frac{-3}{\sqrt{2}}\right) + 1 + \frac{1}{\sqrt{2}}$

$= \cancel{\frac{-1}{\sqrt{2}} - \frac{3\sqrt{3}}{2} + 1}$

$= \frac{-10}{\sqrt{2}} + 1 = -5\sqrt{2} + 1$

$\lambda_1 = \frac{-1}{\sqrt{2}}$; $x = \frac{1}{\sqrt{2}}$, $y = \frac{3}{\sqrt{2}}$, $z = 1 - \frac{1}{\sqrt{2}}$

$f(x,y,z) = 2/\sqrt{2} + 9/\sqrt{2} + 1 - \frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}} + 1 = 1 + 5\sqrt{2}$

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Jacobian :→ If f_1, f_2 be 2 functions of two variables x and y possessing partial derivatives of first order at every point of the domain then

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$ is called Jacobian Matrix.

and $$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$$ is called Jacobian of f_1 & f_2 w.r.t. x_1 & x_2.

and is denoted by $J(f_1, f_2)$ or $\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}$ or $J_f(x_1, x_2)$
(where $f = (f_1, f_2)$)

Ex If $f = x^2 - x\sin y$; $g = x^2y^2 + x + y$. Find $\frac{\partial(f,g)}{\partial(x,y)}$

$$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

$f_x = 2x - \sin y$; $f_y = -x\cos y$; $g_x = 2xy^2 + 1$; $g_y = 2x^2y + 1$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} 2x - \sin y & -x\cos y \\ 2xy^2 + 1 & 2x^2y + 1 \end{vmatrix}$$

$$= (2x - \sin y)(2x^2y + 1) + x\cos y\,(2xy^2 + 1)$$

(2) $f_1(x_1, x_2, x_3) = x_1^3$; $f_2(x_1, x_2, x_3) = e^{x_2}$; $f_3(x_1, x_2, x_3) = x_1 + \sin x_3$
Then find $\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}$

Sol^n $\frac{\partial f_1}{\partial x_1} = 3x_1^2$ $\frac{\partial f_1}{\partial x_2} = 0 = \frac{\partial f_1}{\partial x_3}$

$\frac{\partial f_2}{\partial x_1} = 0$, $\frac{\partial f_2}{\partial x_2} = e^{x_2}$; $\frac{\partial f_2}{\partial x_3} = 0$

$\frac{\partial f_3}{\partial x_1} = 1$; $\frac{\partial f_3}{\partial x_2} = 0$, $\frac{\partial f_3}{\partial x_3} = \cos x_3$

$$\Rightarrow \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} 3x_1^2 & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 1 & 0 & \cos x_3 \end{vmatrix}$$

$$= 3x_1^2 \cdot e^{x_2} \cdot \cos x_3$$

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Que $f(x,y) = (x+y, (x+y)^2)$ find $J_f(x,y)$

Sol^n $f_1(x,y) = x+y$, $f_2(x,y) = (x+y)^2$

$$\frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_1}{\partial y} = 1, \quad \frac{\partial f_2}{\partial x} = 2(x+y); \quad \frac{\partial f_2}{\partial y} = 2(x+y)$$

$$J_f(x,y) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2(x+y) & 2(x+y) \end{vmatrix} = 0.$$

$\Rightarrow f_1$ & f_2 are functionally dependent.

HW (1) $f(x,y,z) = (x \sin y \cos z, \; x \sin y \sin z, \; x \cos y)$
Find $J_f(x,y,z)$

(2) Let $f_1(x,y) = \dfrac{x+y}{1-xy}$; $f_2(x,y) = \tan^{-1}x + \tan^{-1}y$

Find $J_f(x,y)$ or $\dfrac{\partial(f_1, f_2)}{\partial(x,y)}$.

Change of variables / Transformation of Coordinates:→

Let $z = f(x,y)$
where $x = \phi(u,v)$; $y = \psi(u,v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \quad -(1)$$

and
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \quad -(2)$$

From (1) and (2)

$$\frac{\partial f/\partial x}{\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\frac{\partial f}{\partial v}\cdot\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\cdot\frac{\partial x}{\partial v}} = \frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}}$$

$\because$

$$\begin{array}{cccc} \frac{\partial y}{\partial u} & \frac{\partial f}{\partial u} & \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial f}{\partial v} & \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\frac{\partial(f,y)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}$$

IIly $$\frac{\partial f}{\partial y} = -\frac{\frac{\partial(f,x)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}$$

Que $z = f(x,y)$; $x = r\cos\theta$, $y = r\sin\theta$ Then Show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$

Sol^n $$\frac{\partial f}{\partial x} = \frac{\frac{\partial(f,y)}{\partial(r,\theta)}}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(f,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial f}{\partial r}\cdot r\cos\theta - \frac{\partial f}{\partial \theta}\cdot\sin\theta$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{r}\left[r\cos\theta\,\frac{\partial f}{\partial r} - \overset{\sin\theta}{\cancel{\cos\theta}}\,\frac{\partial f}{\partial \theta}\right] = \cos\theta\,\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial f}{\partial \theta}$$

Similary $$\frac{\partial f}{\partial y} = -\frac{\frac{\partial(f,x)}{\partial(r,\theta)}}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \frac{\partial f}{\partial r}\cdot(-r\sin\theta) - \frac{\partial f}{\partial \theta}(\cos\theta)$$

$$= -r\sin\theta\,\frac{\partial f}{\partial r} - \cos\theta\cdot\frac{\partial f}{\partial \theta}$$

$$\Rightarrow \frac{\partial f}{\partial y} = \sin\theta\,\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial f}{\partial \theta}$$

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HW (1) If $z = f(x,y)$, $x = r\cosh\theta$, $y = r\sinh\theta$

Then S.T. $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

(2) If $z = y + f(u)$; $u = \frac{x}{y}$ then $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$.

Sol^n $\cancel{\frac{\partial z}{\partial y} = 1}$; $z = y + f\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$$

$$\frac{\partial z}{\partial y} = 1 + f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

So $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y}\cancel{\left(\frac{1}{y}\right)f'\left(\frac{x}{y}\right)} + 1 - \cancel{\frac{x}{y^2}f'\left(\frac{x}{y}\right)}$

$$= 1.$$

(3) If $u = f(x-y, y-z, z-x)$ Then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

$w_1 = x-y$, $w_2 = y-z$, $w_3 = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial x} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial x} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial x}$$

$$= \frac{\partial u}{\partial w_1}(1) - \frac{\partial u}{\partial w_3}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial y} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial y} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial y}$$

$$= -\frac{\partial u}{\partial w_1} + \frac{\partial u}{\partial w_2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial z} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial z} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial z}$$

$$= -\frac{\partial u}{\partial w_2} + \frac{\partial u}{\partial w_3}$$

So $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

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Differentiation under Integral Sign :→

Leibnitz Integral Rule

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

and
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt + f(x,b(x)).\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x)$$

Ex
$$\frac{d}{dx}\int_{\sin x}^{\cos x} \cosh t^2\,dt$$

$$= \int_{\sin x}^{\cos x} \frac{\partial}{\partial x}(\cosh t^2)\,dt + \cosh(\cos x)^2 . \frac{d}{dx}(\cos x) - \cosh(\sin x)^2 . \frac{d}{dx}(\sin x)$$

$$= 0 + \cosh(\cos^2 x)\,(-\sin x) - \cosh(\sin^2 x).\cos x$$

$$= -\cosh(\cos^2 x)\sin x - \cosh(\sin^2 x).\cos x$$

Ex Using Diff under Integral Sign

P.T. $\int_0^1 \frac{x^y - 1}{\log x}dx = \log(1+y)$

Sol^n Let $I(y) = \int_0^1 \frac{x^y-1}{\log x}\,dx$

Diff w.r.t y

$$\frac{d}{dy}(I(y)) = \frac{d}{dy}\int_0^1 \frac{x^y-1}{\log x}\,dx$$

$$= \int_0^1 \frac{\partial}{\partial y}\left(\frac{x^y-1}{\log x}\right)dx$$

$$= \int_0^1 \frac{\partial}{\partial y}\left(\frac{x^y}{\log x}\right)dx = \int_0^1 \frac{x^y \cancel{\log x}}{\cancel{\log x}}\,dx$$

$$= \int_0^1 x^y\,dx = \left(\frac{x^{y+1}}{y+1}\right)_0^1 = \frac{1}{y+1}$$

$$\Rightarrow I(y) = \int \frac{dy}{y+1} = \log_e(y+1) + \log C = \log_e C(y+1)$$

Now $I(0) = \int_0^1 \frac{x^0 - 1}{\log x} = 0$

$\Rightarrow I(0) = \log c(1) \Rightarrow 0 = \log c$

$\Rightarrow c = 1$

$\Rightarrow I(y) = \log(y+1)$

Thm Let $\phi(y) = \int_a^b f(x,y)\,dx$ where $f(x,y)$ is a cts function of (x,y) in the Rectangle $R: \{(x,y);\ a \le x \le b,\ c \le y < d\}$ and $f_y(x,y)$ Exist and is cts in R then $\phi'(y)$ Exist and is Equal to $\int_a^b \frac{\partial}{\partial y} f(x,y)\,dx$.

i.e. $\phi'(y) = \frac{d}{dy} \int_a^b f(x,y)\,dy = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx.$

(3) $w = f(x,y)$; $x = \sqrt{u^2+v^2}$; $y = \cot^{-1}\left(\frac{v}{u}\right)$

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = ?$$

Sol^n

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{u}{\sqrt{u^2+v^2}} + \frac{\partial f}{\partial y}\left(\frac{-1}{1+\frac{v^2}{u^2}}\right)\left(\frac{-v}{u^2}\right)$$

$$= \frac{u}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{v}{\sqrt{u^2+v^2}}\cdot\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\left(\frac{-1}{1+\frac{v^2}{u^2}}\right)\left(\frac{1}{u}\right)$$

$$= \frac{v}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial x} - \frac{u}{u^2+v^2}\cdot\frac{\partial f}{\partial y}$$

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \frac{u^2+v^2}{u^2+v^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{v^2+u^2}{(u^2+v^2)^2}\left(\frac{\partial f}{\partial y}\right)^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \frac{1}{u^2+v^2}\cdot\left(\frac{\partial f}{\partial y}\right)^2$$

$$= \left[(u^2+v^2)\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]\frac{1}{u^2+v^2}$$

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Que $f(x,y) = (y-x)^4 + (x-2)^4$ has a Minimum at $(2,2)$.

Sol^n Let $f(x,y) = (y-x)^4 + (x-2)^4$

$f_x = -4(y-x)^3 + 4(x-2)^3$;

$f_y = 4(y-x)^3$

$f_{xx} = 12(y-x)^3 + 12$

$f_{yy} = 12(y-x)^2$

$f_{yx} = -12(y-x)^2$

$J_{(2,2)} = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}_{(2,2)}$ $\underline{|J| = 0} \rightarrow$ Test fails.

$f(2,2) = 0$

$f(x,y) - f(2,2) = (y-x)^4 + (x-2)^4 \geq 0$

$\Rightarrow (2,2)$ is pt of Minima.

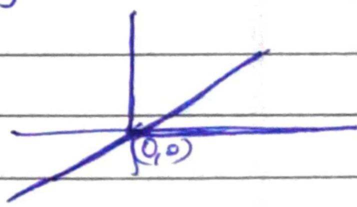
Que $f(x,y) = (x-y)^2 + x^3 - y^3 + x^5 \rightarrow$ at origin.

Sol^n $f(0,0) = 0$ $\hookrightarrow |J| = 0$

$f(x,y) - f(0,0) = (x-y)^2 + (x-y)(x^2+y^2+xy) + x^5$.

along $y=x$ line

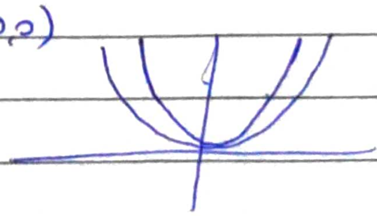
$f(x,y) - f(0,0) = x^5$



$\Rightarrow$ In nbd of $(0,0)$, there are points at which $f(x,y)$ is +ive and there are pts at which $f(x,y)$ is -ive

$\Rightarrow f(x,y)$ has neither maxima nor minima at $(0,0)$.

Que $f(x,y) = 2x^4 + y^2 - 3x^2y$ at $(0,0)$



$= 2x^4 - 2x^2y - x^2y + y^2$

$= 2x^2(x^2-y) - y(x^2-y)$

$\Rightarrow (2x^2-y)(x^2-y)$

Neither Max nor Min at $(0,0)$

Ex $f(x,y) = x^2 - 2xy + y^2 + x^4 + y^4$ at origin.

Sol^n $|J| = 0$ — check

$f(0,0) = 0$

$f(x,y) = (x-y)^2 + x^4 + y^4$

$f(x,y) > 0 \ \forall \ (x,y) \in N_\delta(0,0) - \{0,0\}$.

$\Rightarrow$ $f(x,y)$ has Minimum at $(0,0)$

Ex $f(x,y) = x^2 - 3xy^2 + 2y^4$ at origin.

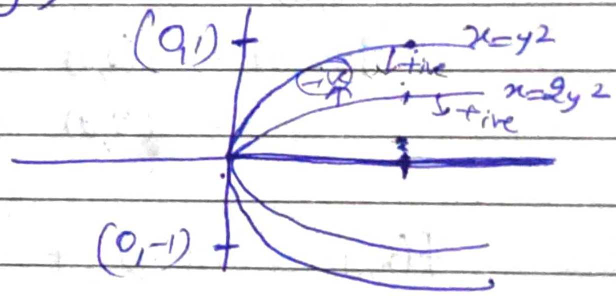
Sol^n $f(x,y) = x^2 - 2xy^2 - xy^2 + 2y^4$

$= x(x-2y^2) - y^2(x-2y^2)$

$= (x-2y^2)(x-y^2)$

$\underline{<0 \quad >0}$

$\underline{<0}$



Practise sheet-01

(1) (a) $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$

Put $y = r\sin\theta$, $x = r\cos\theta$

as $(x,y)\to(0,0)$, $r\to 0$ $\left(\because r = \sqrt{x^2+y^2}\right)$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{r\to 0} \frac{(r^3\cos^3\theta)(r\sin\theta)}{r^6\cos^6\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{r^2(\cos^3\theta \cancel{\times}\sin\theta)}{r^4\cos^6\theta + \sin^2\theta}$$

let $\theta = 0$; $\lim_{r\to 0} \frac{r^2(1)}{r^4(1)} = \lim_{r\to 0} \frac{1}{r^2} = \infty$ $\downarrow$ lim doesnot Exist

(b) $$\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y^2} = \lim_{r\to 0} \frac{r\cos\theta + \sqrt{r\sin\theta}}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{r\cos\theta + \sqrt{r}\sqrt{\sin\theta}}{r^2}$$

$$= \lim_{r\to 0} \frac{1}{r}\cos\theta + \frac{1}{r\sqrt{r}}\sqrt{\sin\theta}$$

let $\theta = 0$; $\lim_{r\to 0} \frac{1}{r} \to \infty \to$ lim does not Exist.

(2) $$f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} ; & (x,y)\neq(0,0) \\ 1/2 ; & (x,y) = (0,0) \end{cases}$$

$f(0,0) = 1/2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} \frac{\sin^{-1}(t)}{\tan^{-1}(2t)}$$

$\left[\because \text{let } t = x+2y,\ t\to 0 \text{ as } (x,y)\to(0,0)\right]$

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$$\lim_{t\to 0} \frac{\sin^{-1} t}{\tan^{-1} 2t} = \lim_{t\to 0} \frac{\frac{\sin^{-1} t}{t} \times t}{\frac{\tan^{-1}(2t)}{2t} \times 2t}$$

$$= \frac{\lim_{t\to 0} \left(\frac{\sin^{-1} t}{t}\right)}{\lim_{t\to 0} \left[\frac{\tan^{-1}(2t)}{2t}\right]} \times \frac{\cancel{t}}{2\cancel{t}}$$

$$= \frac{1}{2} = f(0,0)$$

$\Rightarrow f(x,y)$ is continuous at $(0,0)$

(3) $$f(x,y) = \begin{cases} \frac{x^2+xy+x+y}{x+y} ; & (x,y) \neq (2,2) \\ 4 ; & (x,y) = (2,2) \end{cases}$$

Solⁿ

$f(2,2) = 4$

$$\lim_{(x,y)\to(2,2)} f(x,y) = \lim_{(x,y)\to(2,2)} \frac{x(x+y)+1(x+y)}{(x+y)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{\cancel{(x+y)}(x+1)}{\cancel{(x+y)}} = 3$$

$$\Rightarrow \lim_{(x,y)\to(2,2)} f(x,y) \neq f(2,2)$$

$\Rightarrow f(x,y)$ is not continuous at $(2,2)$.

(4) $$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right) ; & x+y \neq 0 \\ 0 ; & x+y = 0. \end{cases}$$

$f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

$\Rightarrow f$ is continuous at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\cancel{h}\sin\frac{1}{h}}{\cancel{h}}$$

$$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \rightarrow \text{doesnot Exist.}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\cancel{k}\sin\frac{1}{k}}{\cancel{k}}$$

$$= \lim_{k\to 0} \sin\left(\frac{1}{k}\right) \rightarrow \text{does not Exist.}$$

(5) $$f(x,y) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$$

$f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+2y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\theta + 2\sin^3\theta)}{r^2}$$

$$= \lim_{r\to 0} r(\cos^3\theta + 2\sin^3\theta) = 0.$$

$\Rightarrow f$ is continuous at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{2k^3}{k^2}}{k} = 2.$$

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To Check diff. of f at (0,0)

$$f(0+h,0+k) - f(0,0) = A.h + B.k + \sqrt{h^2+k^2}\cdot\phi(h,k)$$

where $A = f_x(0,0)$ & $B = f_y(0,0)$

$$\Rightarrow f(h,k) = h + 2k + \sqrt{h^2+k^2}\cdot\phi(h,k)$$

$$\Rightarrow \frac{h^3+2k^3}{h^2+k^2} = (h+2k) + \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \frac{h^3+2k^3}{h^2+k^2} - \frac{(h+2k)}{1} = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{1}{\sqrt{h^2+k^2}}\left[\frac{h^3+2k^3-(h+2k)(h^2+k^2)}{h^2+k^2}\right]$$

$$= \frac{1}{(h^2+k^2)^{3/2}}\left[\cancel{h^3}+\cancel{2k^3}-\cancel{h^3}-hk^2-2kh^2-\cancel{2k^3}\right]$$

$$= \frac{-hk(k+2h)}{(h^2+k^2)^{3/2}}$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{-hk(k+2h)}{(h^2+k^2)^{3/2}}$$

Put $h = r\cos\theta$, $k = r\sin\theta$

$$\lim_{r\to 0} \frac{-r^2\cos\theta\sin\theta\,(r\sin\theta + 2r\cos\theta)}{(r^2)^{3/2}}$$

$$= \lim_{r\to 0} \frac{-\cancel{r^3}\cos\theta\sin\theta\,(\sin\theta+2\cos\theta)}{\cancel{r^3}}$$

$\rightarrow$ depends on θ only

$\Rightarrow \lim_{(h,k)\to(0,0)} \phi(h,k)$ does not Exist

$\Rightarrow$ f is not diff at (0,0).

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(6) $u = \left(xz + \frac{x}{z}\right)^{y}; \quad z \neq 0.$

Sol^n

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz$$

$$\frac{\partial u}{\partial x} = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$\frac{\partial u}{\partial y} = \left(xz + \frac{x}{z}\right)^{y} \cdot \log\left(xz + \frac{x}{z}\right)$$

$$\frac{\partial u}{\partial z} = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right)$$

$$\therefore du = \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z+\frac{1}{z}\right)dx + y\left(x - \frac{x}{z^2}\right)dz\right] + \left[\left(xz + \frac{x}{z}\right)^{y}\log\left(xz+\frac{x}{z}\right)\right]dy.$$

(7) $w = z\log y + y\log z + xyz$

$x = \sin t; \quad y = t^2+1; \quad z = \cos^{-1}t \quad$ at $t=0.$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = yz; \quad \frac{\partial w}{\partial y} = \frac{z}{y} + \log z + xz$$

$$\frac{\partial w}{\partial z} = \log y + \left(\frac{y}{z}\right) + xy$$

$$\frac{dx}{dt} = \cos t, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\therefore \frac{dw}{dt} = yz.\cos t + \left(\frac{z}{y} + \log z + xz\right)2t - \frac{1}{\sqrt{1-t^2}}\left(\log y + \frac{y}{z} + xy\right)$$

at $t=0, \ x=0, \ y=1, \ z = \pi/2.$

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$$\therefore \frac{dw}{dt} = \frac{\pi}{2} - \frac{2}{\pi}$$

(8) $x = u\cos\alpha - v\sin\alpha \qquad z = f(x,y)$

$y = u\sin\alpha + v\cos\alpha$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x}(\cos\alpha) + \frac{\partial f}{\partial y}(\sin\alpha)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}(-\sin\alpha) + \frac{\partial f}{\partial y}(\cos\alpha)$$

$$\underline{LHS} = \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\cos\alpha\frac{\partial f}{\partial x} + \sin\alpha\frac{\partial f}{\partial y}\right)^2 + \left(-\sin\alpha\frac{\partial f}{\partial x} + \cos\alpha\frac{\partial f}{\partial y}\right)^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\alpha + \sin^2\alpha) + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\alpha + \cos^2\alpha)$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \underline{RHS}.$$

(10) (a) $4x^2 + y^2 - 2x + 1$; R: $2x^2 + y^2 \leq 1$.

$f(x,y) = 4x^2 + y^2 - 2x + 1$

$f_x = 8x - 2 = 0 \Rightarrow x = 1/4$

$f_y = 2y = 0 \Rightarrow y = 0$

Critical point is $\left(\frac{1}{4}, 0\right)$

$f_{xx} = 8$

$f_{xy} = 0 = f_{yx}$

$f_{yy} = 2$

$$J = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$M_{11} = 8 > 0$

$M_{22} = |J| > 0 \Rightarrow$ Positive definite

$\Rightarrow \left(\frac{1}{4}, 0\right)$ is point of Relative Minima

and $f\left(\frac{1}{4}, 0\right) = 4 \times \frac{1}{16} - 2 \times \frac{1}{4} + 1 = \frac{1}{4} - \frac{1}{2} + 1$

$= \frac{1 - 2 + 4}{4} = \frac{3}{4}$.

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On boundary, $2x^2+y^2=1$

$\Rightarrow y^2 = 1-2x^2$

$\therefore f(x,y) = 4x^2+(1-2x^2)+1-2x$

$= 2x^2-2x+2 = g(x)$

$g'(x) = 4x-2 = 0 \Rightarrow x = \frac{1}{2}$

$g''(x) = 4 > 0$

$\Rightarrow x = \frac{1}{2}$ is point of Minima

and $g(\frac{1}{2}) = 2\times\frac{1}{4} - 2\times\frac{1}{2} + 2 = \frac{1}{2} - 1 + 2$

$= \frac{1}{2} + 1 = 3/2.$

at $x = \frac{1}{2}$, $y^2 = 1-2\left(\frac{1}{4}\right) = 1-\frac{1}{2} = \frac{1}{2}$

$\Rightarrow y = \pm\frac{1}{\sqrt{2}}$

$\therefore$ at $\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ Min value of f is $3/2$.

$\Rightarrow$ Absolute Minima occurs at $\left(\frac{1}{4}, 0\right)$ and value is $\frac{3}{4}$.

(b) $4x^2+2y^2+4xy-10x-2y-3$; $R: 0\le x\le 3, -4\le y\le 2$.

Solⁿ $f(x,y) = 4x^2+2y^2+4xy-10x-2y-3$

$f_x = 8x+4y-10 = 0$

$f_y = 4y+4x-2 = 0$

$$\therefore \begin{aligned} 4x + \cancel{2y} &= 5 \\ -2x \mp \cancel{2y} &= -1 \end{aligned}$$

$2x = 4 \Rightarrow \underline{x=2} \Rightarrow y = -3/2$

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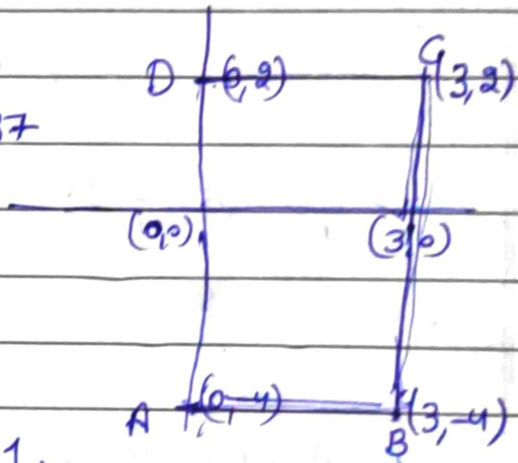
$\Rightarrow$ Critical point is $\left(2, \frac{-3}{2}\right)$

$f_{xx} = 8$
$f_{yy} = 4$
$f_{xy} = 4 = f_{yx}$
$\therefore J = \begin{bmatrix} 8 & 4 \\ 4 & 4 \end{bmatrix}$

$M_{11} = 8 > 0$
$M_{22} = |J| > 0 \Rightarrow$ Positive definite
$\Rightarrow \left(2, \frac{-3}{2}\right)$ is point of Relative Minima

and Min. value of f is $-\frac{23}{2}$.



at $A(0,-4)$; $f(x,y) = 2(16) - 2(-4) - 3$
$= 32 + 8 - 3 = 37$

at $B(3,-4)$; $f(x,y) = -5$

at $C(3,2)$; $f(x,y) = 31$

at $D(0,2)$; $f(x,y) = 8 - 4 - 3 = 1$.

along AB, $y = -4$, $f(x,y) = 4x^2 - 16x - 10x + 37$
$= 4x^2 - 26x + 37 = g(x)$
$g'(x) = 8x - 26 = 0 \Rightarrow x = \frac{13}{4}$

$g''(x) = 8 > 0 \Rightarrow$ pt. of Minima.

and $g\left(\frac{13}{4}\right) = \frac{169}{4} - 26\times\frac{13}{4} + 37 = \boxed{\frac{-11}{4}}$ at $\left(\frac{13}{4}, -4\right)$

along AD; $x = 0$, $f(x,y) = 2y^2 - 2y - 3 = g(y)$
$g'(y) = 4y - 2 \Rightarrow y = 1/2$

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$g''(y) = 4 > 0 \Rightarrow$ pt of minima

at $\left(0, \frac{1}{2}\right)$; $f(x,y) = \frac{2}{4} - \frac{2}{2} + 3 = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}$

along BC, $x = 3$; $f(x,y) = 2y^2 + 10y + 3 = g(y)$

$g'(y) = 4y + 10 = 0 \Rightarrow y = \frac{-5}{2}$

$g''(y) = 4 > 0 \Rightarrow$ pt of minima

at $\left(3, -\frac{5}{2}\right)$, $f(x,y) = 2\left(-\frac{5}{2}\right)^2 + 10\left(-\frac{5}{2}\right) + 3$

$= \frac{25}{2} - \frac{50}{2} + 3 = \boxed{\frac{-19}{2}}$

along CO; $y = 2$; $f(x,y) = 4x^2 - 2x + 1 = g(x)$

$g'(x) = 8x - 2 = 0 \Rightarrow x = 1/4$

$g''(x) = 8 > 0 \Rightarrow$ pt of minima

at $\left(\frac{1}{4}, 2\right)$; $f(x,y) = 4\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right) + 1$

$= \frac{1}{4} - \frac{2}{4} + 1 = \boxed{3/4}$.

$\therefore$ Absolute minima occurs at $\left(2, -\frac{3}{2}\right)$ and value is $-\frac{23}{2}$.

Absolute Maxima occurs at $(0, -4)$ and value is 37.

(11) $a^3x^2 + b^3y^2 + c^3z^2$ s.t. $x^{-1} + y^{-1} + z^{-1} = 1$

Solⁿ $F(x,y,z) = a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$

$F_x = 2a^3x - \frac{\lambda}{x^2} = 0$

$\Rightarrow \lambda = 2a^3x^3$ — (1)

$F_y = 2b^3y - \frac{\lambda}{y^2} = 0$

$\Rightarrow \lambda = 2b^3y^3$. — (2)

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$F_z = 2c^3z - \frac{d}{z^2} = 0$

$\Rightarrow d = 2c^3z^3 \quad -(3)$

from (1), (2), (3)

$d = \underbrace{2a^3x^3 = 2b^3y^3} = \underbrace{2c^3z^3}$

$\Rightarrow x = \frac{b}{a}y$ and $z = \frac{b}{c}y$.

Put in $x^{-1}+y^{-1}+z^{-1}=1$

$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

$\Rightarrow \frac{a}{b}\left(\frac{1}{y}\right)+\left(\frac{1}{y}\right)+\left(\frac{c}{b}\right)\left(\frac{1}{y}\right)=1$

$\Rightarrow \frac{1}{y}\left[\frac{a}{b}+1+\frac{c}{b}\right]=1$

$\Rightarrow \frac{a+b+c}{b} = y$

$\therefore x = \frac{a+b+c}{a}$ and $z = \frac{a+b+c}{c}$

Now $f(x,y,z) = a^3x^2+b^3y^2+c^3z^2$

$= a^3\left(\frac{a+b+c}{a}\right)^2 + b^3\left(\frac{a+b+c}{b}\right)^2 + c^3\left(\frac{a+b+c}{c}\right)^2$

$= (a+b+c)^2\left[\frac{a^3}{a^2}+\frac{b^3}{b^2}+\frac{c^3}{c^2}\right]$

$= (a+b+c)^2\left[a+b+c\right]$

$= (a+b+c)^3.$

$\therefore$ Extreme value is $(a+b+c)^3$ at $\left(\frac{t}{a},\frac{t}{b},\frac{t}{c}\right)$ where $t = a+b+c$.

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Que $f(x,y) = \cancel{\lim} \dfrac{(x-1)^2 \log x}{(x-1)^2 + y^2}$ Find $\lim\limits_{(x,y)\to(1,0)} f(x,y)$

Sol^n Put $x-1 = X$

$$\lim_{(X,Y)\to(0,0)} f(X,Y) = \lim_{(X,Y)\to(0,0)} \frac{X^2 \log(X+1)}{X^2+Y^2}$$

Put $X = r\cos\theta$, $Y = r\sin\theta$

$$\lim_{r\to 0} \frac{\cancel{r^2}\cos^2\theta \log(r\cos\theta+1)}{\cancel{r^2}}$$

$$= \lim_{r\to 0} \cos^2\theta \log(r\cos\theta+1)$$

as $r\to 0$; $r\cos\theta \to 0$ as $\cos\theta$ is bounded

$\therefore \log(r\cos\theta+1) \to \log 1$ as $r\to 0$
$= 0$

$$\therefore \lim_{r\to 0} \cos^2\theta \log(r\cos\theta+1) = 0.$$

Que $\lim\limits_{(x,y)\to(2,-2)} \dfrac{\sqrt{x-y}-2}{x-y-4}$

Sol^n Put $x-y = t$

as $x\to 2$, $y\to -2$, $x-y\to 4$

$$\lim_{t\to 4} \frac{\sqrt{t}-2}{t-4} = \lim_{t\to 4} \frac{\sqrt{t}-2}{(\sqrt{t})^2-(2)^2} = \lim_{t\to 4} \frac{\cancel{\sqrt{t}-2}}{\cancel{(\sqrt{t}-2)}(\sqrt{t}+2)}$$

$$= \frac{1}{4} \text{ Ans}$$

Que $f(x,y) = x^2+2y^2$ S.t. $y-x^2+1=0$

Sol^n $F(x,y) = x^2+2y^2+\lambda(y-x^2+1)$

$F_x = 2x - 2\lambda x = 0 \Rightarrow x(1-\lambda) = 0$

$\Rightarrow \lambda = 1$ or $x = 0$

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Que $f(x,y) = \frac{(x-1)^2 \log x}{(x-1)^2+y^2}$ Find $\lim_{(x,y)\to(1,0)} f(x,y)$

Sol^n Put $x-1 = X$

$\lim_{(X,y)\to(0,0)} f(X,y) = \lim_{(X,y)\to(0,0)} \frac{X^2 \log(X+1)}{X^2+y^2}$

Put $X = r\cos\theta$, $y = r\sin\theta$

$\lim_{r\to 0} \frac{\cancel{r^2}\cos^2\theta \log(r\cos\theta+1)}{\cancel{r^2}}$

$= \lim_{r\to 0} \cos^2\theta \log(r\cos\theta+1)$

as $r\to 0$; $r\cos\theta \to 0$ as $\cos\theta$ is bounded

$\therefore \log(r\cos\theta+1) \to \log 1$ as $r\to 0$
$= 0$

$\therefore \lim_{r\to 0} \cos^2\theta \log(r\cos\theta+1) = 0.$

Qus $\lim_{(x,y)\to(2,-2)} \frac{\sqrt{x-y}-2}{x-y-4}$

Sol^n Put $x-y = t$

as $x\to 2$, $y\to -2$, $x-y\to 4$

$\lim_{t\to 4} \frac{\sqrt{t}-2}{t-4} = \lim_{t\to 4} \frac{\sqrt{t}-2}{(\sqrt{t})^2-(2)^2} = \lim_{t\to 4} \frac{\cancel{\sqrt{t}-2}}{\cancel{(\sqrt{t}-2)}(\sqrt{t}+2)}$

$= \frac{1}{4}$ Ans

Que $f(x,y) = x^2+2y^2$ S.t. $y-x^2+1=0$

Sol^n $F(x,y) = x^2+2y^2+\lambda(y-x^2+1)$

$F_x = 2x-2\lambda x = 0 \Rightarrow x(1-\lambda) = 0$

$\Rightarrow \lambda = 1$ or $x = 0$

$F_y = 4y + \lambda = 0$

$y = \frac{-\lambda}{4}$

$\lambda = 1 \Rightarrow y = \frac{-1}{4}$

Now $y - x^2 + 1 = 0$

$\Rightarrow \frac{-1}{4} + 1 = x^2 \Rightarrow x^2 = \frac{3}{4}$

$\Rightarrow x = \pm \frac{\sqrt{3}}{2}$

$\because$ at $\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{4}\right)$ $f\left(\pm\frac{\sqrt{3}}{2}, \frac{-1}{4}\right) = \frac{3}{4} + 2\left(\frac{-1}{4}\right)^2$

$= \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$.

$x = 0;$ $y = -1$ $(\because y - x^2 + 1 = 0)$

and $\lambda = 4$

at $(0, -1)$, $f(x, y) = 2$

Que→ $f(x,y) = \begin{cases} \dfrac{|x|}{|x|+|y|}\sqrt{x^4+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

Sol^n $f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\cancel{|r|}\,|\cos\theta|}{\cancel{|r|}[|\cos\theta|+|\sin\theta|]} \sqrt{r^4\cos^4\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r \cdot \frac{|\cos\theta|}{|\cos\theta|+|\sin\theta|} \sqrt{r^2\cos^4\theta + \sin^2\theta}$$

$$= 0$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\cancel{|h|}}{\cancel{|h|}}\sqrt{h^4+0}}{h}$$

$$= \lim_{h\to 0} \frac{h^2}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

Que→ $f(x,y) = \begin{cases} \dfrac{x^2y(x-y)}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$ Find $f_{xy} - f_{yx}$?

Sol^n $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0$

$$\frac{\partial}{\partial y}(f_x) = f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}}k(h-k)}{(h^2+k^2)\cancel{h}}$$

$$= 0$$

$$\therefore f_{yx}(0,0) = \frac{0-0}{k} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$$

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$$\frac{\partial}{\partial x}(f_y) = f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \cancel{\lim_{h\to 0}}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{h^2 k(h-k)}{(h^2+k^2)\cancel{k}}$$

$$= h$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$\therefore f_{xy}(0,0) - f_{yx}(0,0) = 1.$$

Que $f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^\alpha} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$ For which value of α, f is cont & diff at $(0,0)$

Sol^n:→ $f(0,0) = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = 0$ for f to be Continuous.

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^\alpha} = \lim_{r\to 0} \frac{r^2 \cdot \cos\theta \sin\theta}{(r^2)^\alpha}$$

$$= \lim_{r\to 0} (r^2)^{1-\alpha} \cos\theta \sin\theta$$

If $1-\alpha = 0$ then lim does not Exist

If $1-\alpha < 0$ then lim does not Exist

If $(1-\alpha) > 0$ Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0. = f(0,0)$

$\Rightarrow f$ is cts at $(0,0)$ for $\alpha < 1$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

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$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K) - f(0,0)}{K} = 0$$

$$f(h,k) - f(0,0) = \sqrt{h^2+k^2}\cdot \phi(h,k)$$

$$\Rightarrow \frac{hk}{(h^2+k^2)^\alpha} = \sqrt{h^2+k^2}\cdot\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{hk}{(h^2+k^2)^{\alpha+1/2}}$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{hk}{(h^2+k^2)^{\alpha+1/2}}$$

$$= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{(r^2)^{\alpha+1/2}}$$

$$= \lim_{r\to 0} (r^2)^{1/2-\alpha}\cos\theta\sin\theta$$

If $\frac{1}{2}-\alpha > 0$ Then $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$

$$\Rightarrow \alpha < \frac{1}{2}$$

So f is diff at $(0,0)$ for $\alpha < \frac{1}{2}$.

HW Que

$$f(x,y) = \begin{cases} \frac{xy^2}{x+y} & \text{If } x+y \neq 0 \\ 0 & \text{If } x+y = 0 \end{cases}$$

Then find the value of $\left(\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y\partial x}\right)$ at $(0,0)$.

Que $f(x,y) = x+y$ and $g(x,y) = xy-16$.

Sol^n^ $F(x,y) = x+y+\lambda(xy-16)$

$F_x = 1+\lambda y = 0 \Rightarrow \lambda = \frac{-1}{y}$

$F_y = 1+\lambda x = 0 \Rightarrow \lambda = \frac{-1}{x}$

$\frac{-1}{y} = \frac{-1}{x} \Rightarrow x = y$

$xy - 16 = 0$

$x^2 = 16 \Rightarrow x = \pm 4$

$y = \pm 4$

$\cancel{(4,-4)}, \cancel{(-4,4)}$; $(4,4), (-4,-4)$

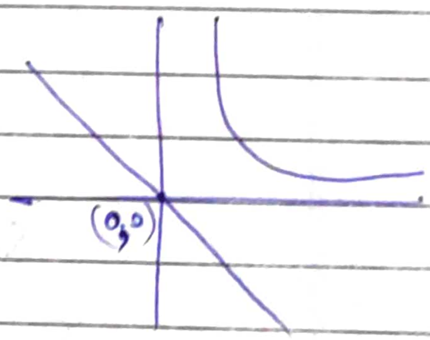
$f(x,y) = \underline{8}, \underline{-8}, \cancel{0, 0}$

$F_{xx} = 0, F_{yy} = 0, F_{xy} = \lambda$

$J = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$

$|J| = -\lambda^2 < 0 \Rightarrow$ Saddle pt.

Now $F(x,y) = xy - 16 + \lambda(x+y)$



$F_x = y+\lambda = 0 \Rightarrow \lambda = -y$

$F_y = x+\lambda = 0 \Rightarrow \lambda = -x$

$\therefore y = x$

$x+y = 0 \Rightarrow x = 0 = y$

$\therefore (0,0)$ critical point.

$F_{xx} = 0 = F_{yy}$; $F_{xy} = 1$, $F_{yx} = 1$

$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |J| < 0$

Que $f(x,y) = xy$ s.t $2x+3y=6$

Sol^n → $F = xy + \lambda(2x+3y-6)$

$F_x = y + 2\lambda = 0 \Rightarrow \lambda = -y/3 \qquad y = -2\lambda$

$F_y = x + 3\lambda = 0 \Rightarrow \lambda = -x/2 \qquad x = -3\lambda$

$\Rightarrow \frac{-x}{2} = \frac{-y}{3} \Rightarrow 3x = 2y$

$\Rightarrow y = \frac{3}{2}x$

$2x + 3\left(\frac{3}{2}x\right) = 6$

$\Rightarrow \left(2 + \frac{9}{2}\right)x = 6 \Rightarrow x = \frac{12}{13}$

$2(-3\lambda) + 3(-2\lambda) = 6$

$\Rightarrow -12\lambda = 6 \Rightarrow \lambda = \frac{-1}{2}$

$x = 3/2, \; y = 1$

$\left(\frac{3}{2}, 1\right)$ – critical point.

$f\left(\frac{3}{2}, 1\right) = \frac{3}{2}$

Que $f(x,y,z) = x + 2y - 2z$ s.t $x^2 + 2y^2 + 4z^2 = 1$

Sol^n $F = x + 2y - 2z + \lambda(x^2 + 2y^2 + 4z^2 - 1)$

$F_x = 1 + 2\lambda x = 0 \qquad x = \frac{-1}{2\lambda}$

$F_y = 2 + 4\lambda y = 0 \qquad y = \frac{-1}{2\lambda}$

$F_z = -2 + 8\lambda z = 0 \qquad z = \frac{1}{4\lambda}$

$\frac{1}{4\lambda^2} + \frac{2}{4\lambda^2} + \frac{4}{16\lambda^2} = 1$

$\frac{1}{4\lambda^2}(4) = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$

$\lambda = 1; \; x = \frac{-1}{2}, \; y = \frac{-1}{2}, \; z = \frac{1}{4}$

$f(x,y,z) = -\frac{1}{2} - \frac{2}{2} - \frac{1}{2} = -\frac{4}{2} = -2$

$\lambda = -1, \; x = \frac{1}{2}, \; y = \frac{1}{2}, \; z = \frac{-1}{4}$

$f = \frac{3}{2} - \frac{1}{2} = 1$

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Qu $f(x,y) = x^2+xy+y^2; \quad x^2+y^2=8$

Solⁿ $F = f + \lambda g$

$= (x^2+xy+y^2) + \lambda(x^2+y^2-8)$

$F_x = \cancel{2x+2\lambda x = 0} \quad 2x + y + 2\lambda x = 0$

$F_y = x + 2y + 2\lambda y = 0$

$2x(1+\lambda) + y = 0$

$x + 2y(1+\lambda) = 0$

$$\begin{vmatrix} 2(1+\lambda) & 1 \\ 1 & 2(1+\lambda) \end{vmatrix} = 0$$

$4(1+\lambda)^2 = 1 \Rightarrow 1+\lambda^2 = \pm \frac{1}{2}$

$\Rightarrow 1+\lambda = 1/2$

$\Rightarrow \lambda = -1/2, \quad 1+\lambda = -\frac{1}{2}$

$\lambda = -3/2.$

$\cancel{\lambda = -\frac{3}{4}};$

$\lambda = -\frac{1}{2} \quad 2x\left(\frac{1}{2}\right) + y = 0 \Rightarrow x + y = 0$

$y = -x$

$x^2 + x^2 = 8$

$2x^2 = 8 \Rightarrow x = \pm 2$

$x = 2, y = -2 \Rightarrow (2,-2) \;\&\; (-2,2)$

$x = -2, y = 2$

$\lambda = -\frac{3}{2}; \quad 2x\left(-\frac{1}{2}\right) + y = 0$

$\Rightarrow -x + y = 0 \Rightarrow x = y$

$(2,2) \; (-2,-2)$

$f(2,2) = 4+4+4 = 12$

$f(-2,-2) = 12$

$f(2,-2) = 4-4+4 = 4$

$f(-2,2) = 4.$

Que $f(x,y) = 5 - 4\sin x + y^2$; $0 < x < 2\pi$, $y \in R$

$f_x = -4\cos x = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = \pi/2,\ 3\pi/2$

$f_y = 2y = 0 \Rightarrow y = 0$

$\left(\frac{\pi}{2}, 0\right)$ $\left(\frac{3\pi}{2}, 0\right)$

$f_{xx} = 4\sin x$

$f_{yy} = 2$

$$\begin{bmatrix} 4\sin x & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Positive definite $\Rightarrow$ Minima