Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 8 - A Gentle Review of Matrix Algebra

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2024

Why matrices?

- Matrices are a way to organize data in a structured way.
- When dealing with complex mathematical models or operations, using "normal" algebra can be cumbersome.
- For instance, consider a four-variable system of equations:

$$
\begin{aligned}
2x + 3y + 4z + 5w &= 10 \\
3x + 4y + 5z + 6w &= 20 \\
4x + 5y + 6z + 7w &= 30 \\
5x + 6y + 7z + 8w &= 40
\end{aligned}
$$

Why matrices?

- If we try to solve this system using algebra, we would have to do a lot of calculations.
- For instance, we may try to solve the first equation for x:

$$
\begin{aligned}
2x &= 10 - 3y - 4z - 5w \\
x &= \frac{10 - 3y - 4z - 5w}{2}
\end{aligned}
$$

- And then substitute this expression into the second equation:

$$3\left(\frac{10 - 3y - 4z - 5w}{2}\right) + 4y + 5z + 6w = 20$$

Why matrices?

- Which would then need to be simplified and solved for y:

$$\frac{30 - 9y - 12z - 15w}{2} + 4y + 5z + 6w = 20$$

- And so on...

Why matrices?

- **Matrices** are rectangular arrays of numbers considered as a single mathematical entity.
- They can help organize a system of equations or complex mathematical operations in a more structured way.
- Through matrix algebra, we can perform operations on matrices quickly and efficiently.

Matrix notation

- We represent a matrix with a capital letter, usually in boldface, e.g., $\mathbf{A}$.
  - In handwritten text, matrices are often represented with capital letters in block letters, e.g., $\mathbb{A}$, or using a different color.
- Consider, for instance, the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

- We usually either use square brackets or round brackets (parentheses) to represent matrices.

Elements

- The numbers in the matrix are called **elements**. They are usually represented with lowercase letters, e.g., a_{ij}.
- We typically define the *dimension* of a matrix as the number of rows and columns it has. Thus, the matrix A above is a 3×4 matrix.
- To refer to a specific element in a matrix, we use the notation a_{ij}, where i is the row number and j is the column number.
  - For instance, in the matrix A above, $a_{23} = 7$.

Matrix general form

- A matrix can be represented in general form as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Where m is the number of rows and n is the number of columns.
- This matrix is said to be an $m \times n$ matrix.
- It is common to see the dimensions of the matrix written as a subscript, e.g., $A_{m \times n}$. or $\underset{k \times n}{A}$

Vectors

- A vector is a special case of a matrix that has only one column or one row.
- Vectors which have only one column are called **column vectors**.
- Vectors which have only one row are called **row vectors**.
- For instance, consider the following column vector:

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

The main diagonal

- The **main diagonal** of a matrix is the set of elements that run from the top left to the bottom right of the matrix.
- More formally, the main diagonal of a matrix A is the set of elements a_{ij} where $i = j$.
- For instance, consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- The main diagonal of this matrix is $\{1, 5, 9\}$.
- Sometimes we refer to the main diagonal as only the "diagonal" of the matrix.

Special matrices

- There are several types of special matrices that are commonly used, and you should be familiar with them.
- **Square matrix**: A matrix where the number of rows is equal to the number of columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is a square matrix.}$$

Special matrices

- **Diagonal matrix**: A square matrix where all elements off the main diagonal are equal to 0.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ is a diagonal matrix.}$$

- **Note**: Sometimes, everything that is not in the main diagonal is referred to as the “off-diagonals”.

Special matrices

- **Identity matrix**: A square matrix where all elements on the main diagonal are equal to 1 and all other elements are equal to 0.
- This matrix is usually represented by the symbol I_n, where n is the dimension of the matrix.
  - For instance, I_3 is a 3×3 identity matrix, as it always a square matrix.
- The identity matrix is analogous to the number 1 in normal algebra.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special matrices

- **Zero matrix**: A matrix where all elements are equal to 0.
- This matrix is usually represented by the symbol $O_{m \times n}$, where m is the number of rows and n is the number of columns.

Matrix operations