

Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 2 - Sigma Notation and Further Topics on Algebra

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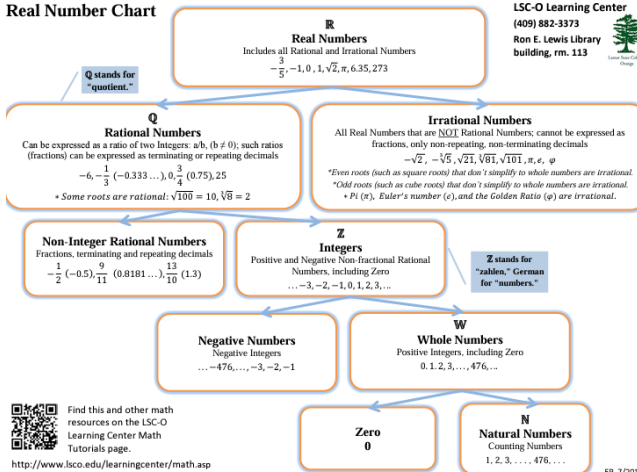
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A quick recap on real numbers

- The set of real numbers is denoted by \mathbb{R} .

Real Number Chart

LSC-O Learning Center
(409) 882-3373
Ron E. Lewis Library
building, rm. 113



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Summation

- In some cases we will be interested in the sum of a sequence of numbers.
- For example, the sum of the first 10 natural numbers is given by:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

- What if we want to sum the first 100 natural numbers?
 - It is a lot to write down! Let's use an *ellipsis* to denote the sequence.

$$1 + 2 + 3 + \dots + 98 + 99 + 100 = 5,050$$

' - However, this is still a lot to write down!

Sigma or summation notation

- We can use the Greek letter Σ (capital sigma) to denote the sum of a sequence of numbers.
- The sum of the first 10 natural numbers can be written as:

$$\sum_{i=1}^{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

- Notice that the index i represents the starting point of the sum.
 - The upper number denotes the last number to be added.

Sigma or summation notation

- More generally, the sum of the first n natural numbers can be written as:

$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n$$

- In other situations, i simply represents the index of the summands. For example:

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_n$$

This would be the sum of any sequence of numbers $a_1, a_2, a_3, \dots, a_n$.

Averages and sums

- The arithmetic mean (or average) is the sum of a group of numbers, divided by the count of numbers. For example:

$$\text{Average} = \frac{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10}{10} = 5.5$$

- More generally, any average can be written as:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

where x_i are the numbers in the sequence and n is the count of numbers.

Properties of sums

- The sum of a constant times a sequence of numbers is equal to the constant times the sum of the sequence.

$$\sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$$

This is a re-estatement of the distributive property of addition. For example, if $c = 2$:

$$2(1 + 2 + 3 + 4 + 5) = 2 + 4 + 6 + 8 + 10 = 30$$

Properties of sums

- The sum of two sequences of numbers is equal to the sum of the individual sequences.

$$\sum_{i=1}^n x_i + \sum_{i=1}^n y_i = \sum_{i=1}^n (x_i + y_i)$$

- This is a re-statement of the associative property of addition. See below:

$$(1 + 2) + (3 + 4) = 1 + 2 + 3 + 4 = 10$$

Exponents

- Exponents are a shorthand notation for repeated multiplication.

$$x^n = x \cdot x \cdot x \cdot \dots \cdot x (\text{n times})$$

- Properties of exponents:

- $x^0 = 1$ for any $x \neq 0$.
- $x^1 = x$ for any x .
- $x^m \cdot x^n = x^{m+n}$.
- $(x^m)^n = x^{m \cdot n}$.
- $(x \cdot y)^n = x^n \cdot y^n$.
- $\frac{x^m}{x^n} = x^{m-n}$.
- $x^{-n} = \frac{1}{x^n}$.

Common mistakes with exponents

$$-x^m \cdot y^n = x^{m+n}$$

$$-(x + y)^n = x^n + y^n$$

$$\blacksquare (x^m)^n = x^{m+n}$$

Radicals

- The n -th root of a number x is denoted by $\sqrt[n]{x}$.
- Most common roots are the square root ($n = 2$) and the cube root ($n = 3$).
- In english, you may hear the square root of x as “the sqrt” of x .
 - This might be confusing in Spanish, where “la raíz” is used.
 - Emerges from a coding context in many mathematical/statistical software: `sqrt(x)`.
 - A calculator will typically understand `sqrt(4)` as 2, and so on.
- Properties of radicals:
 - $\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y}$.
 - $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$.
 - $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.
 - $\sqrt[n]{x^n} = x$.

Common mistakes with radicals

- When faced with squared variables, it is common to think that the square root of x^2 is x . For example:

$$\begin{aligned}x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ x &= 2\end{aligned}$$

- Stating that only $x = 2$ is a mistake. The correct answer is $x = \pm 2$. This is since both 2 and -2 squared are equal to 4.
- To avoid this mistake, always attach the \pm sign when taking the square root of a squared variable in an equation.

Quadratic equations

- A quadratic equation is a polynomial equation of the form:

$$ax^2 + bx + c = 0$$

- For example, the equation $3x^2 - 2x - 1 = 0$ is a quadratic equation.
- If you get $x^2 = 4$, you have that $a = 1$, $b = 0$, and $c = -4$.
- We are often interested in finding the solutions to a quadratic equation. These are the values of x that make the equation true.
 - Also called the roots of the equation, or the zeros of the equation.

Zeros of a quadratic equation

- The solutions to a quadratic equation can be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Alternatively, you can factor the quadratic equation and solve for x .
- For example, the equation $x^2 - 5x + 6 = 0$ can be factored as:

$$(x - 2)(x - 3) = 0$$

- Take each factor $(x - 2)$ and $(x - 3)$ and set them equal to zero
 - This is often necessary when the quadratic formula is not practical.
 - You will need to do this often in optimization problems.
- Taking square roots of both sides might work, but only if the equation allows you to do so, and you're careful with the \pm sign.

When are solutions not real?

- Sometimes the solutions to a quadratic equation are not real numbers.
- This happens when the discriminant $b^2 - 4ac$ is negative.
- For example, the equation $x^2 + 1 = 0$ has no real solutions.
- For our purposes, we will not be interested in complex numbers.