Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 4 - Logarithms and related topics

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Logarithms

- The logarithm of a number is the power to which a base must be raised to obtain that number.

$$\log_b(x) = y \iff b^y = x$$

* $\iff$ means "if and only if".

- The most common logarithms are base 10 and base e (Euler's number).
- The natural logarithm is the logarithm with base e.
  - It is denoted $\ln(x)$.
- Euler's number is approximately 2.71828.
  - It is an irrational number.
  - It was discovered by the Swiss mathematician Leonhard Euler while studying compound interest (percentages).

Examples

- $\log_{10}(100) = 2$ because $10^2 = 100$.
- $\log_{15}(225) = 2$ because $15^2 = 225$.
- $\log_e(e) = 1$ because $e^1 = e$.
- $\ln(e^2) = 2$ because $e^2 = e^2$.

Properties of Logarithms

Product rule: $\log_b(xy) = \log_b(x) + \log_b(y)$

Quotient rule: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

Power rule: $\log_b(x^y) = y\log_b(x)$

Change of base formula: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$

Logarithm of 1: $\log_b(1) = 0$

Exponent rule: $b^{\log_b(x)} = x$

Examples of Properties

- $\log_{10}(1000) = \log_{10}(10 \times 100) = \log_{10}(10) + \log_{10}(100) = 1 + 2 = 3$
- $\log_{10}(1000) = \log_{10}(10^3) = 3\log_{10}(10) = 3$
- $\log_{10}(1000) = \log_2(1000)\dfrac{\log_2(1000)}{\log_2(10)} = 3$

Logarithms are our friends!

- As icky as they might seem, logarithms are our friends **because of their properties**.
- When dealing with variable exponentials, logarithms can help us simplify the problem.
  - Use the power rule to bring down the exponent and easily differentiate.
- Natural logarithms are particularly useful for the social sciences.
  - They are used in growth models because of the Euler's number relationship to percent growth.
- Exponential growth shows up in real life in many ways.
  - Population growth, compound interest, and the spread of diseases are all examples of exponential growth.
  - Logarithms are tools to deal with these phenomena.

Logarithmic equations

- Logarithmic equations are equations that involve logarithms
  - Include an unknown variable inside the logarithm.
- To solve a logarithmic equation, we need to use the properties of logarithms to simplify the equation.
- Once the equation is simplified, we can solve for the unknown variable using algebraic techniques.

Example

$$\log_{10}(x) = 2$$

- To solve this equation, we need to remember that $\log_{10}(100) = 2$.
- Therefore, $x = 100$.

Equations which use the exponent rule

- The reason why logarithms are useful is that they allow us to solve equations that involve exponentials.
- For example:

$$2^x = 8$$

- To solve this equation, we can take the logarithm of both sides.

$$\log_2(2^x) = \log_2(8)$$

- Using the exponent rule, we get $x = 3$.

Exponential growth

- Exponential growth is a process that increases at a constant rate over time.
- It is characterized by a constant percentage growth rate.
- Exponential growth is often used to model population growth, compound interest, and the spread of diseases.
- The formula for exponential growth is $y = a(1 + r)^t$.

Example

- A population of 1000 people grows at a rate of 5% per year.
- The population after 10 years is given by the formula $y = 1000(1 + 0.05)^{10}$.
- The population after 10 years is $1000(1.05)^{10} = 1628.89$.

Example

- In some cases, we might want to know how long it will take for a population to reach a certain size.
- For example, how long will it take for a population to double if it is growing at a rate of 5% per year?
- We can use the formula $y = a(1+r)^t$ and solve for t.
- If the population doubles, then $2a = a(1+r)^t$.
- Therefore, $2 = (1+0.05)^t$.
- Taking the logarithm of both sides, we get $\log(2) = t\log(1.05)$.
- Therefore, $t = \dfrac{\log(2)}{\log(1.05)} \approx 14.21$ years.

Logarithmic scaling

- Logarithmic scales are used when there is a large range of values.
  - They compress the scale to make it easier to read.
  - Often allow to observe trends that would be hidden in a natural scale.
- Commonly, the y-axis is in logarithmic scale.
  - A base 10 and a natural logarithm scale are the most common.

Base 10 logarithmic scale

- The base 10 logarithmic scale is used whenever the data is very large
  - It should work well with a non-log scale that is a power of 10, i.e., 10, 100, 1000, etc.
- The scale compresses the data to make it easier to read.
- We interpret the values in the scale as powers of 10, since $\log(10^x) = x$.

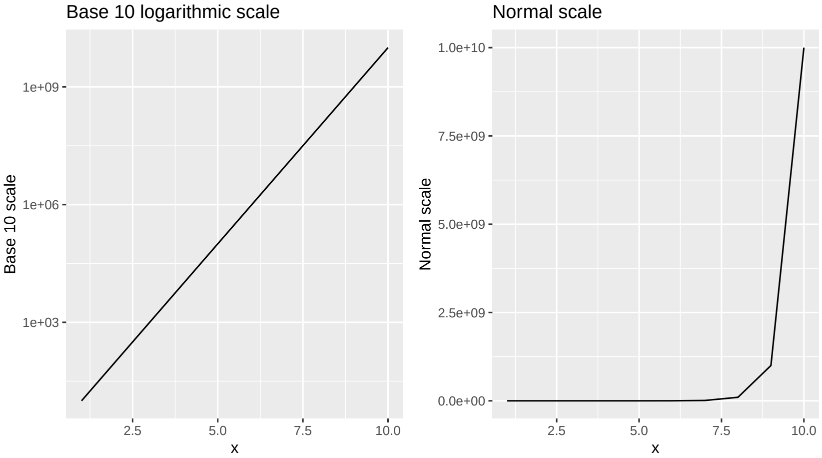
Example of base 10 logarithmic scale

- Below, a dataset is shown with values that increase exponentially.

x	y
1	10
2	100
3	1,000
4	10,000
5	100,000
6	1,000,000
7	10,000,000
8	100,000,000
9	1,000,000,000
10	10,000,000,000

Example of base 10 logarithmic scale

- The following plot shows the same data in a base 10 logarithmic scale and a regular scale.



Natural logarithmic scale

- The natural logarithmic scale is often used when the data shows a exponential or percent growth pattern.
- Powers of e are used to interpret the values in the scale, but this can be approximated to percentages.
- If a variable grows exponentially, the trend will not be linear, however, in a natural logarithmic scale, it will look linear.
- Often used in economics, biology, and other fields where exponential growth is common.

Example of natural logarithmic scale

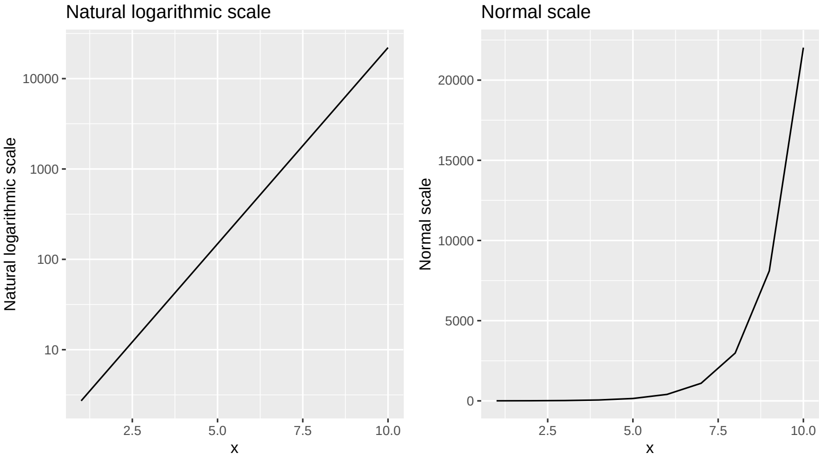
- Below, a dataset is shown with values that increase exponentially.

x	y
1	2.7
2	7.4
3	20.1
4	54.6
5	148.4
6	403.4
7	1,096.6
8	2,981.0
9	8,103.1
10	22,026.5

- Notice that the values increase exponentially
  - The percent growth is around 171.8% for each step.

Example of natural logarithmic scale

- The following plot shows the same data in a natural logarithmic scale and a regular scale.



$\ln(x)$ is a *very* good friend!

- The assumption of linearity in the social sciences comes up often
- This would mean forcing real world data of, say, population, to fit a constant increase of certain people per year.
- This is not realistic, as populations rarely grow in this way.
- Assuming that at a certain point, the population will grow at a constant *percentage* rate is more realistic.
- Sticking a natural logarithm on the data will make it “log-linear”, which makes the linearity assumption more realistic.
  - This way we evade using complex statistical models!