Math for the Social Sciences Module - Young Researchers Fellowship

Lecture 2 - Sigma Notation and Further Topics on Algebra

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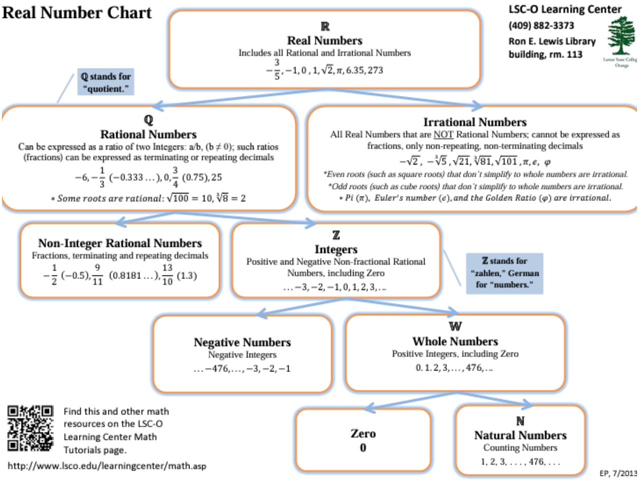
A quick recap on real numbers

- The set of real numbers is denoted by $\mathbb{R}$.



Summation

- In some cases we will be interested in the sum of a sequence of numbers.
- For example, the sum of the first 10 natural numbers is given by:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

- What if we want to sum the first 100 natural numbers?
    - It is a lot to write down! Let's use an *ellipsis* to denote the sequence.

$$1 + 2 + 3 + ... + 98 + 99 + 100 = 5,050$$

' - However, this is still a lot to write down!

Sigma or summation notation

- We can use the Greek letter Σ (capital sigma) to denote the sum of a sequence of numbers.
- The sum of the first 10 natural numbers can be written as:

$$\sum_{i=1}^{10} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

- Notice that the index i represents the starting point of the sum.
  - The upper number denotes the last number to be added.

Sigma or summation notation

- More generally, the sum of the first n natural numbers can be written as:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n$$

- In other situations, i simply represents the index of the summands. For example:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

This would be the sum of any sequence of numbers $a_1, a_2, a_3, \dots, a_n$.

Averages and sums

- The arithmetic mean (or average) is the sum of a group of numbers, divided by the count of numbers. For example:

$$\text{Average} = \frac{1+2+3+4+5+6+7+8+9+10}{10} = 5.5$$

- More generally, any average can be written as:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where x_i are the numbers in the sequence and n is the count of numbers.

Properties of sums

- The sum of a constant times a sequence of numbers is equal to the constant times the sum of the sequence.

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

This is a re-estatement of the distributive property of addition. For example, if $c = 2$:

$$2(1+2+3+4+5) = 2+4+6+8+10 = 30$$

Properties of sums

- The sum of two sequences of numbers is equal to the sum of the individual sequences.

$$\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (x_i + y_i)$$

- This is a re-statement of the associative property of addition. See below:

$$(1+2)+(3+4) = 1+2+3+4 = 10$$

Exponents

- Exponents are a shorthand notation for repeated multiplication.

$$x^n = x \cdot x \cdot x \cdot ... \cdot x (\text{n times})$$

- Properties of exponents:
  - $x^0 = 1$ for any $x \neq 0$.
  - $x^1 = x$ for any x.
  - $x^m \cdot x^n = x^{m+n}$.
  - $(x^m)^n = x^{m \cdot n}$.
  - $(x \cdot y)^n = x^n \cdot y^n$.
  - $\frac{x^m}{x^n} = x^{m-n}$.
  - $x^{-n} = \frac{1}{x^n}$.

Common mistakes with exponents

-$x^m \cdot y^n = x^{m+n}$

-$(x+y)^n = x^n + y^n$

- $(x^m)^n = x^{m+n}$

Radicals

- The n-th root of a number x is denoted by $\sqrt[n]{x}$.
- Most common roots are the square root ($n = 2$) and the cube root ($n = 3$).
- In english, you may hear the square root of x as "the sqrt" of x.
  - This might be confusing in Spanish, where "la raíz" is used.
  - Emerges from a coding context in many mathematical/statistical software: `sqrt(x)`.
  - A calculator will typically understand sqrt(4) as 2, and so on.
- Properties of radicals:
  - $\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{x \cdot y}$.
  - $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$.
  - $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.
  - $\sqrt[n]{x}^{n} = x$.

Common mistakes with radicals

- When faced with squared variables, it is common to think that the square root of x^2 is x. For example:

$$\begin{aligned} x^2 &= 4 \\ \sqrt{x^2} &= \sqrt{4} \\ x &= 2 \end{aligned}$$

- Stating that only $x = 2$ is a mistake. The correct answer is $x = \pm 2$. This is since both 2 and -2 squared are equal to 4.
- To avoid this mistake, always attach the $\pm$ sign when taking the square root of a squared variable in an equation.

Quadratic equations

- A quadratic equation is a polynomial equation of the form:

$$ax^2 + bx + c = 0$$

- For example, the equation $3x^2 - 2x - 1 = 0$ is a quadratic equation.
- If you get $x^2 = 4$, you have that $a = 1$, $b = 0$, and $c = -4$.
- We are often interested in finding the solutions to a quadratic equation. These are the values of x that make the equation true.
  - Also called the roots of the equation, or the zeros of the equation.

Zeros of a quadratic equation

- The solutions to a quadratic equation can be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Alternatively, you can factor the quadratic equation and solve for x.
- For example, the equation $x^2 - 5x + 6 = 0$ can be factored as:

$$(x-2)(x-3) = 0$$

- Take each factor $(x-2)$ and $(x-3)$ and set them equal to zero
  - This is often necessary when the quadratic formula is not practical.
  - You will need to do this often in optimization problems.
- Taking square roots of both sides might work, but only if the equation allows you to do so, and you're careful with the $\pm$ sign.

When are solutions not real?

- Sometimes the solutions to a quadratic equation are not real numbers.
- This happens when the discriminant $b^2 - 4ac$ is negative.
- For example, the equation $x^2 + 1 = 0$ has no real solutions.
- For our purposes, we will not be interested in complex numbers.

Absolute value

- The absolute value of a number x is denoted by $|x|$.
- The absolute value of a number is the distance of the number from zero on the number line.
- The absolute value of a number is always positive.
- For example, $|3| = 3$ and $|-3| = 3$.
- The absolute value of a number can be thought of as the "positive" version of the number.

Properties of absolute value

- The absolute value of a number is zero if and only if the number is zero.
- The absolute value of a number is positive if the number is not zero.
- The absolute value of a number is the same as the number if the number is positive.
- The absolute value of a number is the negative of the number if the number is negative.
- The absolute value of a sum is less than or equal to the sum of the absolute values.

$$|x + y| \leq |x| + |y|$$

Cobb-Douglas production functions

- The Cobb-Douglas production function is a common functional form used in economics.
- The production function is given by:

$$Q = AL^{\alpha}K^{\beta}$$

where Q is the quantity of output, L is the quantity of labor, K is the quantity of capital, and A, α, and β are parameters.

- The exponents α and β are often between 0 and 1.

Particularities of the Cobb-Douglas production function

- We cannot apply the properties of exponents directly to the Cobb-Douglas production function, since L and K are not the same base nor raised to the same power.
- The exponents α and β are often interpreted as the relative importance of labor and capital in the production process.
- A numerical example: $Q = 2L^{0.5}K^{0.5}$.
- Notice that fractional exponents are common in the Cobb-Douglas production function
  - This means that we are essentially working with radicals

Cobb-Douglas optimal input levels

- Economic theory predicts that a firm under Cobb-Douglas technology (and other assumptions) will choose capital and labour in the following way:

$$\frac{K}{L} = \frac{\beta w}{\alpha r}$$

where w is the wage rate and r is the rental rate of capital.

Example: Cobb-Douglas optimal input levels

- With the knowledge that we have up to this point, we can solve for the optimal input levels of labor and capital in a Cobb-Douglas production function.
- Suppose that a firm works under a Cobb-Douglas production function given by

$$Q = 3L^{\alpha}K^{\beta}$$

The relative importance of labor and capital is 30% and 70%, respectively. The wage is $w = 10$ and the rental rate of capital is $r = 20$.

- For you: what are the optimal input levels of labor and capital?