Math for the Social Sciences

Extra Assignment

LIDE Young Researchers Fellowship

September 2, 2024

1. Write down the gradient and the Hessian for each of the following functions:

(a) $f(x, y) = x^4 + x^2 - 6xy + 3y^2$

(b) $f(x, y) = xy^2 + x^3y - xy$

(c) $f(x, y, z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

Note: The Hessian is the matrix of second partial derivatives, as follows:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

2. Find the matrix product AB of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}.$$

Then find the product BB'.

3. For matrix B above:

   (a) Find the determinant of B (show all steps and do research on how to calculate the determinant of a 3x3 matrix, if needed).

   (b) Find the inverse of B, using a computer program or calculator.

   (c) Perform the hand calculation of $B^{-1}B$, to verify that the result is the identity matrix I.

   (d) Can you extend the results for matrix A above? Why or why not? Clearly explain.

4. Consider the function $f(x) = -\frac{x^3}{3} + x^2 - x + 4$.

   (a) Draw the graph of $f(x)$.

   (b) Find the critical points of $f(x)$.

   (c) Determine the concavity of $f(x)$.

   (d) Find any local maxima or minima of $f(x)$, if they exist.

   (e) If I were to restrict the domain of $f(x)$ to $[3,\ 3]$ what would be the maximum and minimum values of $f(x)$? Are these values local maxima or minima?

5. Consider demand for hospital beds in a city, given by $Q = 100 - 2P$, where Q is the quantity of hospital beds demanded and P is the price of a hospital bed.

   (a) When COVID-19 hit the city, the demand for hospital beds increased and shifted right to $Q = 200 - 2P$. Draw the graph of the demand curve before and after COVID-19.

   (b) If the supply of hospital beds is given by $Q = 50{+}3P$, find the equilibrium price and quantity of hospital beds before and after COVID-19.

   (c) If the city government wants to increase the quantity of hospital beds to 300, what should the price be?

   (d) A reduced form estimation of demand for hospital beds in the city is given by $Q = 100 - 2P + 0.5I$, where I is the income of the city. Calculate the partial price elasticity of demand for hospital beds in the city at equilibrium, with an Income I of 100. Interpret the result.

   (e) Assume the government intervenes and increases the price by 20%. Calculate the percentage change in quantity demanded of hospital beds. Interpret the result.

6. Consider the sum of squared errors, a statistical measure that will be covered later in the program. It is expressed as

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

   where y_i is the actual value of the dependent variable, $\hat{y}_i$ is the predicted value of the dependent variable, and n is the number of observations.

   (a) Using properties of sums, show that the sum of squared errors can be expressed as

$$SSE = \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} y_i\hat{y}_i + \sum_{i=1}^{n} \hat{y}_i^2$$

(b) Consider now y as a vector of actual values and $\hat{y}$ as a vector of predicted values. Show that the sum of squared errors can be expressed as

$$SSE = y'y - 2y'\hat{y} + \hat{y}'\hat{y}$$

, starting from the vector equivalent of the sum of squared errors,

$$SSE = y - \hat{y}'(y - \hat{y})$$

. Use properties of matrices and vector as needed.

(c) In the equation above, substitute $y = X\beta + \epsilon$, where X is the matrix of independent variables, and β and ϵ are vectors. Show that the sum of squared errors can be expressed as

$$SSE = \epsilon'\epsilon$$