Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 1 - Introduction and foundational concepts

Daniel Sánchez Pazmiño

Laboratorio de Investigación para el Desarrollo del Ecuador

2024

Introduction: Mathematics and the Social Sciences

- Mathematics per se **not** a distinct field in the social sciences
- We use mathematics **as an approach** to study reality
- It is not hard to think about social science models which were initially not mathematical
  - Freud's model of the id, ego, and superego
  - Downs' of rational voting through utility maximization
  - Schumpeter's model of creative destruction
- Modern researchers represent models using math
  - Allows to extend the model using mathematical "laws"
  - Makes models less subject to interpretation
  - Allows for statistical testing of hypotheses

Is it so challenging?

- Math is often feared because of its complexity
- Often the reason people fail is poor foundations
- Must master basic concepts before moving to more complex ones
  - This means that math takes time and few have the patience for it
  - Comparing yourself to others is **not useful**
  - Some have better foundations, some have inherent talent, some have more time

Order of operations

- The order of operations defines the priority of operations in an expression
- This is a consensus which everyone agreed to so we can all understand each other
  - Calculators and statistical software follow this order
  - As basic as this is, this is tested on the GRE (graduate school entrance exam)
- **PEMDAS**: **P**arentheses, **E**xponents, **M**ultiplication and **D**ivision, **A**ddition and **S**ubtraction
- If two operations have the same priority, we go from left to right.
- No sign means multiplication

Exercise

Order of operations

Calculate the following expression:

$$3 + 4 \times 2^2(3 + 1 \times 3 \div 3) + 4 - 5 + 5$$

Solution

Order of operations

- Start with the parentheses. Apply PEMDAS within too.

$$4 \times 1 = 4$$

- Substitute into the full expression, follow order, and solve

$$\begin{aligned} 3 + 4 \times 2^2(4) + 4 - 5 + 5 \\ 3 + 4 \times 4(4) + 4 - 5 + 5 \\ 3 + 4 \times 16 + 4 - 5 + 5 \\ 3 + 64 + 4 - 5 + 5 \\ 67 + 4 - 5 + 5 \\ 71 - 5 + 5 \\ 71 \end{aligned}$$

Thoughts

- As useless this might be with numbers, it is crucial when working with variables
  - E.g. What does $\lambda_1(x_1 + x_2) \times \lambda_1 x_1 + x_2^2(\tau \cdot \gamma)$ simplify to?
- Calculators typically don't work with variables
- Not understanding PEMDAS may lead you to code the wrong formula in your software
  - It is hard to debug this as it not a syntax error

Variables and the greek alphabet

- Variables are used to represent unknowns in equations
  - Technically, they are placeholders for numbers
- x is commonly used, y and z are common too
- Scientists need more variables as they study more complex systems
  - The greek alphabet is used for this
- Get comfortable seeing and using the greek alphabet

The Greek alphabet

Alpha - Α α	Eta - Η η	Nu - Ν ν	Tau - Τ τ
Beta - Β β	Theta - Θ θ	Xi - Ξ ξ	Upsilon - Υ υ
Gamma - Γ γ	Iota - Ι ι	Omicron - Ο ο	Phi - Φ φ
Delta - Δ δ	Kappa - Κ κ	Pi - Π π	Chi - Χ χ
Epsilon - Ε ε	Lambda - Λ λ	Rho - Ρ ρ	Psi - Ψ ψ
Zeta - Ζ ζ	Mu - Μ μ	Sigma - Σ σ/ς	Omega - Ω ω

Figure 1: Greek alphabet. Source: University of Northern Colorado

- Commonly, the same letter is used for different purposes in different fields and contexts
  - β represents the slope coefficient, but also the discount factor in finance

Working with percentages and proportions

- Percentages are a way to express a fraction of a whole (the symbol being %)
- The word "percent" means "per hundred"; meaningful for calculation

$$\text{Percentage}\% = \frac{\text{Part}}{\text{Whole}} \times 100\%$$

- A useful way of thinking about percentages is as *proportions*: the decimal representation of the percentage
  - Divide by 100 and drop the % sign
  - 50% is 0.50, 25% is 0.25, 100% is 1.00
- This is useful for calculations so you don't use the above formula all the time
- Example: A 25% discount on a $75 item is $0.25 \times 75 = 18.75$

Working with percentages and proportions

- Example: What is the final price of a \$75 item with a 25% discount?
- You may be tempted to use the rule of three, but this is not necessary (it takes longer)
- Use proportions to understand that 1 reprents the full price (the \$75) and 0.25 represents the discount.

$$\$75 \cdot (1 - 0.25) = \$75 \cdot 0.75 = \$56.25$$

- Notice that the new final price is 75% of the original price, so this makes sense!
  - Don't be afraid to use your intuition
  - Social scientists are not math machines, we use intuition to guide us

Percentage change and subindices

- It is common to express increases and decreases in percentages
- This involves introducing the concept of **variation**
  - Typically, we express change or variation as Δ
- Often, we use subindices to represent the old and new values
  - E.g. x_0 is the old value, x_1 is the new value
- The formula for percentage change is

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100\% = \frac{x_1 - x_0}{x_0} \times 100$$

- A shorthand for this is dividing the final value by the initial value and subtracting 1
  - A negative value means a decrease, a positive value means an increase

Solving for a final value after a percentage change

- Example: The central government has proposed a 10% increase in the minimum wage. If the current minimum wage is \$460, what would be the new minimum wage?
- Typically, one would calculate the 10% of \$460 and add it to the original value.

$$x_0 + (x_0 \cdot \Delta\%) = x_1$$

- It is faster to represent $\Delta\%$ as a proportion, add it to 1, and multiply by the original value $(1 + 0.10 = 1.10)$
- Doing this is a simplification from the formula above
- However, it is also intuitive, since you're simply finding a value which is 10% larger than the whole (which is 1).
- So, the new minimum wage is $\$460 \times 1.10 = \$506$

Solving for a final value after a percentage change

- The same can be done for percentage *decreases*
- Example: The average wage of an economics graduate has decreased by 5%. If the average wage was \$500, what is the new average wage?
- The new average wage is \$500 $\times$ (1-0.05) $=$ \$500 $\times$ 0.95 $=$ \$475
- Once again, this is intuitive: the new wage is only 95% of the original wage

An exercise for you

- Example: A Juan Valdez iced matcha latte costs \$3.50. What is the added value tax being paid by the consumer?

Solution

- The current tax is 15% of the price
- In Ecuador, law mandates that the tax is included in the price
- The price after tax can be calculated as $P_{\text{bf}} \cdot 1.15 = 3.50$, so it is $3.50/1.15 \approx 3.04$
- The tax is the difference between the price before and after tax, so it is $3.50 - 3.04 = 0.46$
- Notice that if I had calculated 15% of $3.50, I would have gotten $0.525, which is not the same
- This is because the tax is calculated on the price before tax, not the price after tax.
- You have to be very careful when defining the "whole" in the percentage formula!

Ratios

- Ratios are a way to express the relationship between two parts of a whole
  - Not the same as the percentage, which is the relationship of a part to the whole
- They are often expressed as 1 to 2, 1:2.
- These can be a bit confusing, but context is helpful.
  - A ratio of 1:2 means that for every 1 unit of the first part, there are 2 units of the second part
- Example: A production function using a ratio of 1:2 for labor and capital means that for every 1 unit of labor, 2 units of capital are used.

Application: tax rate changes and percentage change interpretations

- Recently, the Ecuadorian government increased the value-added tax (VAT) rate from 12% to 15%
- Question for you: which the following statements is correct?
  1. There was a 3% increase in the VAT rate
  2. There was a 25% increase in the VAT rate
  3. There was a 15% increase in the VAT rate

Percentage points and percent changes

- When we are analyzing something is measured in percentages itself, we need to be careful
- We call the the absolute change for these cases a **percentage point change**
- This is different from a **percent change**, which is the relative change from a value
- It might or might not be useful to use percentage changes for these cases, as it can be misleading
- This is when expressing everything as a proportion might be useful.

Ratios

- You can work out the the relationship from one part to the whole by adding the parts of the ratio.
    - This may or may not make sense depending on the context
- With a ratio of 1:2 for labor and capital, the whole is 3 units, but this is not meaningful
- However, if you're told that in a hospital the ratio of doctors to nurses is 1:2, then doctors are $1/3$ of the whole staff

Equations

- An equation is a statement that two expressions are equal, typically with an unknown.
  - E.g. $x + 2 = 5$
- We often care about solving equations, which means finding the value of the unknown that makes the equation true (i.e. satisfies the equation)
  - For above, $x = 3$ will make the equation true (5 = 5)
- Equations can be solved by isolating the unknown on one side of the equation, and doing the math.

Equations

- There are rules for solving equations:
  - You can add or subtract the same value to both sides
  - You can multiply or divide both sides by the same value
  - Exponents must be raised to the same power on both sides
  - You can take the square root of both sides by adding $\pm$
  - We will see more rules as we go along

Functions

- A function is a rule that assigns a unique output to each input
  - Think of it as a machine that takes an input and gives an output
  - We stop caring about solving for the unknown, and care about the relationship between the input and output
- Functions are often represented as $f(x)$, where f is the function and x is the input
  - The value of the function at x is $f(x)$
- A function doesn't necessarily have to be a mathematical formula, but it is often represented as one
- Social scientists use functions to represent social phenomena
  - E.g. the relationship between income and happiness
  - E.g. the relationship between education and income
- Functions can be linear, quadratic, exponential, logarithmic, etc.

Functions

- The simplest function is the constant function, which always returns the same value
  - E.g. $f(x) = 5$
- Notice how this challenges our equation solving skills
  - There is no unknown to solve for
  - If we input 3, we get 5, if we input 10, we get 5
- A linear function is the next simplest.
  - E.g. $f(x) = 2x + 3$
  - Here we do have an input x which goes into the function
  - The function multiplies the input by 2, adds 3, and returns the result
  - It's called linear because the graph of the function is a straight line

Functions: domain and range

- The **domain** of a function is the set of all possible inputs
  - For $f(x) = 2x + 3$, the domain is all real numbers ($\mathbb{R}$)
- The **range** of a function is the set of all possible outputs
  - For $f(x) = 2x + 3$, the range is all real numbers ($\mathbb{R}$)
- The domain and range are important to understand the behavior of the function
- Some functions have a restricted domain because of the way the equation is defined
  - E.g. $f(x) = \frac{1}{x}$ has a domain of all real numbers except 0
  - The function is not defined at 0 because there is no such thing as division by 0
- The domain and range are often represented as intervals
  - E.g. $x \in (-\infty, 0) \cup (0, \infty)$
  - E.g. $f(x) \in (-\infty, 0) \cup (0, \infty)$

Inequalities

- Inequalities are statements that two expressions are not equal
  - E.g. $x > 5$
- When we have more elaborate expressions, we can have more complex inequalities
  - E.g. $2x + 3 > 5$
  - Simplify using the rules of equations to find which values of x make the inequality true.
  - E.g. $2x > 2$, $x > 1$. This means that all values of x greater than 1 make $2x + 3 > 5$ true.
- Use the same rules as equations to solve inequalities, except when multiplying or dividing by a negative number
  - In this case, the inequality sign flips
  - E.g. $-2x < 6$, $x > -3$
- Inequalities are often used to represent constraints in optimization problems

Application: returns to scale in a production function

- A production function is a mathematical model of how a country, a firm or any economic agent transform resources (inputs) into goods and services (outputs)
- In certain contexts, researchers care about whether the production function can be scaled
  - This means asking whether doubling the inputs will double the outputs?
  - This is called **returns to scale**
- This can be worked out by combining all of our knowledge so far

Application: returns to scale in a production function

- A production function is given by $Q = 2L + 3K$, where Q is the output, L is labor, and K is capital.
  - Think of it as a firm that produces goods (can be pizza) using workers and ovens.
- What happens to output if we double both labor and capital? Can the pizza firm double pizza production?
- We can work this out by plugging in values to the production function.
  - We need a base value of output. Let us see what happens when $L = 1$ and $K = 1$.
  - $Q = 2(1) + 3(1) = 2 + 3 = 5$
- Now, let us see what happens when $L = 2$ and $K = 2$.
  - $Q = 2(2) + 3(2) = 4 + 6 = 10$
- How to see if output has doubled? We can calculate the percentage

Application: returns to scale in a production function

- If the percentage change in output is less than the percentage change in inputs, we say that the production function has decreasing returns to scale.
  - If you double the inputs, you get less than double the outputs
- If the percentage change in output is more than the percentage change in inputs, we say that the production function has increasing returns to scale.
  - If you double the inputs, you get more than double the outputs
- Returns to scale are important for understanding the behavior of firms and countries.
  - What kind of production function do you think Ecuador has?
  - How do you think researchers can estimate the production function of a country?