Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 3 - Equation Systems and Graphing

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Equation systems

- A set of of equations that share the same variables is called an *equation system*.
- For example:

$$x + y = 3 \tag{1}$$
$$2x - y = 1 \tag{2}$$

- Because both (1) and (2) share x and y, they form an equation system.
- We usually want to *solve* the system, i.e., find the values of x and y that satisfy both equations.

Solving equation systems

- There are several methods to solve equation systems.
  - Substitution
  - Elimination
  - Graphing
  - Matrices (we will see this later)
- Substitution is typically the most "mechanical" method.
  - Express one variable in terms of the other and substitute in the other equation.
- Elimination is more algebraic.
  - Add or subtract the equations to eliminate one variable.
  - Might involve multiplying one or both equations by a constant.

Solving the example system

- Let's solve the example system:

$$x + y = 3$$
$$2x - y = 1$$

- We can solve this system by substitution.
  - From (1), we have $y = 3 - x$.
  - Substitute this into (2):

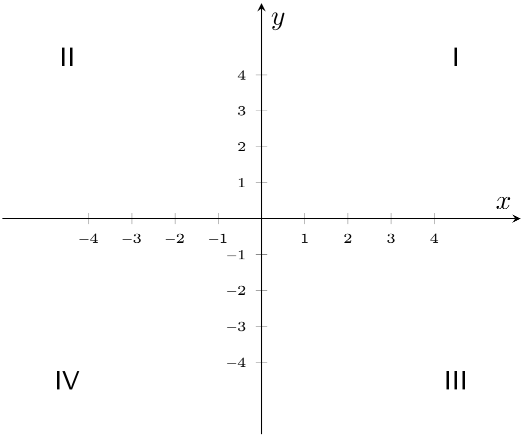
$$2x - (3 - x) = 1$$

- Solve for x and then substitute back to find y.

The Cartesian plane

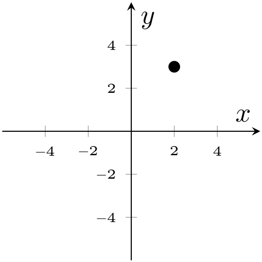
- The Cartesian plane is a two-dimensional space where we can plot points.
- It is formed by two perpendicular lines, the *x-axis* and the *y-axis*.
- The point where the axes intersect is called the *origin*.
- The axes divide the plane into four *quadrants*.

The Cartesian plane



Plotting points

- To plot a point, we use an ordered pair (x, y).
  - x is the distance from the y-axis.
  - y is the distance from the x-axis.
- For example, the point $(2, 3)$ is 2 units to the right and 3 units up from the origin. See below:



Linear equations

- The equations we've seen so far are *linear* equations.
  - They represent straight lines in the Cartesian plane.
- Linear equations can be written in the form $y = mx + b$.
  - m is the *slope* of the line.
  - b is the *y-intercept*.

The Slope

- The ratio of the vertical change to the horizontal change.
  - It tells us how steep the line is.
  - The bigger the slope, the steeper the line.
- Given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- Requires two points (call them P_1 and P_2) on the line, with coordinates (x_1, y_1) and (x_2, y_2).

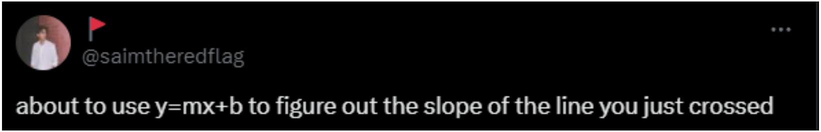


Figure 1: A meme

Intercepts

- The *y-intercept* is the point where the line crosses the y-axis.
  - This happens when $x = 0$.
  - So, we set $x = 0$ in the equation and solve for y.
  - In the equation $y = mx + b$, the y-intercept is $(0, b)$.
- The *x-intercept* is the point where the line crosses the x-axis.
  - This happens when $y = 0$.
  - So, we set $y = 0$ in the equation and solve for x.

Graphing linear equations

- To graph a linear equation, we need to find two points on the line.
  - The easiest points are the intercepts.
  - We can also use the slope to find a second point.
- Example: graph the line $y = 2x + 1$.
  - It might be useful to draw a table of values.

x	y
0	1
1	3
-1	-1

Graphing the line

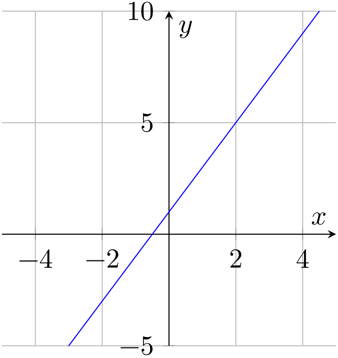


Figure 2: Plot of the equation $y = 2x + 1$

Upward-sloping and downward-sloping lines

- If $m > 0$, the line is "upward-sloping" or increasing.
  - As x increases, y also increases.
- If $m < 0$, the line is "downward-sloping" or decreasing.
  - As x increases, y decreases.

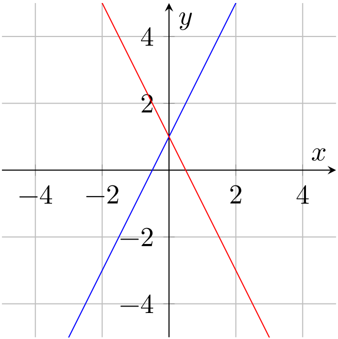


Figure 3: Upward-sloping and downward-sloping lines

Properties of slopes

- If $m = 0$, the line is horizontal.
  - y does not change as x changes.
- If $m = \infty$, the line is vertical.
  - x does not change as y changes.
- If $m = 1$, the line has a 45-degree angle.
- Lines with the same slope are parallel.
- Lines with slopes that multiply to -1 are perpendicular.
  - This means that $m_1 \cdot m_2 = -1$, or that $m_1 = -\frac{1}{m_2}$ (the negative reciprocal).

How to find the equation of a line

1. If you know the slope m and a point (x_1, y_1) on the line, you can use the point-slope form:
   - $y - y_1 = m(x - x_1)$
2. If you know two points (x_1, y_1) and (x_2, y_2) on the line, you can use the slope formula to find m and then use $y = mx + b$ to find b.
3. If you know the slope m and the y-intercept b, you can use $y = mx + b$ directly (this is the slope-intercept or point-slope form).

Graphing equation systems

- To solve an equation system graphically, we graph both equations and find the point where they intersect.
- The point of intersection is the solution to the system.
- Example: graph the system

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 1 \end{aligned}$$

Graphing the system

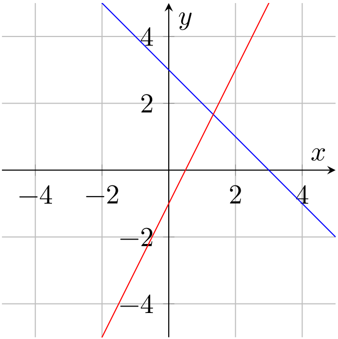


Figure 4: Graph of the system

Systems with no solutions

- Sometimes, the lines are parallel and do not intersect.
  - This means that the system has no solution.
- How can we tell if two lines are parallel?
  - They have the same slope.
  - The coefficients of x and y in the equations are proportional.

Systems with infinite solutions

- Sometimes, the lines coincide and intersect at every point.
    - This means that the system has infinite solutions.
- How can we tell if two lines coincide?
    - They have the same slope and the same y-intercept.
    - The coefficients of x and y in the equations are proportional, and the constants are equal.

Graphic representations - systems with no solutions

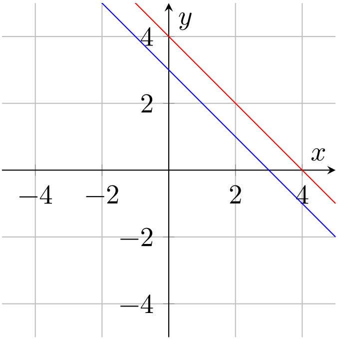
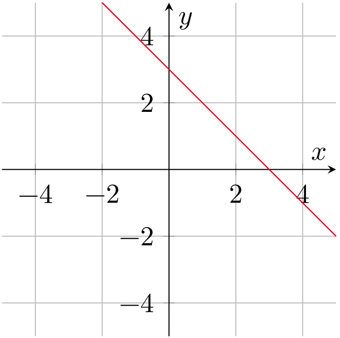


Figure 5: Graph of a system with no solutions

Graphic representations - systems with infinite solutions



What happens when I try to solve a system that can't be solved?

- If you try to solve a system that has no solution, you will get a contradiction.
  - For example, you might find $0 = 1$.
- If you try to solve a system that has infinite solutions, you will get an identity.
  - For example, you might find $0 = 0$.
- In both cases, you cannot reach something like we're used to, like $x = 3$ and $y = 2$.

More than two equations

- We can extend the concept of equation systems to more than two equations.
  - For example, a system of three equations in three variables.
- The same principles apply
  - Elimination
  - Substitution
  - Graphing
  - Matrices (typically used for larger systems)
- For graphing, we need to consider more dimensions.
  - For a system of three equations in three variables, we need a three-dimensional space (3D), with x, y, and z axes.

Quadratic equations revisited

- Quadratic equations are equations of the form $y = ax^2 + bx + c$.
  - They represent parabolas in the Cartesian plane.
- The vertex of the parabola is given by $x = -\dfrac{b}{2a}$.
  - This is where the parabola reaches its maximum or minimum.
  - The vertex is the point where the parabola changes direction.
  - It has coordinates $\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)$.
- The parabola opens upwards if $a > 0$ and downwards if $a < 0$.

Graphing quadratic functions

- To graph a quadratic function, the vertex is a good starting point.
- If there are any roots or x-intercepts, they are also useful.
  - The roots are the points where the parabola crosses the x-axis.
  - They are given by the solutions to the equation $ax^2 + bx + c = 0$.
- Intercepts with the y-axis might be useful too.
  - The y-intercept is the point where the parabola crosses the y-axis.
  - Solved by setting $x = 0$ in the equation.
  - They don't necessarily exist.
- Ultimately, might need to use a table of values to plot the parabola.

Concave parabola

- A parabola that opens upwards is called *concave*.
  - It has a minimum at the vertex.

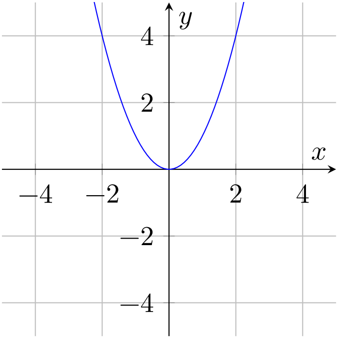


Figure 6: Concave parabola

Convex parabola

- A parabola that opens downwards is called *convex*.
    - It has a maximum at the vertex.

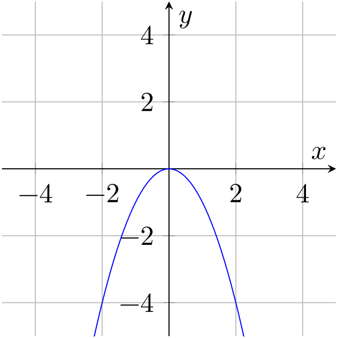


Figure 7: Convex parabola

Cubic equations/functions

- Cubic equations are equations of the form $y = ax^3 + bx^2 + cx + d$.
  - They represent cubic functions in the Cartesian plane.
- The graph of a cubic function is a curve that can have multiple inflection points.
- The inflection points are points where the curve changes concavity.
  - Convex to concave or vice versa.

Graph of a cubic function

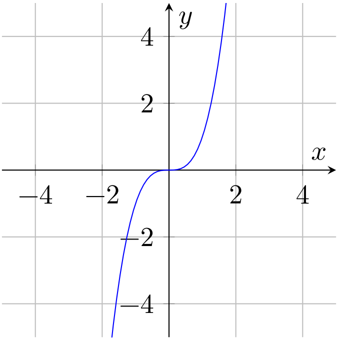


Figure 8: Graph of x^3