Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 5 - Single Variable Calculus

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2024

What is Calculus?

- Calculus is the mathematical study of change.
- It has two main branches:
  - **Differential calculus**: studies the rate at which quantities change.
  - **Integral calculus**: studies the accumulation of quantities.
- Calculus is used in many fields, including physics, engineering, economics, and social sciences.

Differential Single Variable Calculus

- Single variable calculus deals with functions of one variable.
- Functions of one variable are functions that take a single input and produce a single output. This is usually denoted as:

$$f : \mathbb{R} \to \mathbb{R}$$

- Differential calculus studies the rate at which functions change. Hence, rates of change are the main focus of this branch of calculus.

Rates of Change

- Remember variations Δx and Δy? Rates of change for functions are not too different from these.
- The rate of change of a function $f(x)$ at a point $x = a$ represents how much the function changes as x changes by a small amount Δx around a.
- The concept of slope is crucial to understand rates of change, as it is the ratio of the change in y to the change in x.
- The slope is not constant for all functions. It can change depending on the point at which it is calculated.

Slopes and Tangent Lines

- The slope of the tangent line to the graph of $f(x)$ at $x = a$ is the rate of change of $f(x)$ at $x = a$.

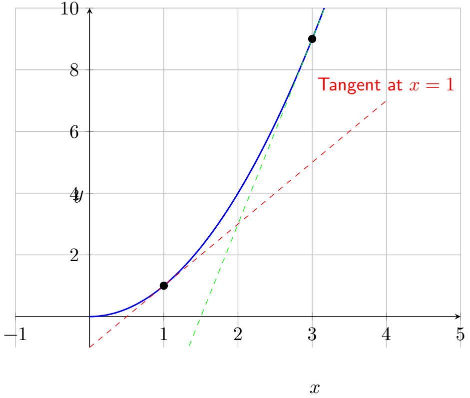


Figure 1: Curve $f(x) = x^2$ and its tangent lines at $x = 1$ and $x = 3$

Derivatives

- Because rates of change are not constant, it makes sense to define a function that gives the rate of change of another function at any point.
- This function is called the **derivative** (or the first derivative). It is denoted as $f'(x)$ or $\frac{df}{dx}$.
- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the graph of $f(x)$ at $x = a$.
- The derivative of a function $f(x)$ is itself a function that gives the rate of change of $f(x)$ at any point.

A brief mention of limits

- The derivative of a function $f(x)$ at a point $x = a$ is defined as the limit of the average rate of change of $f(x)$ as x approaches a.
- Limits are a fundamental concept in calculus. They are used to define derivatives, integrals, and many other concepts in calculus.
- Limits are an operator which gives the value that a function **approaches** as the input approaches a certain value. It makes sense to think of them as the value that a function gets closer and closer to as the input gets closer and closer to a certain value.
- There are many rules and properties of limits that are used to calculate them. We will not cover them in this lecture, but they are crucial to understand calculus, and you should review them if you want to learn more about calculus.

The Derivative of a Function

- The derivative of a function $f(x)$ at a point $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- This definition gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.
- The derivative of a function $f(x)$ is itself a function that gives the rate of change of $f(x)$ at any point.
- The derivative of a function $f(x)$ is denoted as $f'(x)$ or $\frac{df}{dx}$.

Derivatives without limits

- Thankfully, we don't really need limits to calculate derivatives for most functions.
- The rules of differentiation allow us to calculate the derivative of a function without using limits, but rather relatively simple algebraic manipulations.

1. **Power rule**: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
2. **Sum rule**: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
3. **Product rule**: If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.
4. **Quotient rule**: If $f(x) = \dfrac{g(x)}{h(x)}$, then
$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}.$$
5. **Chain rule**: If $f(x) = g(h(x))$, then $f'(x) = g'(h(x))h'(x)$.

More properties of derivatives

- The derivative of a constant is zero: If $f(x) = c$, then $f'(x) = 0$.
- The derivative of the identity function is one: If $f(x) = x$, then $f'(x) = 1$.
- For the exponential function $f(x) = a^x$, the derivative is $f'(x) = a^x \ln(a)$.
- For the natural logarithm function $f(x) = \ln(x)$, the derivative is $f'(x) = \dfrac{1}{x}$.
- For the trigonometric functions, the derivatives are as follows:
  - $\sin(x)$: $\cos(x)$
  - $\cos(x)$: $-\sin(x)$
  - $\tan(x)$: $\sec^2(x)$
  - $\cot(x)$: $-\csc^2(x)$
  - $\sec(x)$: $\sec(x)\tan(x)$
  - $\csc(x)$: $-\csc(x)\cot(x)$

Example: Derivative of a power function

- Let's calculate the derivative of the function $f(x) = x^2$.
- Using the power rule, we have:

$$f'(x) = 2x^{2-1} = 2x$$

- Therefore, the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Example: Derivative of a sum

- Let's calculate the derivative of the function $f(x) = x^2 + 3x$.
- Using the sum rule, we have:

$$f'(x) = (x^2)' + (3x)' = 2x + 3$$

- Therefore, the derivative of $f(x) = x^2 + 3x$ is $f'(x) = 2x + 3$.

Example: Derivative of a product

- Let's calculate the derivative of the function $f(x) = x^2 \cdot \sqrt(x)$
- Using the product rule and the power rule, we have:

$$f'(x) = (x^2)' \cdot \sqrt(x) + x^2 \cdot (\sqrt(x))'$$

$$f'(x) = 2x \cdot \sqrt(x) + x^2 \cdot \frac{1}{2\sqrt(x)}$$

$$f'(x) = 2x\sqrt(x) + \frac{x^2}{2\sqrt(x)}$$

Example: Applying the chain rule

- Let's calculate the derivative of the function $f(x) = (x^2 + 1)^3$.
- The chain rule involves two functions: the outer function $g(x) = x^3$ and the inner function $h(x) = x^2 + 1$.
- Using the chain rule, we have:

$$f'(x) = 3(x^2 + 1)^2 \cdot (2x)$$

$$f'(x) = 6x(x^2 + 1)^2$$

Applications: the concept of marginality

- In many fields, the derivative of a function has a special meaning.
- The concept of **margin** is crucial in many fields, denoting the **additional** or **incremental** change in a quantity, given a small change in another quantity.
- The derivative of a function represents the marginal change in the function with respect to the input.
- Marginal products, marginal costs, marginal revenues, and marginal utilities are all examples of marginal quantities that are calculated using derivatives.

Example: marginal products

Consider the Cobb-Douglas production function:

$$Q = L^{0.3}K^{0.7}$$

with L representing labor and K representing capital, and with Q representing output. In the short run, capital is fixed at $K = 100$.

Example: marginal products

What is the marginal product of labor? This means we need to calculate the derivative of the production function with respect to labor.

First, we need to substitute $K = 100$ into the production function:

$$Q = L^{0.3} 100^{0.7} = 100^{0.7} L^{0.3}$$

Now, we can calculate the derivative of Q with respect to L:

$$\frac{dQ}{dL} = 0.3 \cdot 100^{0.7} L^{-0.7} = 0.3 \cdot 100^{0.7} L^{-0.7}$$

Single variable optimization

- Optimization is a key concept in the social sciences and many other fields.
- Optimization involves finding the maximum or minimum value of a function, sometimes subject to constraints.
- Calculus, and specifically derivatives, are crucial tools for optimization.
- In single-variable optimization, we are interested in finding the maximum or minimum value of a function of one variable.
- For now, we will focus on finding the maximum or minimum value of a function without constraints.

Critical points

- Critical points are points at which the derivative of a function is zero or undefined.
- Critical points are important because they can be local maxima, local minima, or saddle points.
  - Local maxima are points where the function reaches a maximum value in a small neighborhood.
  - Local minima are points where the function reaches a minimum value in a small neighborhood.
  - Saddle points are points where the function is neither a maximum nor a minimum.
- To find the critical points of a function, we need to find the values of x for which $f'(x) = 0$ or $f'(x)$ is undefined.

Example: Finding critical points

Consider the function $f(x) = x^3 - 3x^2 + 2x$.

To find the critical points of $f(x)$, we need to find the values of x for which $f'(x) = 0$ or $f'(x)$ is undefined.

First, we need to calculate the derivative of $f(x)$:

$$f'(x) = 3x^2 - 6x + 2$$

Now, we need to find the values of x for which $f'(x) = 0$:

$$3x^2 - 6x + 2 = 0$$

Example: Finding critical points

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3}$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{6 \pm \sqrt{12}}{6}$$

$$x = \frac{6 \pm 2\sqrt{3}}{6}$$

$$x = 1 \pm \frac{\sqrt{3}}{3}$$

Example: Finding critical points

- Therefore, the critical points of the function $f(x) = x^3 - 3x^2 + 2x$ are $x = 1 + \frac{\sqrt{3}}{3}$ and $x = 1 - \frac{\sqrt{3}}{3}$.
- To determine whether these points are local maxima, local minima, or saddle points, we need to analyze the behavior of the function around these points.
- There are four ways to do this:
  - Use the second derivative test (we will cover this later)
  - Look at the chart of the function around the critical points.

Example: Finding critical points

- Let's first look at the graph of $f(x) = x^3 - 3x^2 + 2x$ around the critical points.

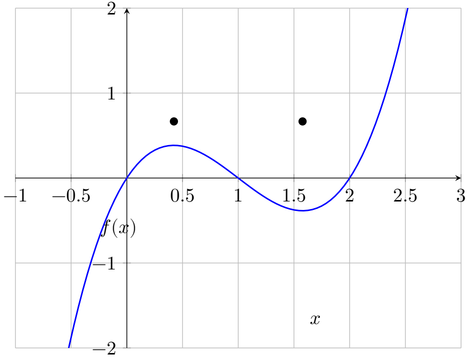


Figure 2: Graph of the function $f(x) = x^3 - 3x^2 + 2x$

Example: Finding critical points

- From the graph, we can see that the function has a local maximum at $x = 1 + \frac{\sqrt{3}}{3}$ and a local minimum at $x = 1 - \frac{\sqrt{3}}{3}$.
- Therefore, the critical points of the function $f(x) = x^3 - 3x^2 + 2x$ are $x = 1 + \frac{\sqrt{3}}{3}$ and $x = 1 - \frac{\sqrt{3}}{3}$.

Maximum and minimum values with the second derivative test

- Once we have found the critical points of a function, we can determine whether they are local maxima, local minima, or saddle points.
- To determine whether a critical point is a local maximum or a local minimum, we can use the second derivative test apart from looking at the graph.
- The second derivative is calculated as the derivative of the first derivative of the original function $f(x)$.
- The second derivative test states that if $f''(x) > 0$ at a critical point $x = a$, then $f(a)$ is a local minimum. If $f''(x) < 0$ at a critical point $x = a$, then $f(a)$ is a local maximum.

Example: Using the second derivative test

Consider the function $f(x) = x^3 - 3x^2 + 2x$.

We have already found the critical points of this function: $x = 1 + \dfrac{\sqrt{3}}{3}$ and $x = 1 - \dfrac{\sqrt{3}}{3}$.

Now, we need to calculate the second derivative of $f(x)$:

$$f'(x) = 3x^2 - 6x + 2$$

$$f''(x) = 6x - 6$$

Example: Using the second derivative test

Now, we need to evaluate $f''(x)$ at the critical points:

$$f''(1+\frac{\sqrt{3}}{3}) = 6(1+\frac{\sqrt{3}}{3}) - 6 = 6 + 2\sqrt{3} - 6 = 2\sqrt{3} > 0$$

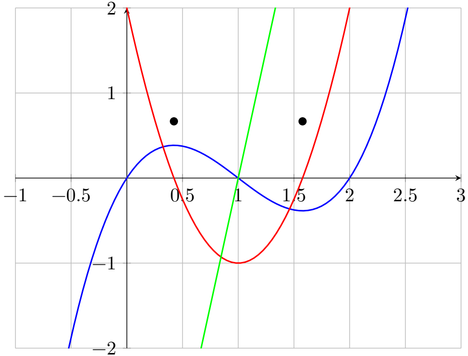
$$f''(1-\frac{\sqrt{3}}{3}) = 6(1-\frac{\sqrt{3}}{3}) - 6 = 6 - 2\sqrt{3} - 6 = -2\sqrt{3} < 0$$

Example: Using the second derivative test

- Therefore, the critical point $x = 1 + \frac{\sqrt{3}}{3}$ is a local minimum, and the critical point $x = 1 - \frac{\sqrt{3}}{3}$ is a local maximum.
- This means that the function $f(x) = x^3 - 3x^2 + 2x$ has a local minimum at $x = 1 + \frac{\sqrt{3}}{3}$ and a local maximum at $x = 1 - \frac{\sqrt{3}}{3}$.

Visualizing the function and derivatives

- Let's visualize the function $f(x) = x^3 - 3x^2 + 2x$ and its first and second derivatives.
  - *Blue* represents the function $f(x)$.
  - *Red* represents the first derivative $f'(x)$.
  - *Green* represents the second derivative $f''(x)$.



Local and global maxima and minima

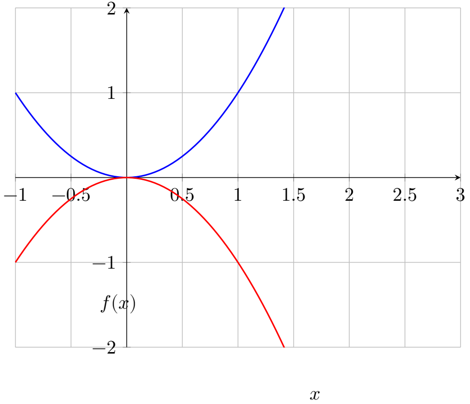
- As you've noticed, the function above had both a local maximum and a local minimum.
- We call these points local since the function goes up and down around them, so in principle they are not the maximum or minimum values of the function.
- We then say that the function has no **global** maximum or minimum.
- Some functions may have local and global maxima or minima, while others may have only local maxima or minima. Others may have neither.
- The existence of global maxima and minima is a more complex problem that involves analyzing the behavior of the function at the boundaries of its domain, and we will not cover it in this lecture.

Maximization and minimization

- We are interested in maximizing as social scientists when functions exhibit the existence of at least a local maximum (minimum for minimization).
- Whether a function has existing maxima or minima is a question of interest in many fields, and has to do with the shape of the function and the behavior of its derivatives.
- We often talk about concavity and convexity of functions to determine whether they have maxima or minima.
- As a “rough” rule, if a function is concave up, it has a minimum, and if it is concave down, it has a maximum.

Concavity and convexity - graphical intuition

- A function is **concave up** if it curves upwards, like a smiley face.
- A function is **concave down** if it curves downwards, like a frown. This is convexity.



Optimization notation

- A real world example of optimization is maximizing or minimizing a function that represents a quantity of interest.
    - For example, a government may want to maximize public welfare given a certain budget amount.
    - Firms want to maximize profits.
    - An epidemiologist may want to minimize the spread of a disease.
- In all these cases, optimization involves finding the maximum or minimum value of a function. This is done by finding the critical points of the function and analyzing their behavior.
- The notation of these optimization problems is usually as follows:

$$\max f(x) \quad \text{or} \quad \min f(x)$$

Optimization notation

- The objective function $f(x)$ represents the quantity we want to maximize or minimize.
- The objective variable x represents the variable that we can change to maximize or minimize the function.
- We often say that we maximize or minimize a function **with respect to** (w.r.t.) the objective variable. In notation, this is written as:

$$\max_x f(x) = \max_x x^2 + x + 5$$