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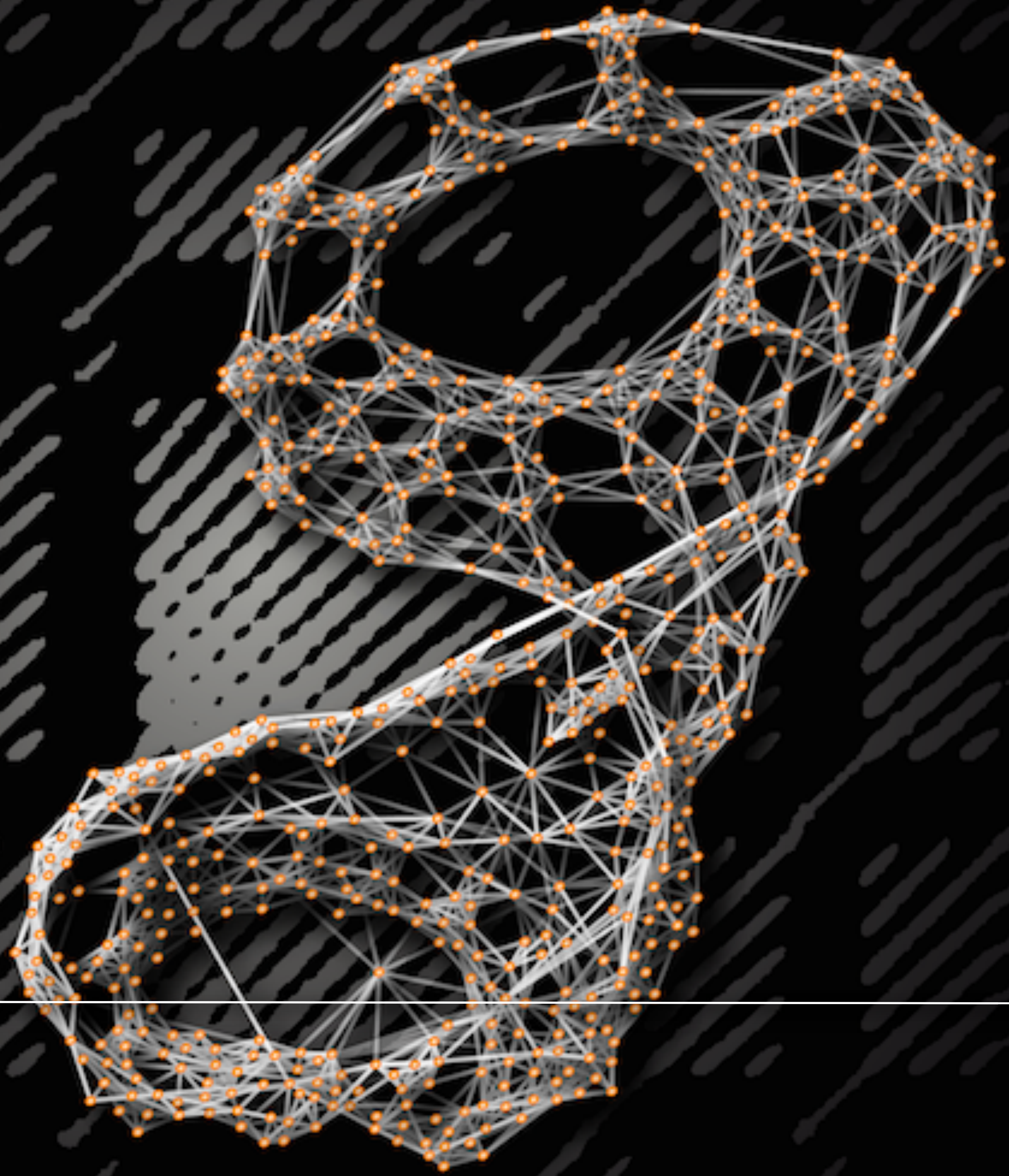
Potsdam Institute for Climate Impact Research

Norbert Marwan

Hauke Krämer

Recurrence Plots Workshop



Leibniz-Gemeinschaft

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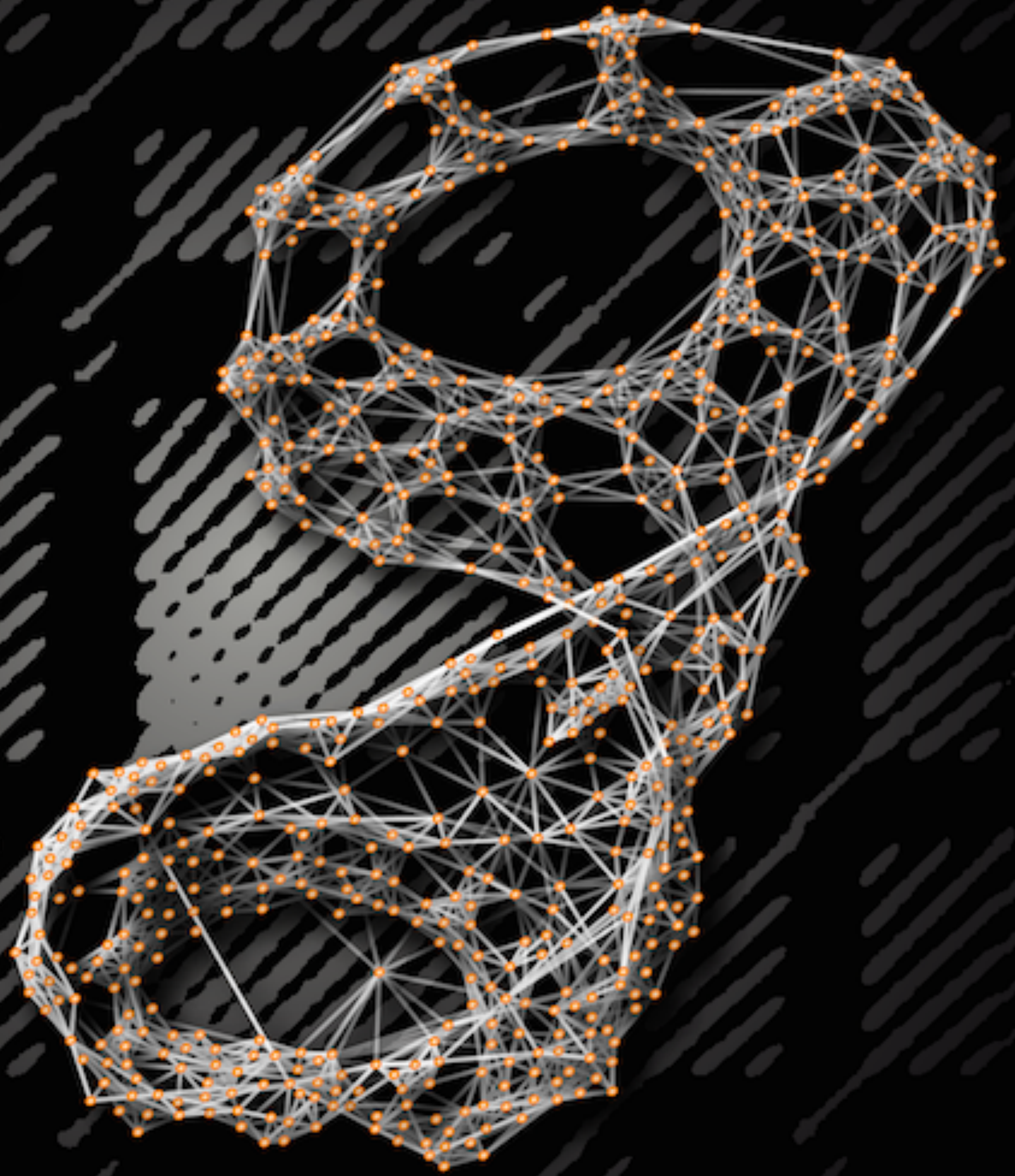
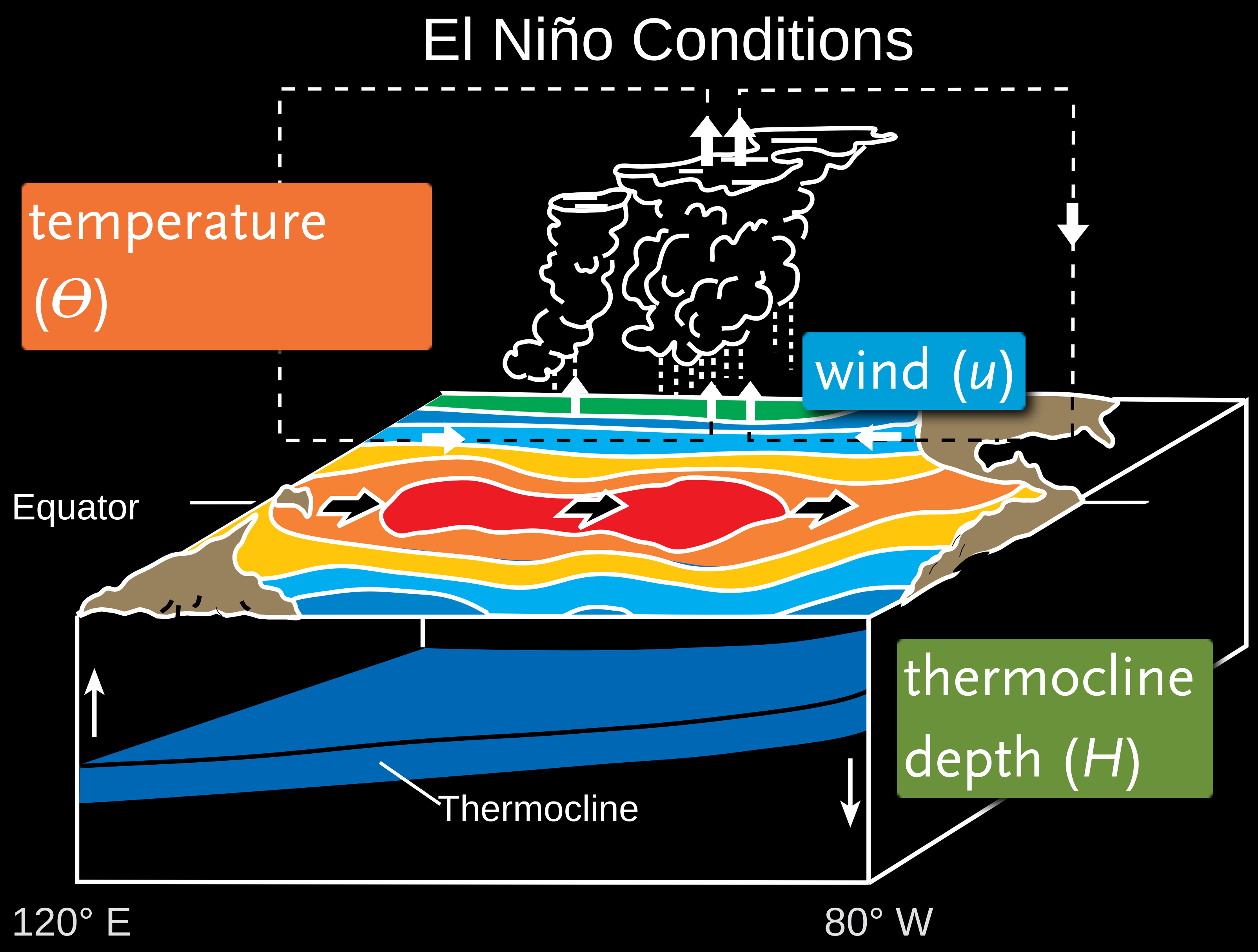
Recurrence Plots Workshop

1. Phase space
2. Visual interpretation
3. Sensitivity on Parameters
4. Recurrence quantification analysis
5. Coupling analysis



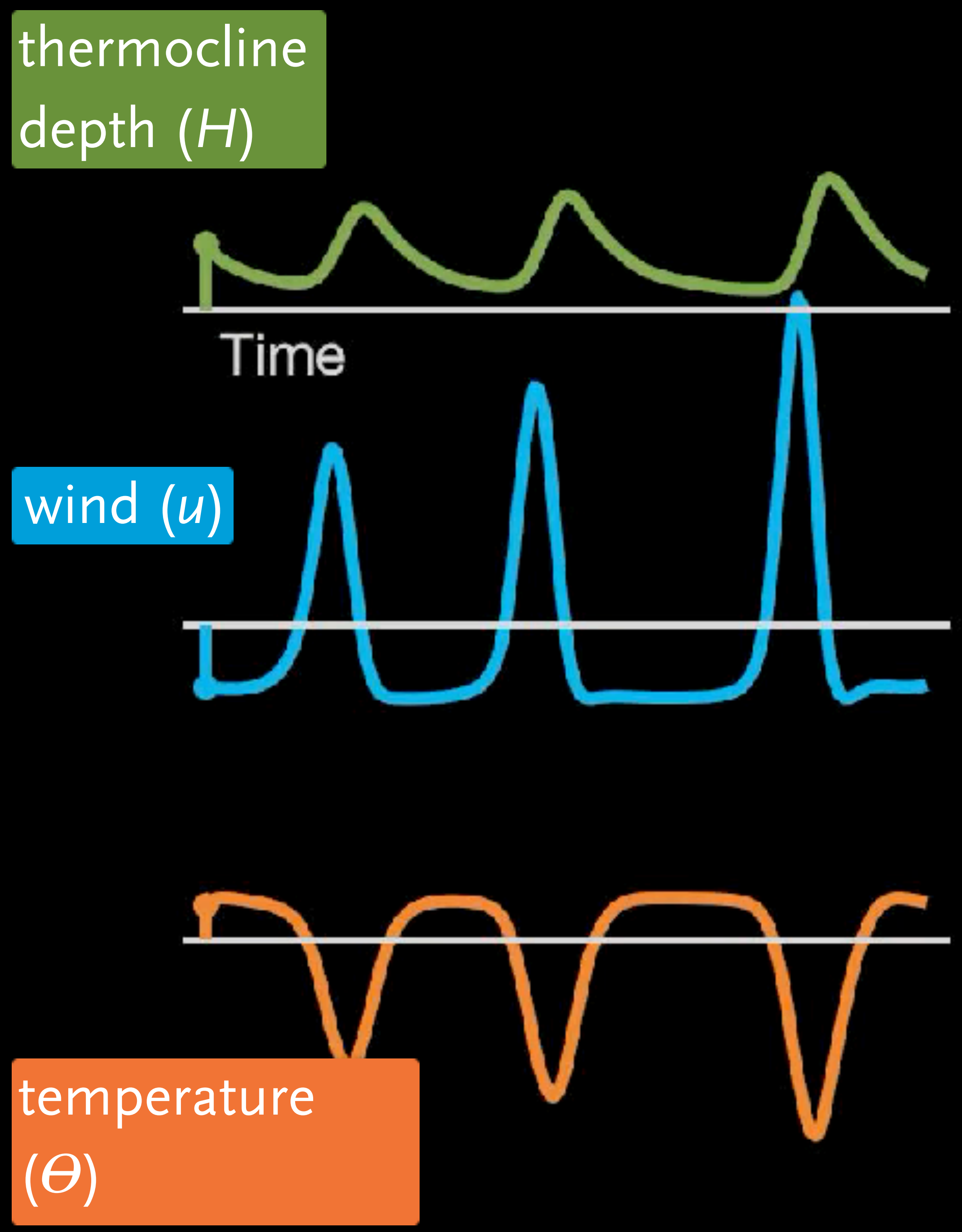
Phase Space

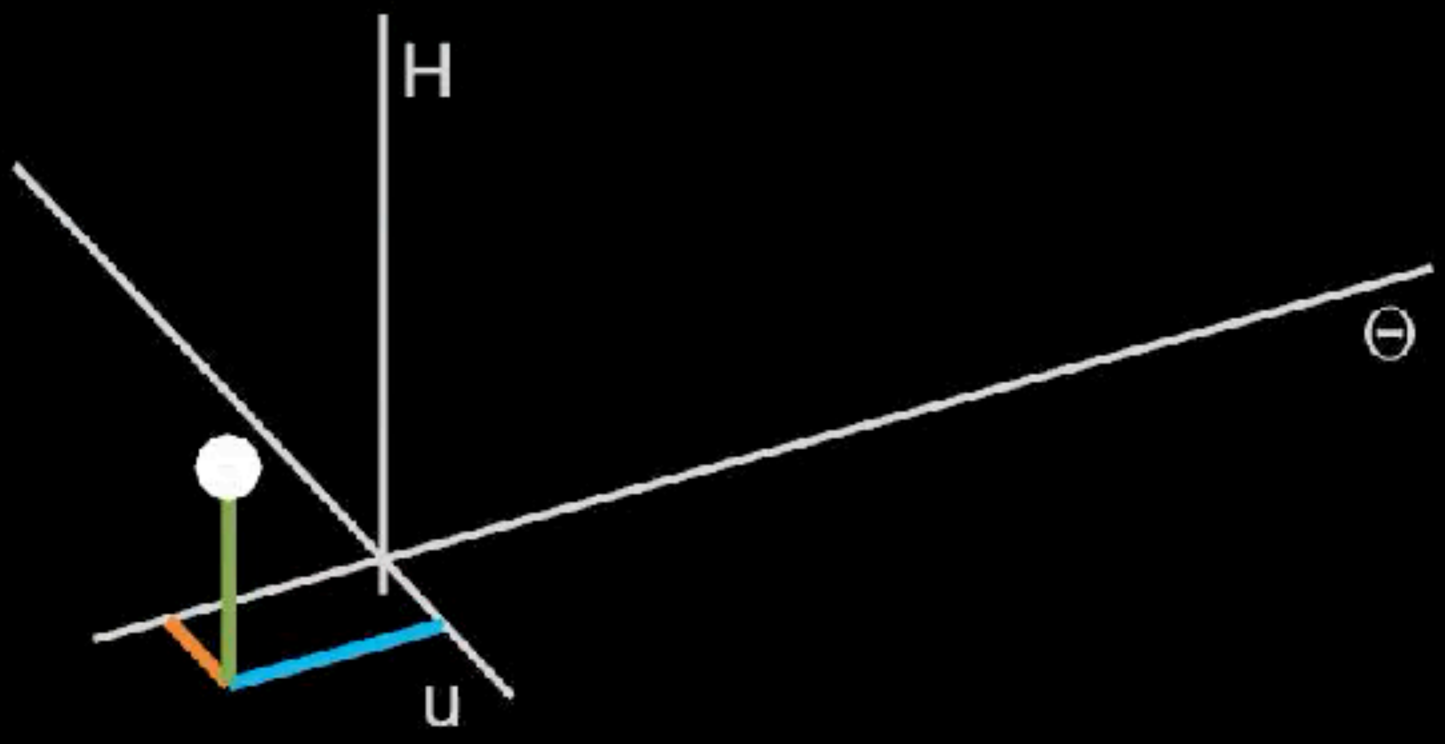
Phase Space

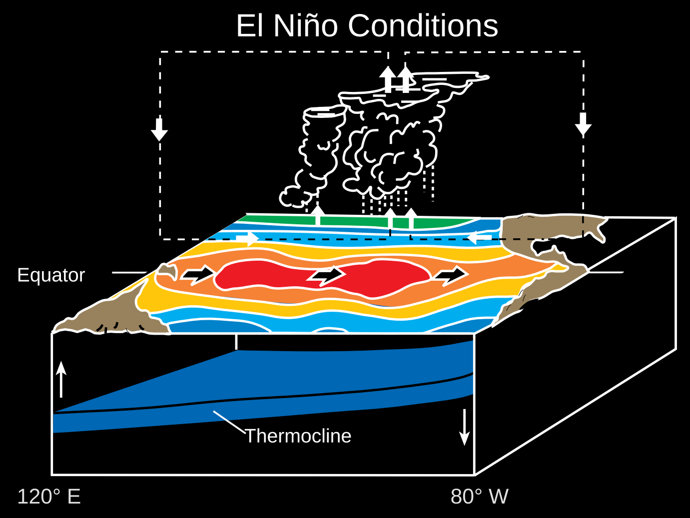


- State of the system:
  3 state variables (simplified)
  Θ, H, u
- Describe the system state at each time point
- Used to model the dynamics

Phase Space


thermocline depth (H)
Time
wind (u)
temperature (Θ)


H
Θ
u

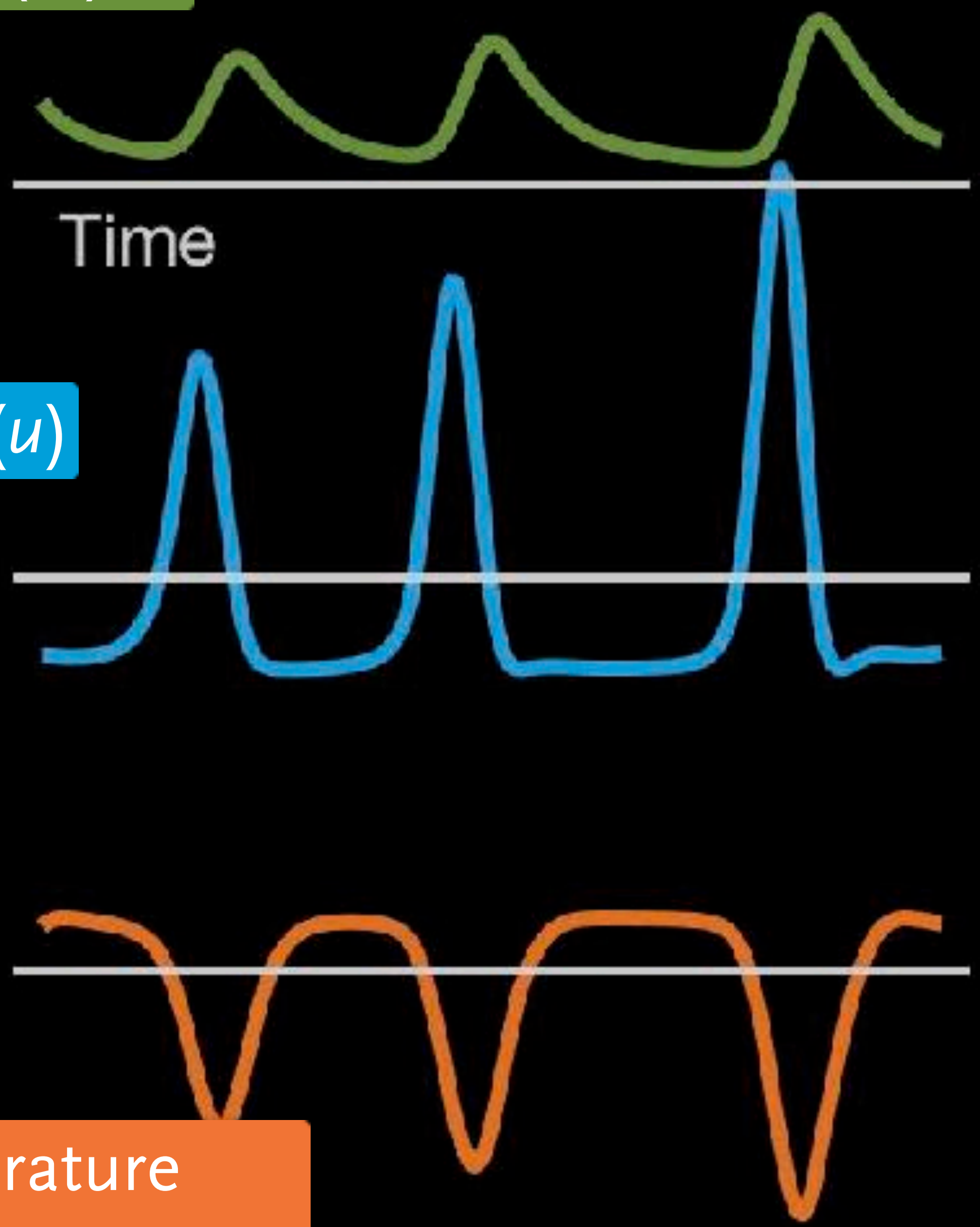

El Niño Conditions
Equator
Thermocline
120° E
80° W

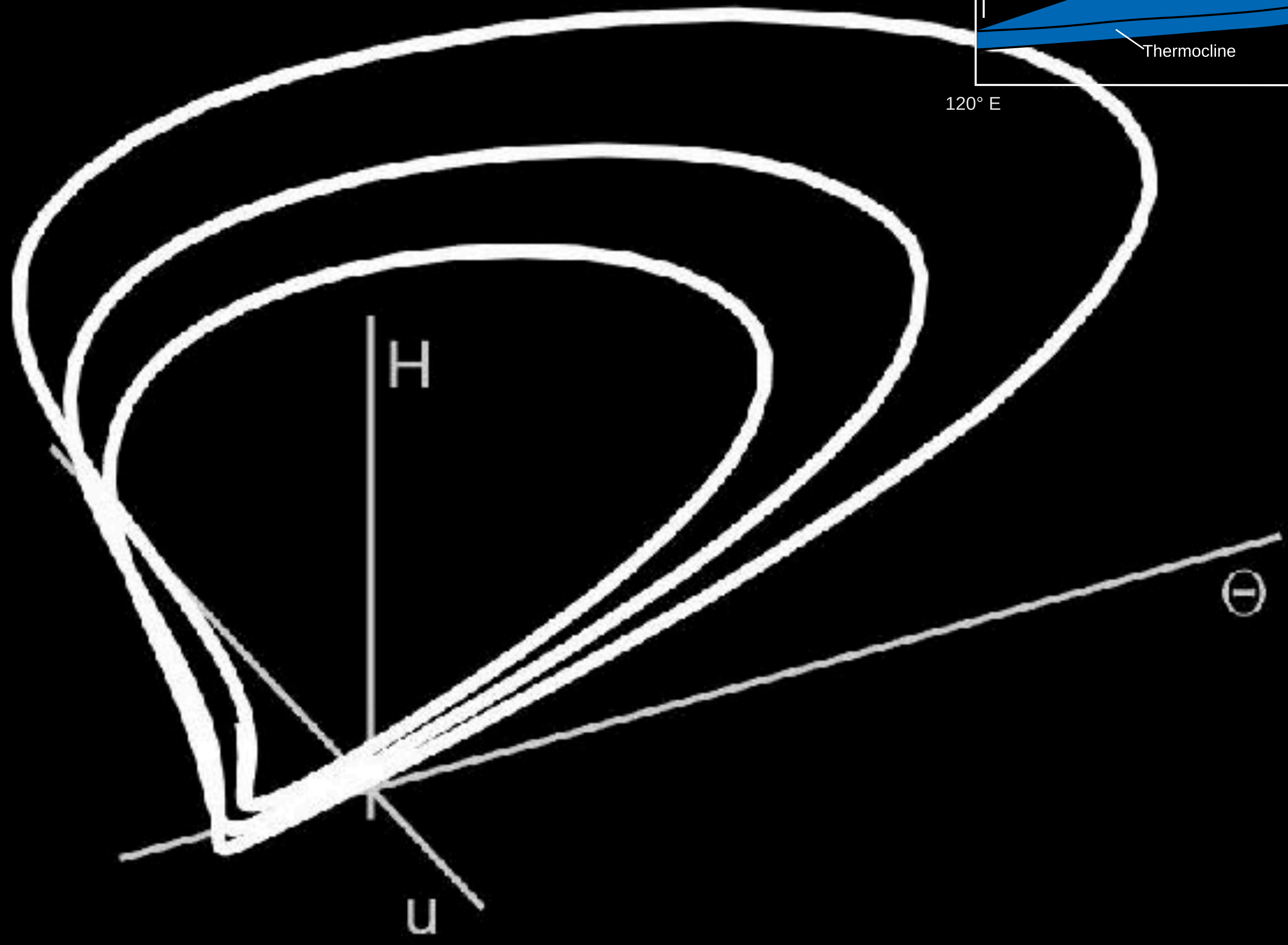
Phase Space

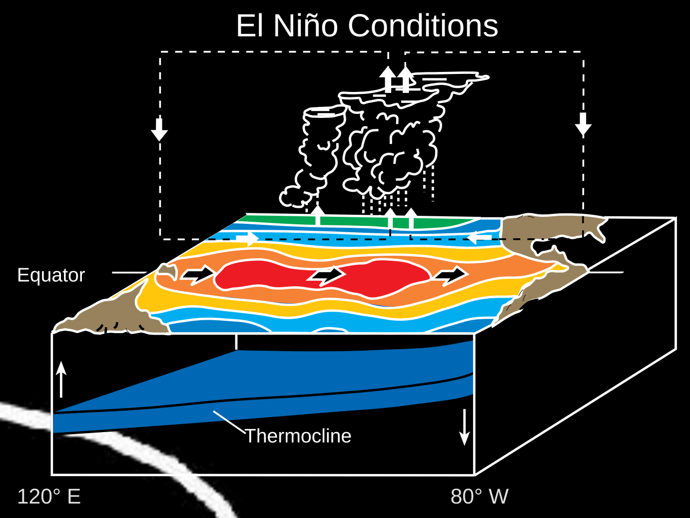
thermocline depth (H)

wind (u)

temperature (Θ)





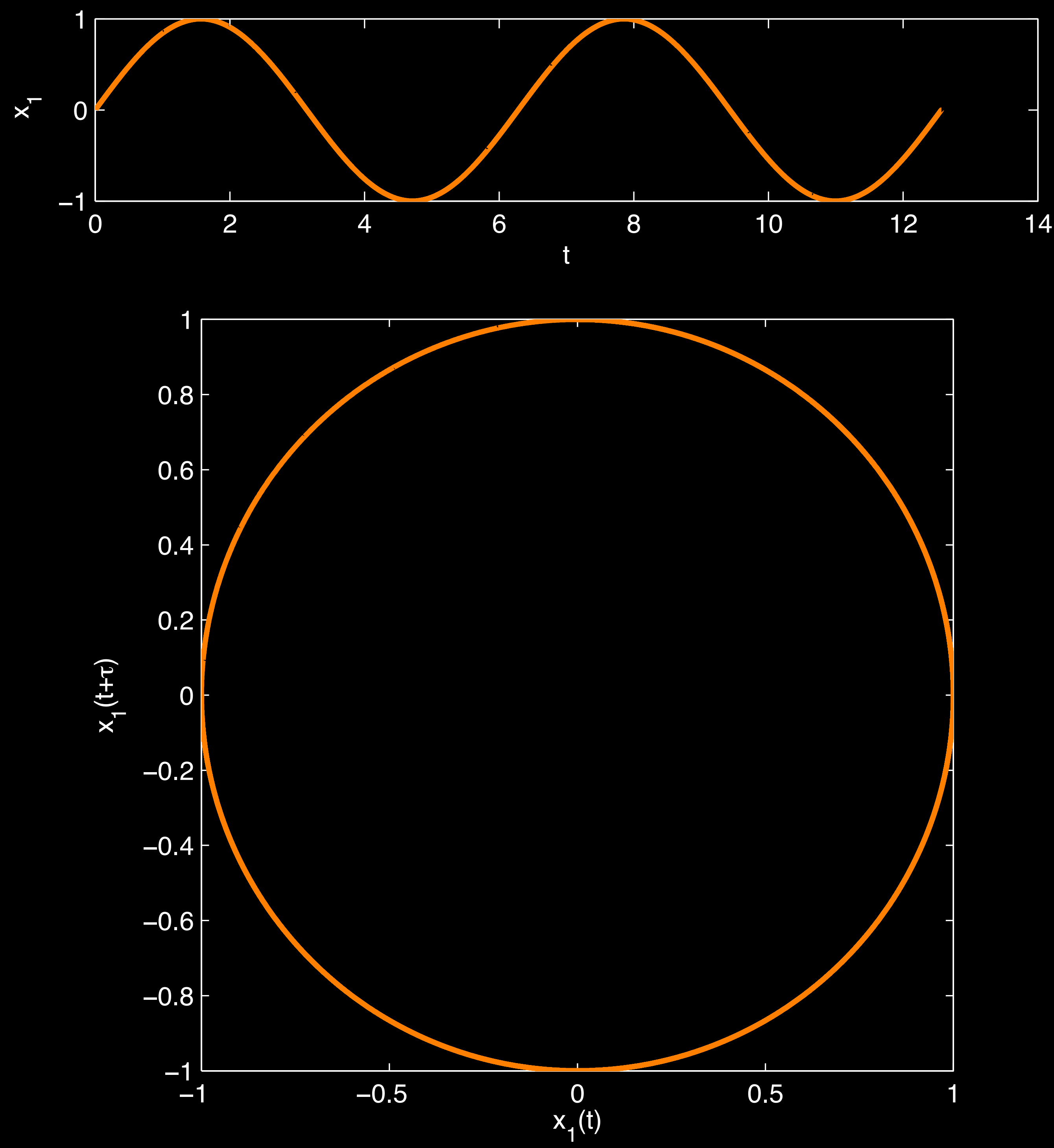


Phase Space Reconstruction

- However:
  often only one variable measured (or available)
- Because:
  not accessible, not known, too expensive to measure it ...

Methods for Phase Space Reconstruction

- Auto-correlation, mutual information (time delay)
- False nearest neighbours (embedding dimension)
- More advanced: Cao, Pecora, Nichkawde, PECUZAL, ...

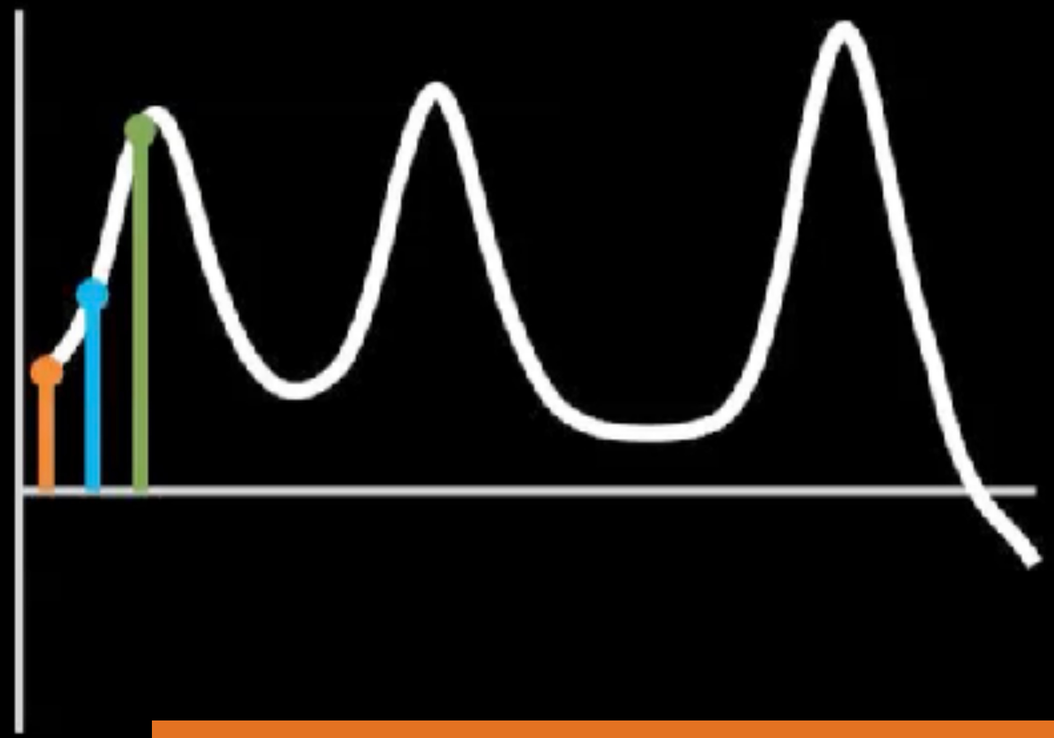


Kraemer et al, New J Phys 23, 2021

Time Delay Embedding

- Packard et al, 1980; Takens, 1981
- Use time shifted copies of the time series

$$\vec{x}(i) = \left(u_i, \underbrace{u_{i+\tau}}_{\text{time delay}}, u_{i+2\tau}, \ldots, \underbrace{u_{i+(m-1)\tau}}_{\text{embedding dimension}}\right)$$

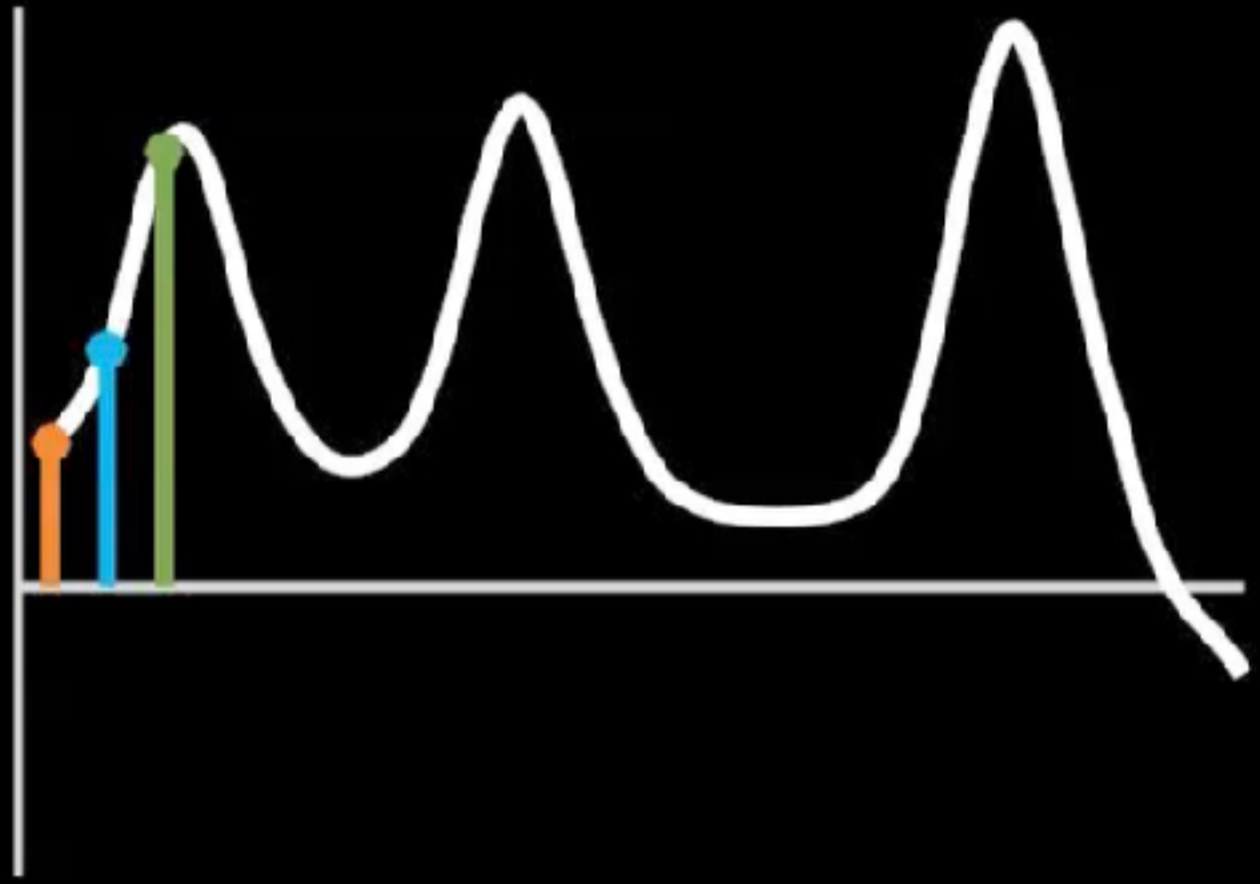


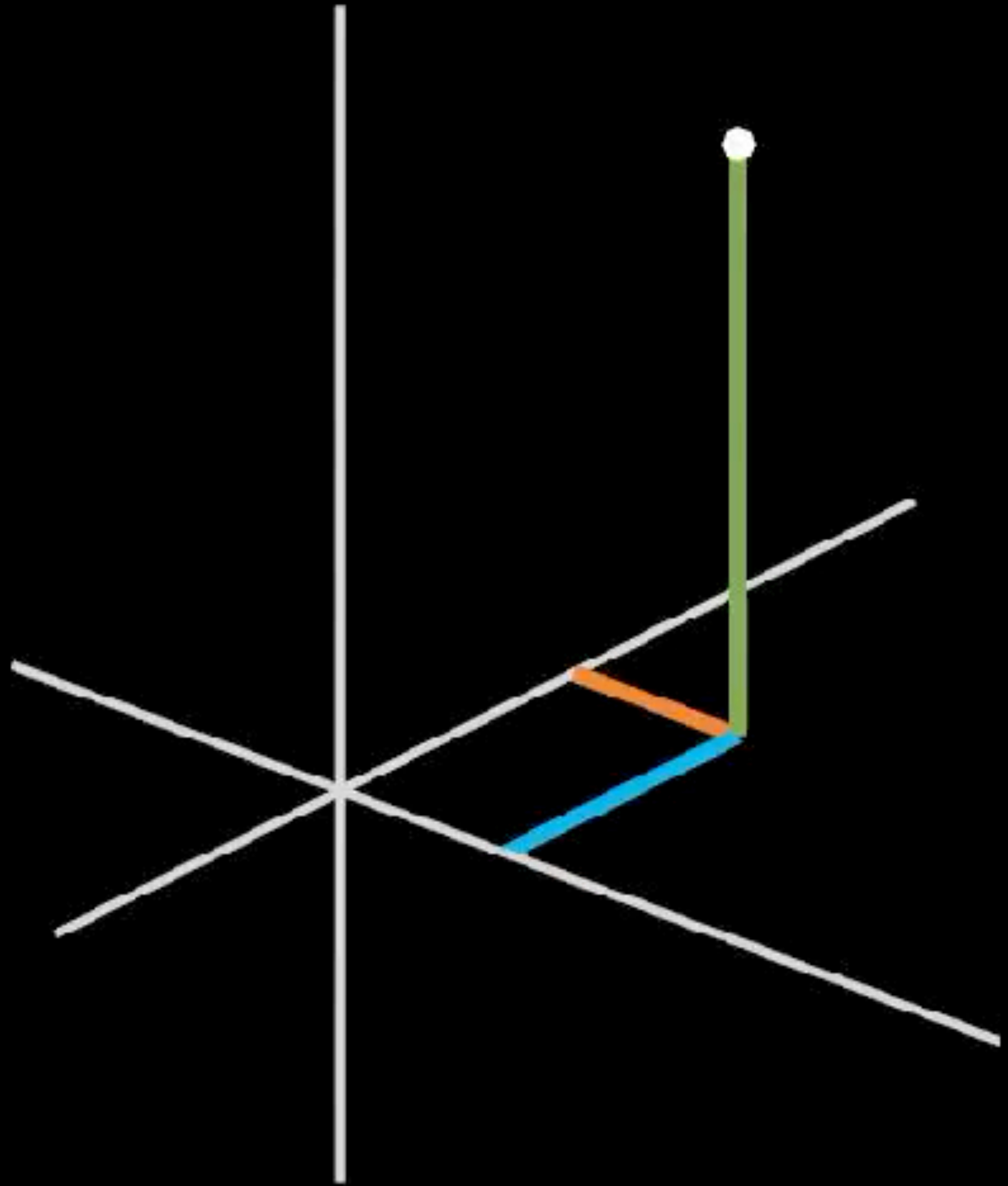
positive/ negative delay: check carefully with your research question

Note: Phase space trajectory is shorter by $(m\text{-}1)\tau$

Time Delay Embedding

$$\vec{x}(i) = \left(u_i, u_{i+\tau}, u_{i+2\tau}, \ldots, u_{i+(m-1)\tau}\right)$$





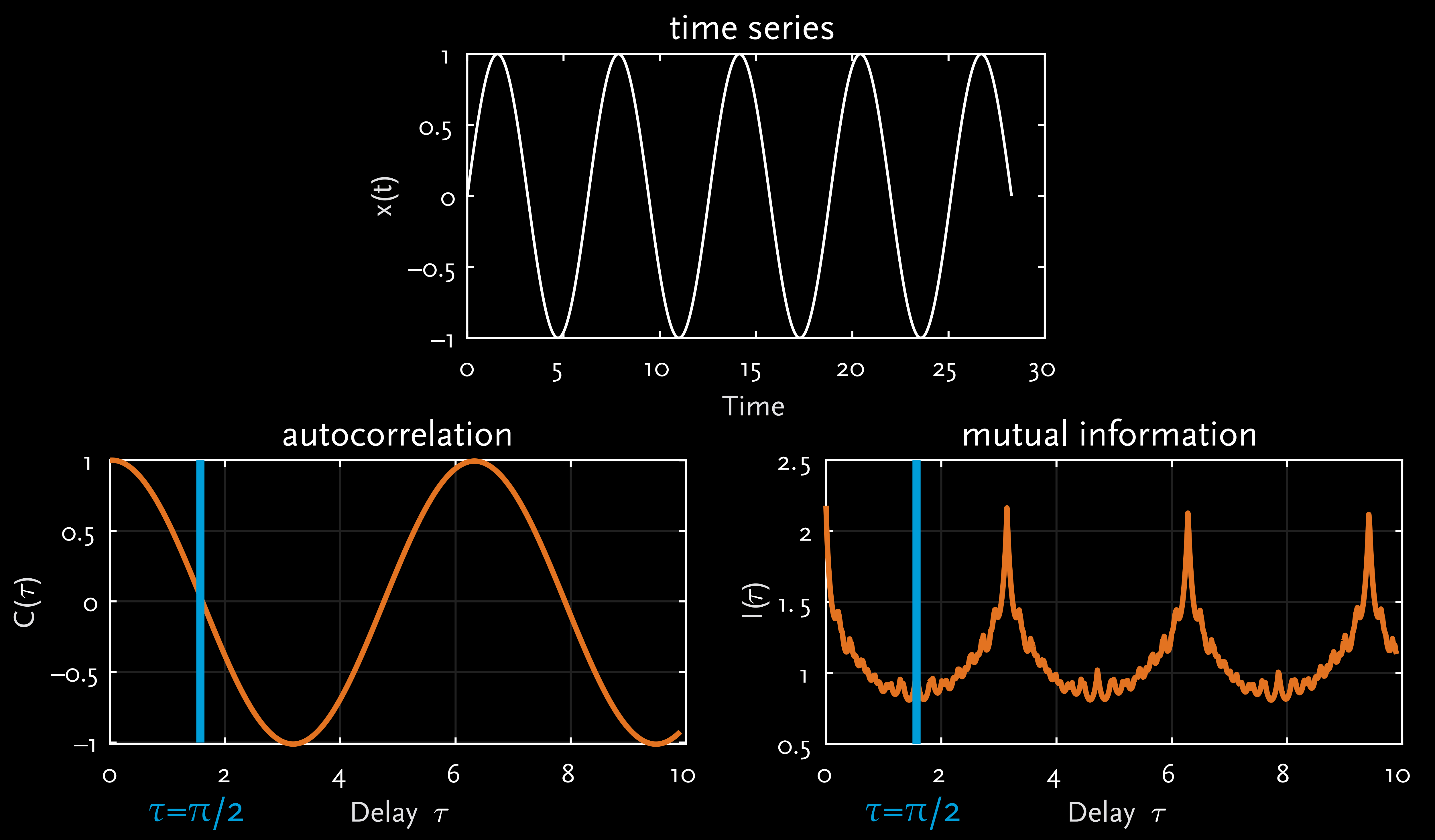
Embedding Delay

$$\vec{x}(i) = (u_i, u_{i+\tau}, u_{i+2\tau}, \ldots, u_{i+(m-1)\tau})$$

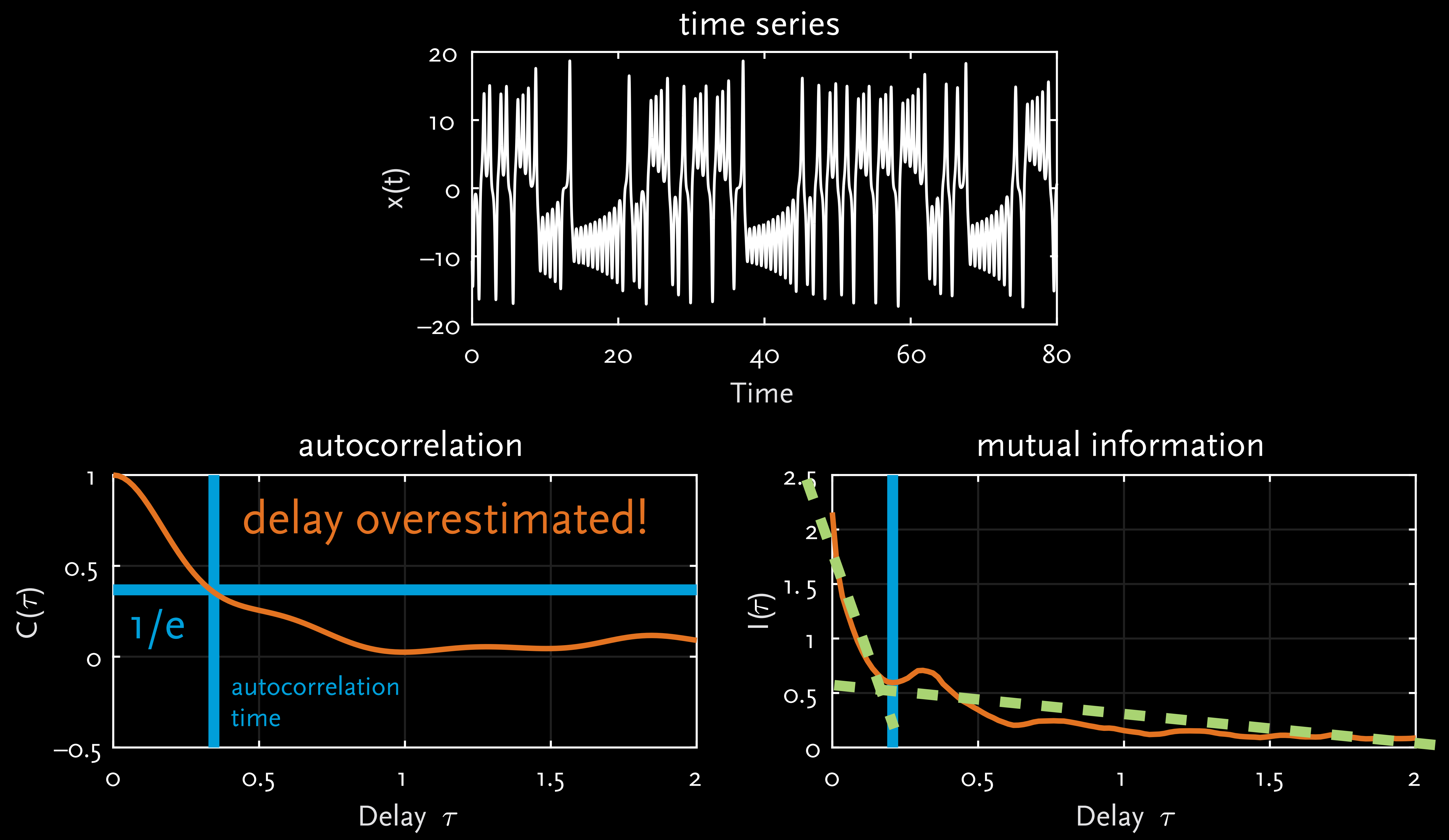
Independent components

	AUTOCORRELATION	MUTUAL INFORMATION
	$C(\tau) = \frac{\sum_{t=1}^{N-\tau}(x_t - \bar{x})(x_{t+\tau} - \bar{x})}{(N-\tau)\sigma^2}$	$I(\tau) = \sum_x \sum_x p(x_t, x_{t+\tau}) \log\left(\frac{p(x_t, x_{t+\tau})}{p(x_t)\, p(x_{t+\tau})}\right)$
clear cycles:	first zero crossing	first minimum
no clear cycles:	decorrelation time	point where steep decrease changes to flat decrease

Embedding Delay



Embedding Delay



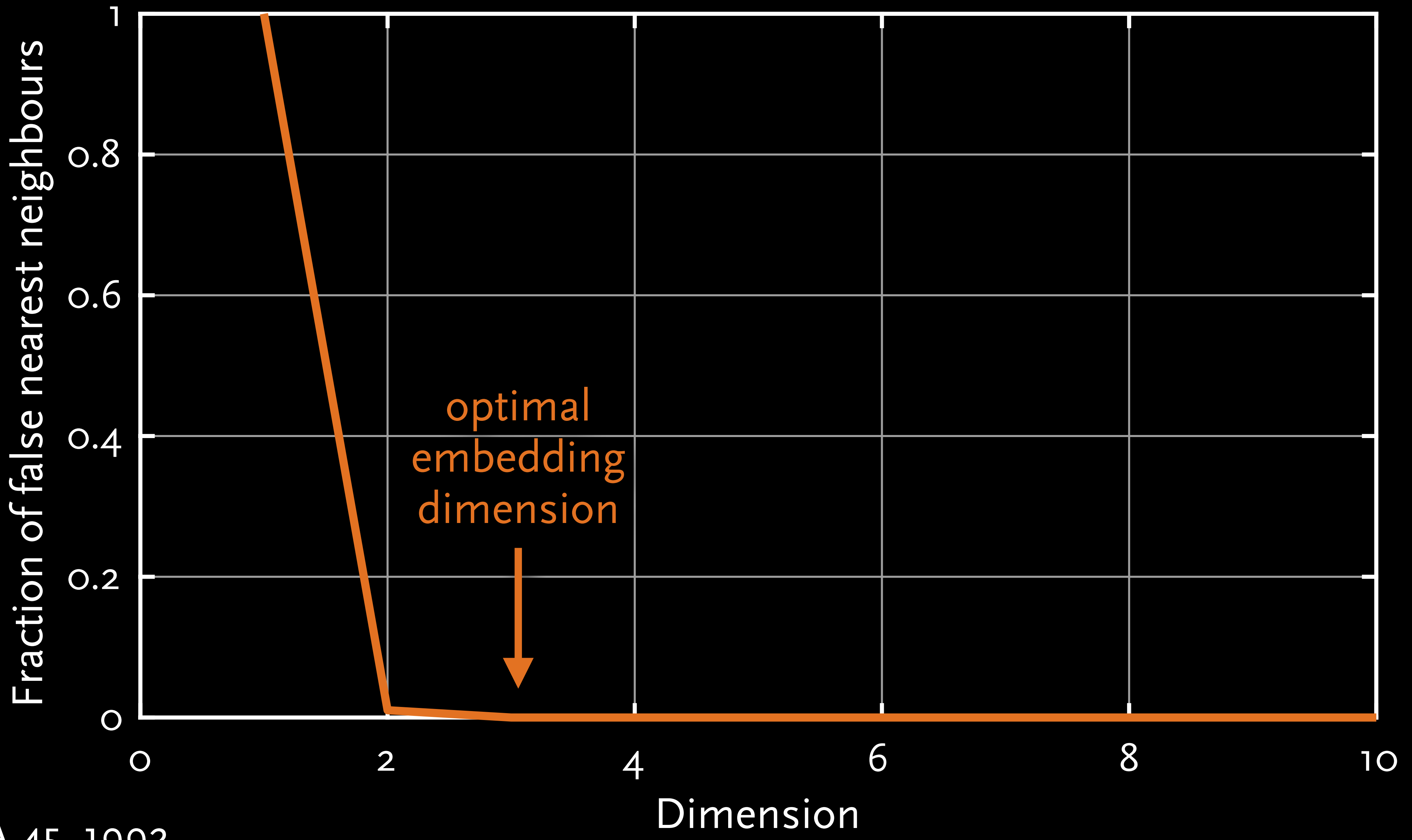
Embedding Dimension

$$\vec{x}(i) = \left(u_i, u_{i+\tau}, u_{i+2\tau}, \ldots, u_{i+(m-1)\tau}\right)$$

- Embedding dimension = number of components of phase space vector
- Aim: unfold the attractor
- Too high = state space becomes more sparse; increases correlations between phase space vectors
- False nearest neighbours (Kennel et al., 1992)

False Nearest Neighbours

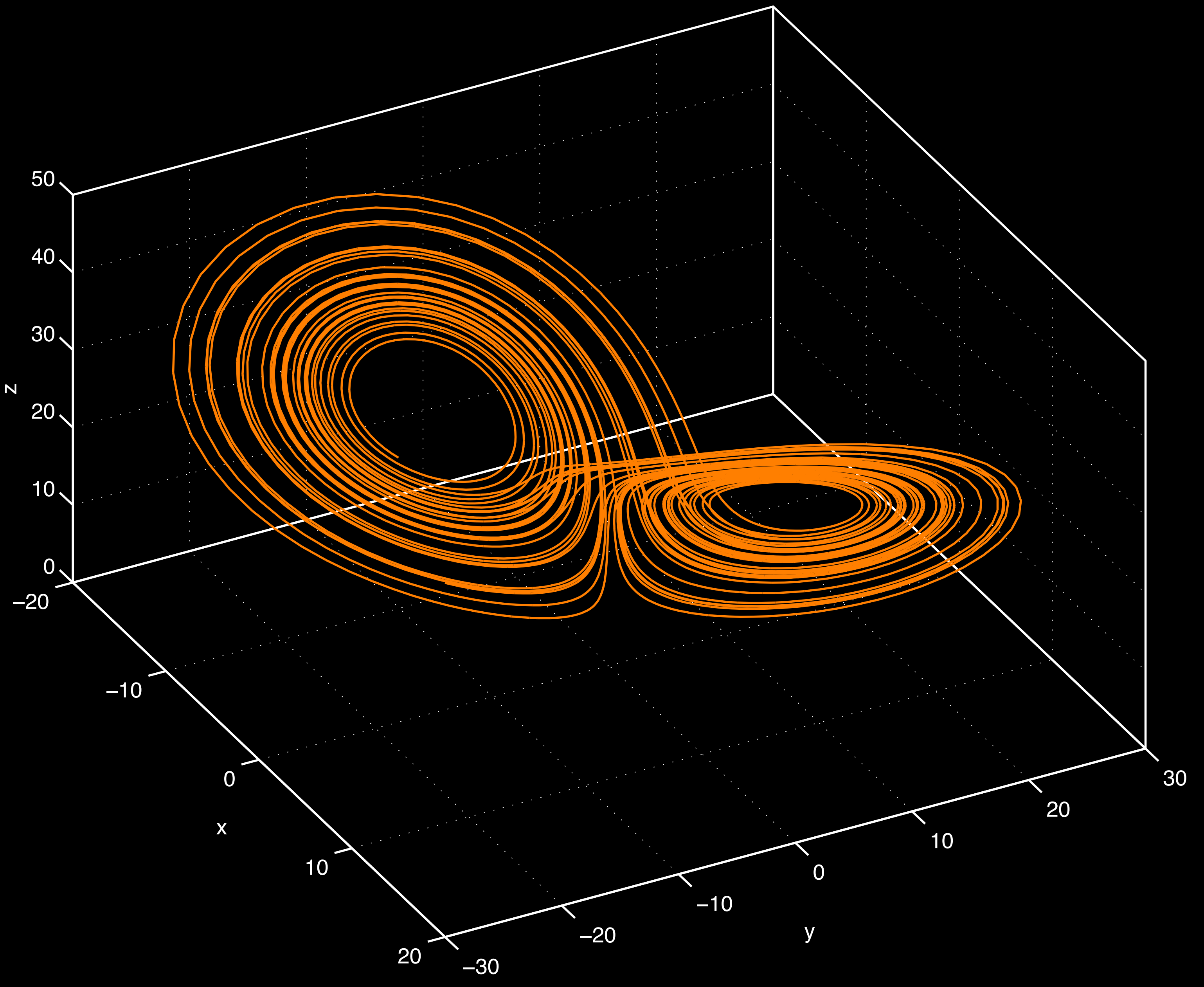
- Embedding dimension: m where fraction of false neighbours vanishes



Kennel et al, Phys Rev A 45, 1992

(Reconstructed) Phase Space

- Representation of the system‘s variables (= dimension)
- Trajectory represents the dynamics of the system
- Many properties of the dynamical system can be derived from the phase space trajectory



Visual Interpretation

Recurrence Plot

- To visualise the phase space trajectory by its recurrences
- Recurrence matrix:
  - binary
  - symmetric

$$R_{i,j} = \begin{cases} 1 : \vec{x}_i \approx \vec{x}_j \\ 0 : \vec{x}_i \not\approx \vec{x}_j \end{cases}$$

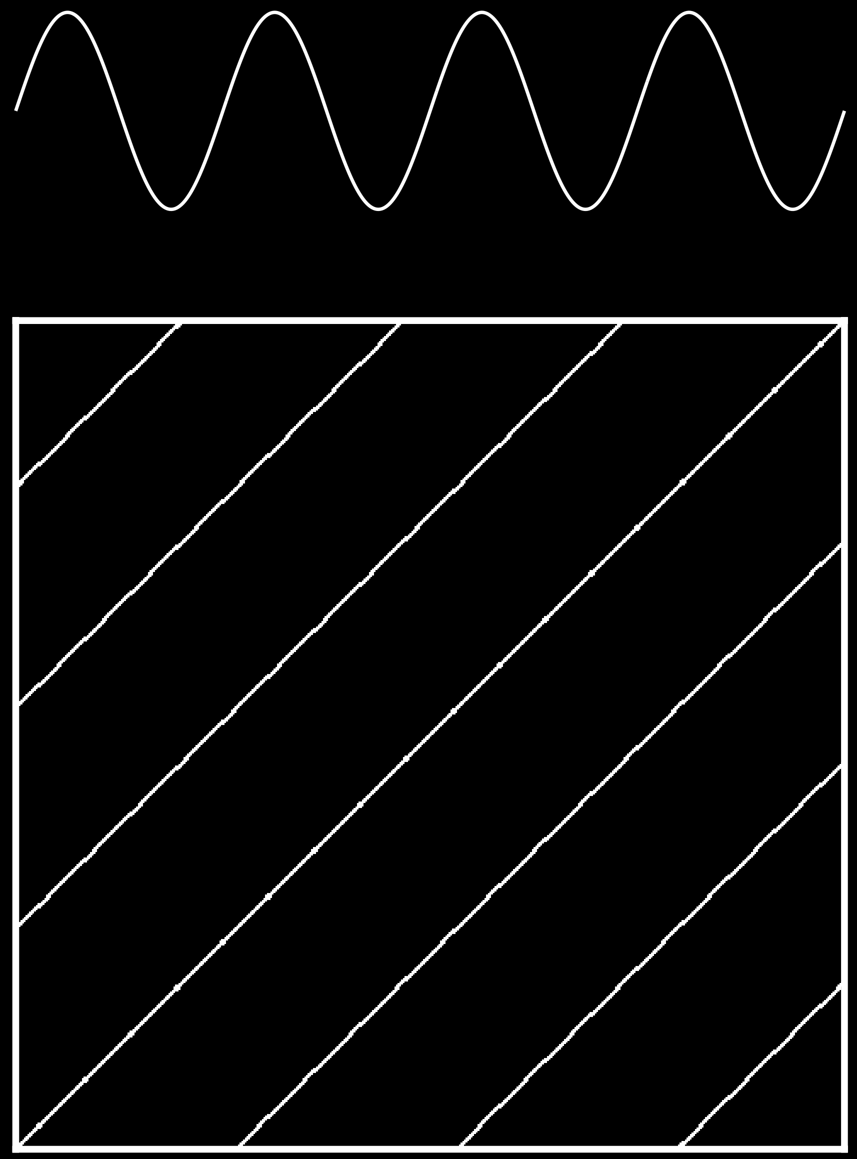
$$R_{i,j} = \Theta\left(\varepsilon - \|\vec{x}_i - \vec{x}_j\|\right)$$

$$\Theta(x \geq 0) = 1;\ \Theta(x < 0) = 0$$

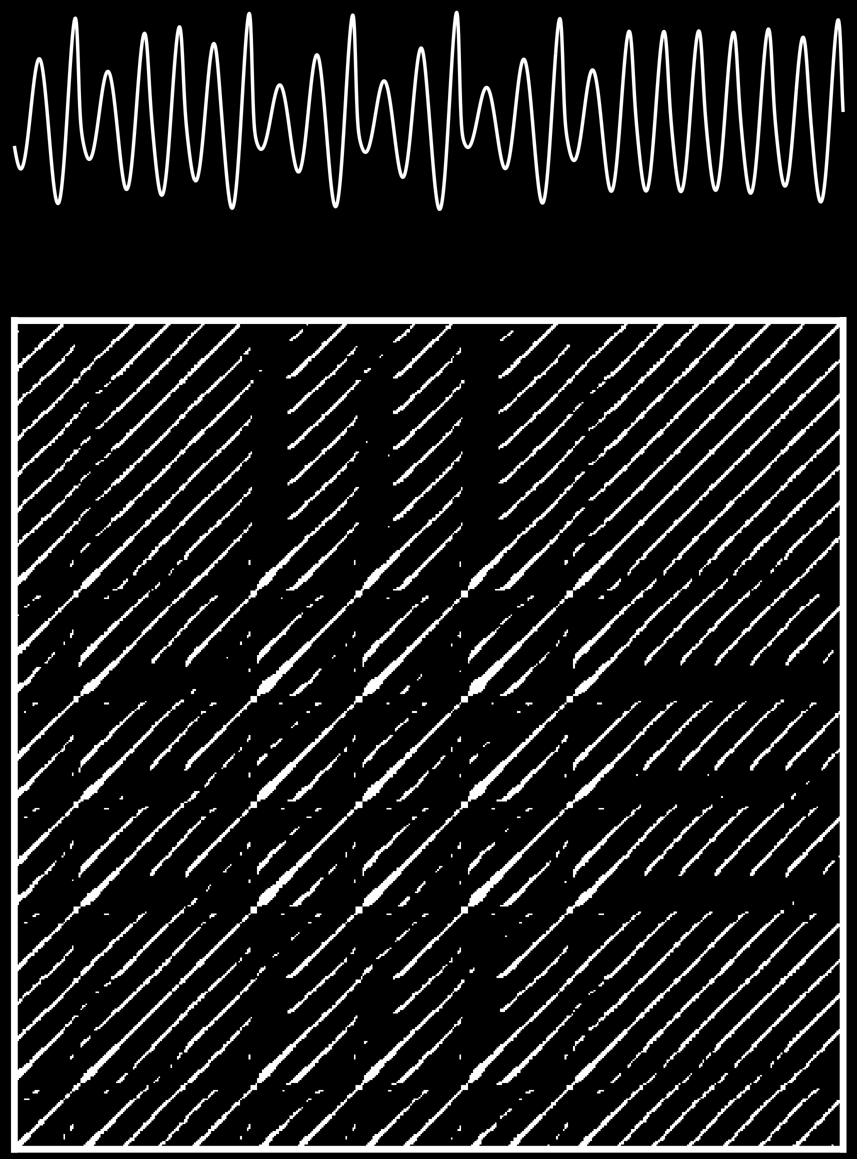
$$\mathbf{R} = \begin{array}{|c|c|c|c|c} \hline 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \end{array}$$

Different Dynamics: Different Patterns

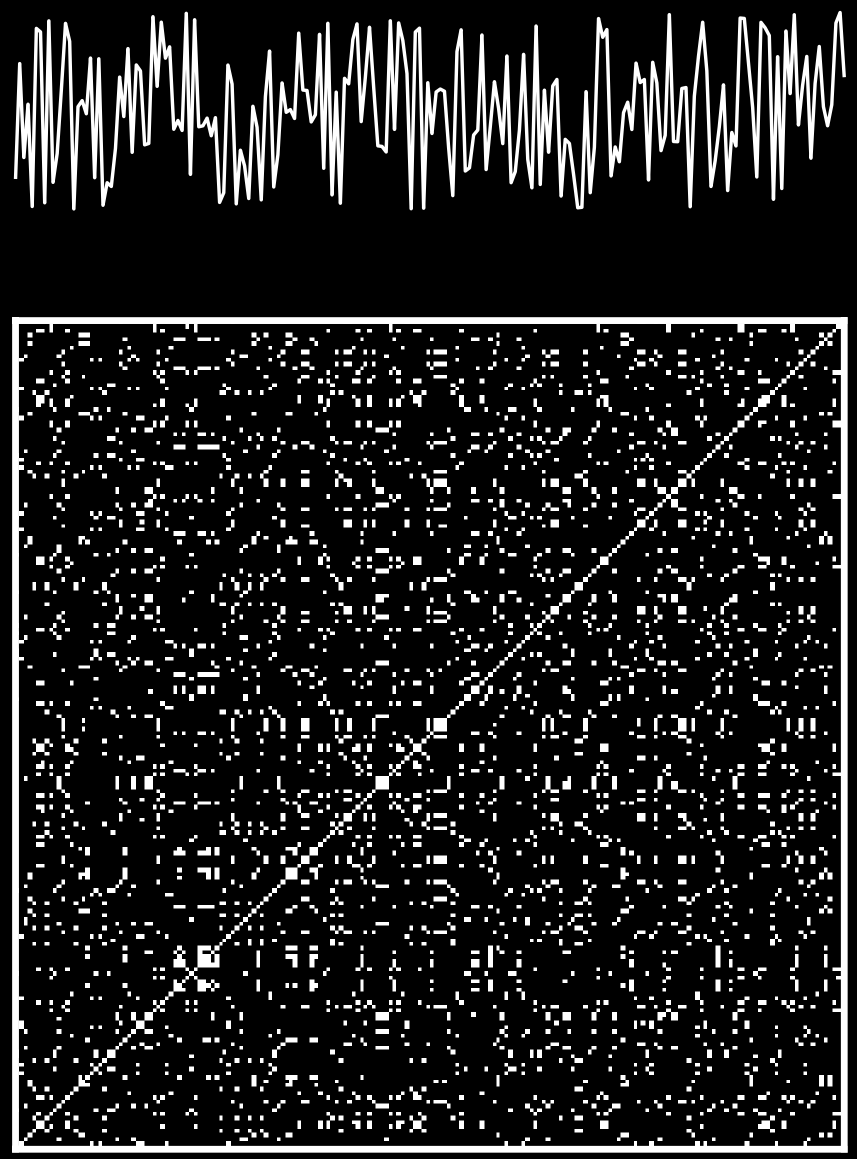
Periodic



Chaotic



Random



Visual Interpretation

Typology (large scale appearance)	homogeneous	periodic	trend	disrupted, separated into blocks
Texture (small scale structures)	only single points	mainly diagonal lines	vertical lines, black blocks	curved pattern
Characteristic pattern	noninterrupted diagonal lines, periodic separation of diagonal lines	sparsely occupied columns/ bars	missing recurrences	local changes in recurrence point density

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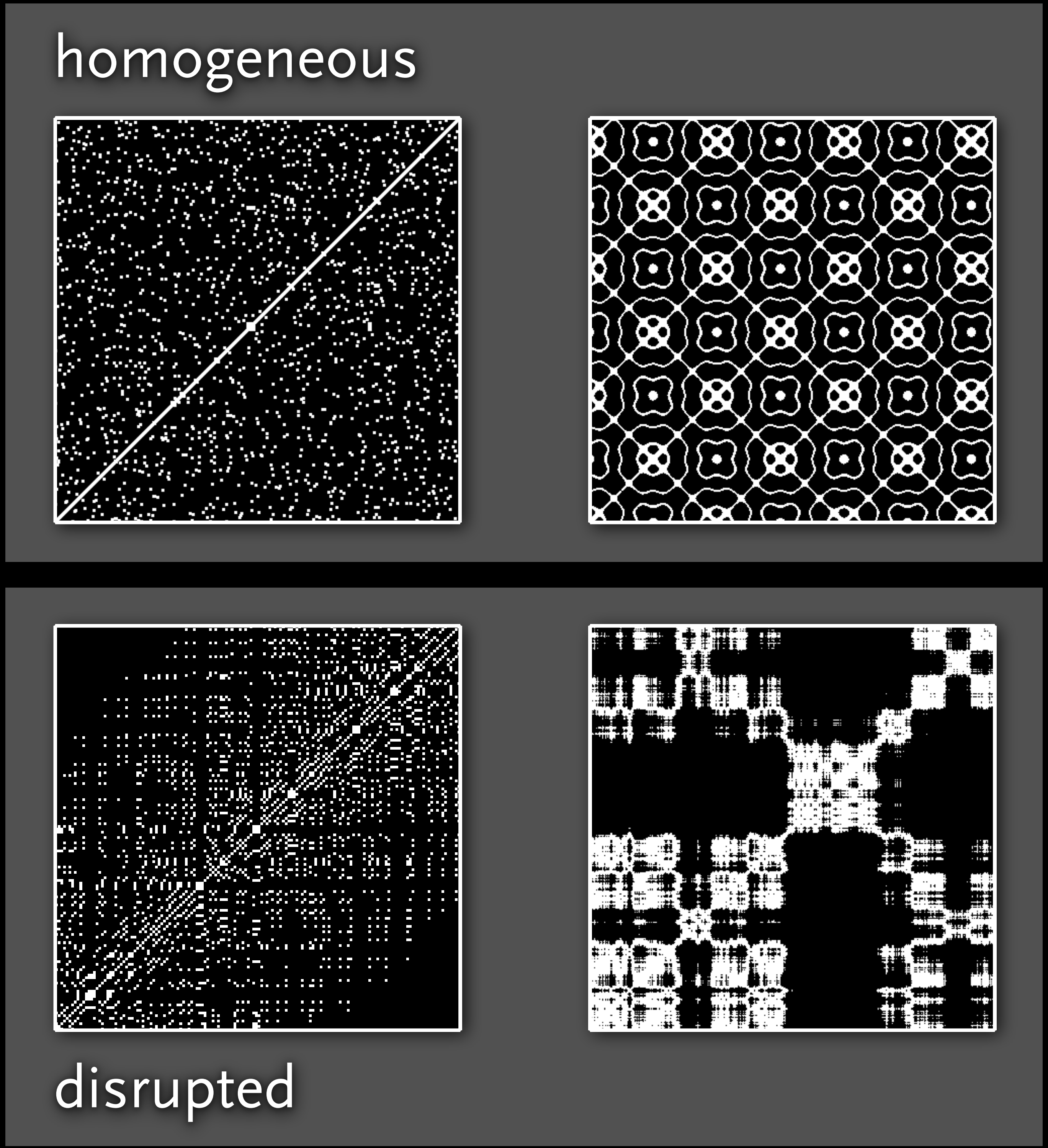
Homogeneous Typology

- Homogeneous:
  time series is stationary
- Heterogeneous:
  time series is not stationary
  (yet, process could be stationary!)

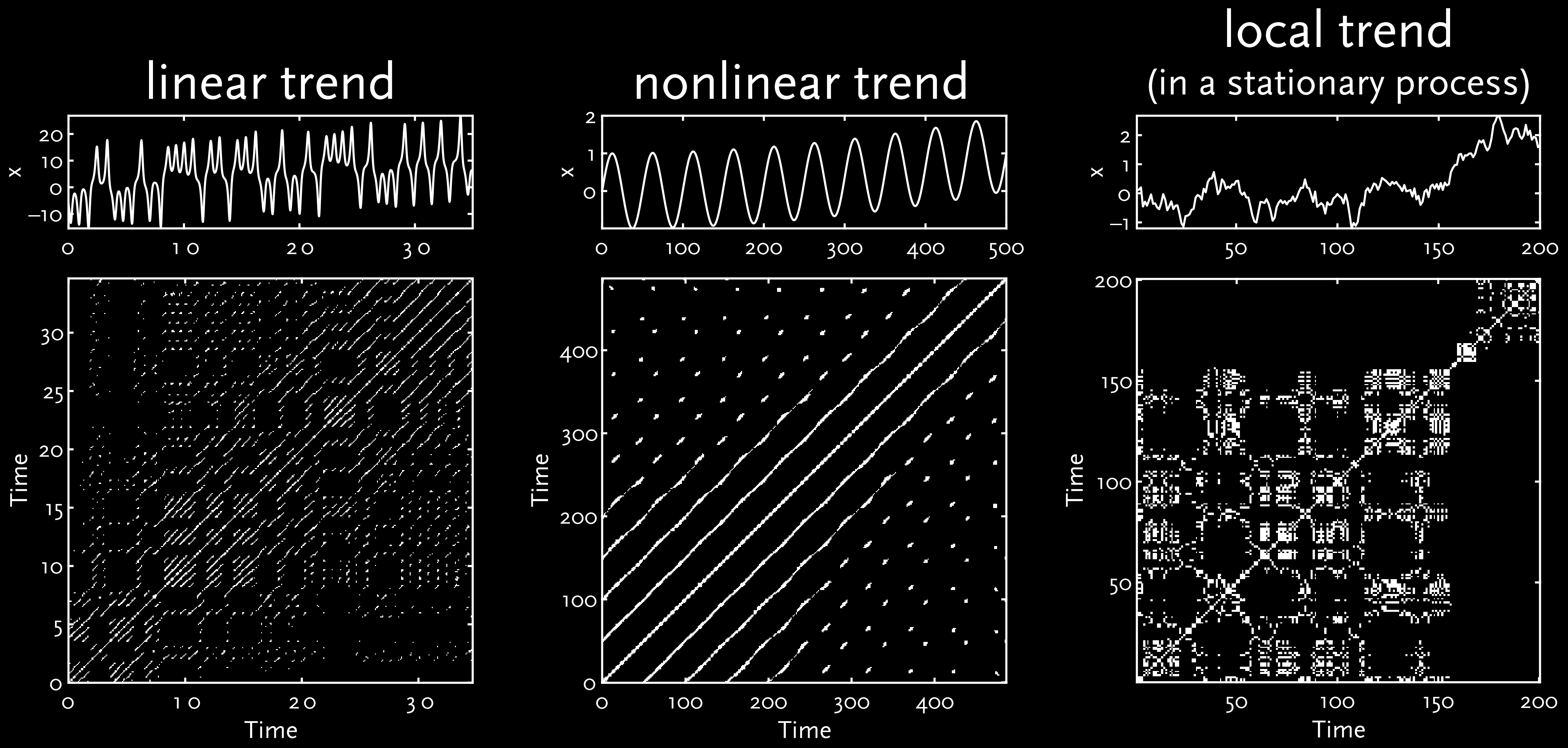
Stationarity:
distribution of x does not change with
temporal shifts $p(x_t) = p(x_{t+s})$

Weak stationarity:
mean and autocorrelation do not
change with temporal shift, finite variance



Fading to the Upper Left and Lower Right Corners

- Non-stationary data
- Process contains (at least a local) trend or shift



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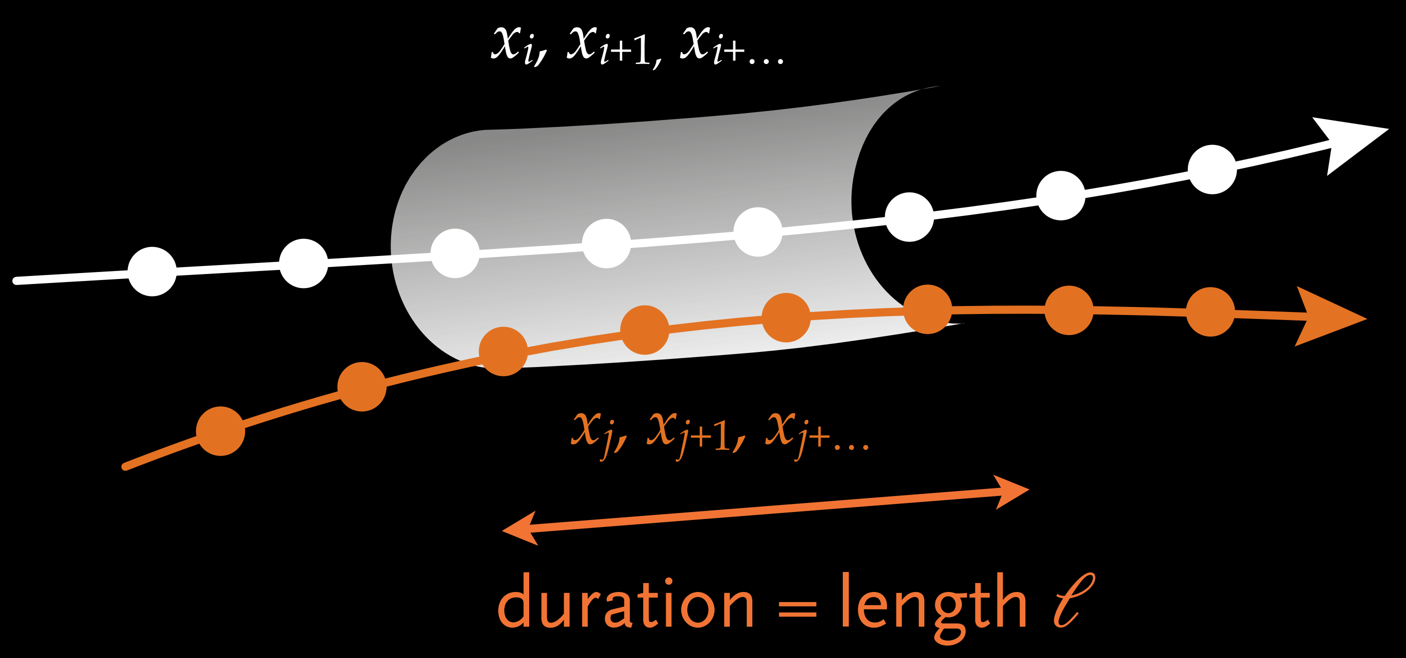
Diagonal Lines Parallel to Main Diagonal

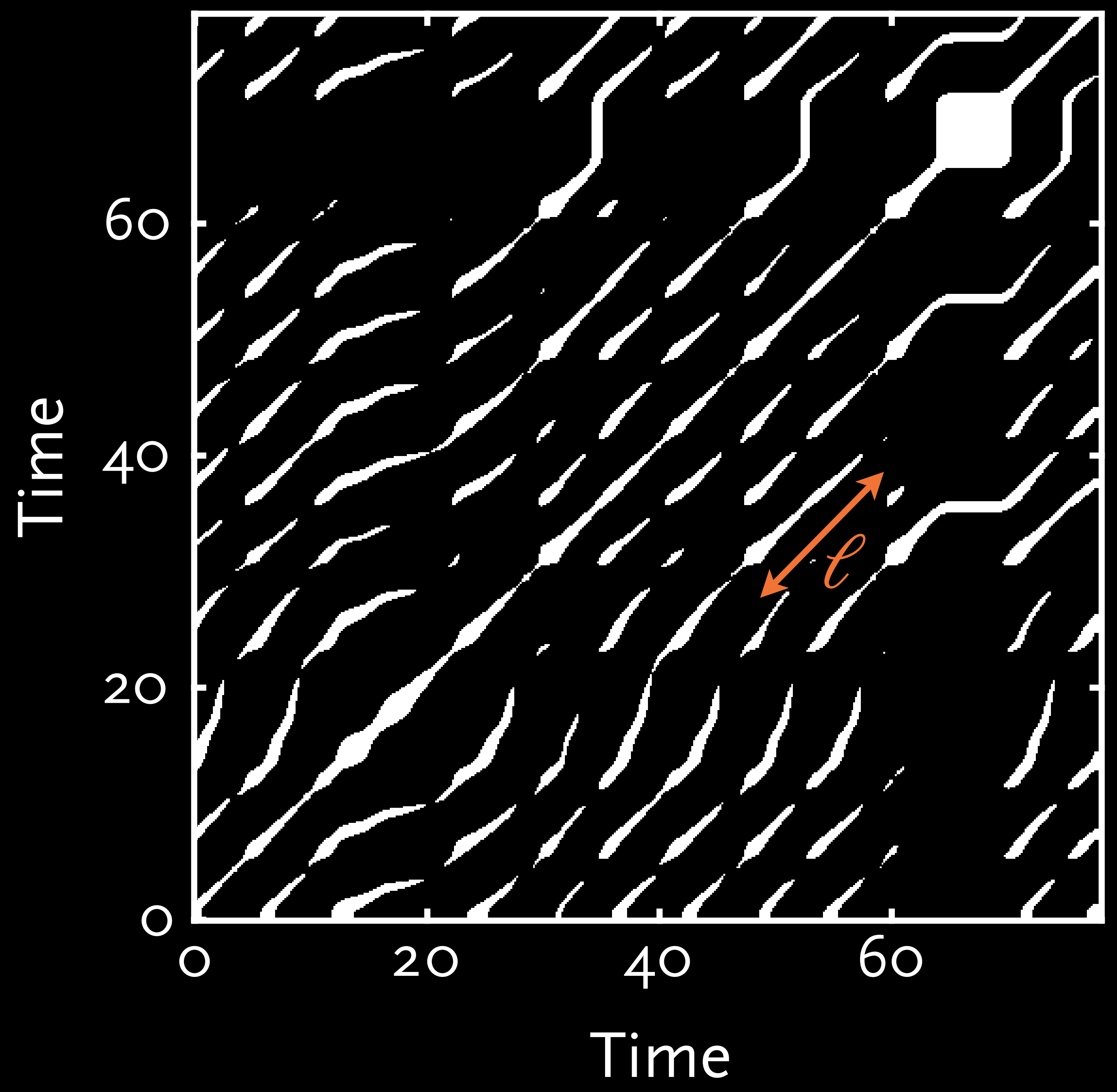


$$\vec{x}_i \approx \vec{x}_j$$

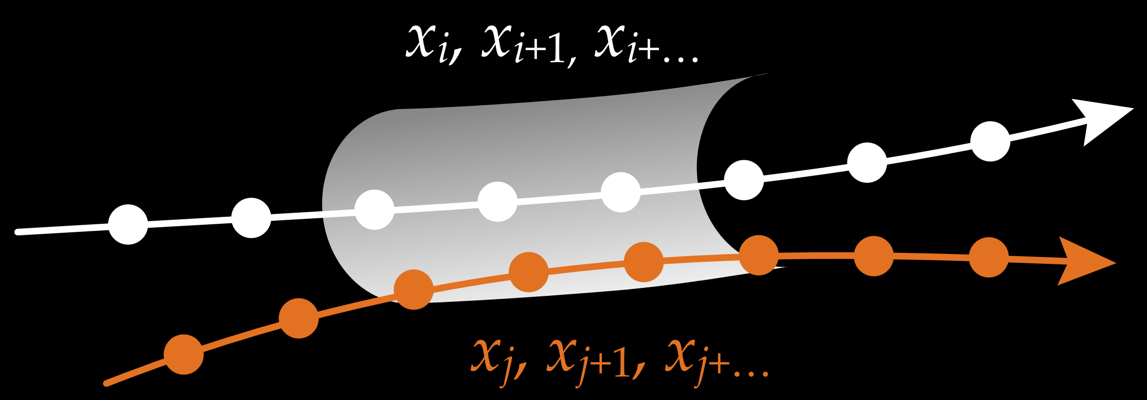
$$\vec{x}_{i+1} \approx \vec{x}_{j+1}$$

$$\vdots$$

$$\vec{x}_{i+\ell} \approx \vec{x}_{j+\ell}$$



Diagonal Lines Parallel to Main Diagonal

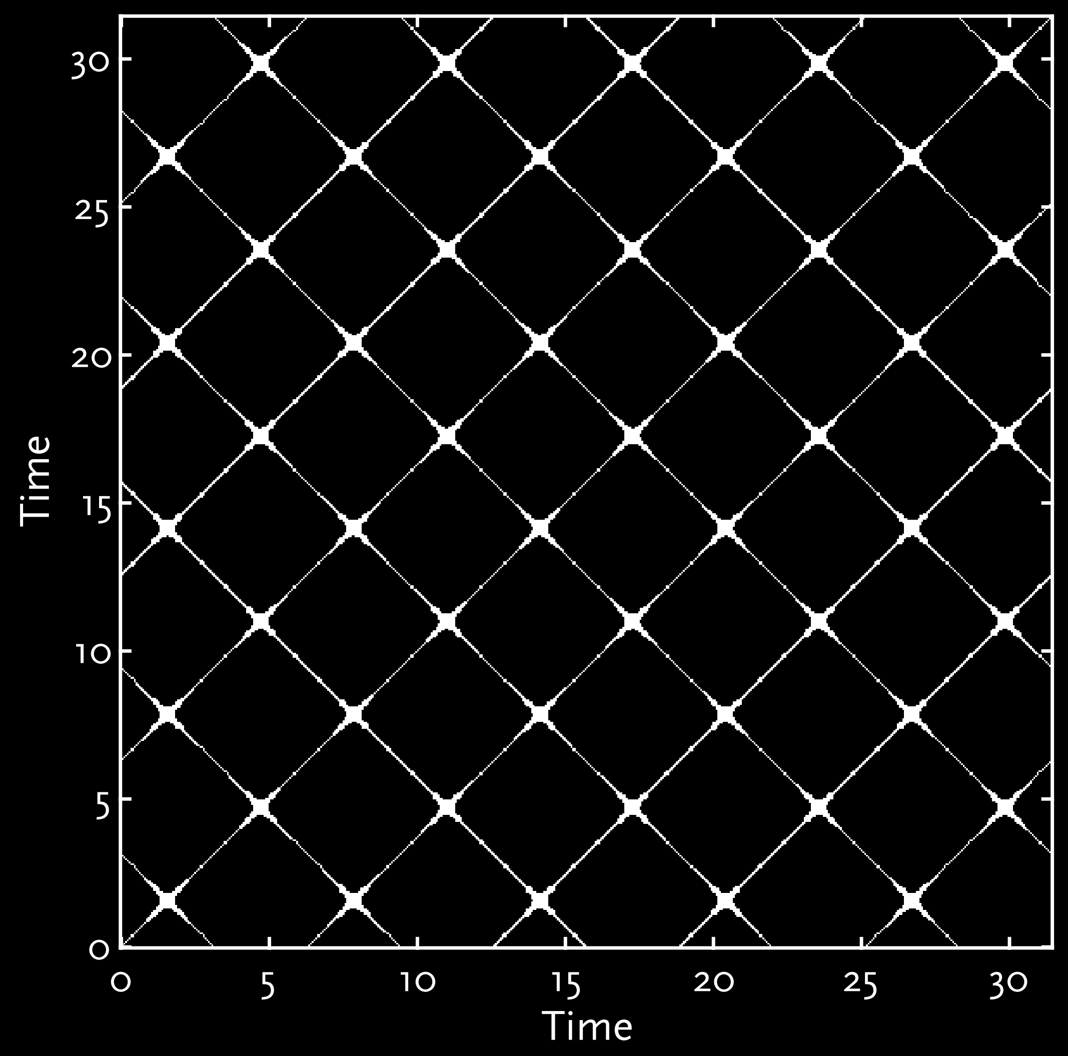


- Evolution of states is similar at different epochs
- Process could be deterministic
- Diagonal lines beside single isolated points: process could be chaotic
- Separation of diagonal lines periodic: unstable periodic orbits

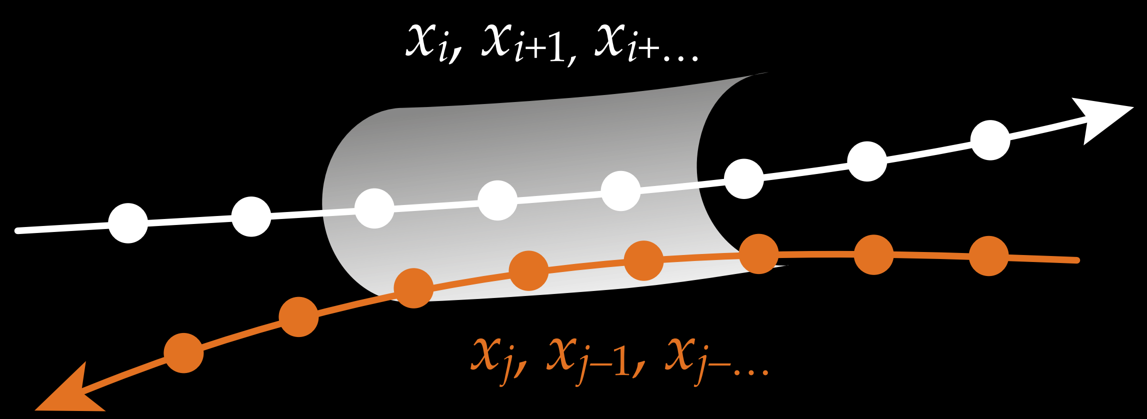
QuizZ

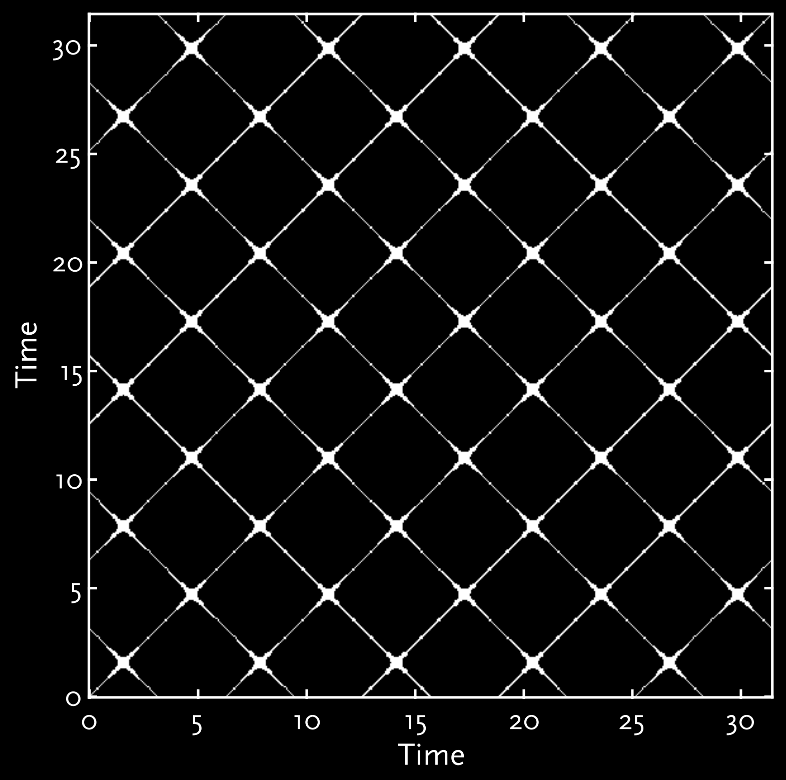
QuizZ

- What does the perpendicular diagonals mean?



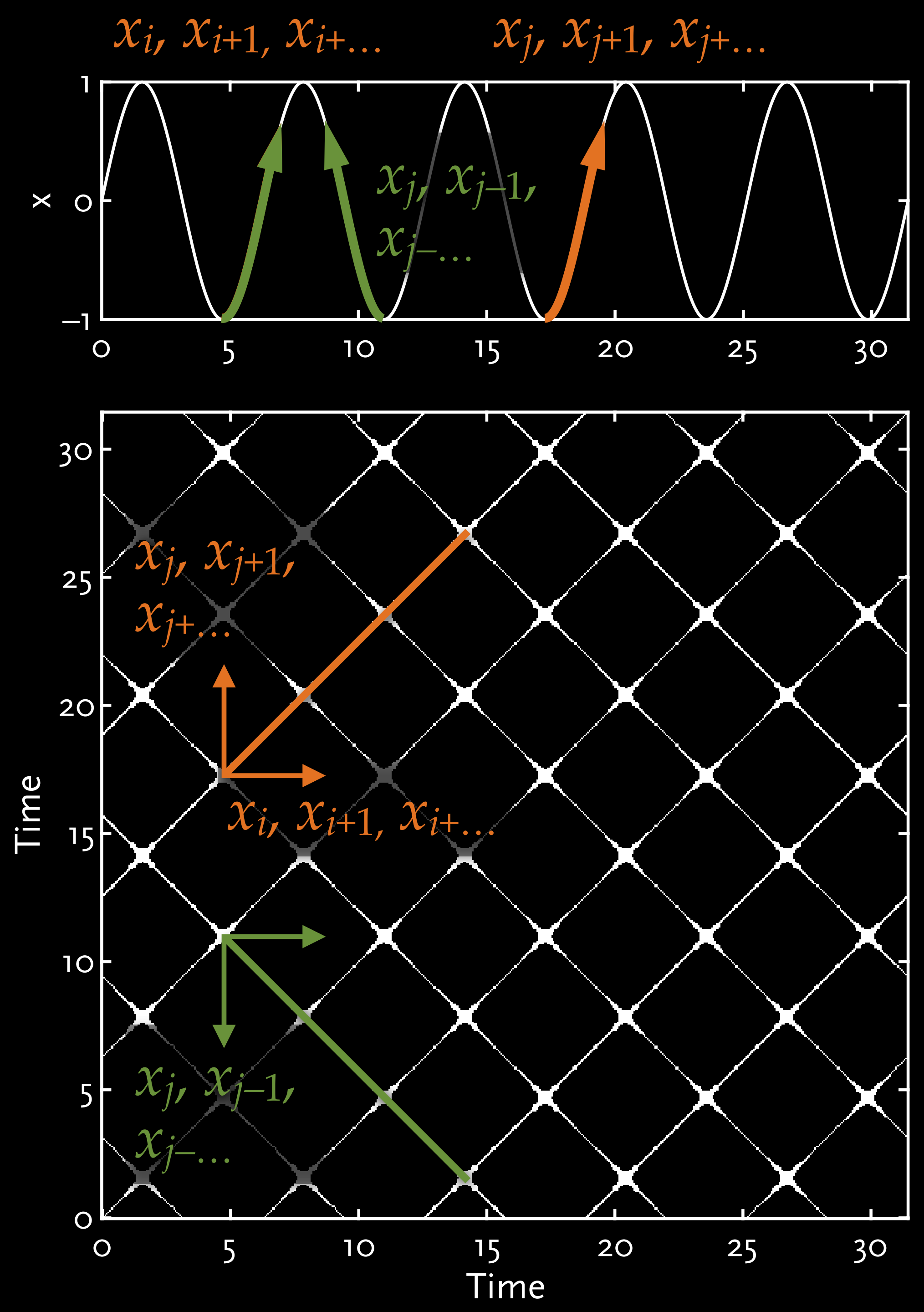
Diagonal Lines Orthogonal to Main Diagonal





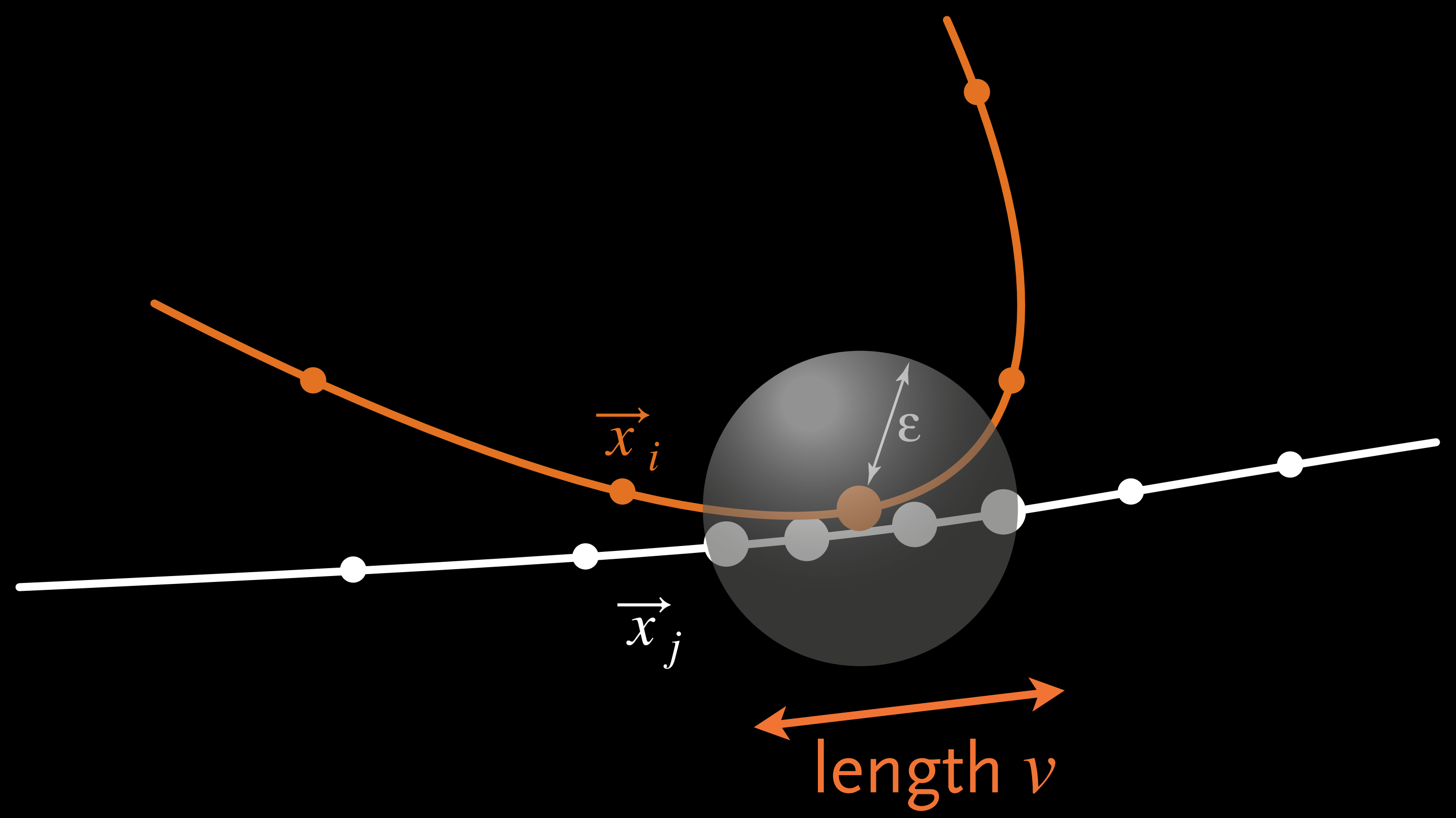
- Evolution of states is similar at different times but with reverse temporal order
- Indication for an insufficient embedding

Diagonal Lines Orthogonal to Main Diagonal



- Evolution of states is similar at different times but with reverse temporal order
- Indication for an insufficient embedding

Vertical Lines, Blocks

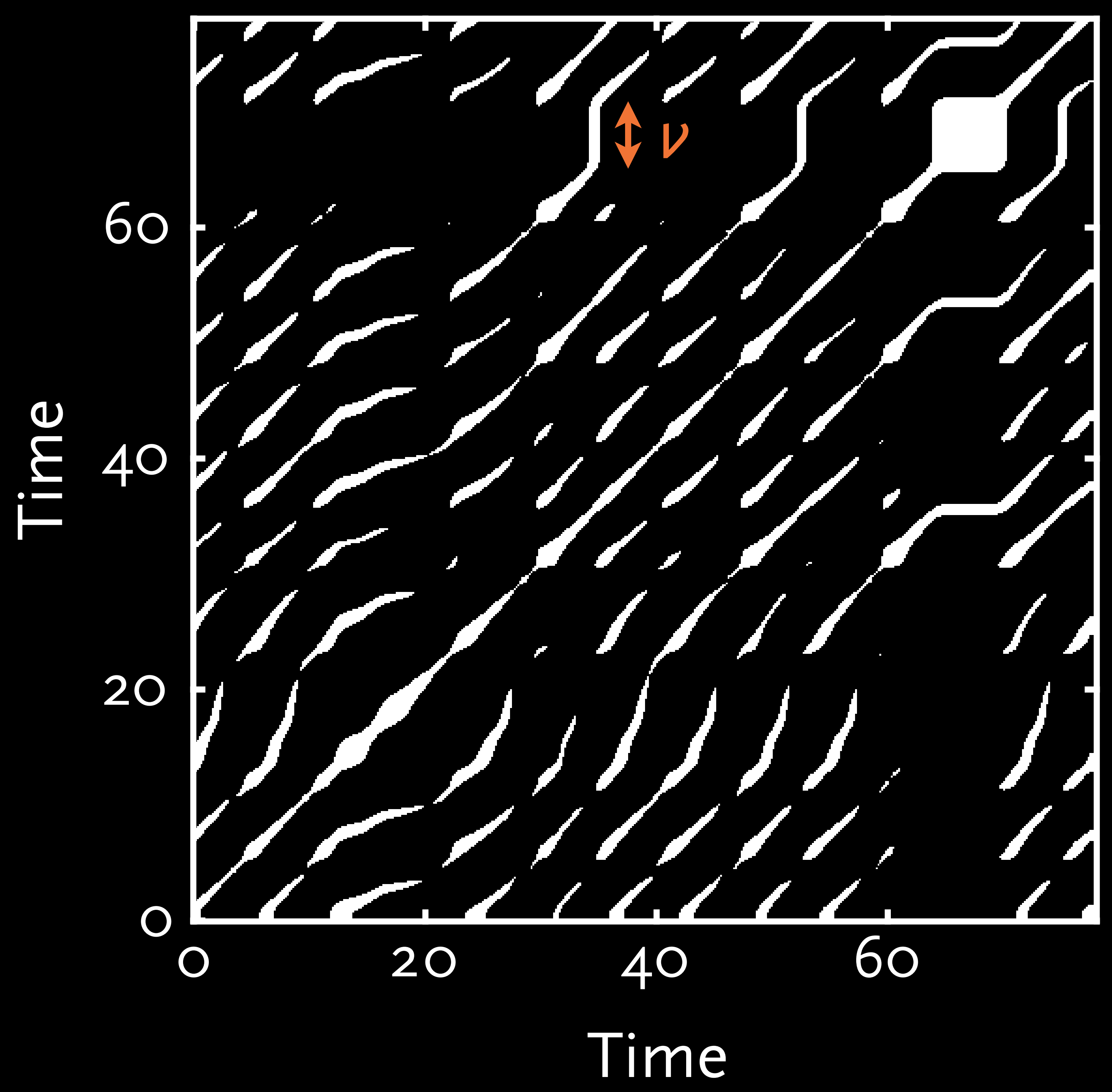


$$\vec{x}_i \approx \vec{x}_j$$

$$\vec{x}_i \approx \vec{x}_{j+1}$$

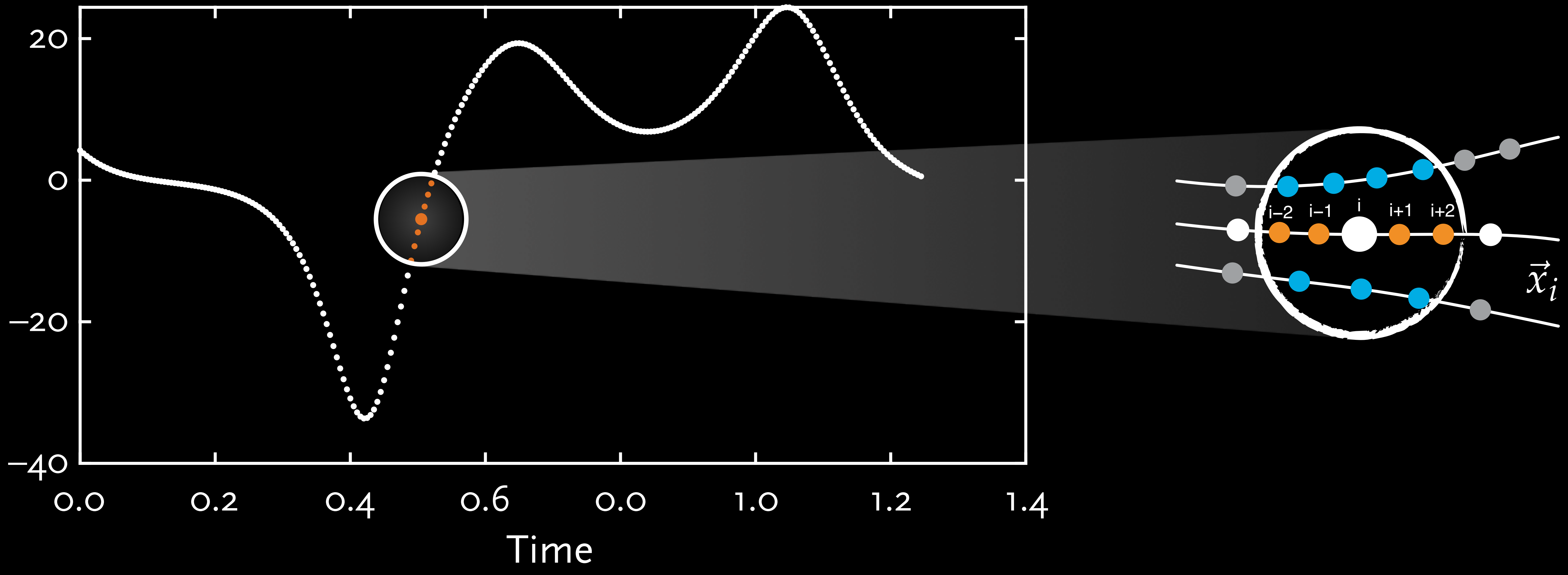
$$\vdots$$

$$\vec{x}_i \approx \vec{x}_{j+v}$$



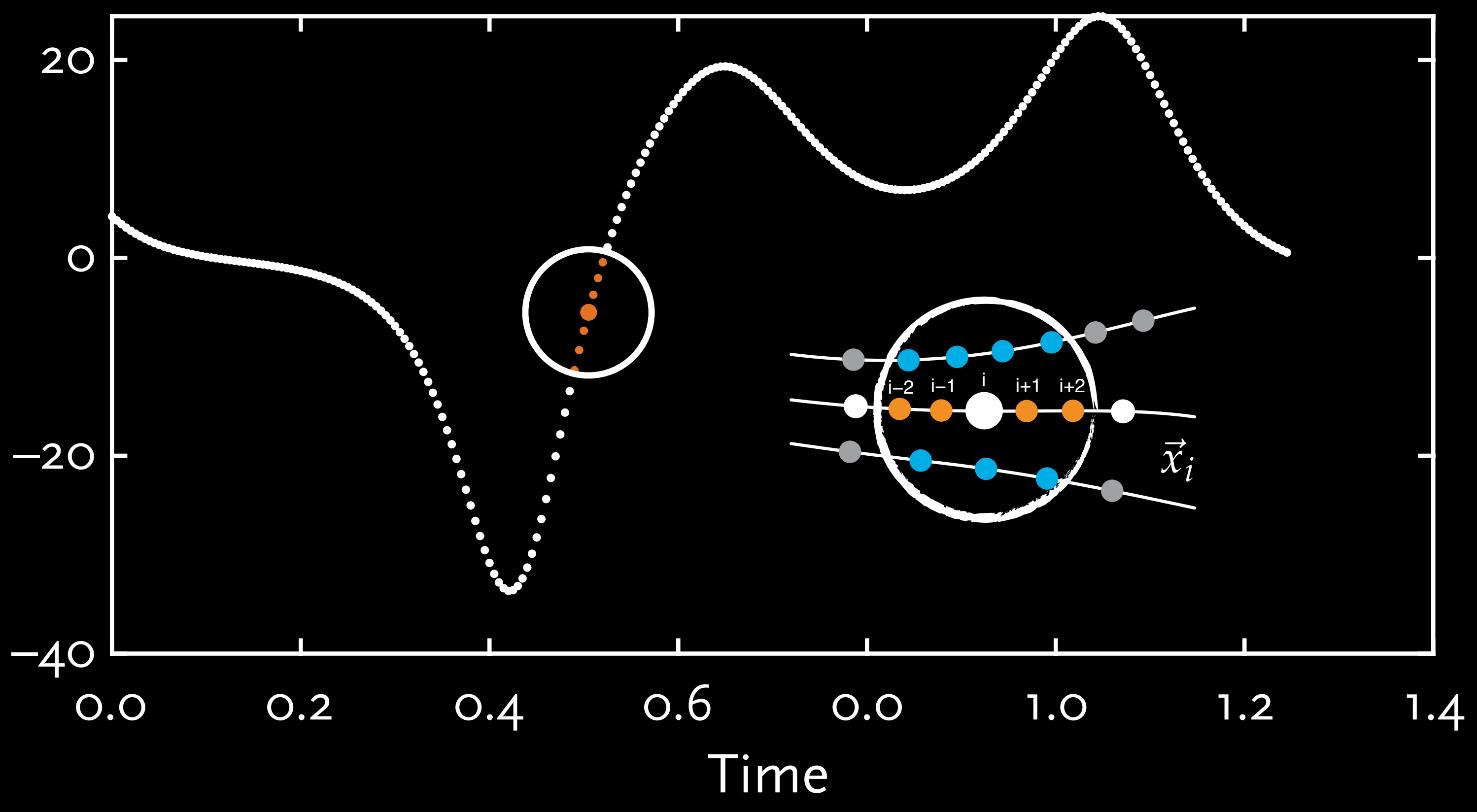
Vertical Lines, Blocks

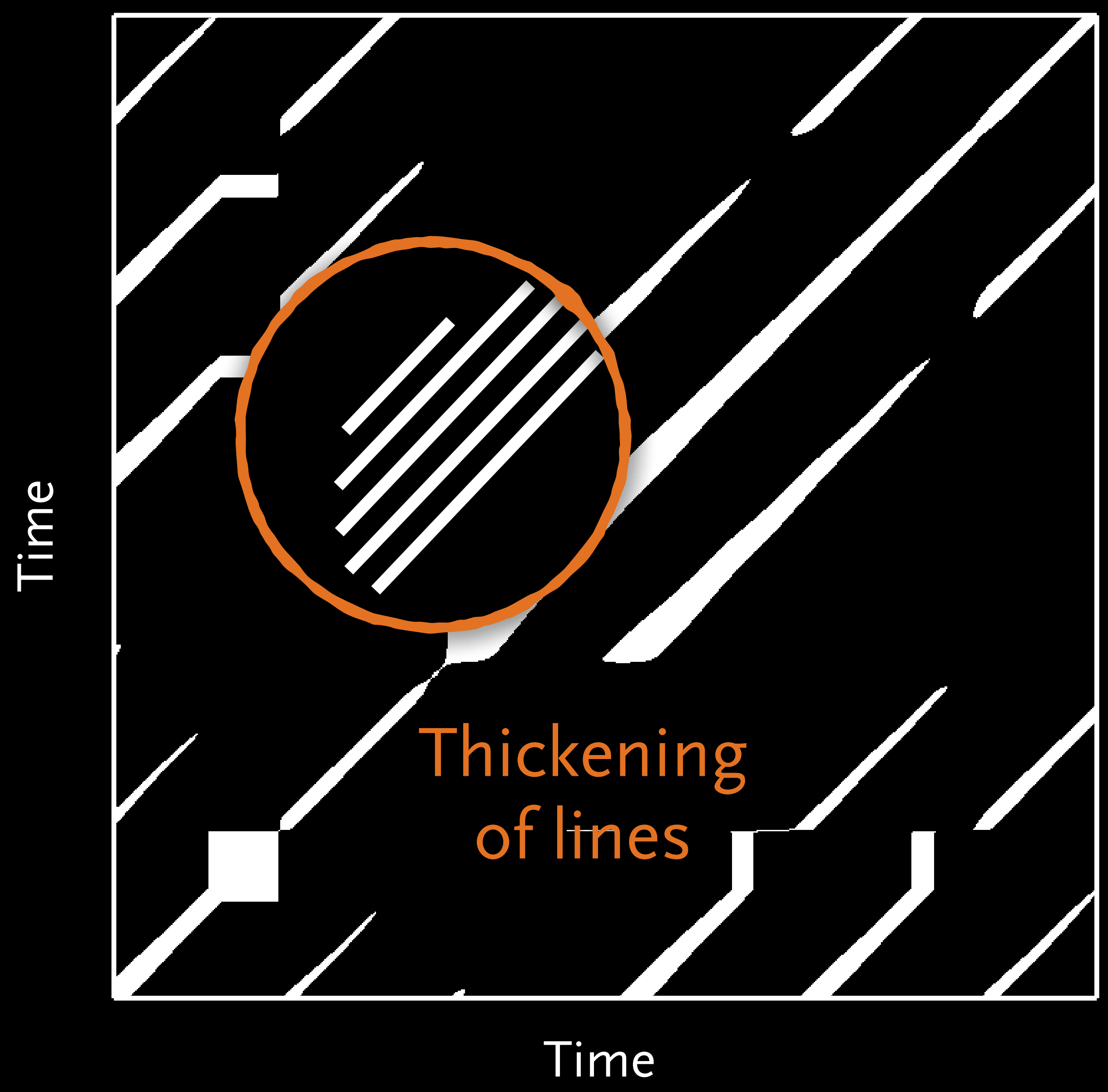
- Close in time in discretized time series or phase space



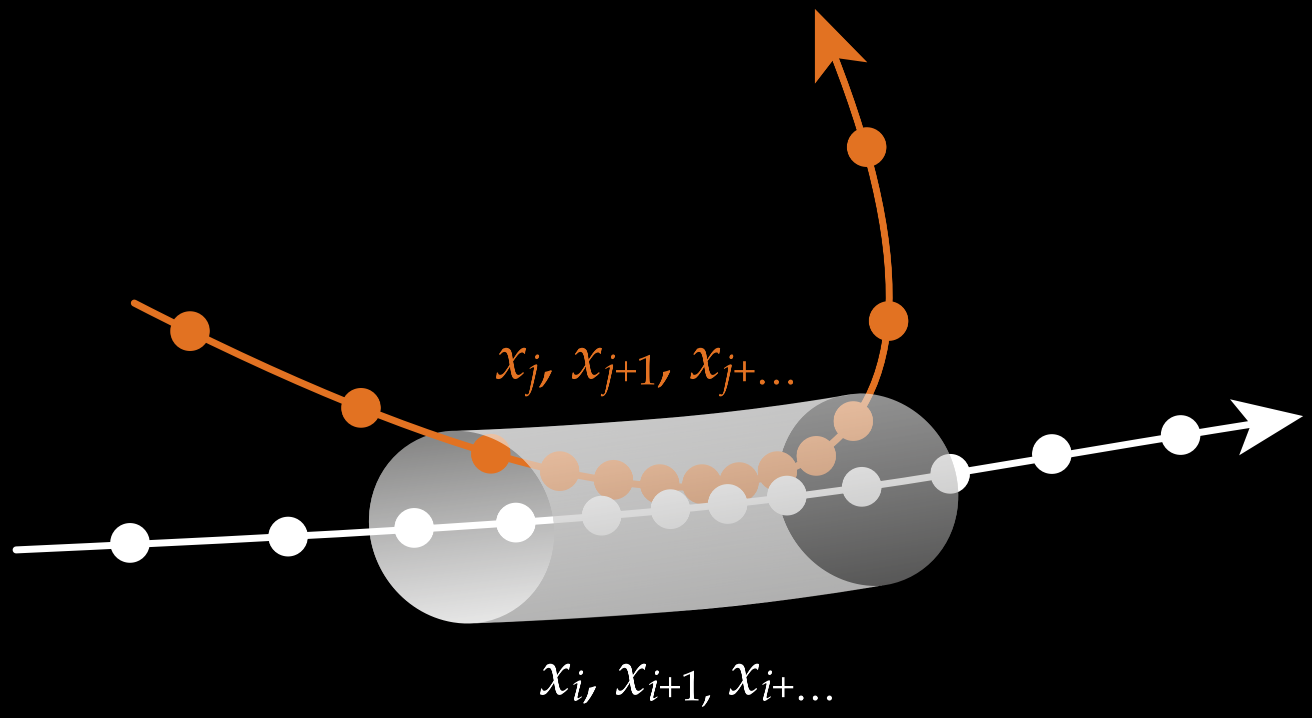
Vertical Lines, Blocks

- Sojourn points (“tangential motion”)
- Affected by sampling time and “smoothness” of the signal



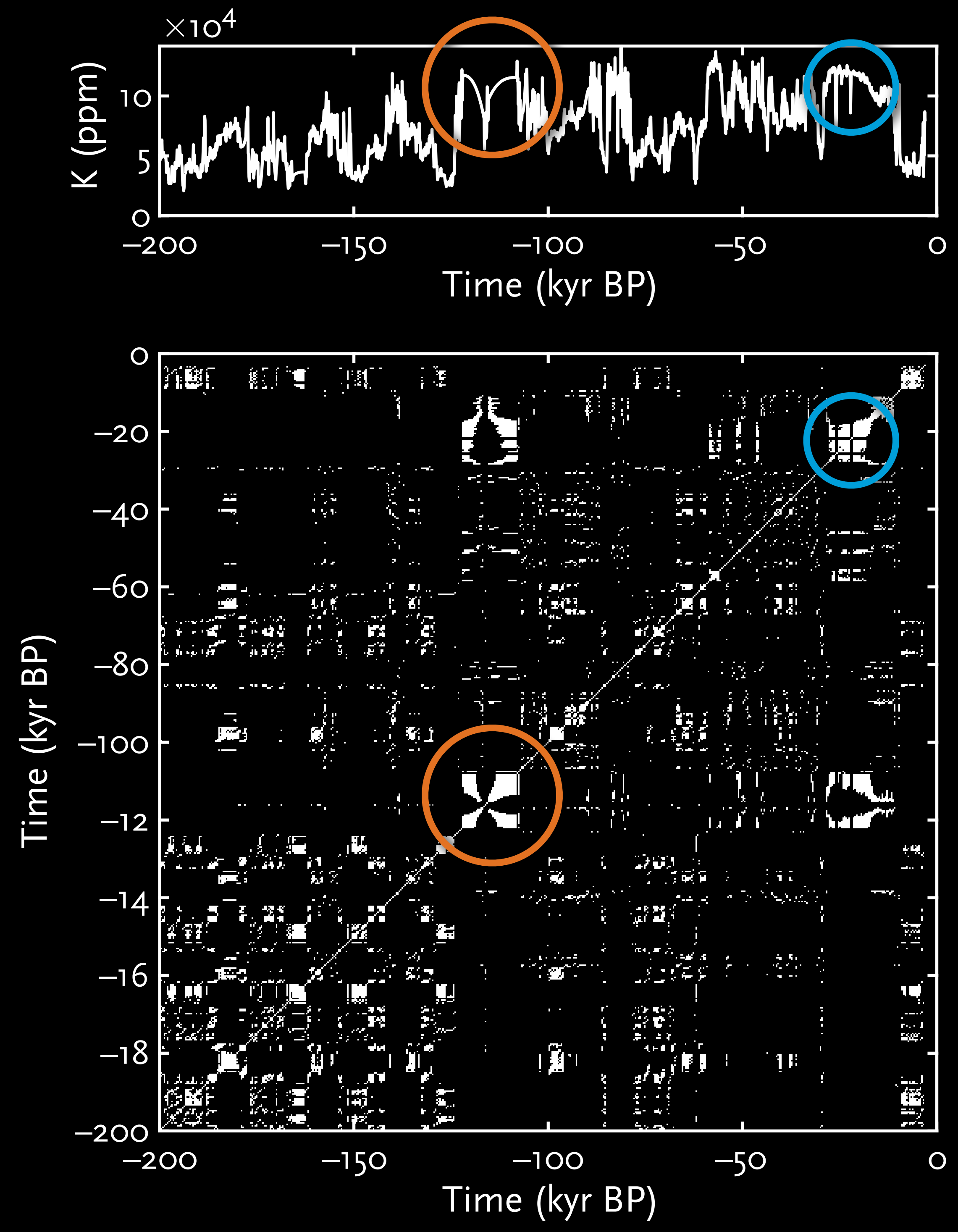


Vertical Lines, Blocks



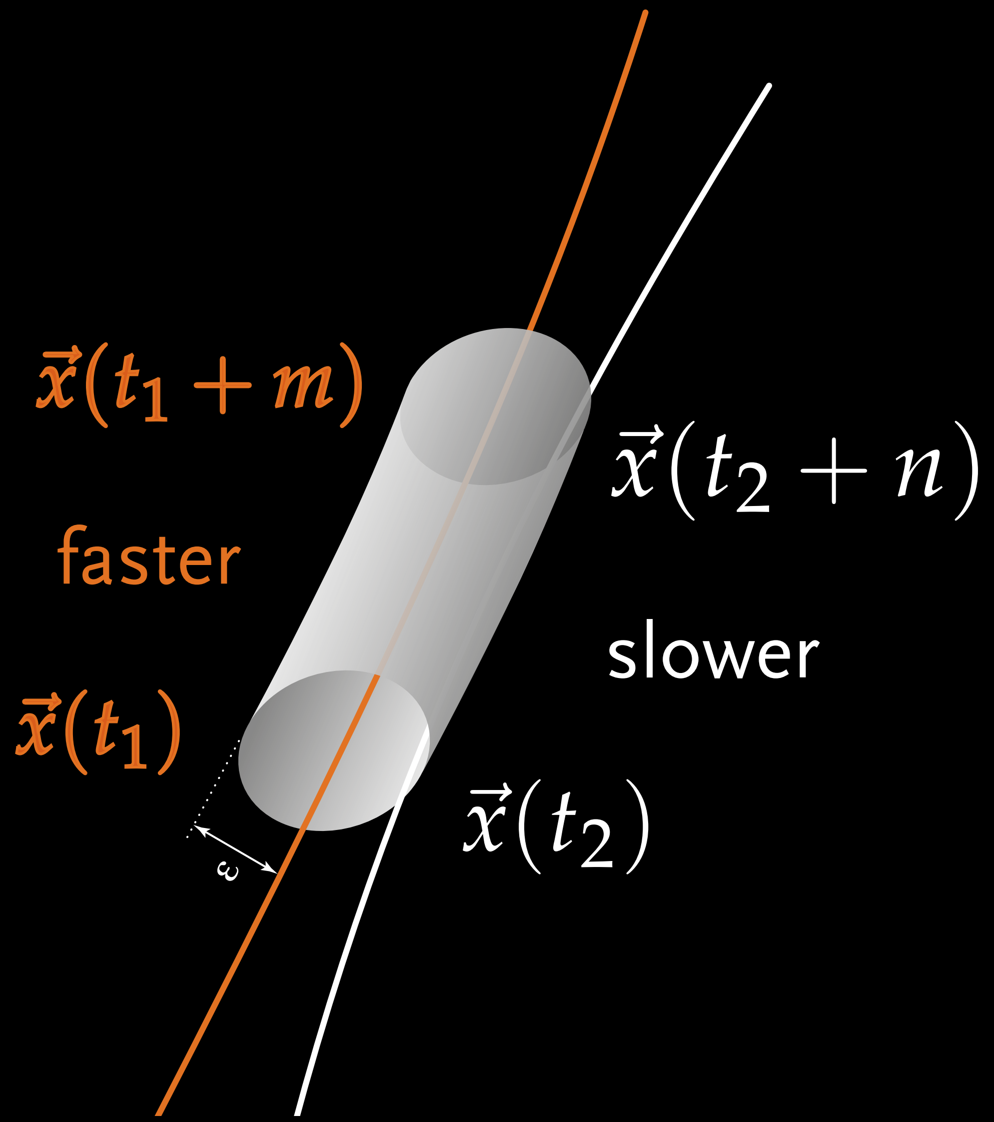
- System is trapped for some time, persistence
- Intermittency/ transition from one regime to another one
- Sojourn points

Vertical Lines, Blocks



- Example: (lake sediment) proxy indicating wet climate in East Africa
- Persistence of states
- Check carefully when interpolation is used

Non-Diagonal and Bowed Lines



$m < n$

- Recurrent epoch:

$$\overrightarrow{x}(t_1, t_1 + \Delta t_1, \ldots, t_1 + m) \approx \overrightarrow{x}(t_2, t_2 + \Delta t_2, \ldots, t_2 + n)$$

Marwan & Kurths, Phys Lett A 336, 2005

Non-Diagonal and Bowed Lines

- Time transfer function:

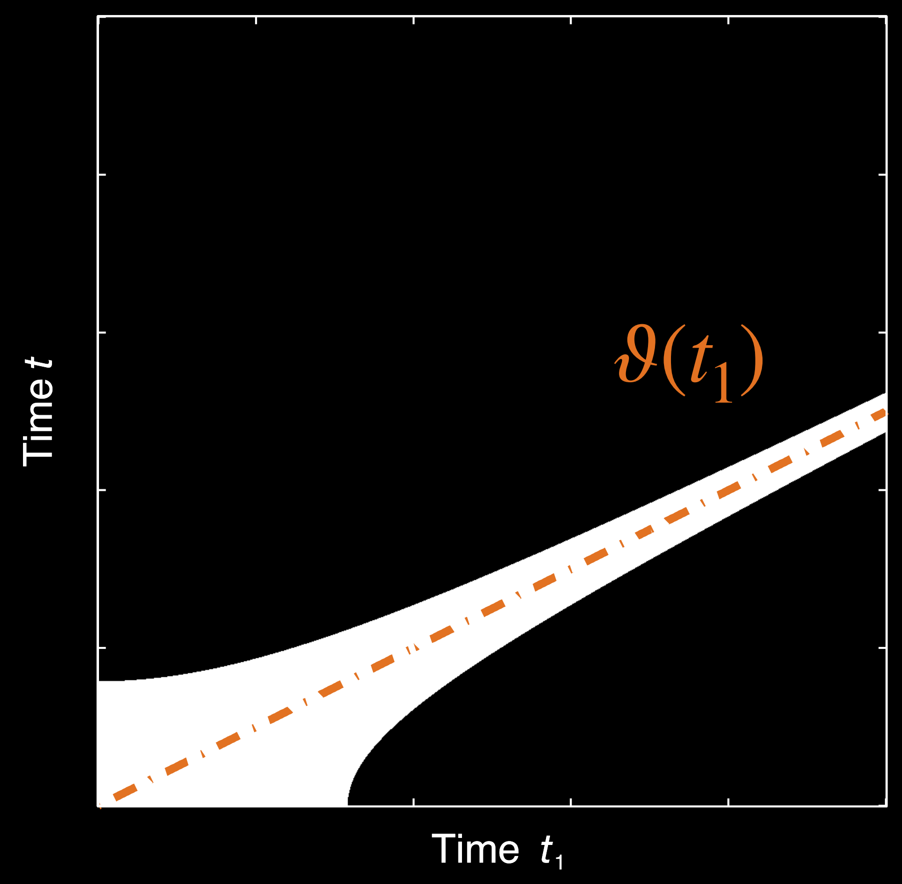
$$t_1 = T_2(t)$$

$$t = T_2^{-1}(t_1) = \vartheta(t_1)$$

- More general:

$$\vartheta(t_1) = T_2^{-1}(t_1)$$

$$\vartheta(t) = T_2^{-1}\left(T_1(t)\right)$$



Marwan & Kurths, Phys Lett A 336, 2005

Non-Diagonal and Bowed Lines

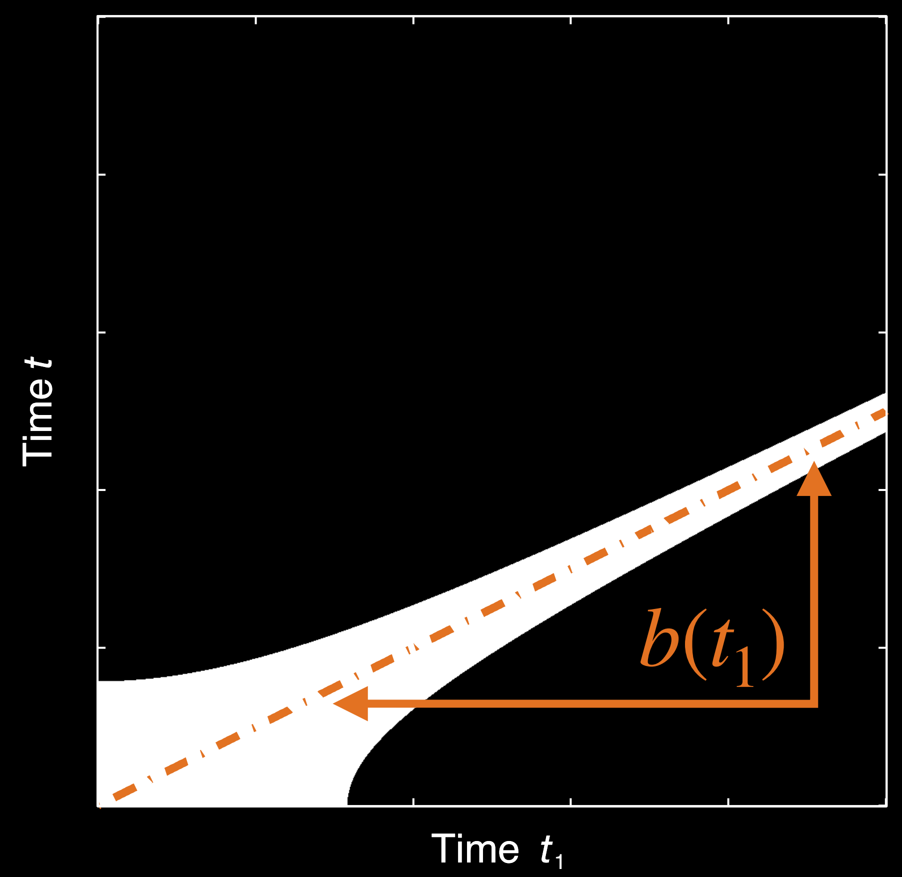
- Time transfer function:

$$t_1 = T_2(t)$$

$$t = T_2^{-1}(t_1) = \vartheta(t_1)$$

- Slope:

$$b(t_1) = \frac{d}{dt} T_2^{-1}(t_1)$$



Slope depends only on local transformation of time scale!

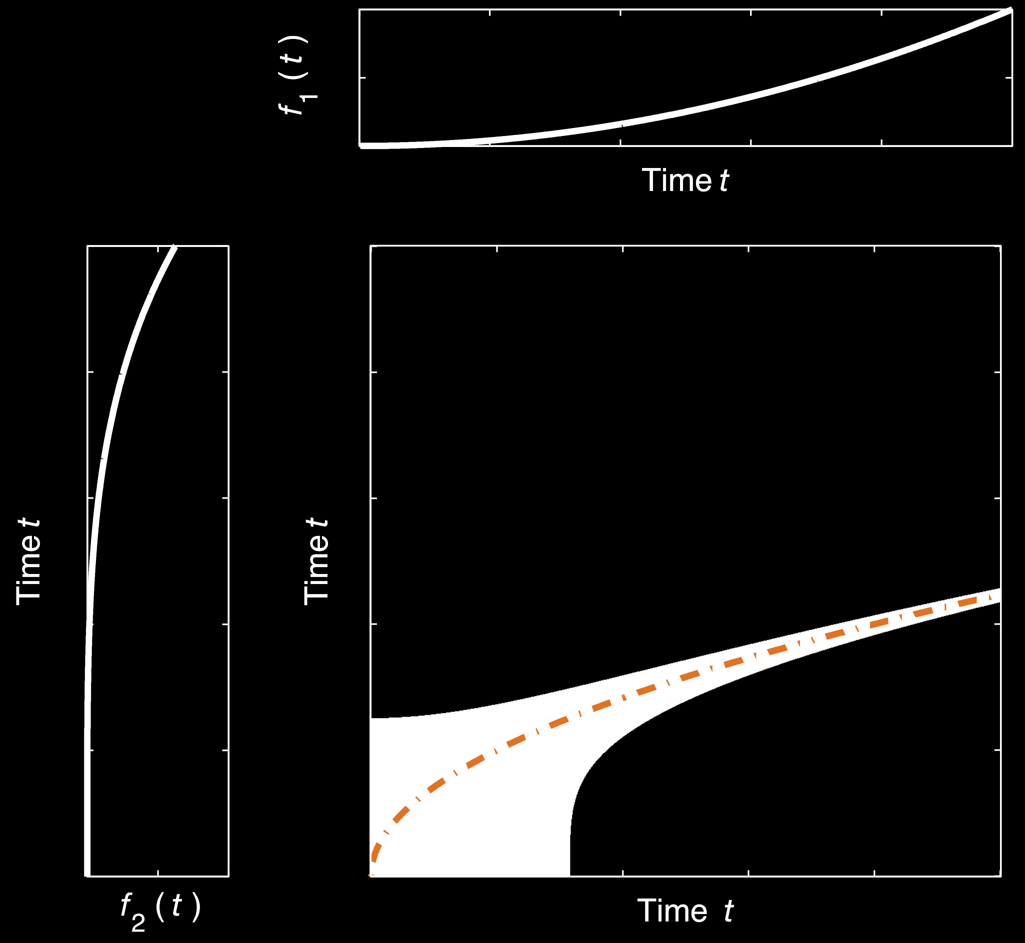
i.e. change in velocity of trajectory, but not on the attractor itself

Marwan & Kurths, Phys Lett A 336, 2005

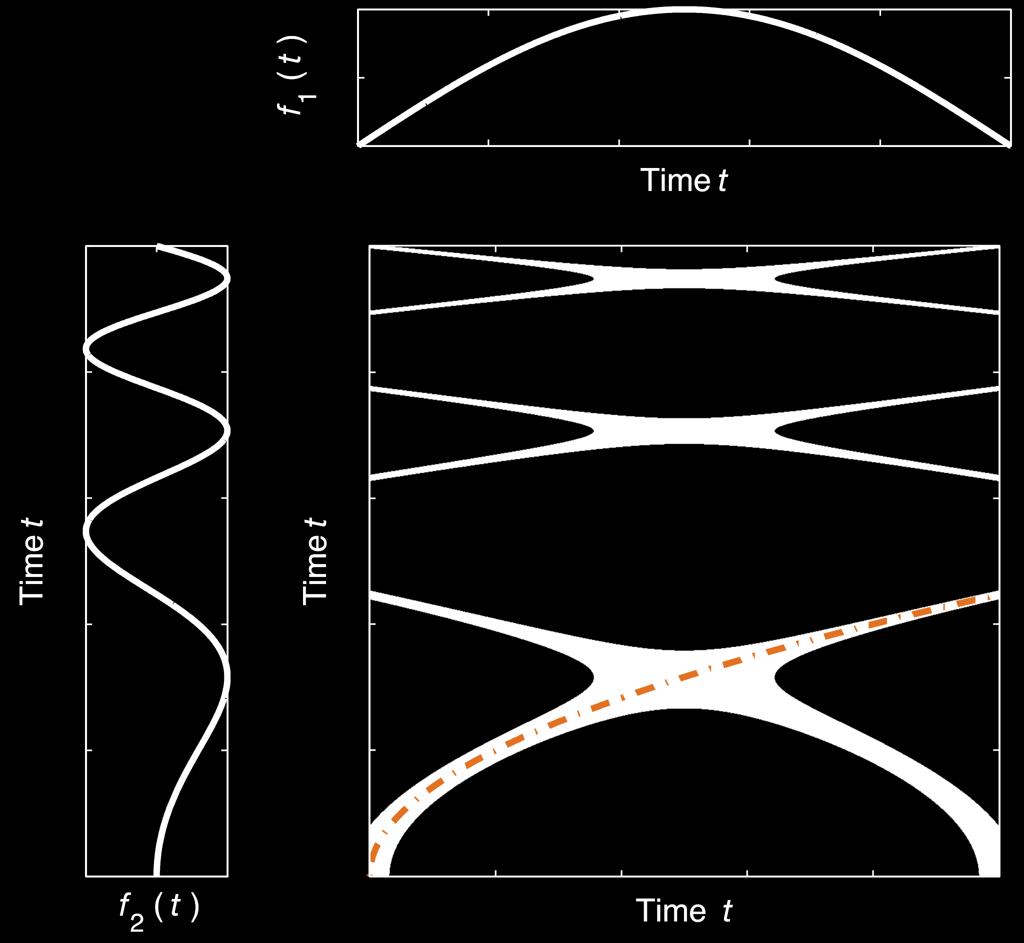
Non-Diagonal and Bowed Lines

$$T_1 = t \qquad T_2 = 5t^2$$

$f(t) = t^2$

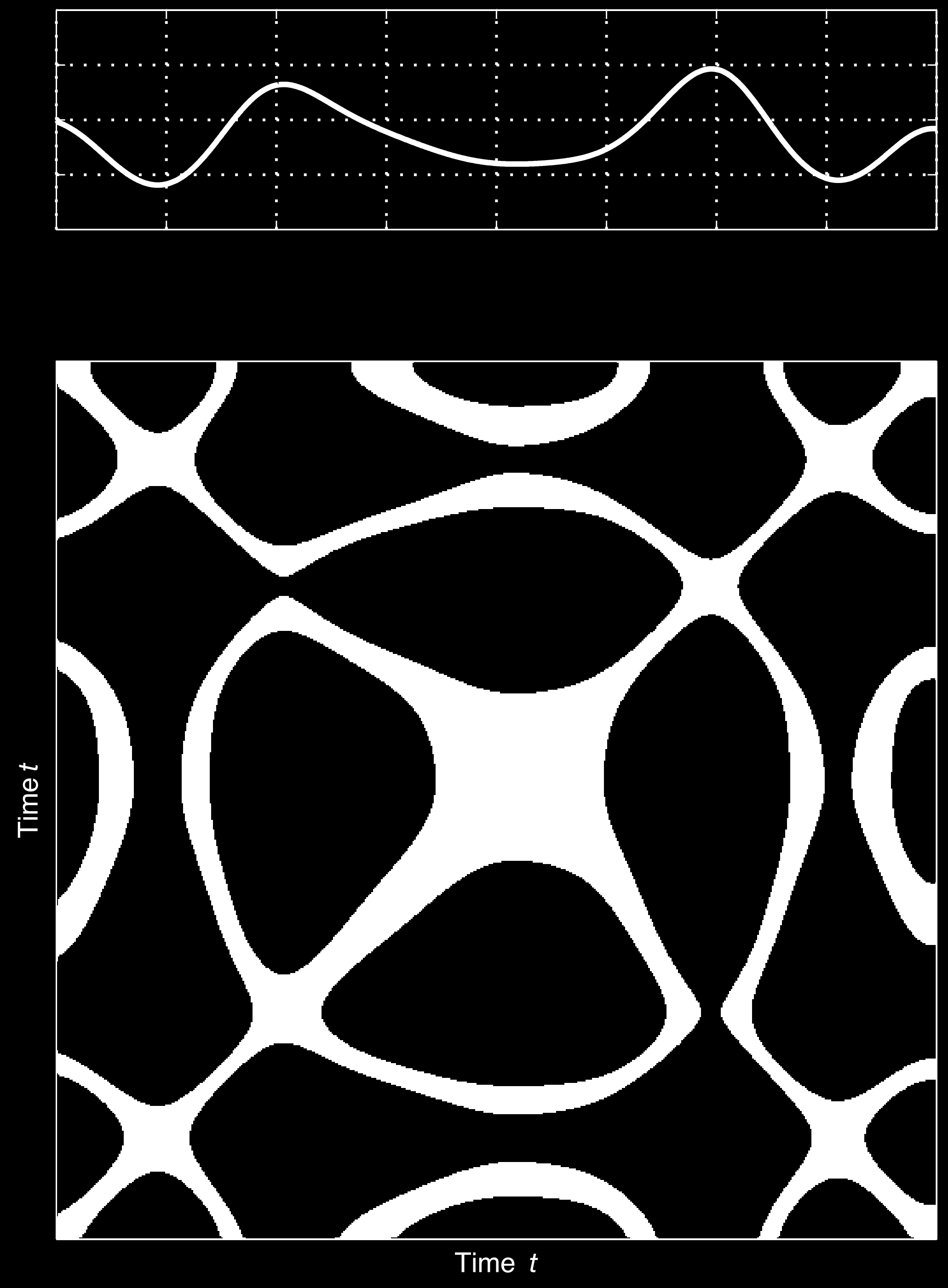


$f(t) = \sin(\pi t)$



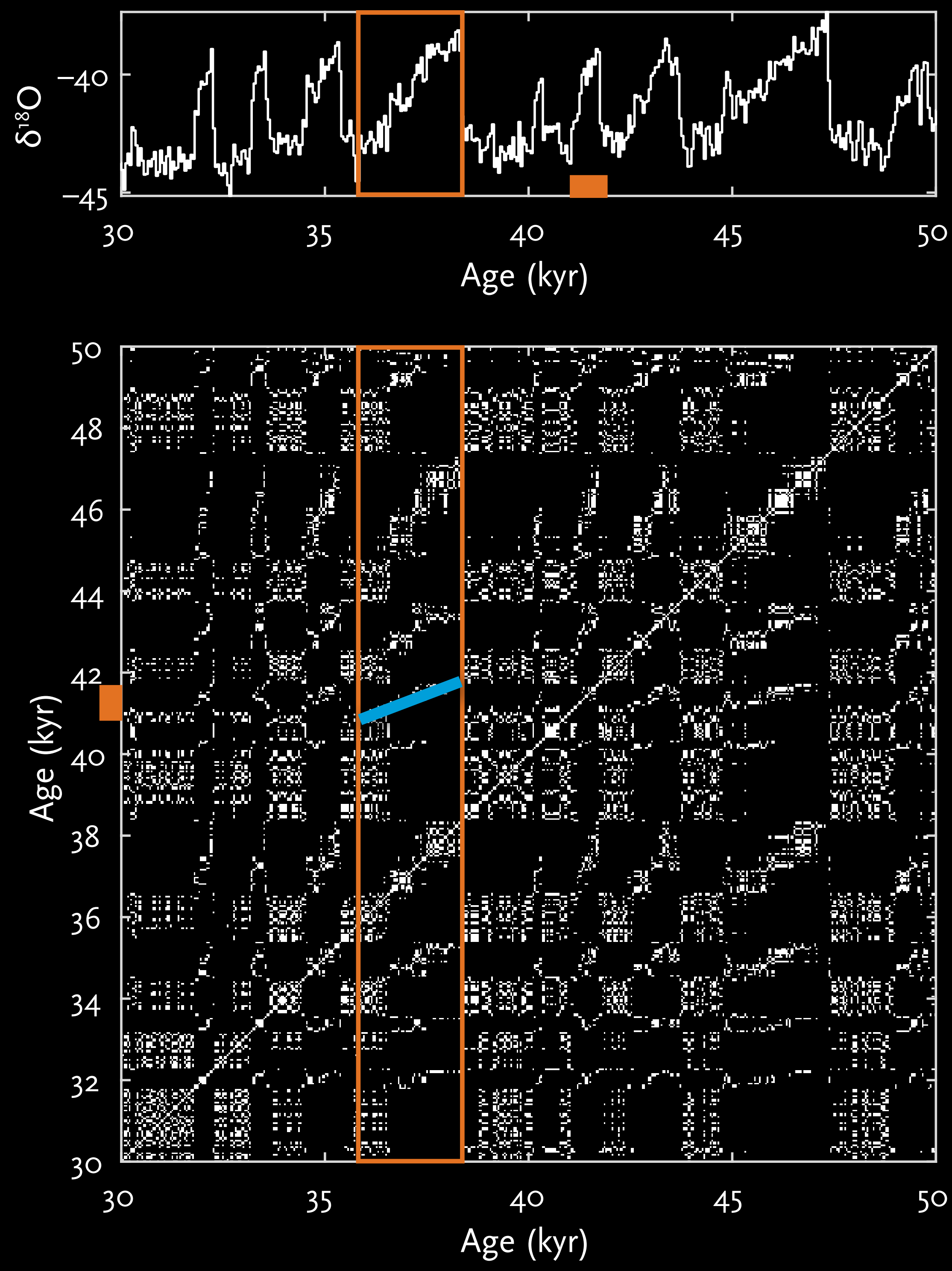
$$\vartheta(t) = T_2^{-1}\left(T_1(t)\right) \qquad \vartheta(t) = \sqrt{0.2t}$$

Non-Diagonal and Bowed Lines



- Evolution of states is similar at different epochs but with different velocity
- Dynamics of the system could be changing

NGRIP Ice Core Record



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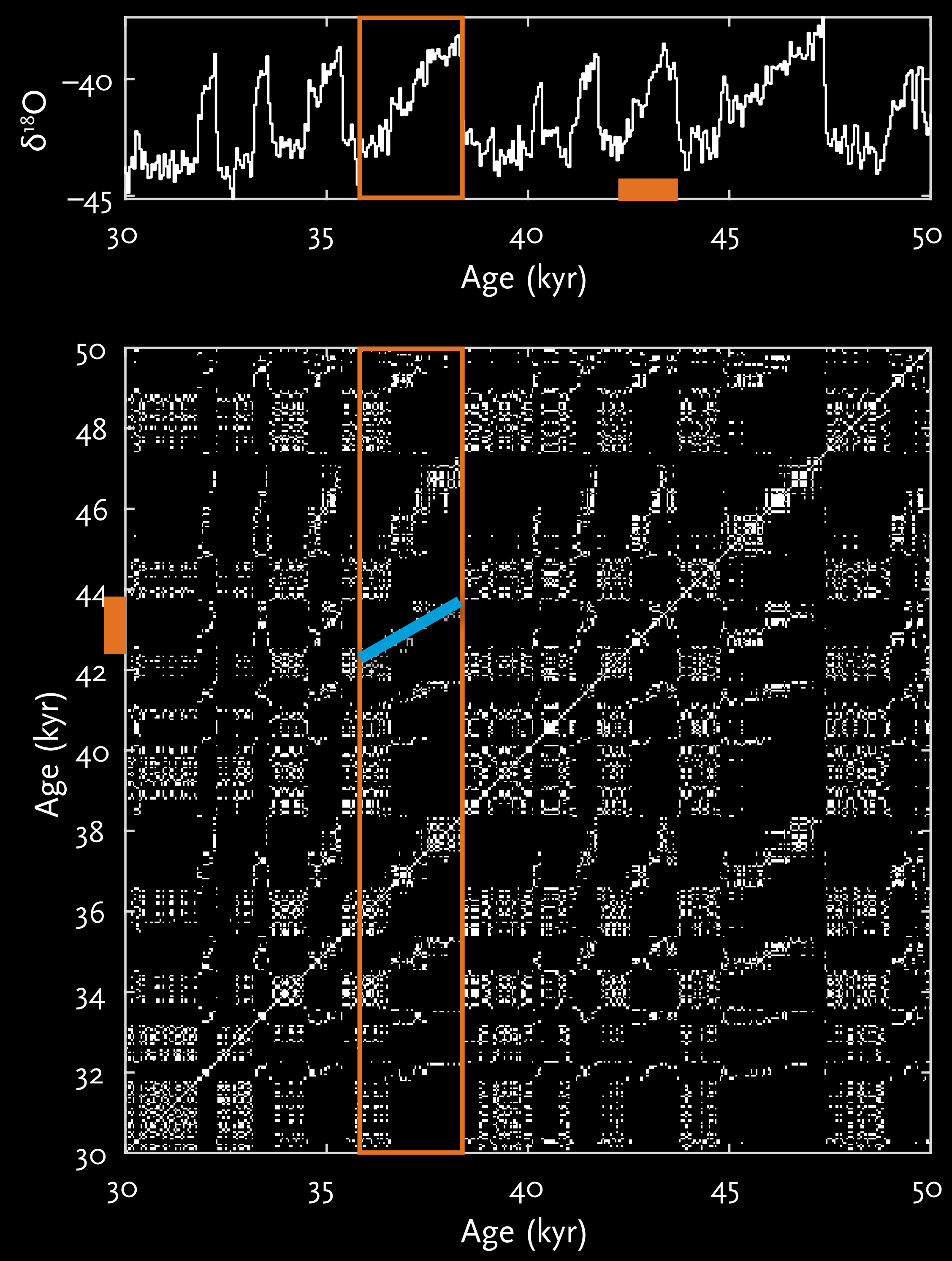

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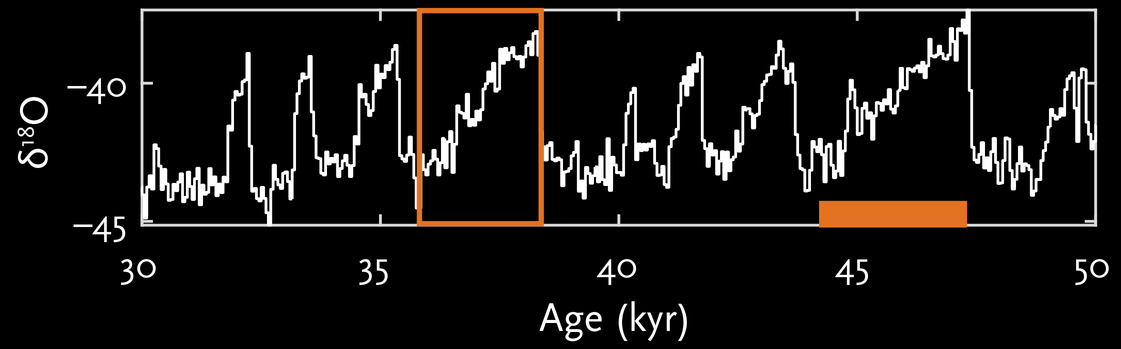
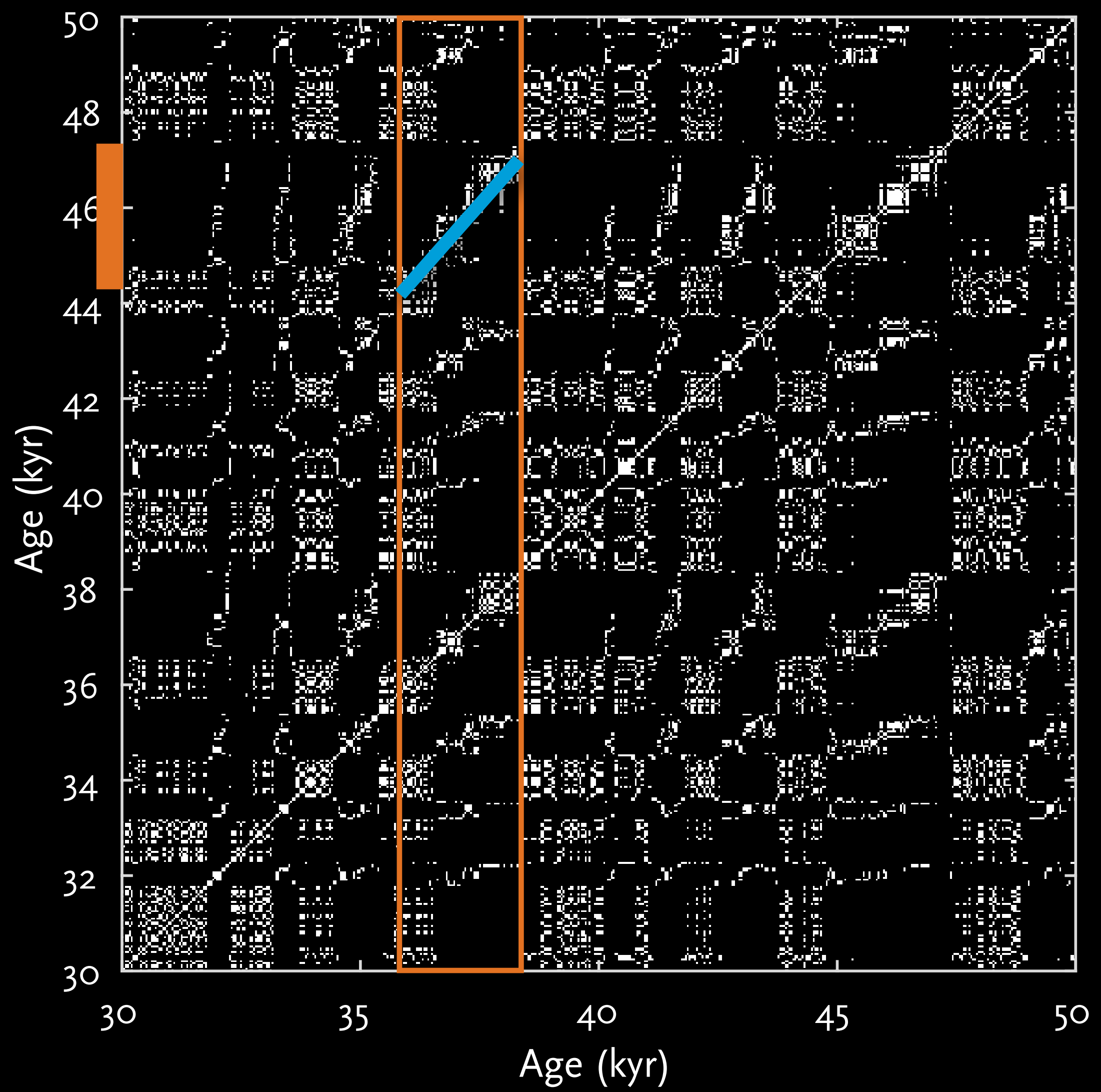
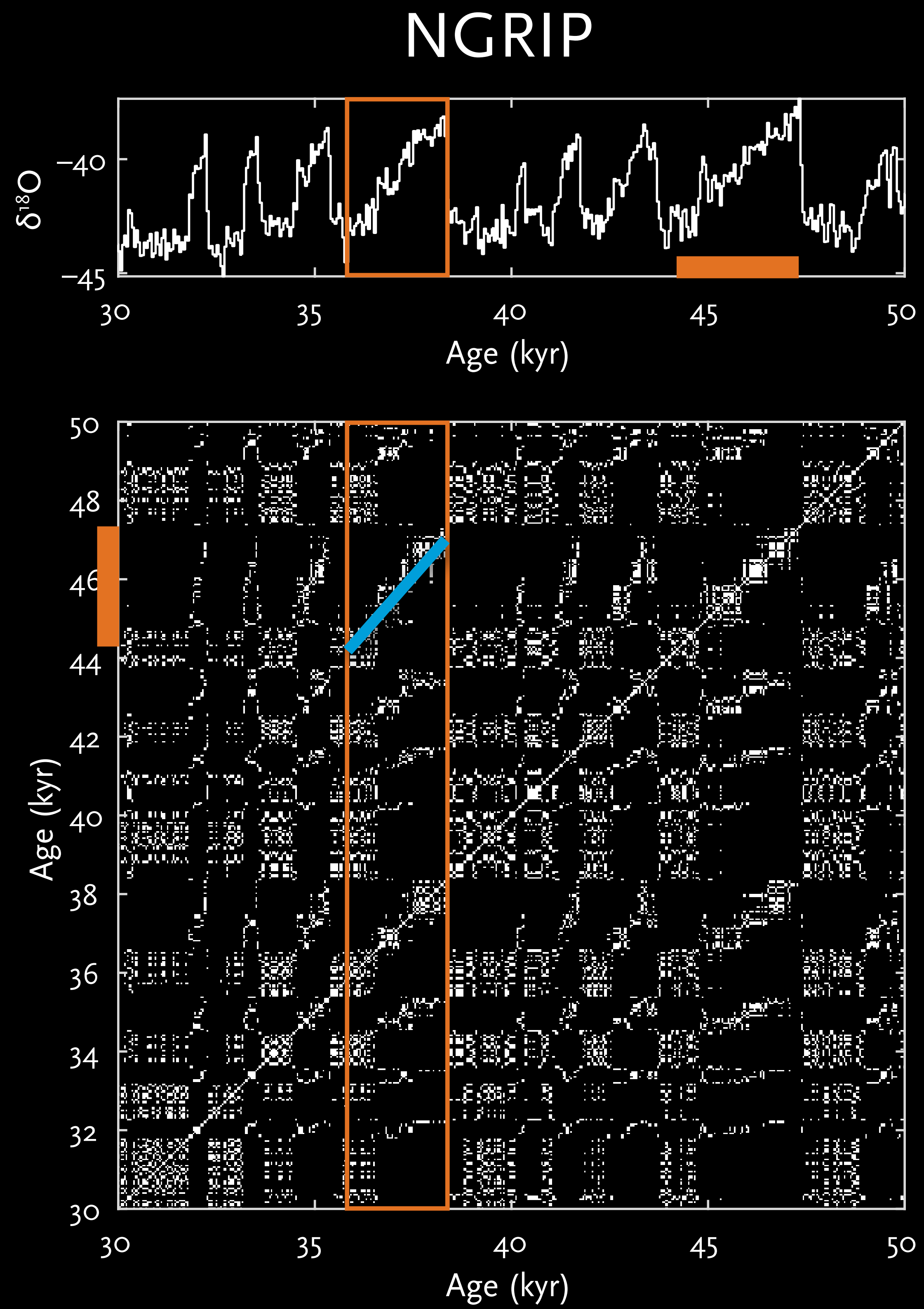


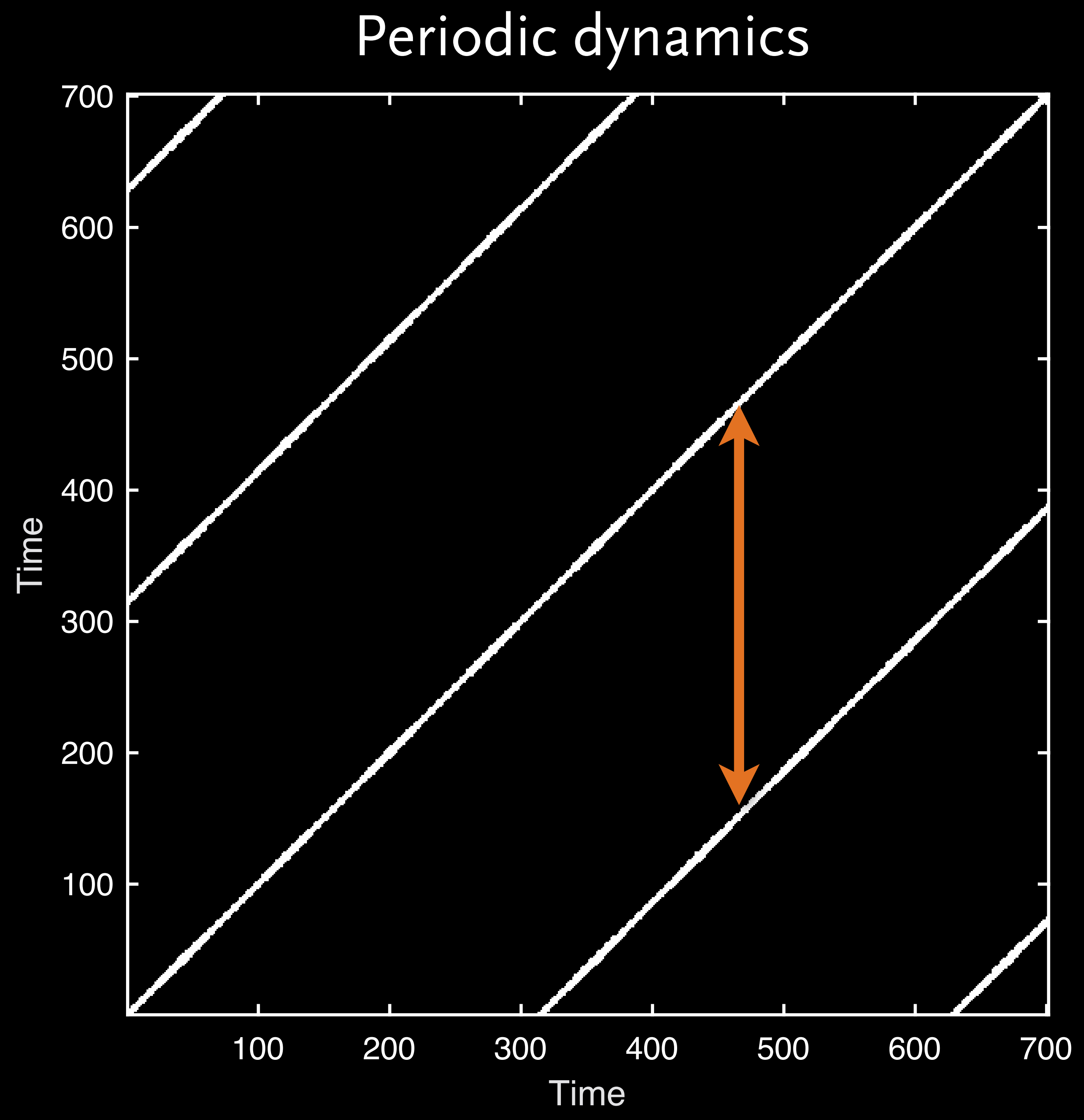


Non-Diagonal and Bowed Lines

- Lines with slopes $\not\approx 45°$ are interesting!
- Change of temporal resolution (sampling issues!)
- Change of dynamics
- System with different time scales (e.g., high and low frequencies)
- Dynamics with self-similarity(?)

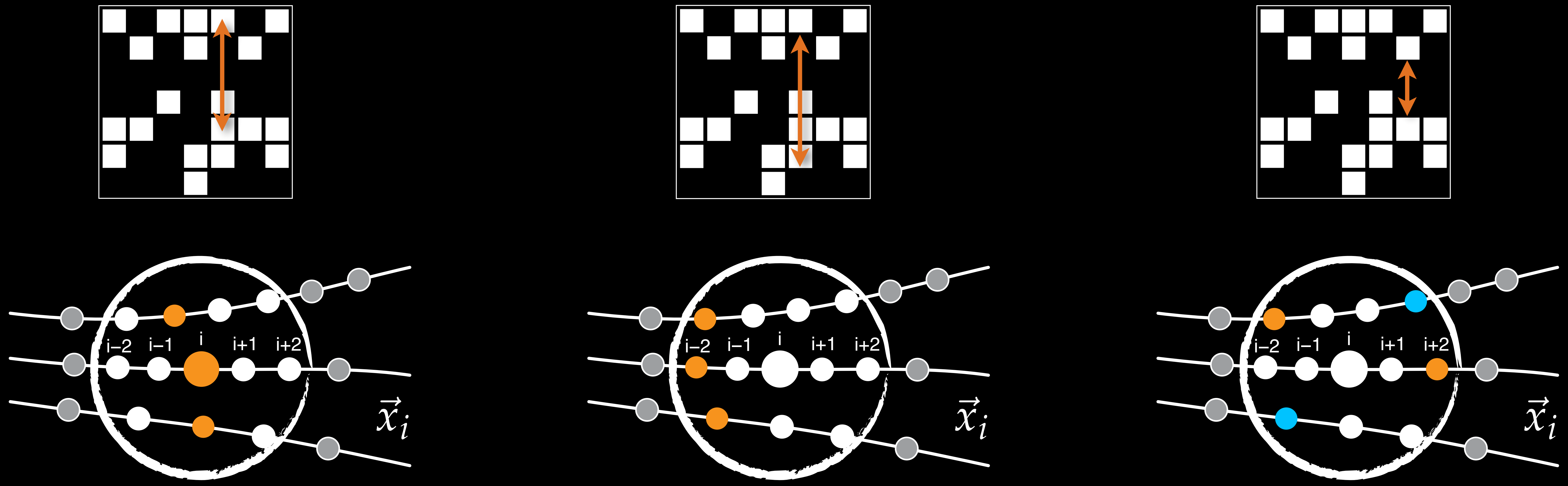


Recurrence Times



- Periodic length:
  vertical distance between recurrence points
  (vertical “empty” lines)

Recurrence Times

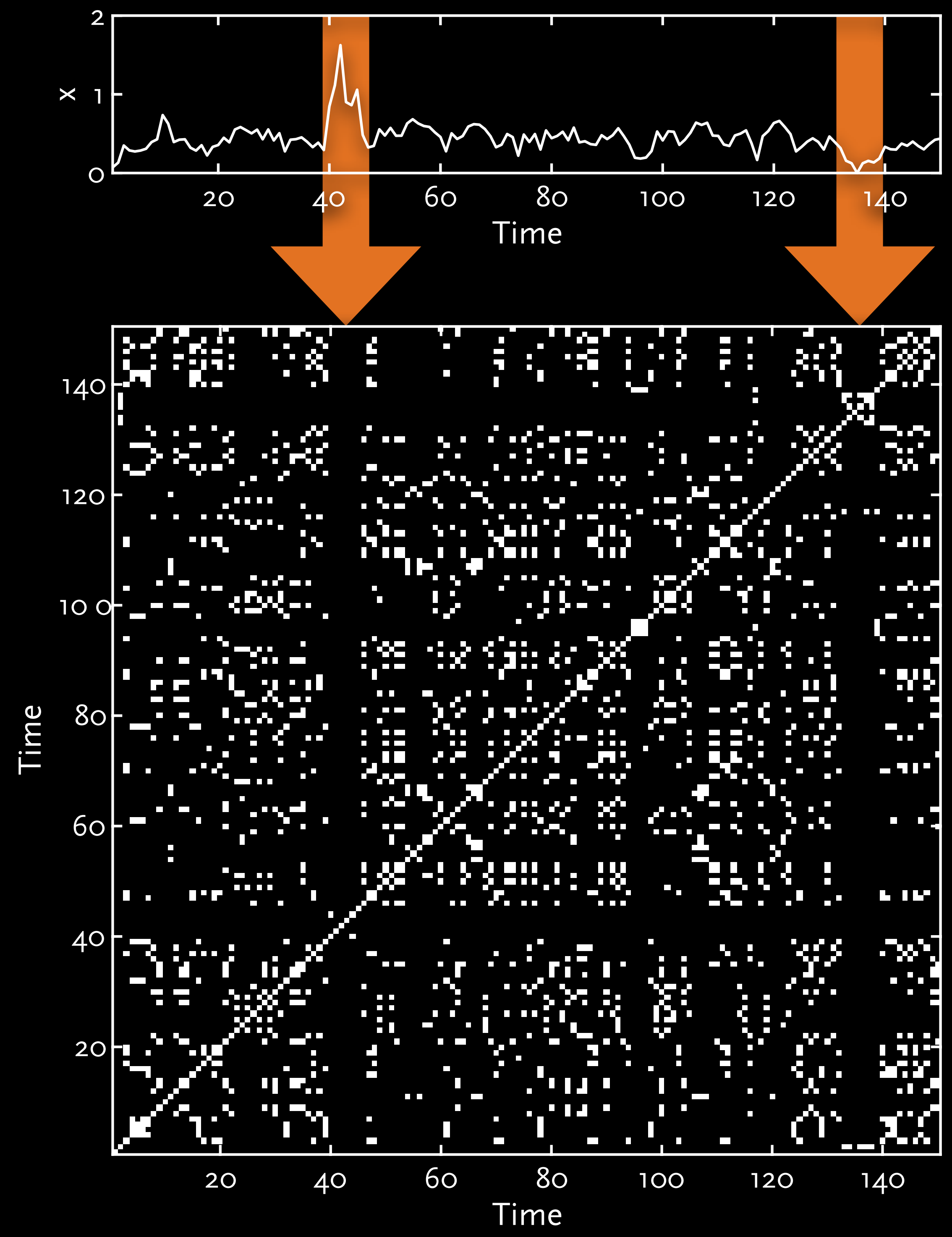


- Recurrence time distribution $P(T)$
- Complexity of the time scale of recurrence (recurrence time entropy)

Visual Interpretation

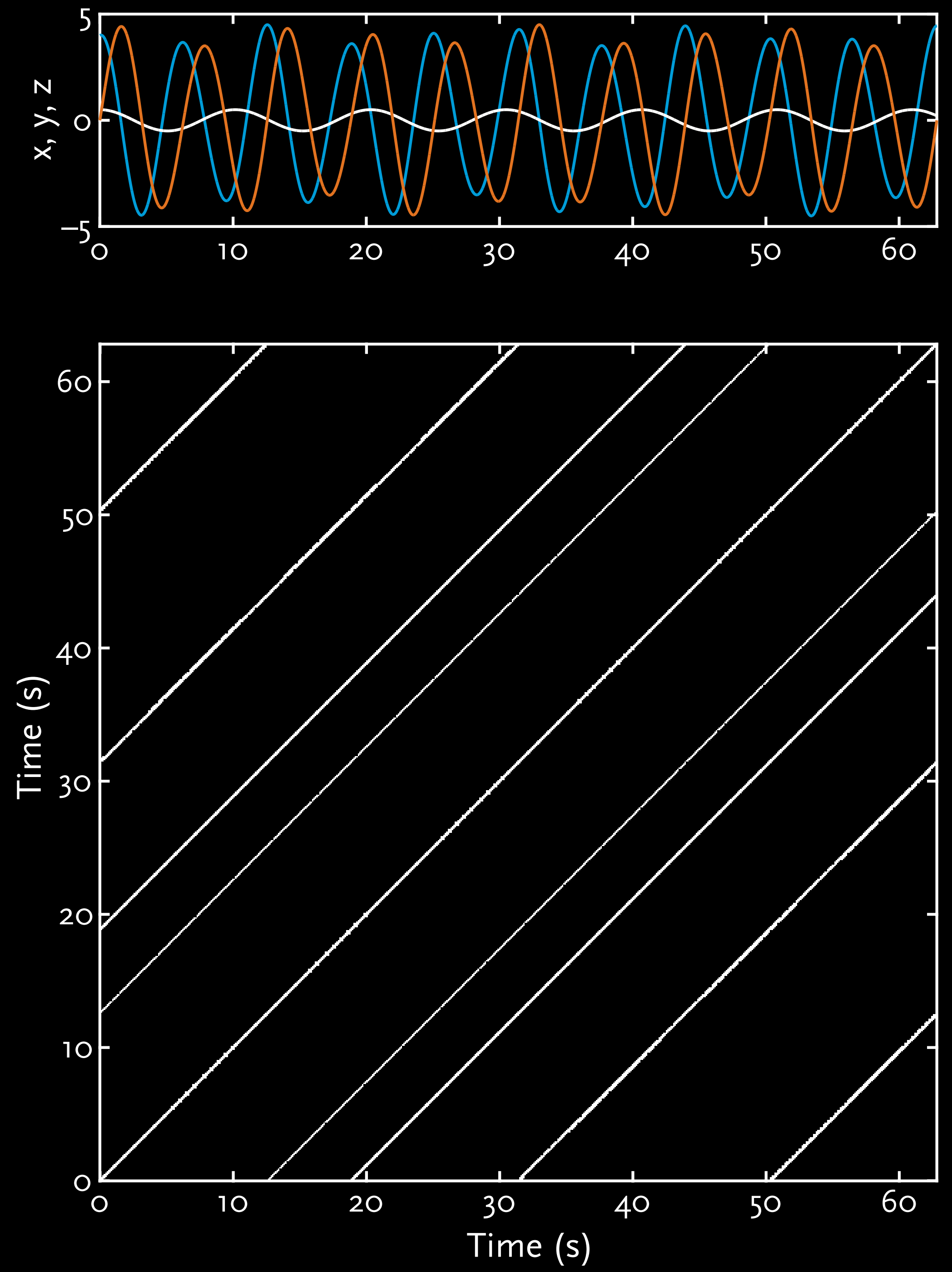
Typology (large scale appearance)	homogeneous	periodic	trend	disrupted, separated into blocks
Texture (small scale structures)	only single points	mainly diagonal lines	vertical lines, black blocks	curved pattern
Characteristic pattern	noninterrupted diagonal lines, periodic separation of diagonal lines	sparsely occupied columns/ bars	missing recurrences	local changes in recurrence point density

Disruptive Structures in Recurrence Plots



- Empty/ sparsely occupied columns
- Rare or extreme events
- Divide RP into block-like sub-regions

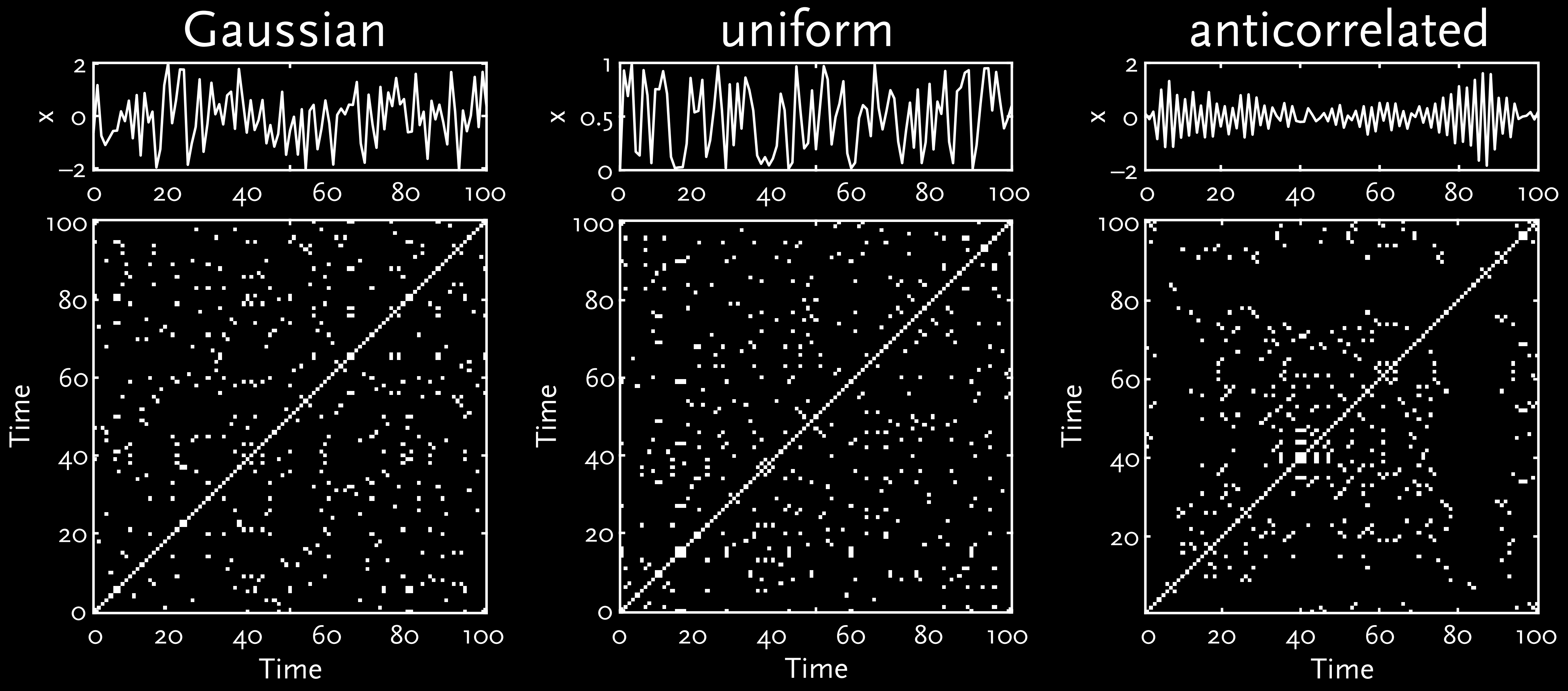
Periodic and Quasi-Periodic Pattern



- Cyclicities in the process
- Time distance between periodic patterns (e.g. lines) corresponds to the periods
- Different distances between long diagonal lines reveal quasi-periodic processes

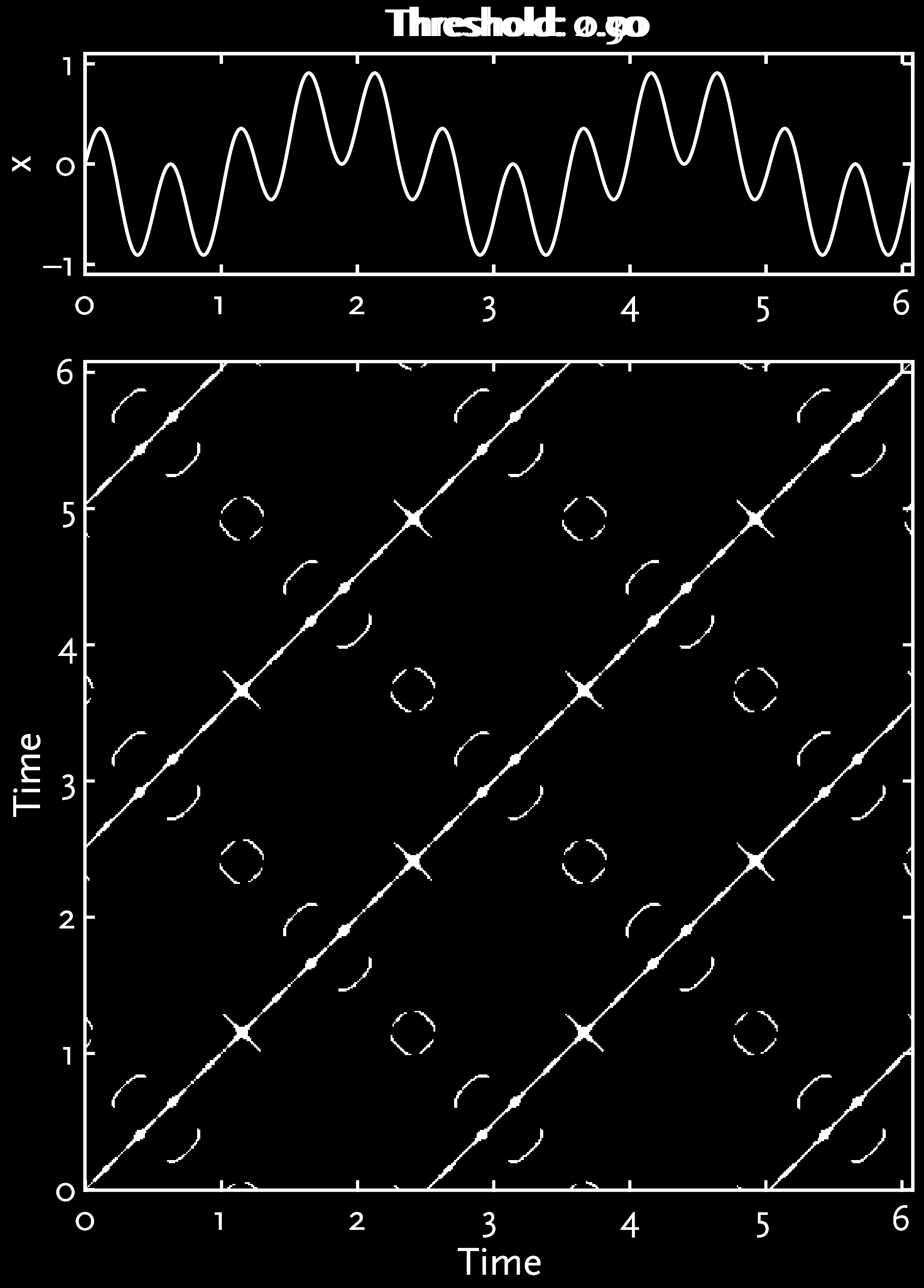
Single Isolated Points

- Strong fluctuation in the process
- If only single isolated points occur, the process may be an uncorrelated random or even anti-correlated process



Sensitivity on Parameters

Recurrence Threshold



- Too small:
  - no points, no essential structures
- Too large:
  - interesting structures are merged, thick lines, completely full
- Optimal selection:
  - large variety of structures, thin lines, different time scales visible

Recurrence Threshold: Quantile Approach

- Preselect RR (e.g., to have $RR = 10\,\%$)
  $RR(\varepsilon) \to \varepsilon$

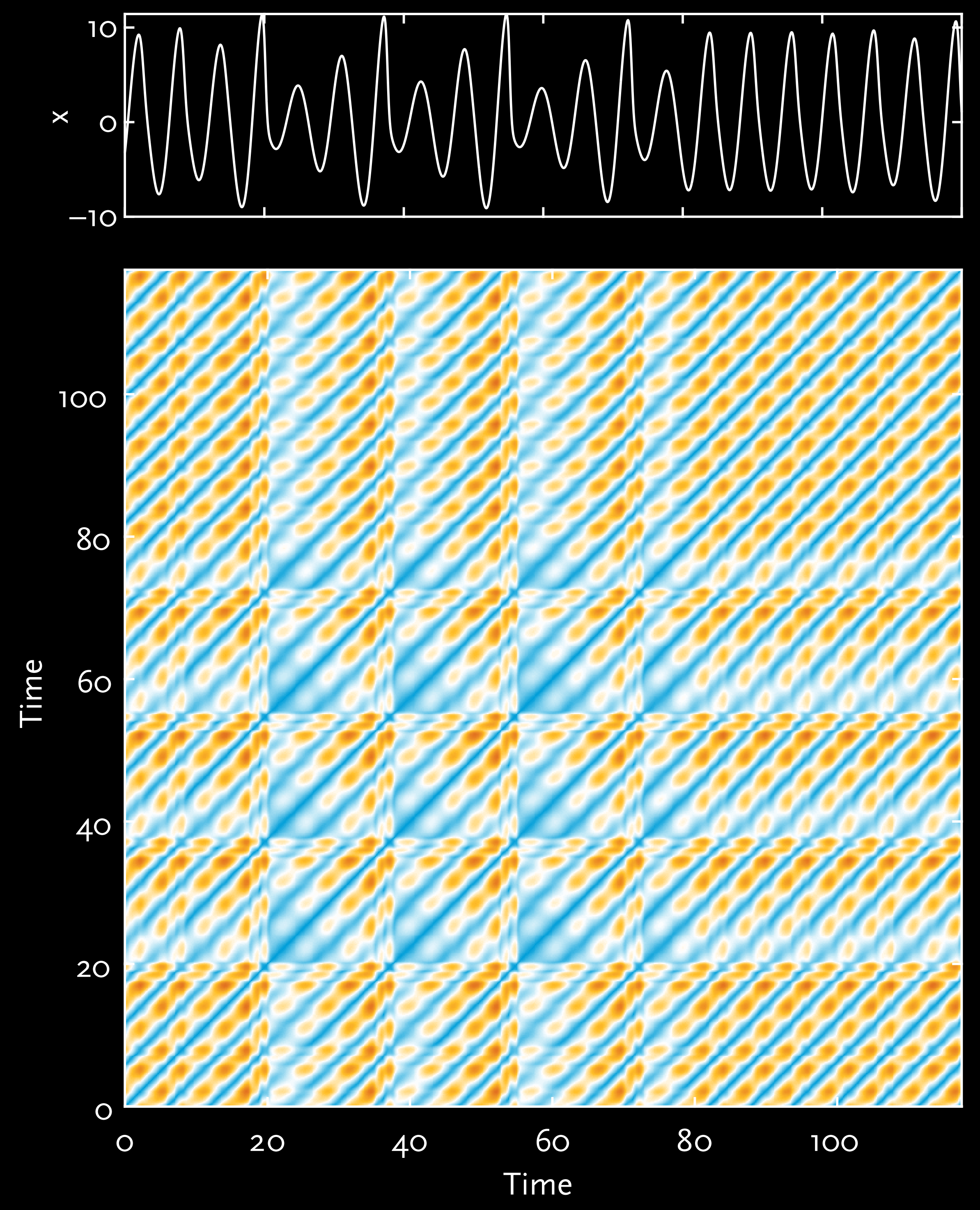
- Recurrence rate:

$$RR(\varepsilon) = \frac{1}{N(N-1)} \sum_{i,j,i \neq j}^{N} R_{i,j(\varepsilon)}$$

- Distance matrix:

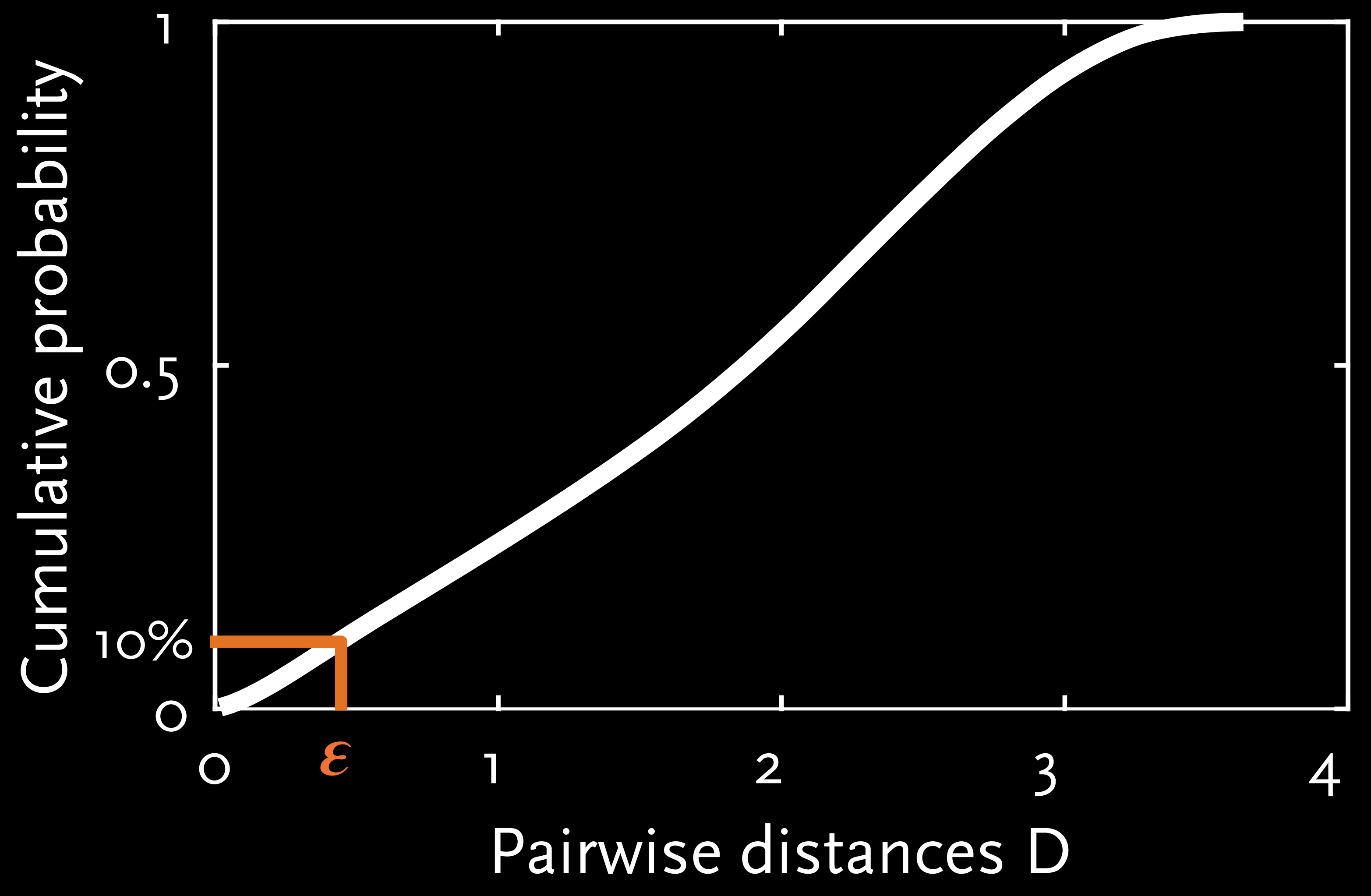
$$R_{i,j}(\varepsilon) = \Theta\left(\varepsilon - D_{i,j}\right)$$

$$D_{i,j} = \| \overrightarrow{x}_i - \overrightarrow{x}_j \|$$



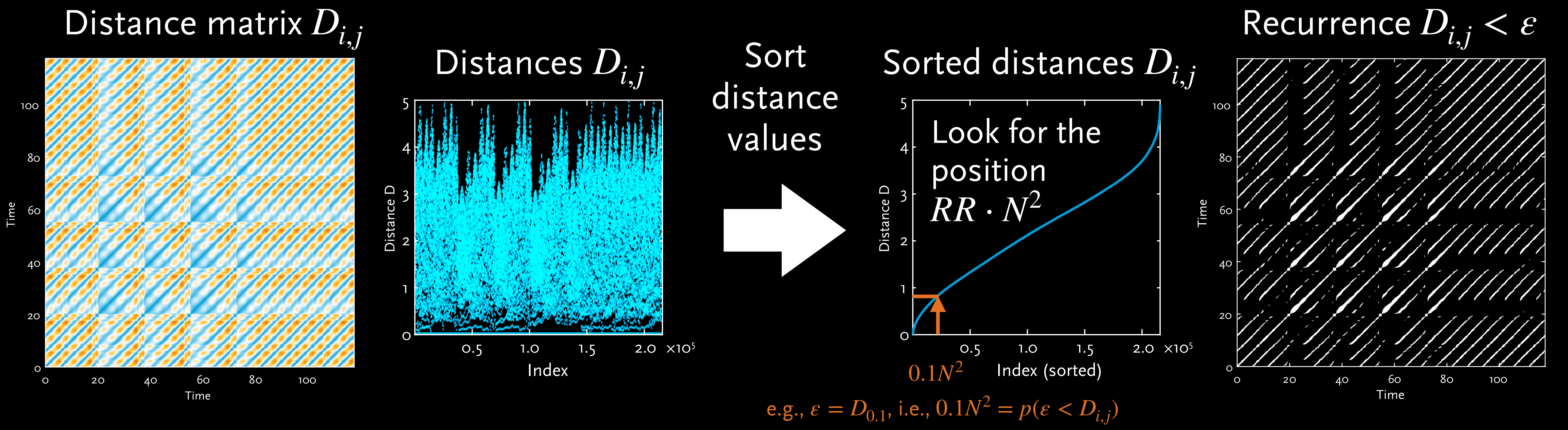
Recurrence Threshold: Quantile Approach

- Preselect RR
  $RR(\varepsilon) \rightarrow \varepsilon$
- Quantile of distance distribution
  $p(x < D_{i,j})$
- e.g., $\varepsilon = D_{0.1}$
  i.e., $0.1 = p(\varepsilon < D_{i,j})$
- Quantile = recurrence rate



Kraemer et al, Chaos 28, 2018

Recurrence Threshold: Quantile Approach



- Robust recurrence characteristics in different embedding dimensions

Kraemer et al, Chaos 28, 2018

Recurrence Threshold: Connectedness

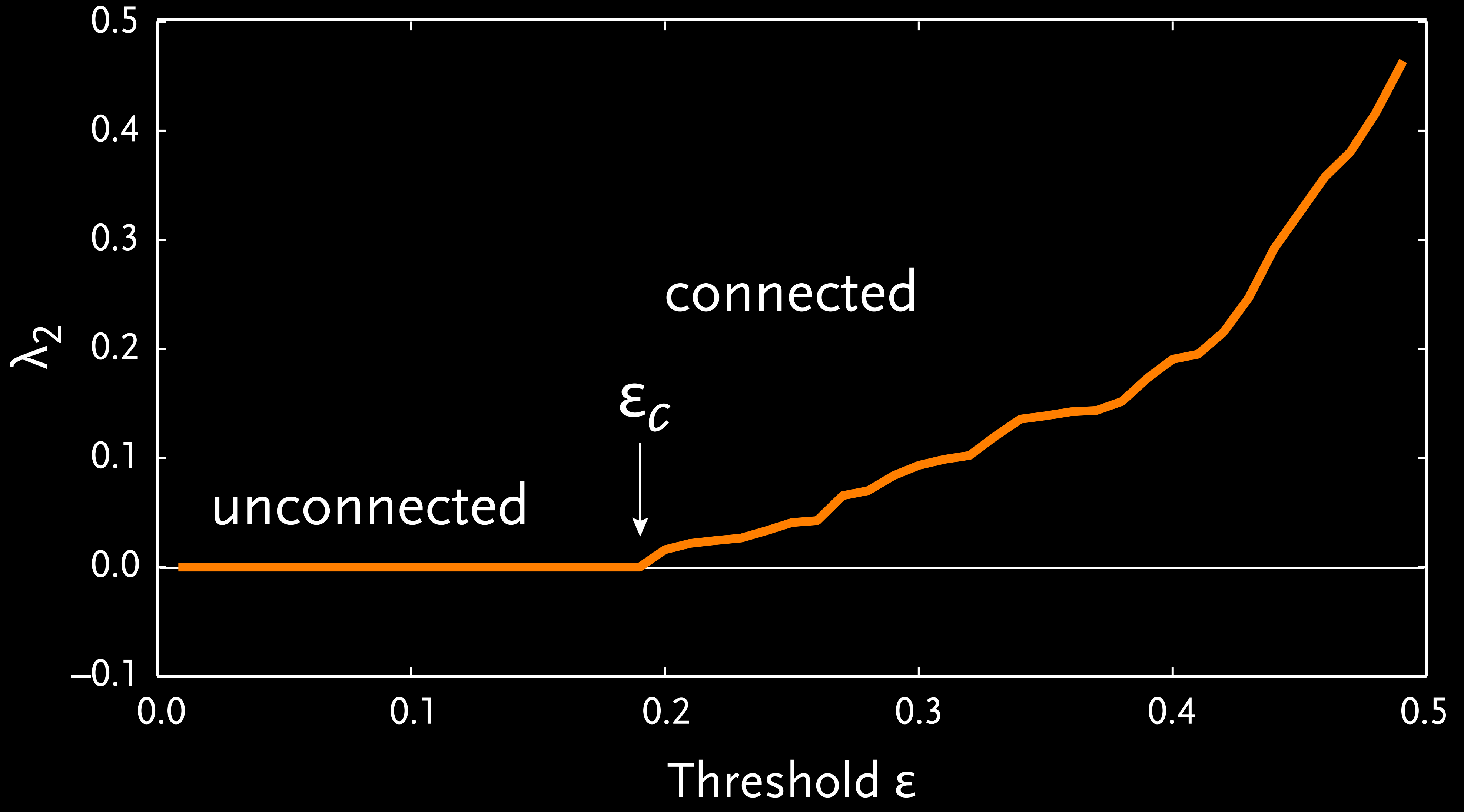
- Connectivity of recurrence matrix

$$A_{i,j} = R_{i,j} - \delta_{i,j}$$

- Laplace matrix

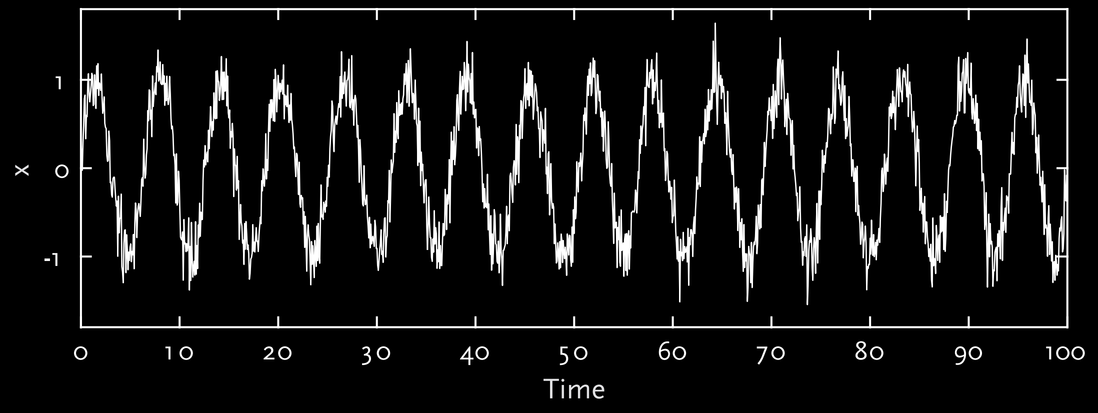
$$L_{i,j} = \delta_{i,j} \sum_j R_{i,j} - R_{i,j}$$

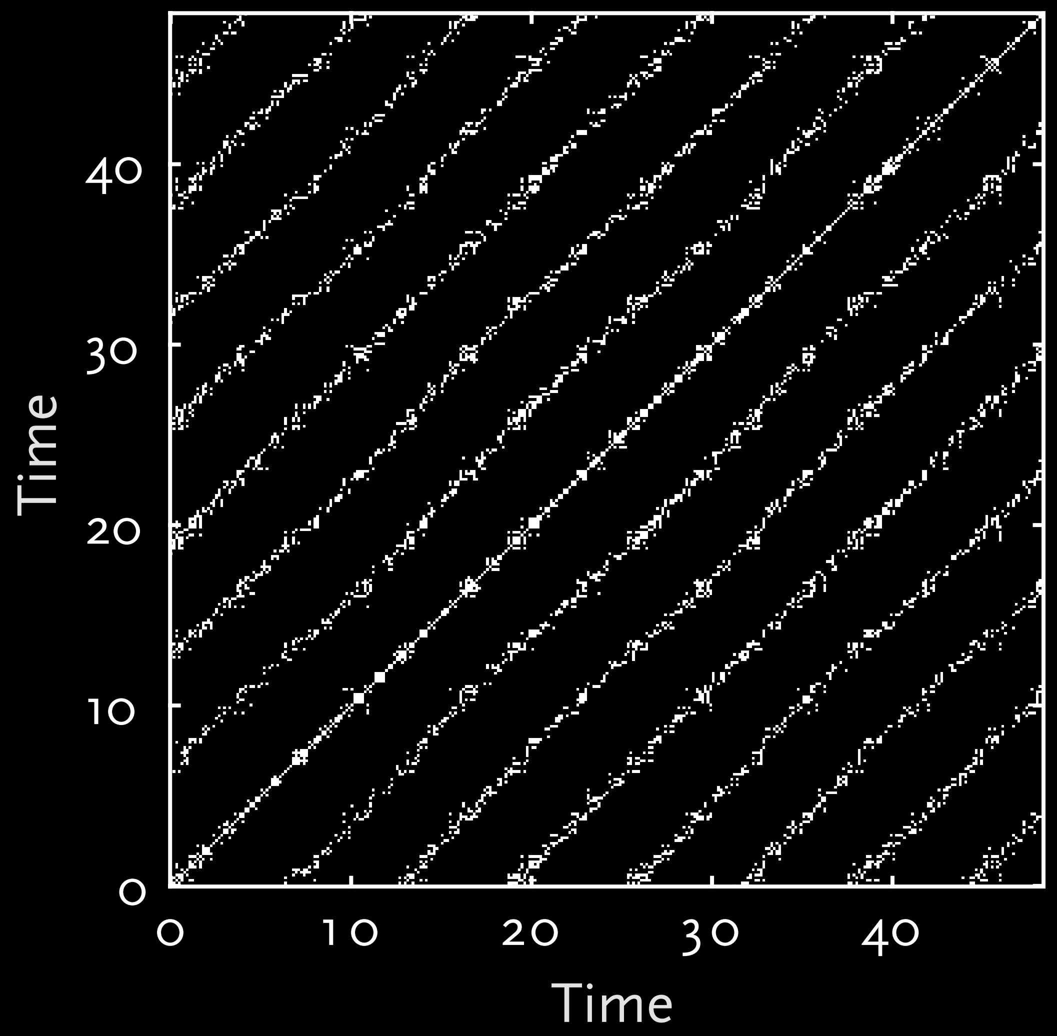
- $\varepsilon \longrightarrow$ 2nd smallest Eigenvalue $\lambda_2 > 0$



Eroglu et al, Nonlin Proc Geophys 21, 2014

Recurrence Threshold: Optimizing Diagonals

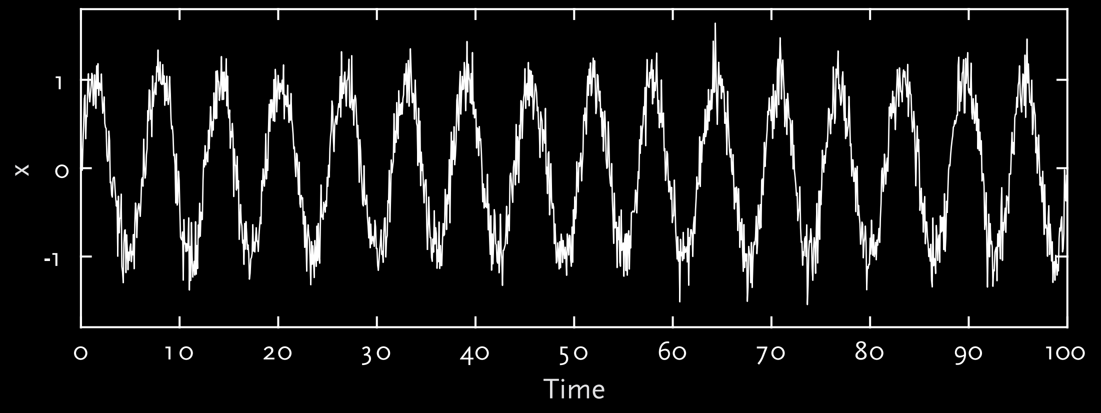


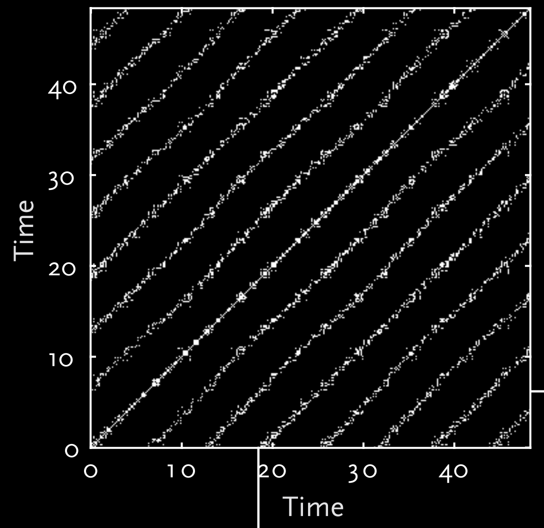


- Minimizing the fragmentation and thickness of the diagonal lines (for cyclical signals)

Matassini et al, Phys Rev E 65, 2002

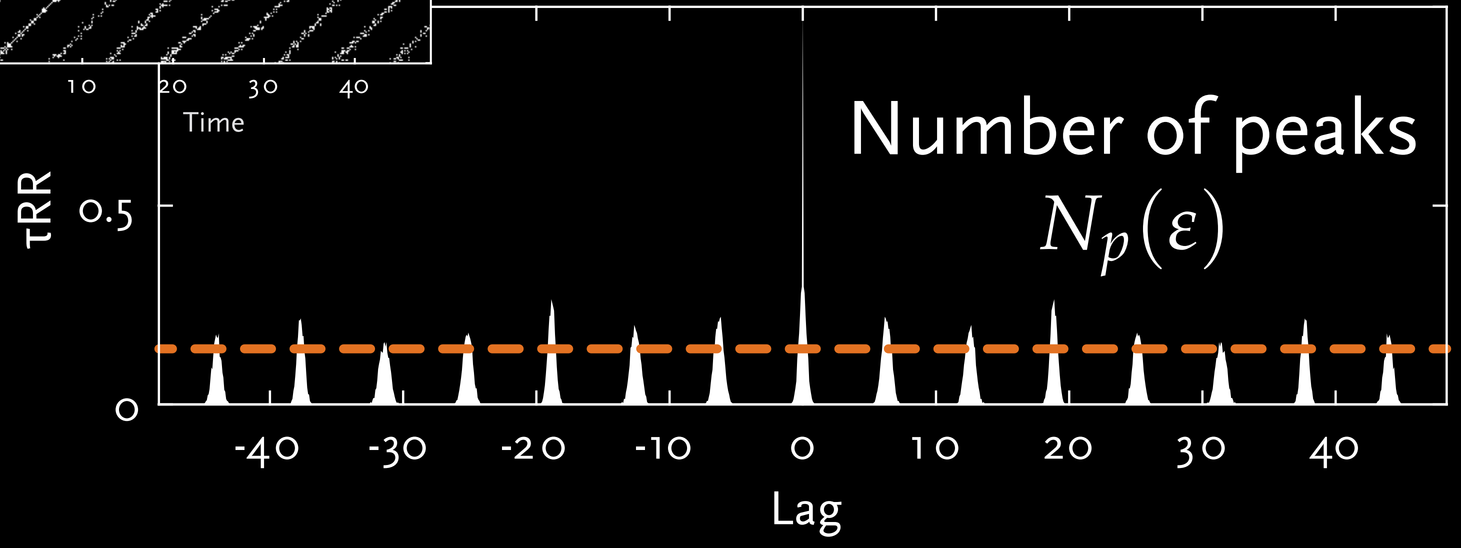
Recurrence Threshold: Optimizing Diagonals





Number of neighbours

$$N_n(\varepsilon) = \frac{1}{N}\sum_{i,j} R_{i,j}(\varepsilon)$$



Number of peaks

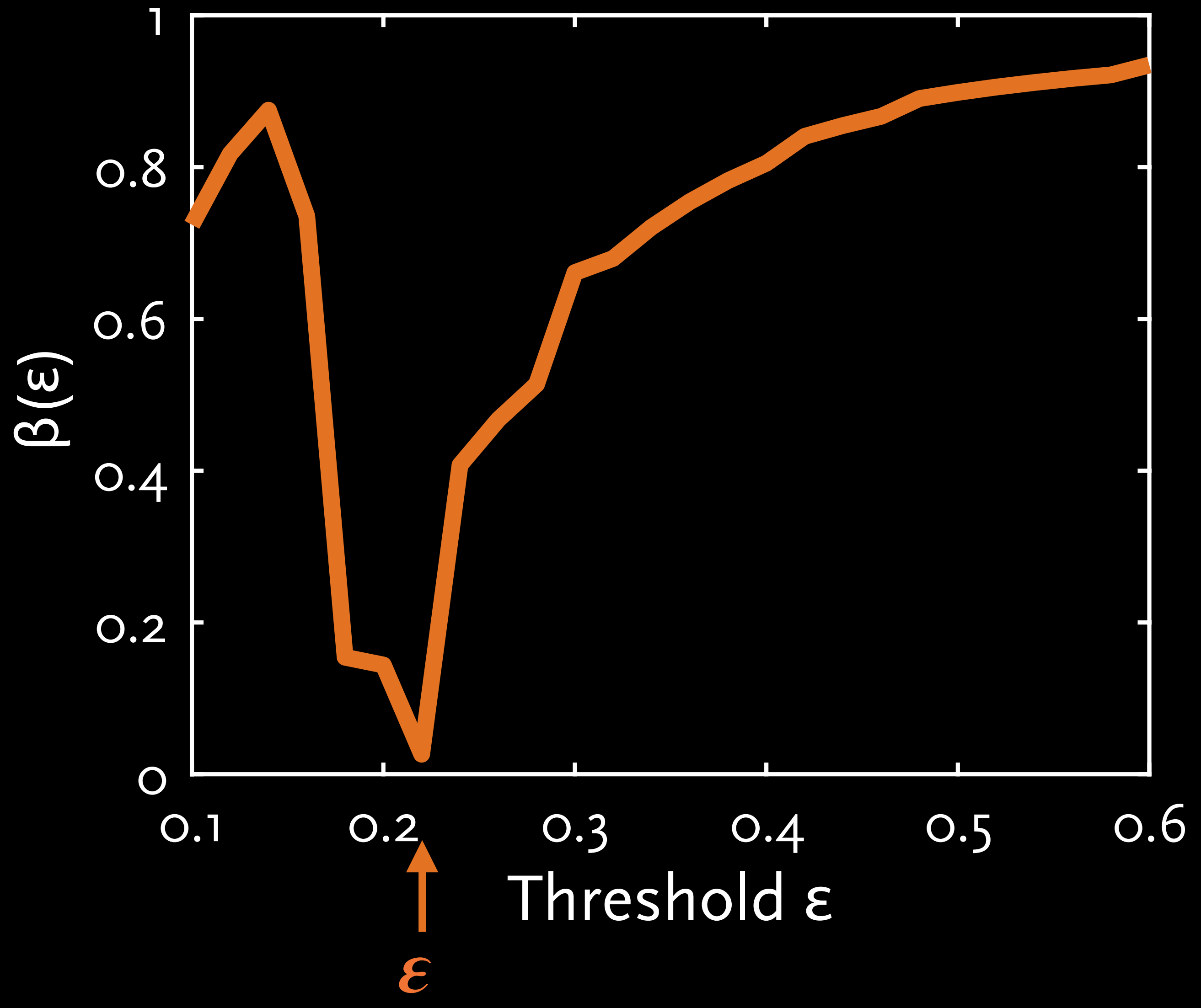
$N_p(\varepsilon)$

- Minimizing the fragmentation and thickness of the diagonal lines (for cyclical signals)
- N_p maximal
- N_n close to N_p

$$\beta(\varepsilon) = \frac{|N_n(\varepsilon) - N_p(\varepsilon)|}{N_n(\varepsilon)}$$

Matassini et al, Phys Rev E 65, 2002

Recurrence Threshold: Optimizing Diagonals



- Minimizing the fragmentation and thickness of the diagonal lines (for cyclical signals)
- N_p maximal
- N_n close to N_p

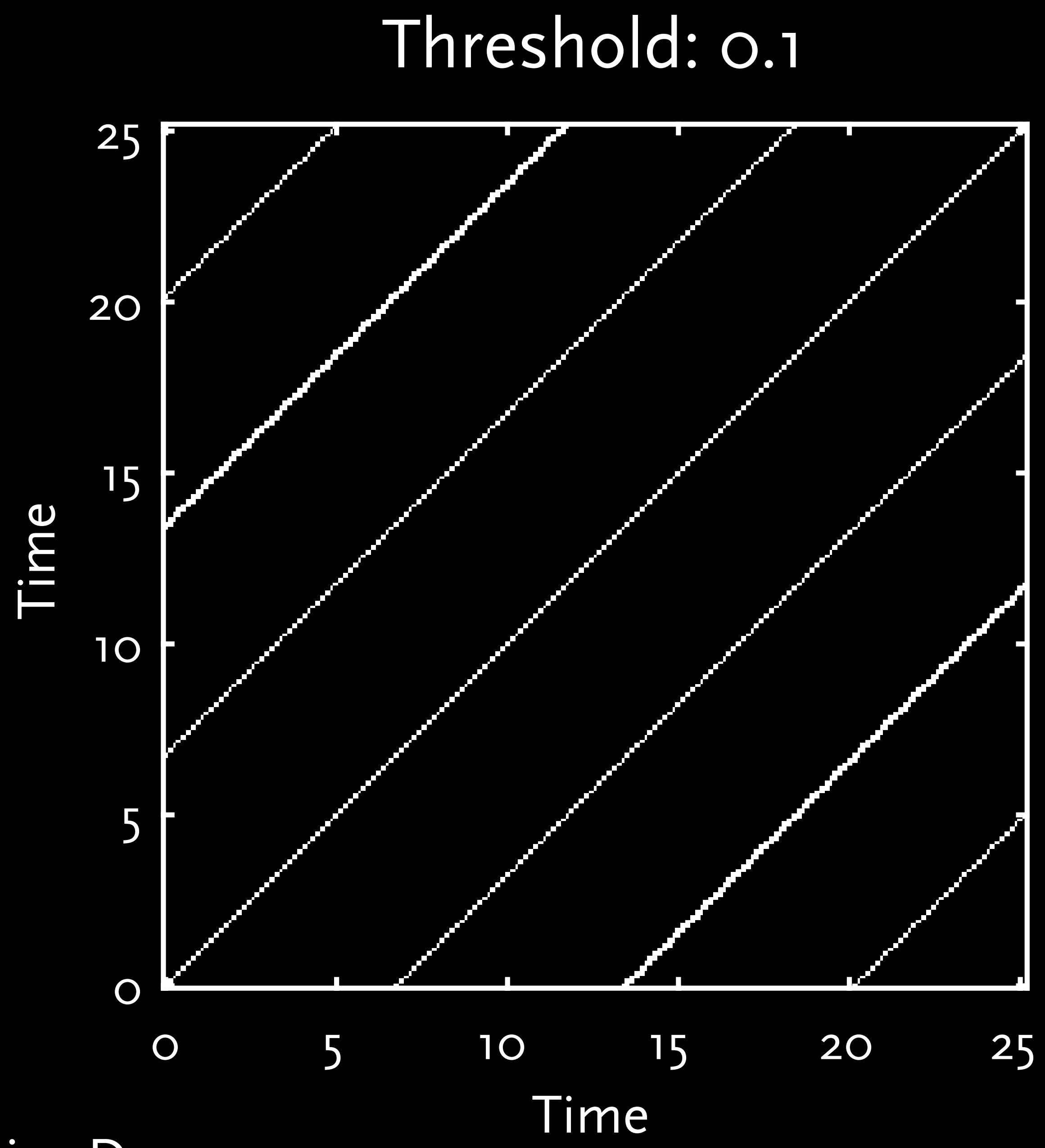
$$\beta(\varepsilon) = \frac{|N_n(\varepsilon) - N_p(\varepsilon)|}{N_n(\varepsilon)}$$

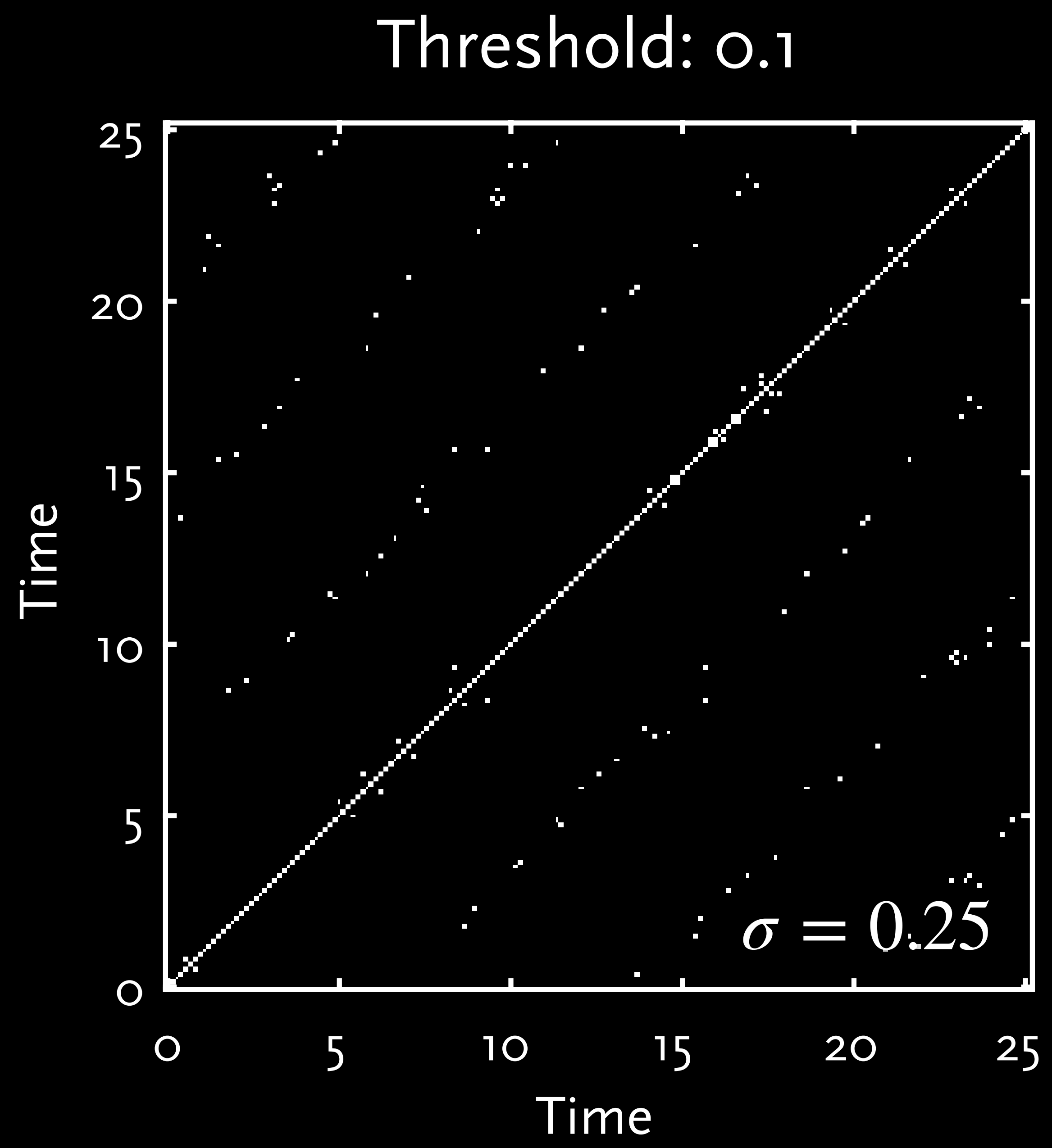
$$\Rightarrow \varepsilon = \arg\min_{\varepsilon} \beta(\varepsilon)$$

Matassini et al, Phys Rev E 65, 2002

Recurrence Threshold: Observational Noise

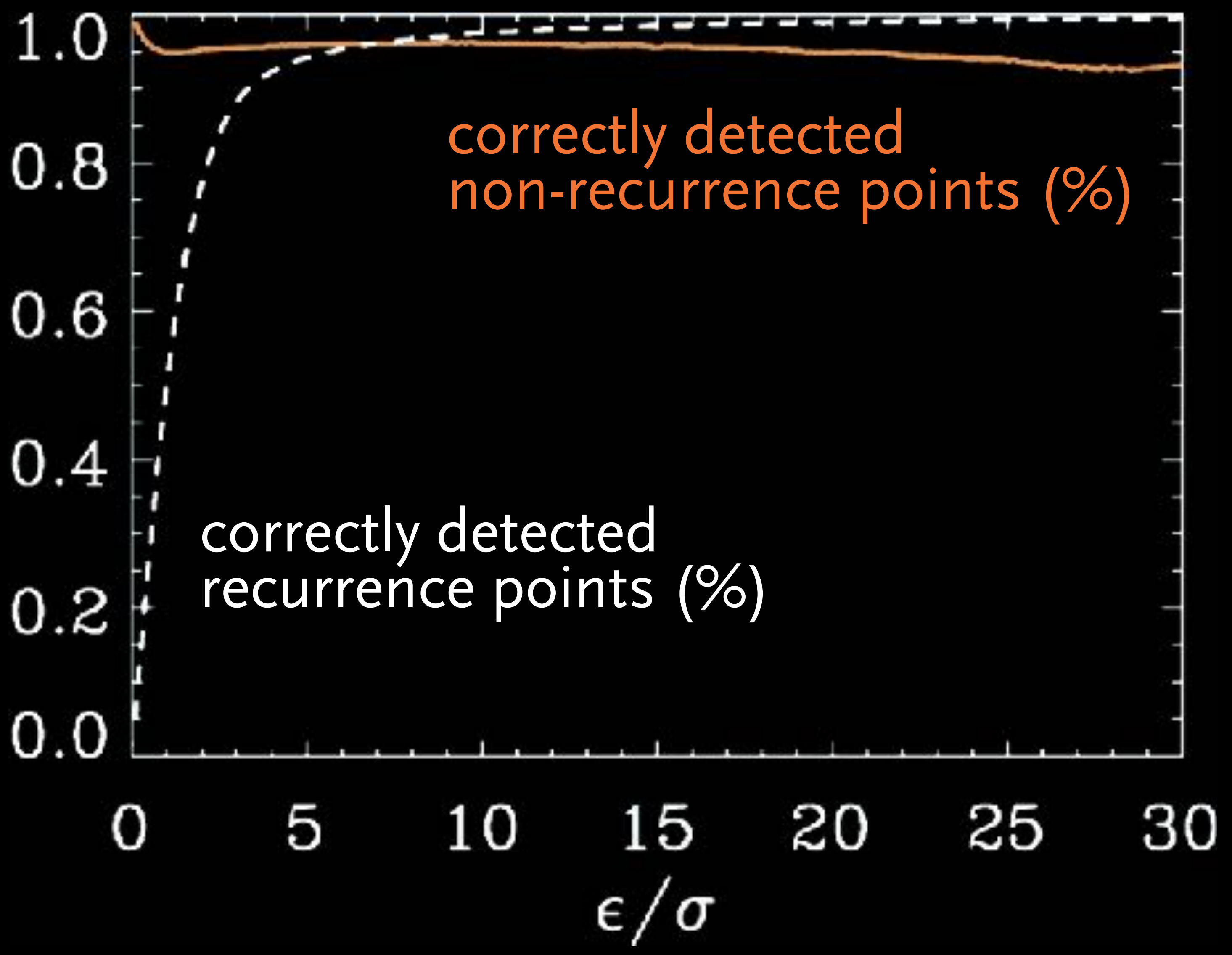
$$x(t) = \hat{x}(t) + \sigma\xi$$





Thiel et al, Physica D 171, 2002

Recurrence Threshold: Observational Noise

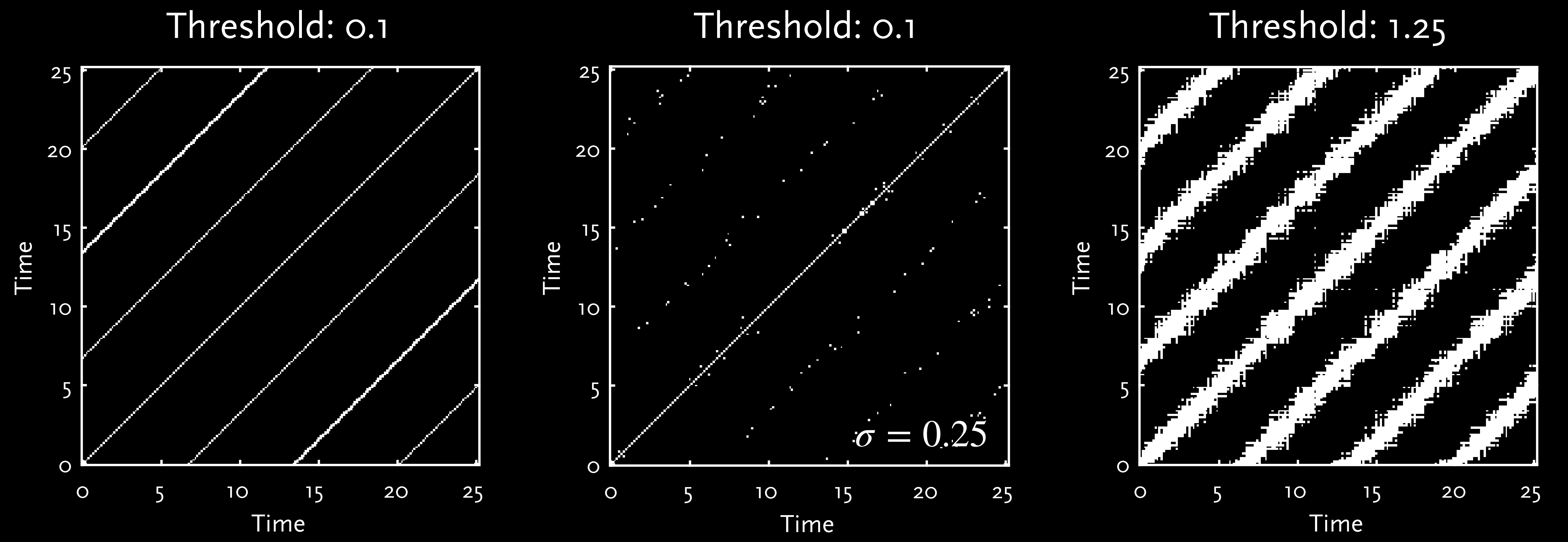


- Observational noise σ

➡ threshold $\varepsilon > 5\sigma$

Thiel et al, Physica D 171, 2002

Recurrence Threshold: Observational Noise



Thiel et al, Physica D 171, 2002

Recurrence Threshold: Topological Similarity

Vega et al., Phys A 445, 2016

- Similarity in number and size of network modules

- Number of modules

$$C\left(\mathbf{R}(\varepsilon - \Delta\varepsilon)\right) = C\left(\mathbf{R}(\varepsilon)\right) = \\ = C\left(\mathbf{R}(\varepsilon + \Delta\varepsilon)\right) > 1$$

- Size of module k

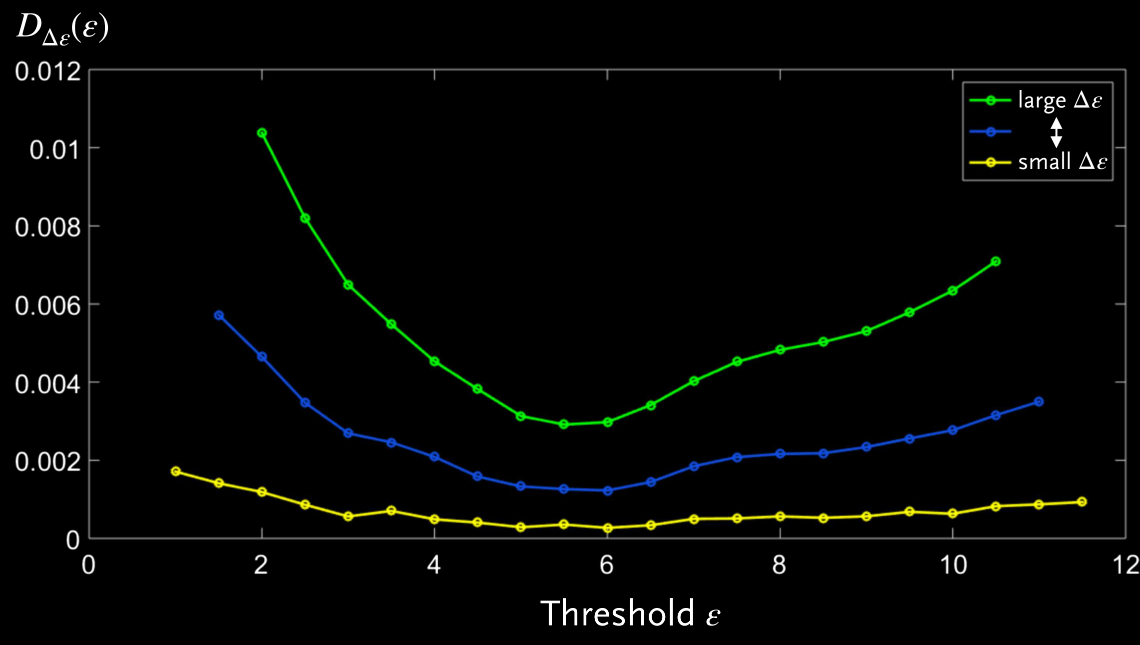
$$\left|C_k\left(\mathbf{R}(\varepsilon + \Delta\varepsilon)\right)\right| - \left|C_k\left(\mathbf{R}(\varepsilon)\right)\right| < \xi$$

Andreadis et al., Chaos 30, 2020

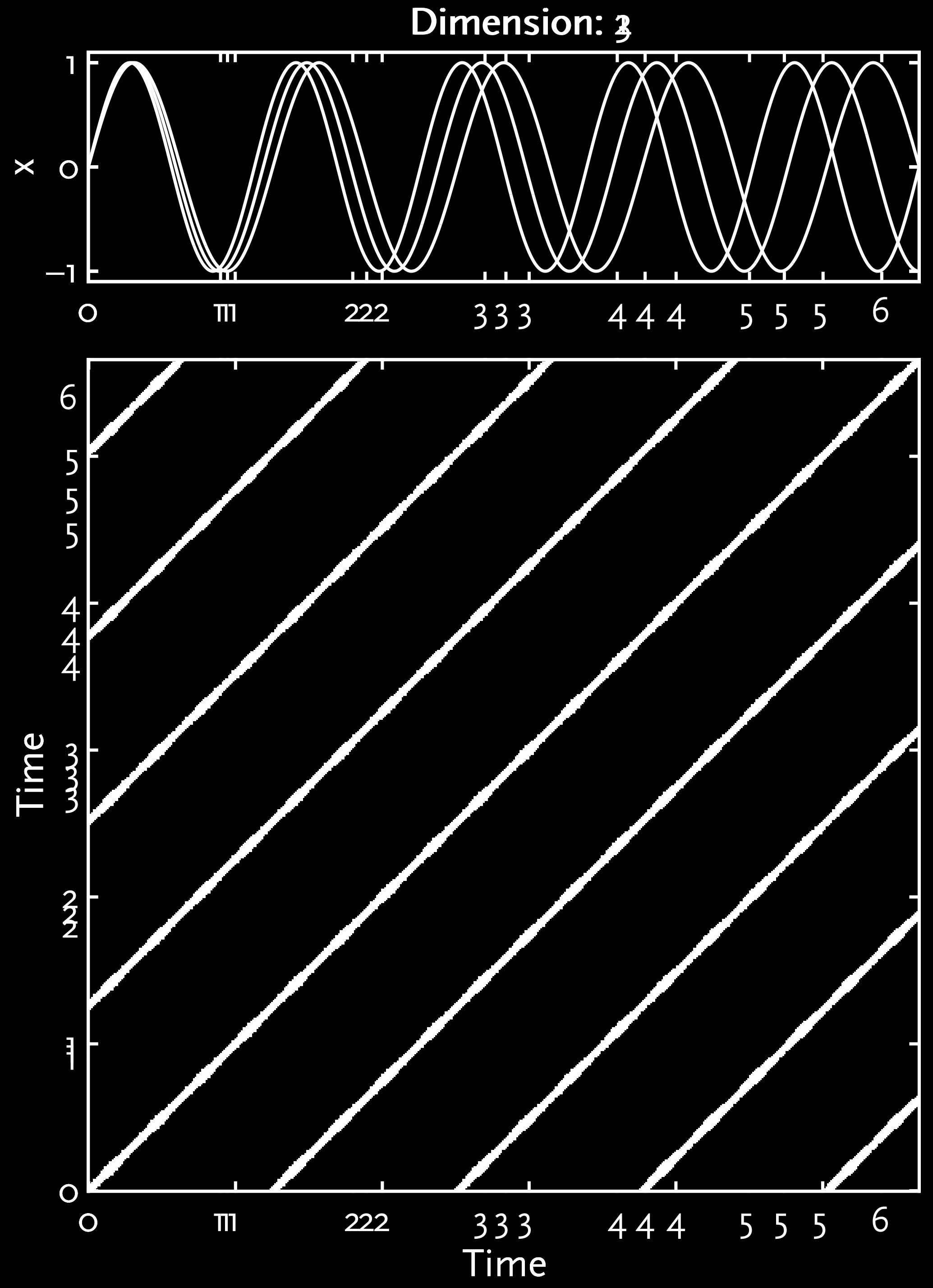
- Minimise deviation of overall recurrences

- Compare $\mathbf{R}(\varepsilon)$ with $\mathbf{R}(\varepsilon + \Delta\varepsilon)$:

$$D_{\Delta\varepsilon}(\varepsilon) = \left|\operatorname{dist}\left(\mathbf{R}(\varepsilon + \Delta\varepsilon), \mathbf{R}(\varepsilon - \Delta\varepsilon)\right)\right|$$

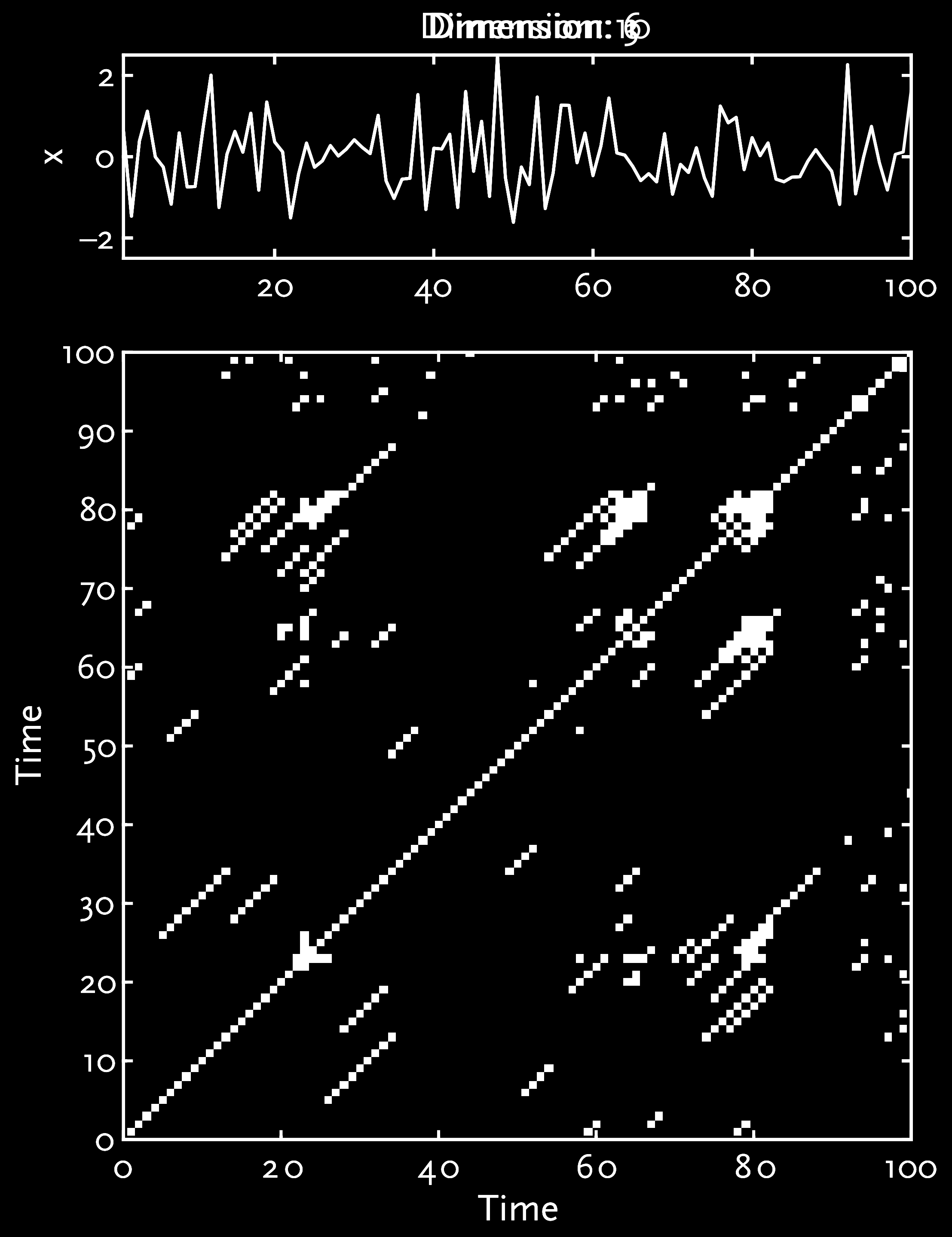


Influence of Embedding Dimension

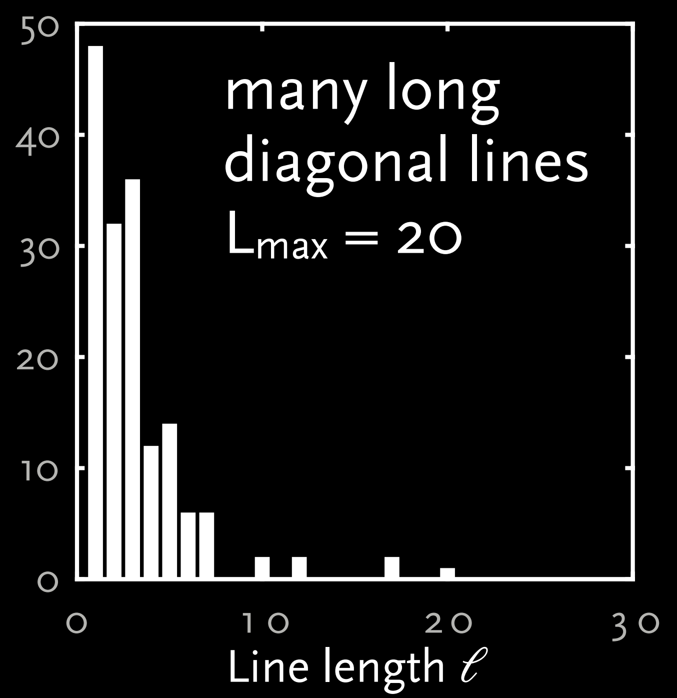


- Too low embedding dimension:
  - attractor (phase space trajectory) is not enough unfolded
  - trajectory segments cross eachother or even evolve in opposite direction
- Too high embedding dimension:
  - over-embedding usually not a problem (within limits)

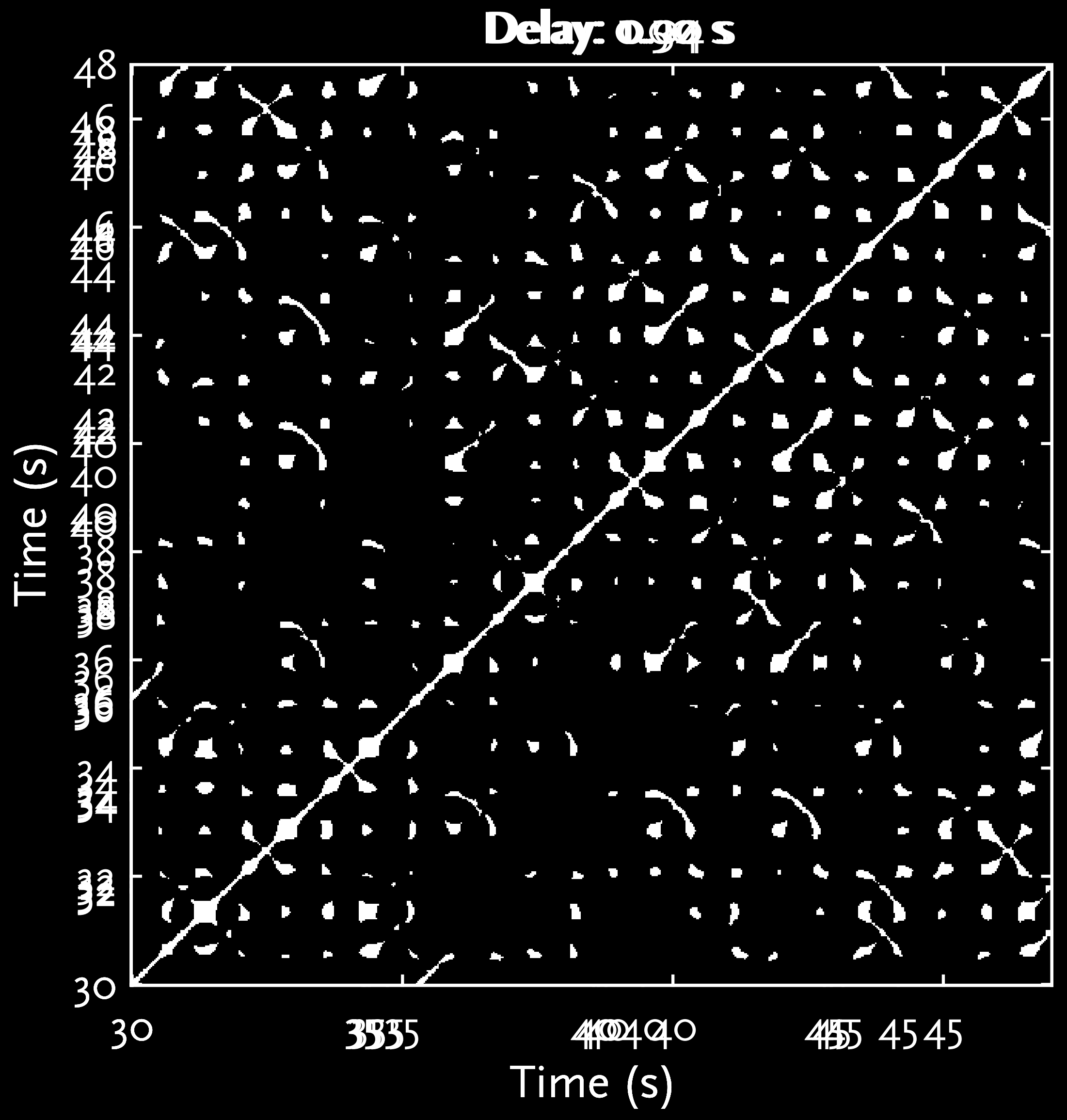
Influence of Embedding Dimension



- Embedding can cause spurious correlations
- Artificially increase of diagonal line lengths (when using adaptive threshold)



Influence of Embedding Delay



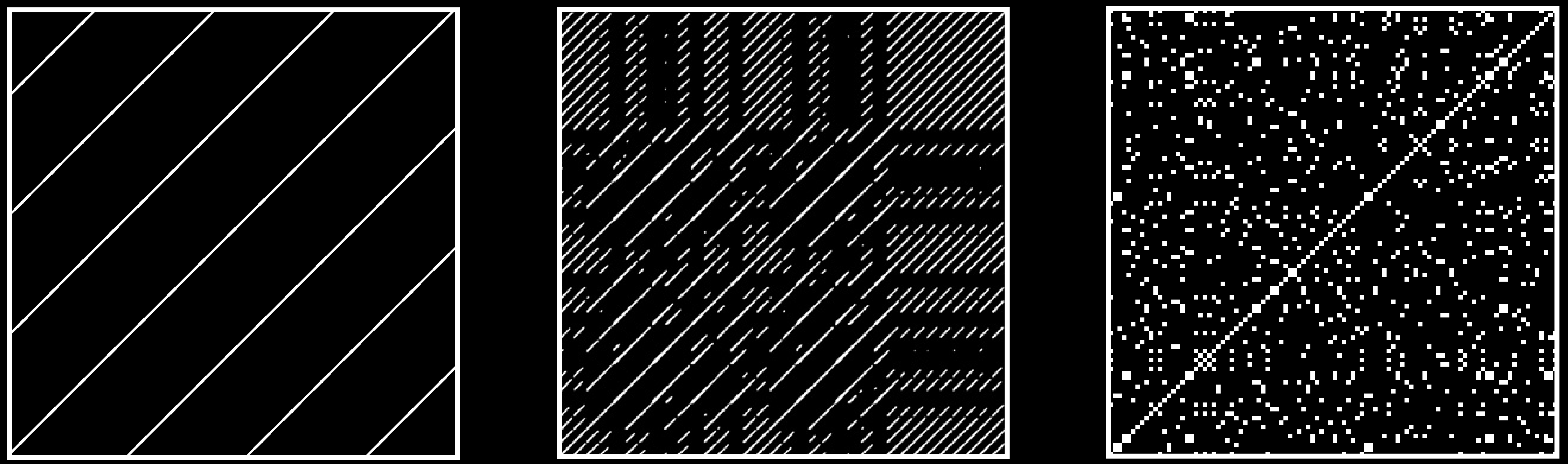
Lorenz system

- Embedding delay necessary to unfold attractor
- Can destroy line structures
- Optimal delay:
  mainly non-interrupted line structures

Recurrence Quantification Analysis

Recurrence Quantification Analysis

- 1992, 1994, Webber & Zbilut:
  Quantify the visually apparent differences in the RP



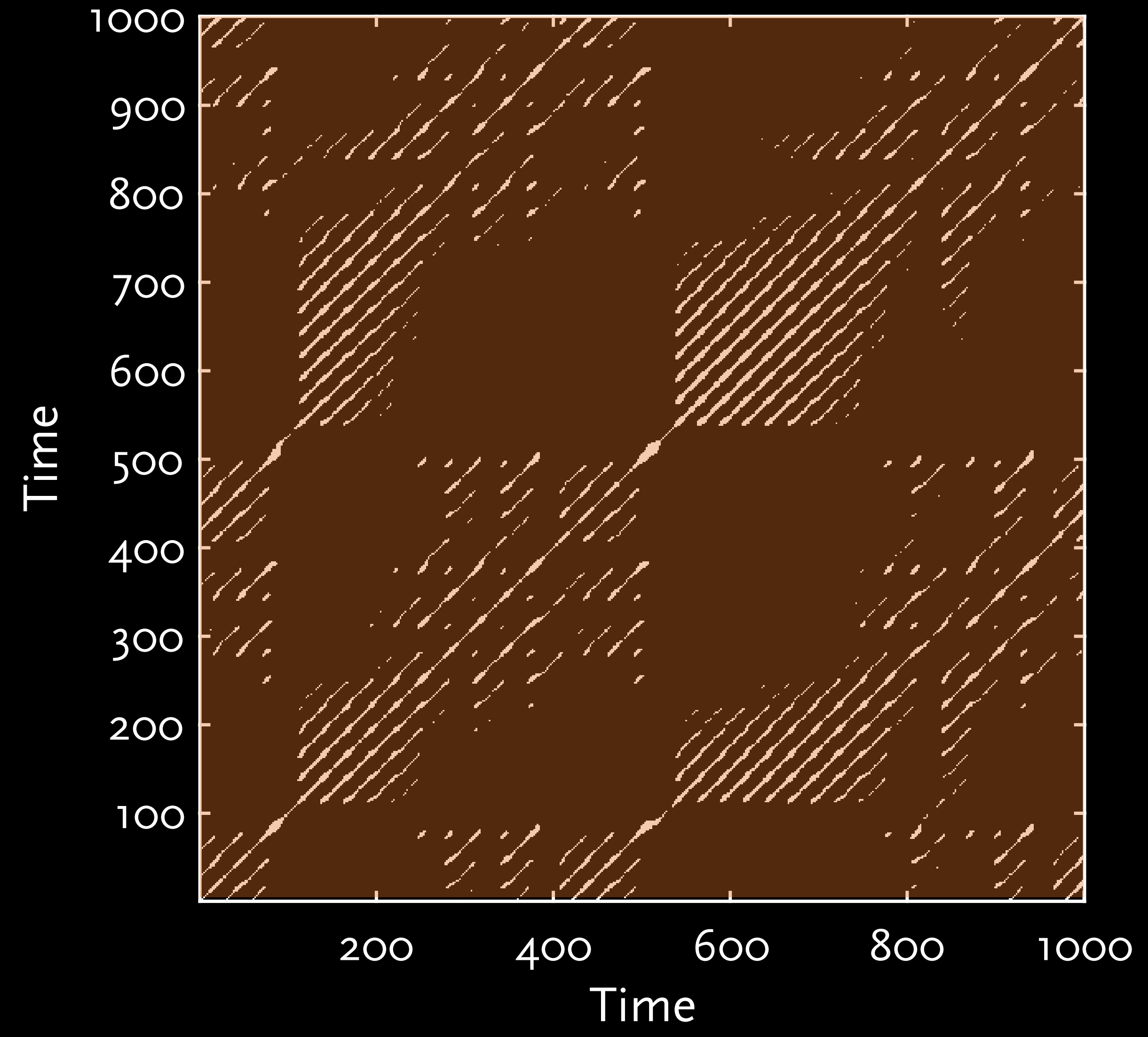
Recurrence Rate

- Density of recurrence points in whole RP

$$RR(\varepsilon, m) = \frac{1}{N_m(N_m - 1)} \sum_{i=1}^{N_m} \sum_{\substack{j=1 \\ j \neq i}}^{N_m} R_{i,j}(\varepsilon, m)$$

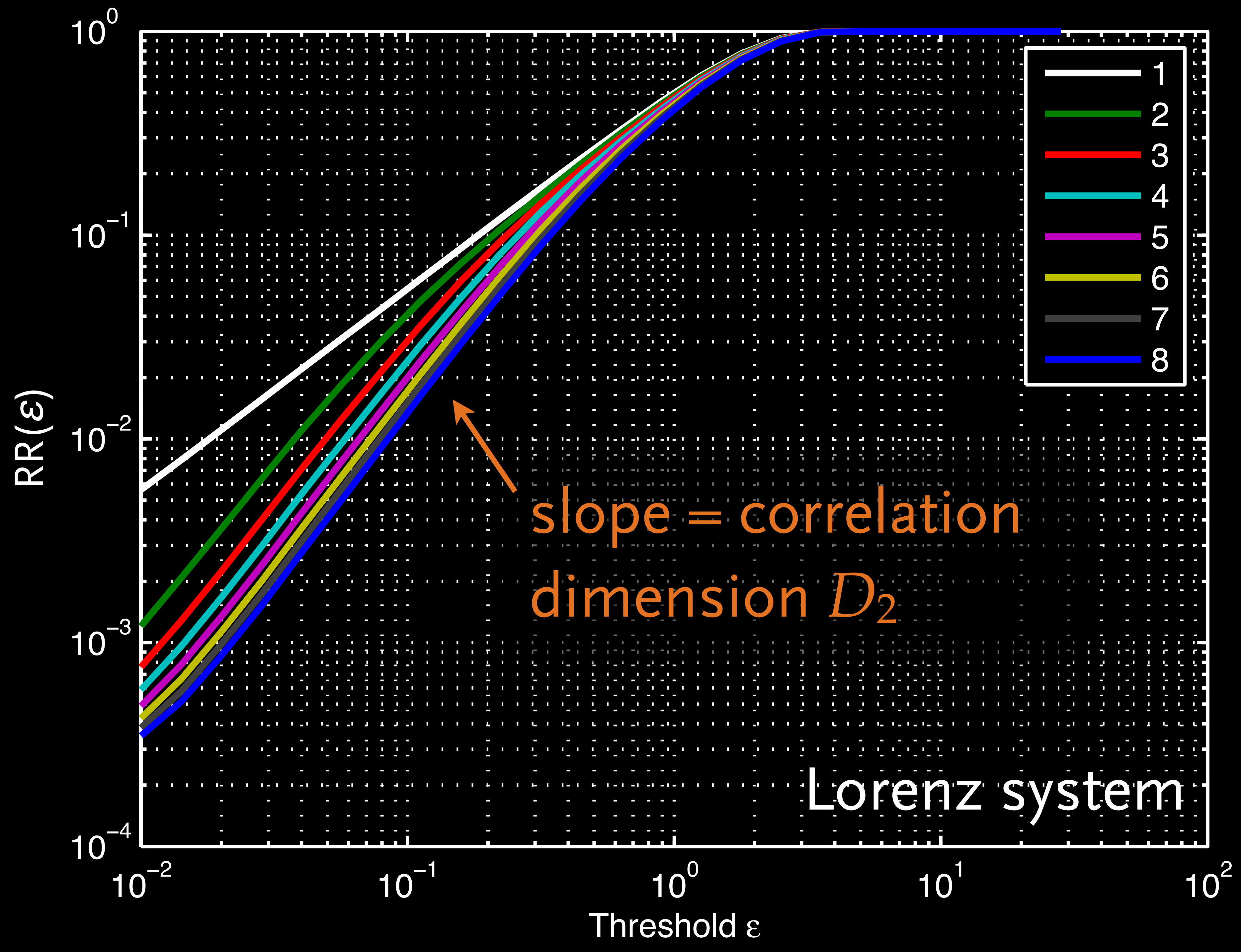
Probability that the system will return to any (randomly selected) former state

- Correlation sum



Grassberger & Procaccia, Phys Rev Lett 50, 1983

Recurrence Rate



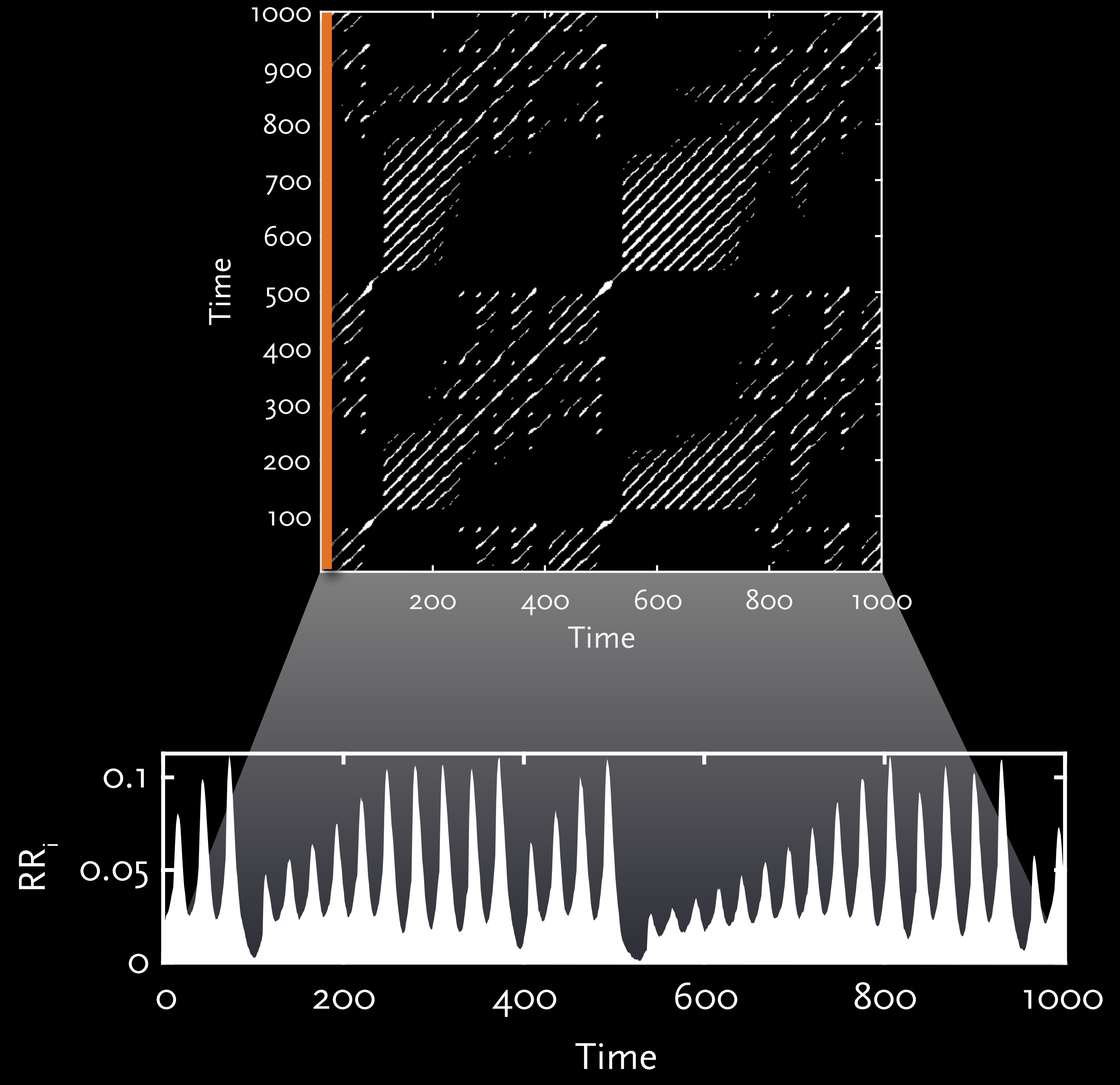
- Useful when testing scaling behaviour
- RR alone closely related to variance of the signal

Column-wise Recurrence Rate

- Density of recurrence points per column

$$RR_i(\varepsilon, m) = \frac{1}{N_m - 1} \sum_{\substack{j=1 \\ j \neq i}}^{N_m} R_{i,j}(\varepsilon, m)$$

Probability of state at time point i



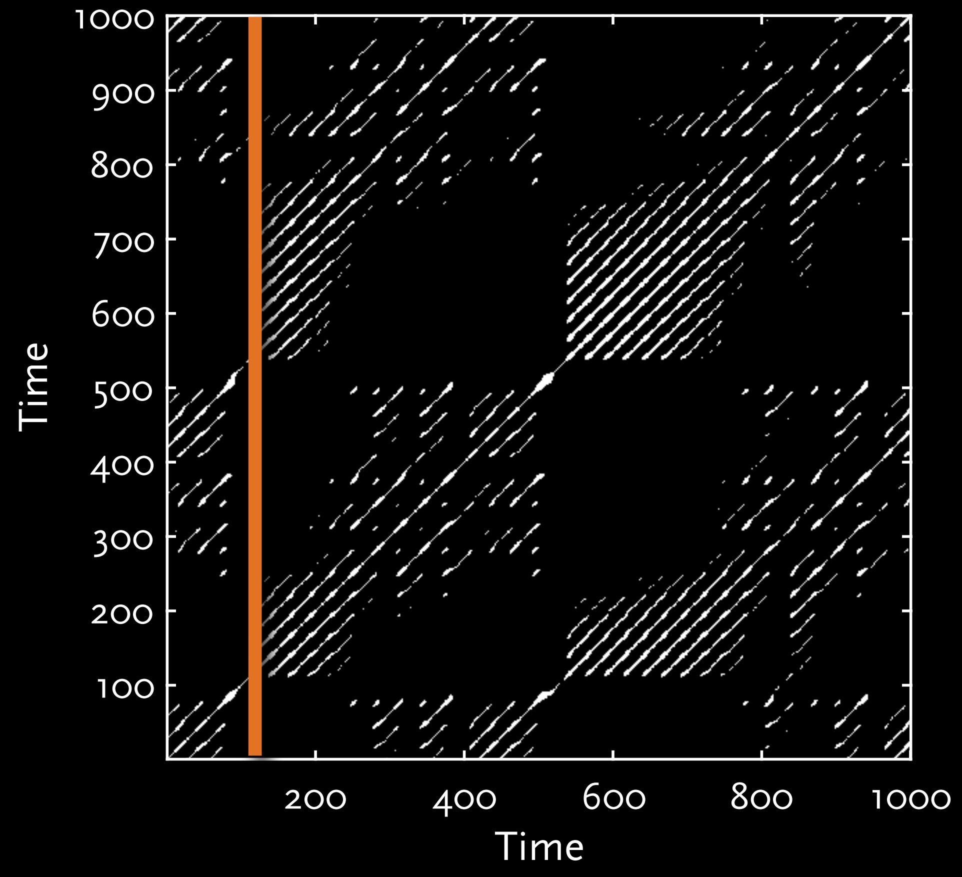
Column-wise Recurrence Rate

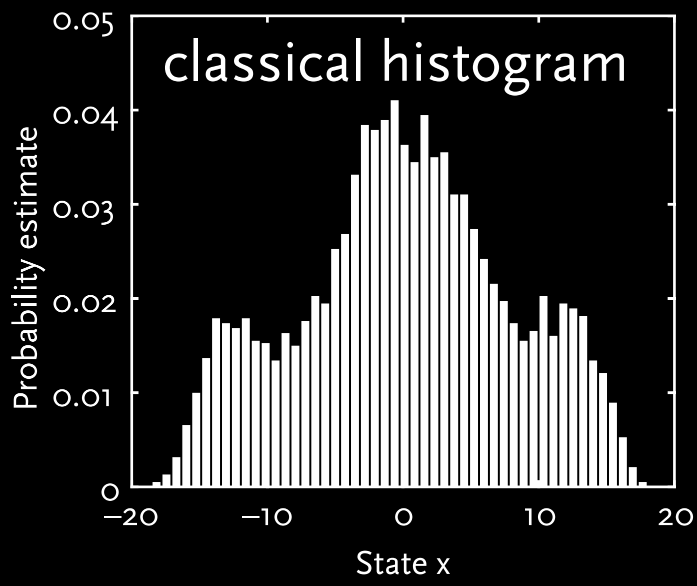
- Density of recurrence points per column

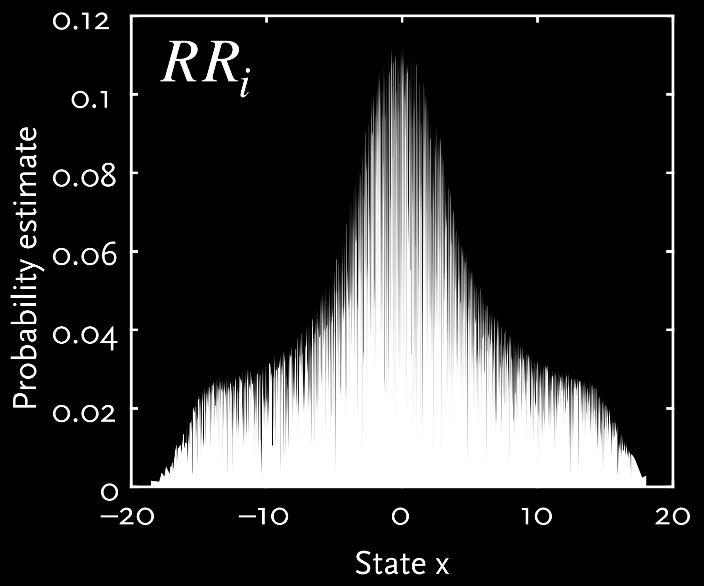
$$RR_i(\varepsilon, m) = \frac{1}{N_m - 1} \sum_{\substack{j=1 \\ j \neq i}}^{N_m} R_{i,j}(\varepsilon, m)$$

Probability of state at time point i

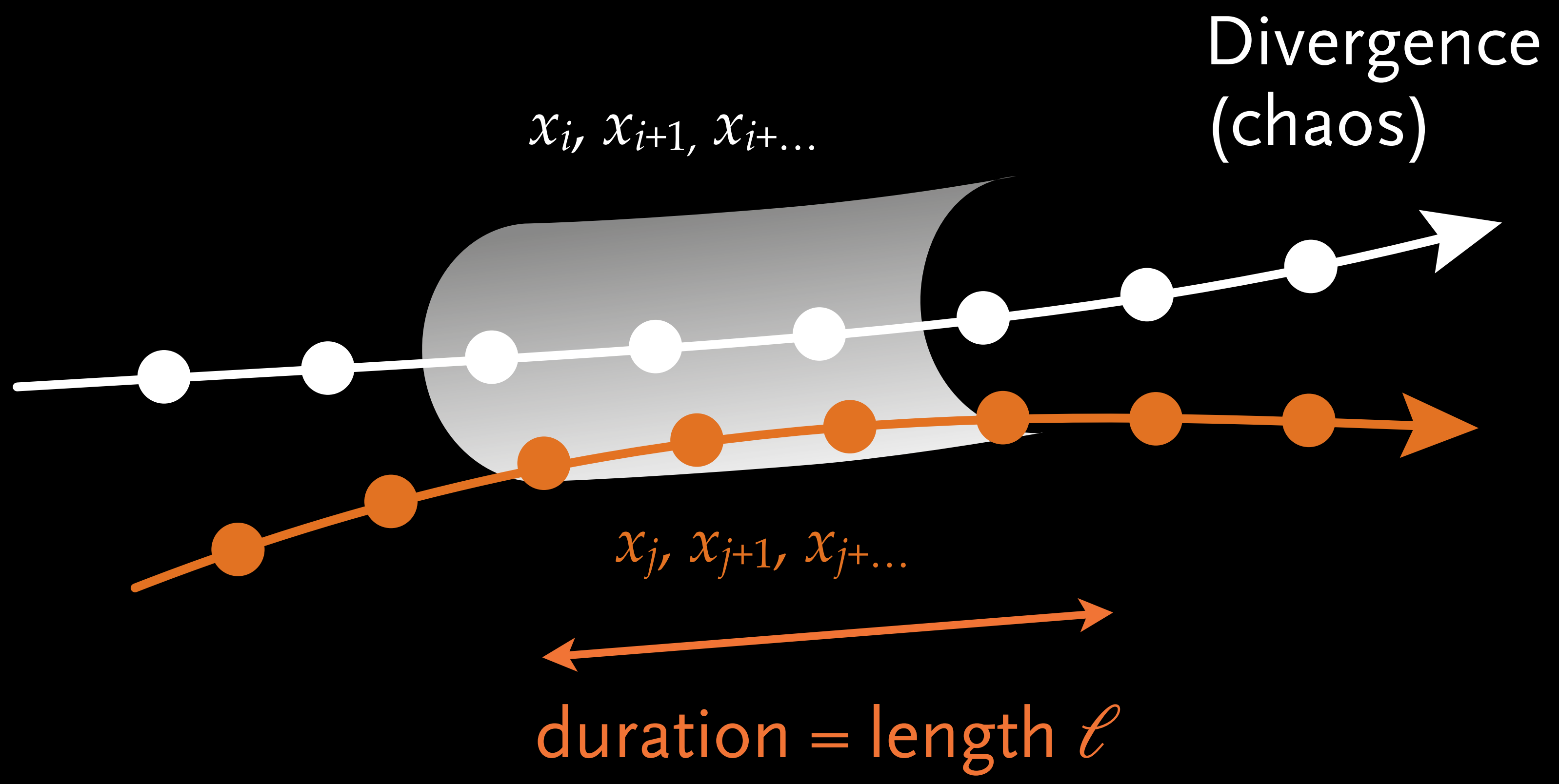
- Alternative probability estimator for many probability based methods, such as mutual information

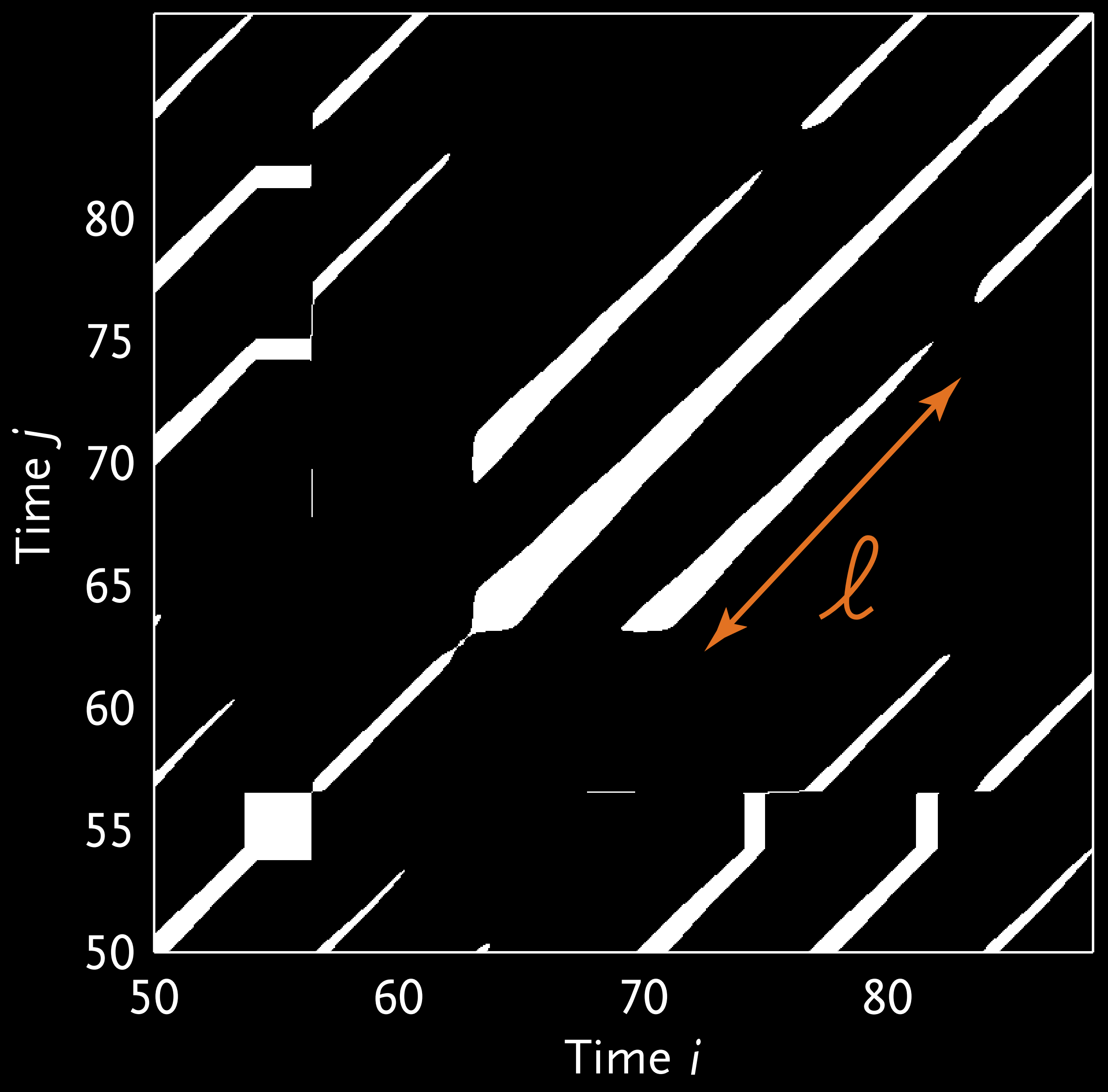




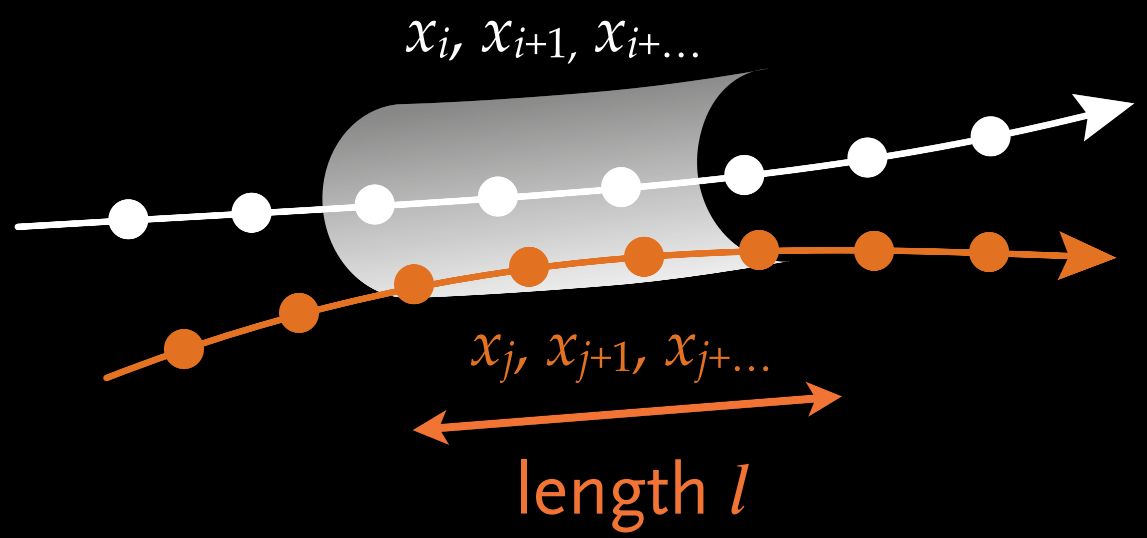


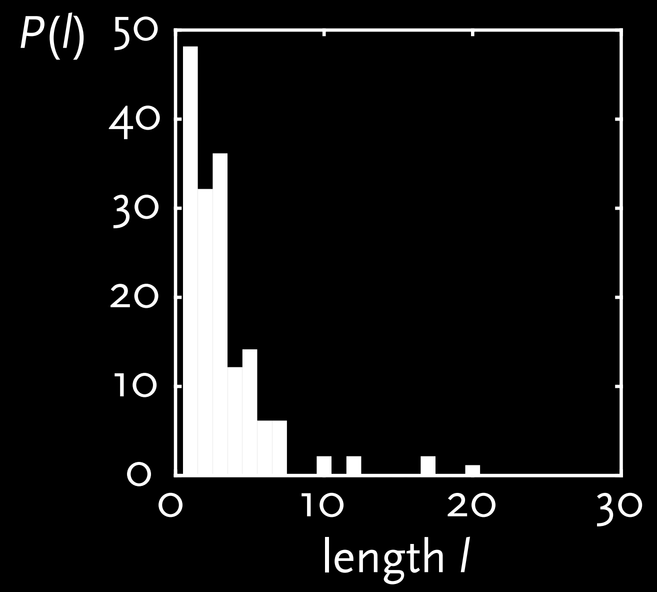
Diagonal Lines





Diagonal Line Entropy





- Shannon entropy of diagonal line length distribution

$$S = -\sum_{l=1}^{N} p(l) \ln p(l)$$

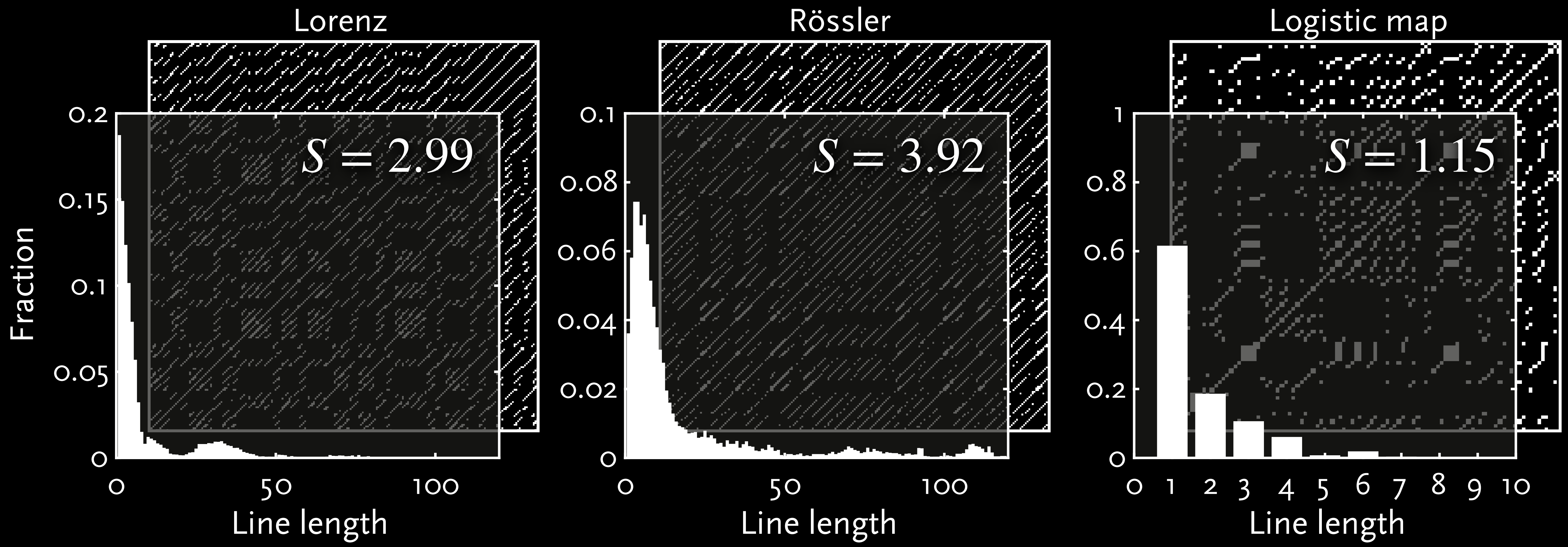
"... the more complex the deterministic structure of the recurrence plot, the larger the [entropy]."

Webber & Zbilut, J Appl Phys 76, 1994

Diagonal Line Entropy

- Entropy of line length distribution $p(l)$

$$S = -\sum_{l=1}^{N} p(l) \ln p(l)$$

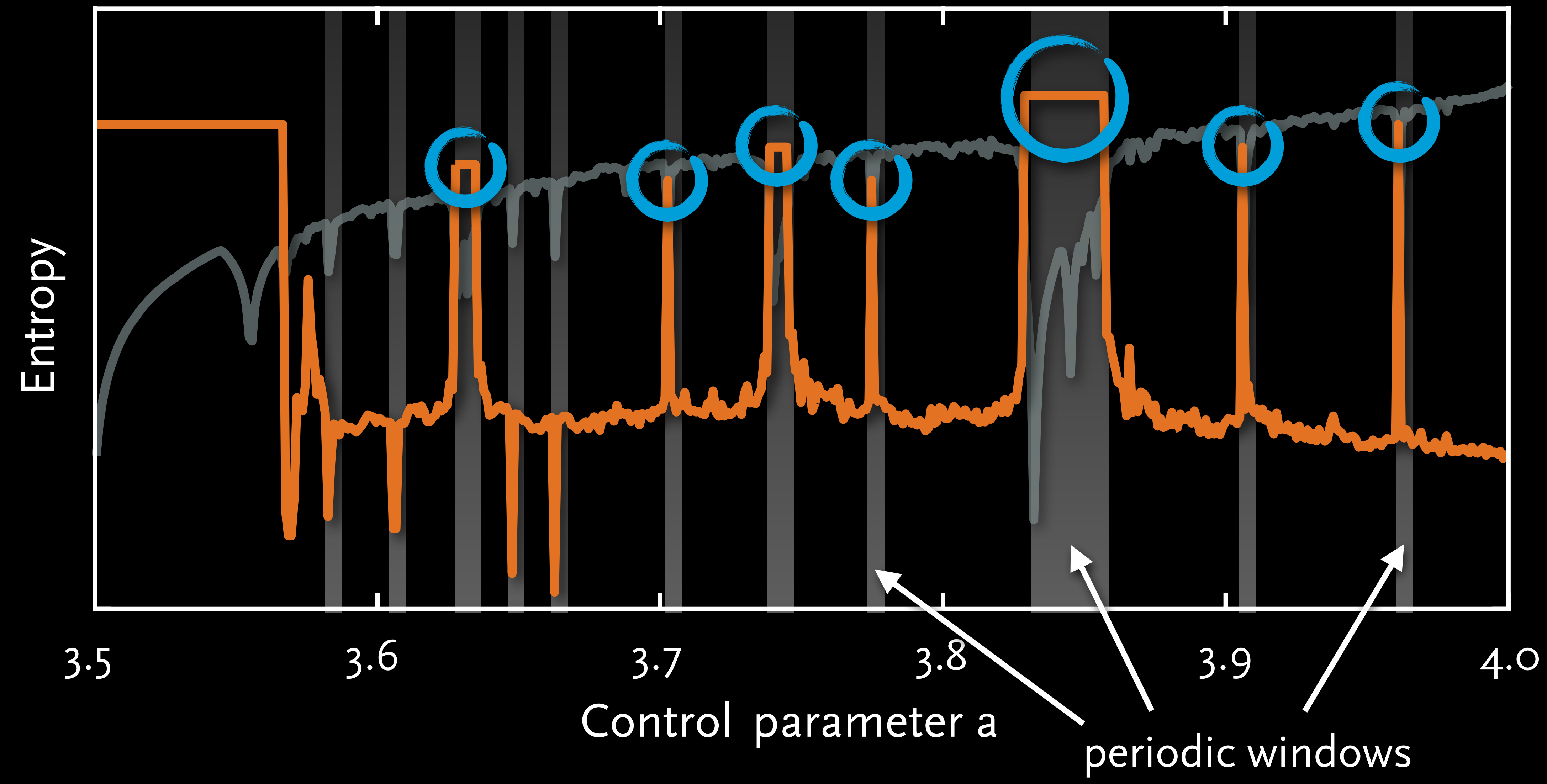


Diagonal Line Entropy

Larger entropy = more complex dynamics

Counterintuitive results?

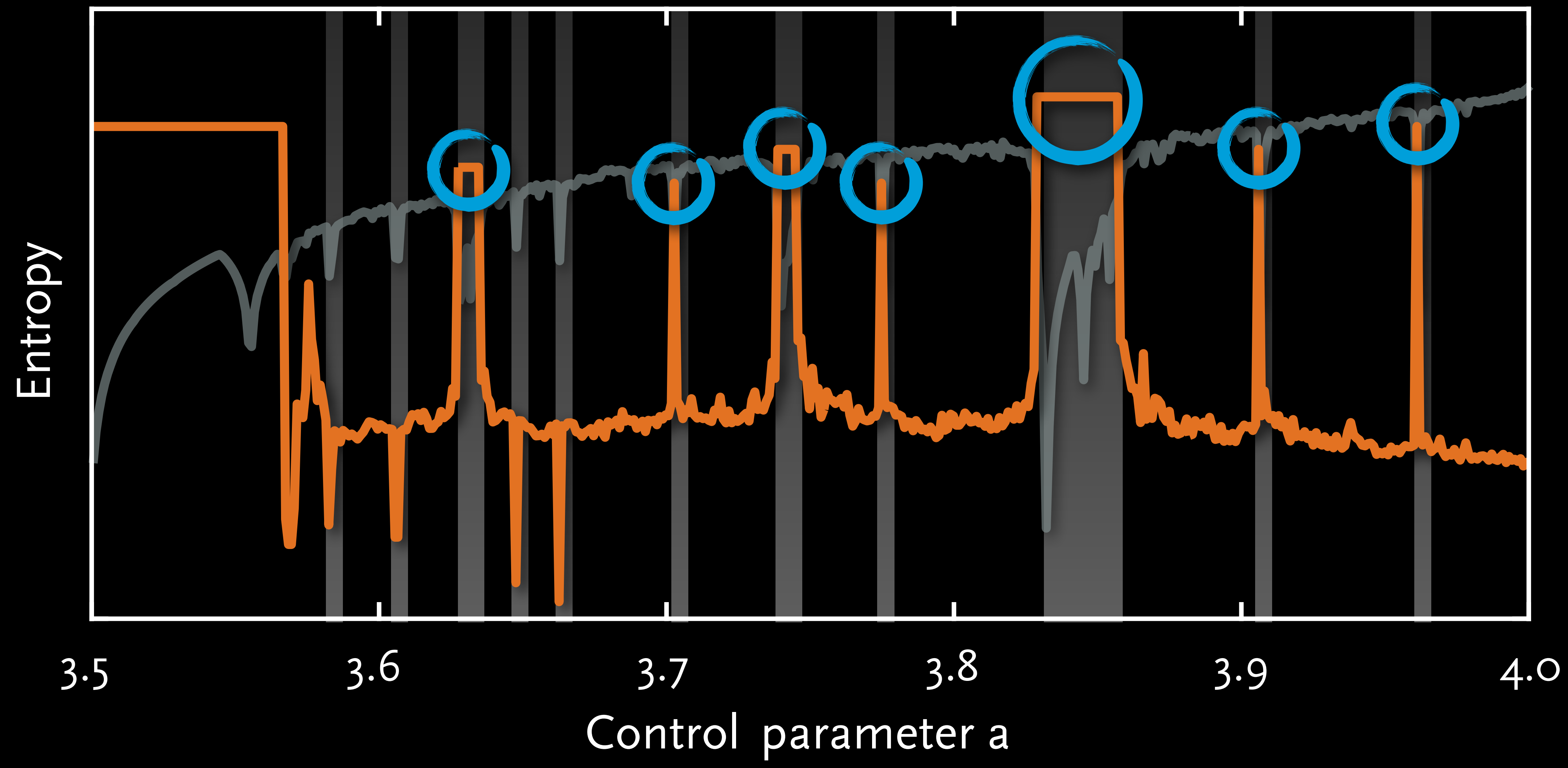
Large values for periodic dynamics



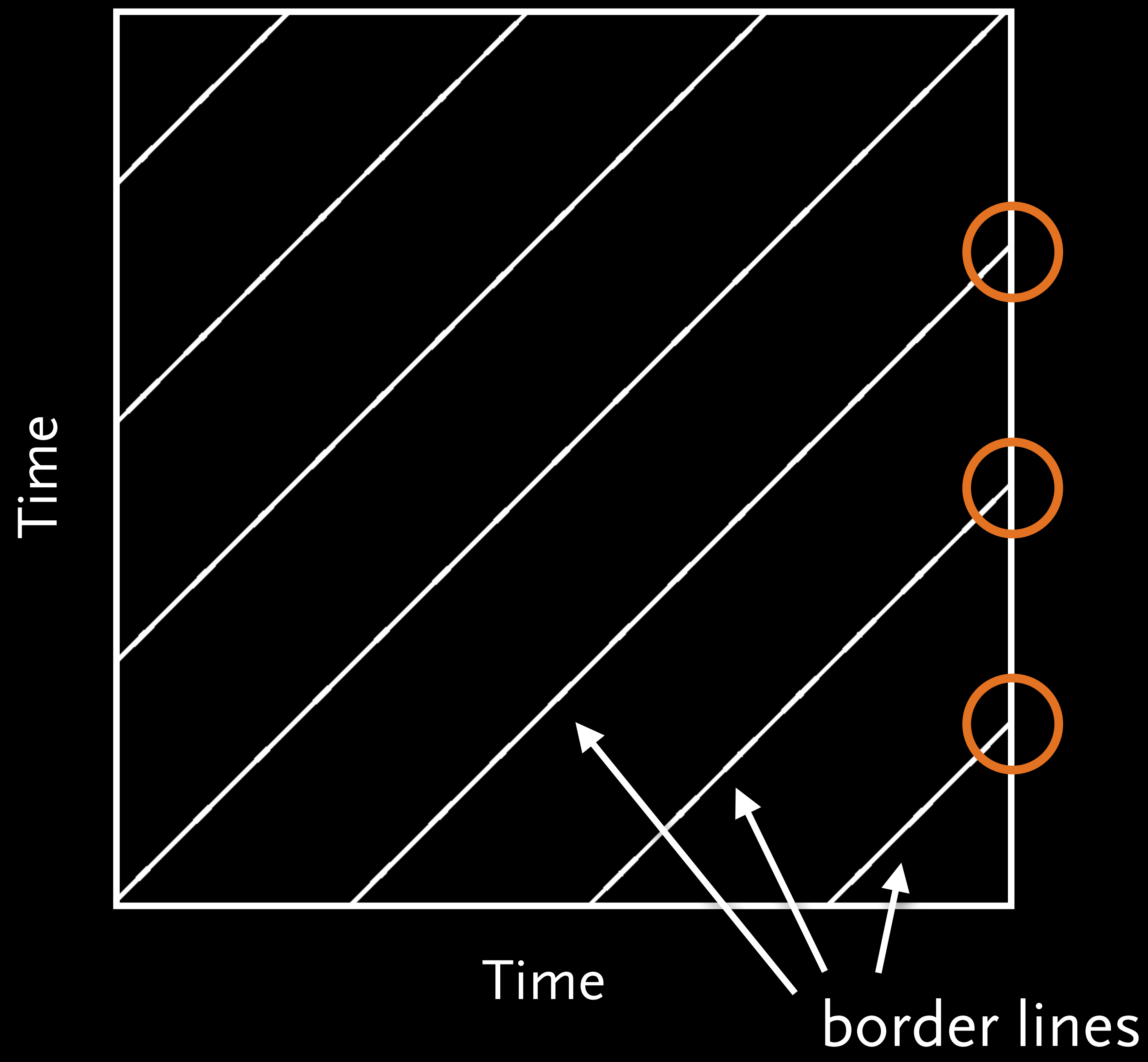
QuizZ

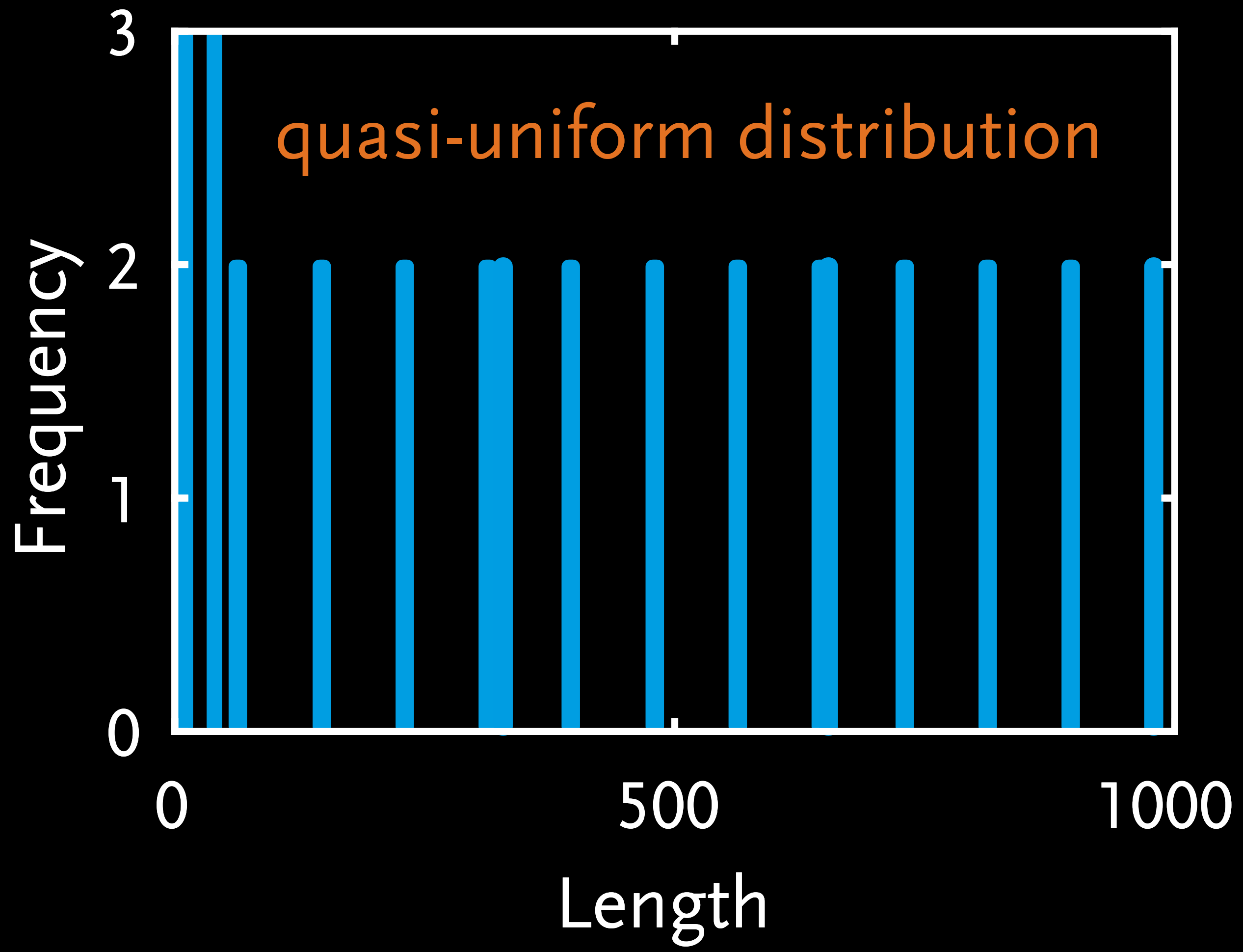
Quizz

- Why large entropy values for periodic regimes?

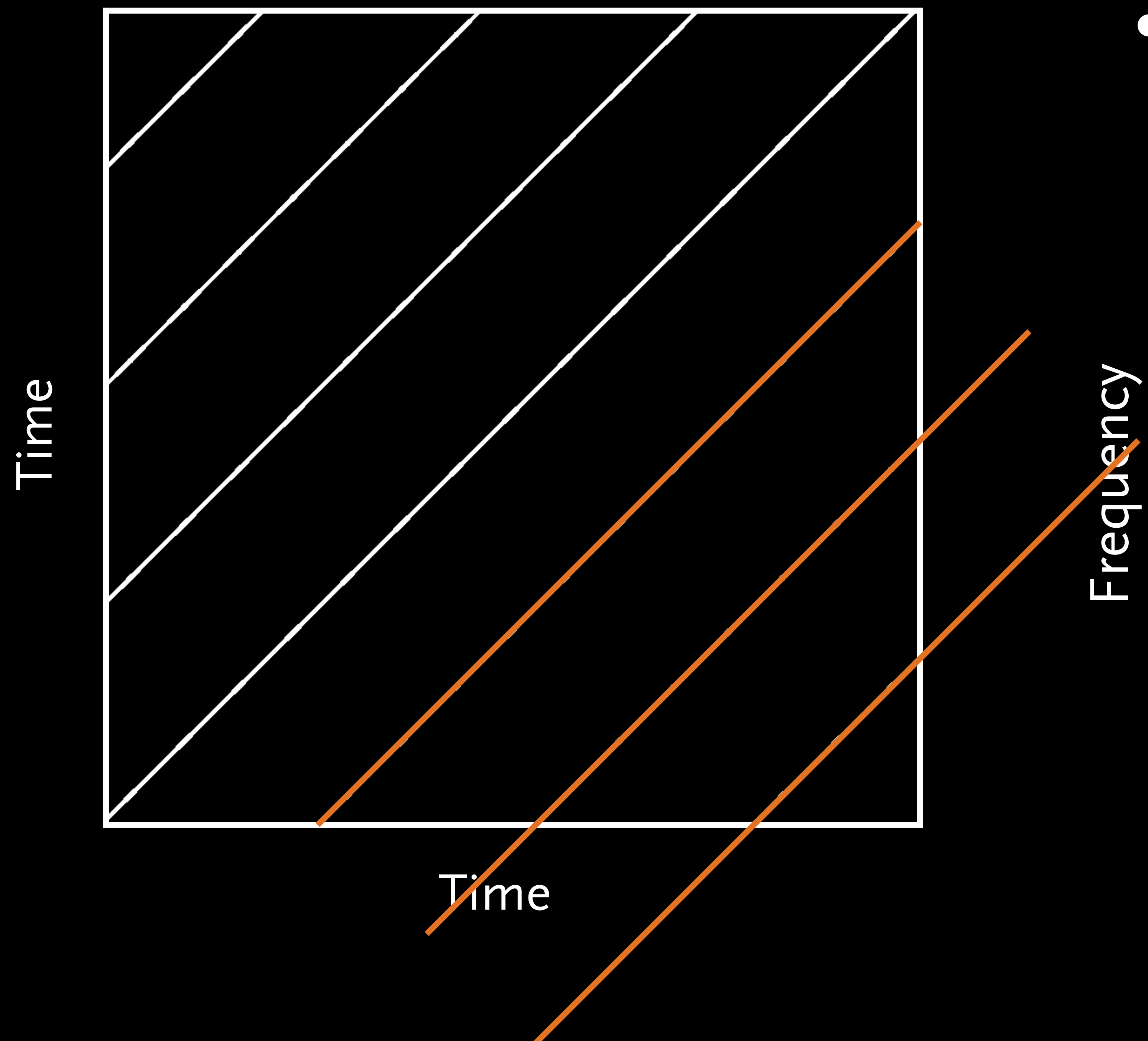


Diagonal Lines for Periodic Dynamics

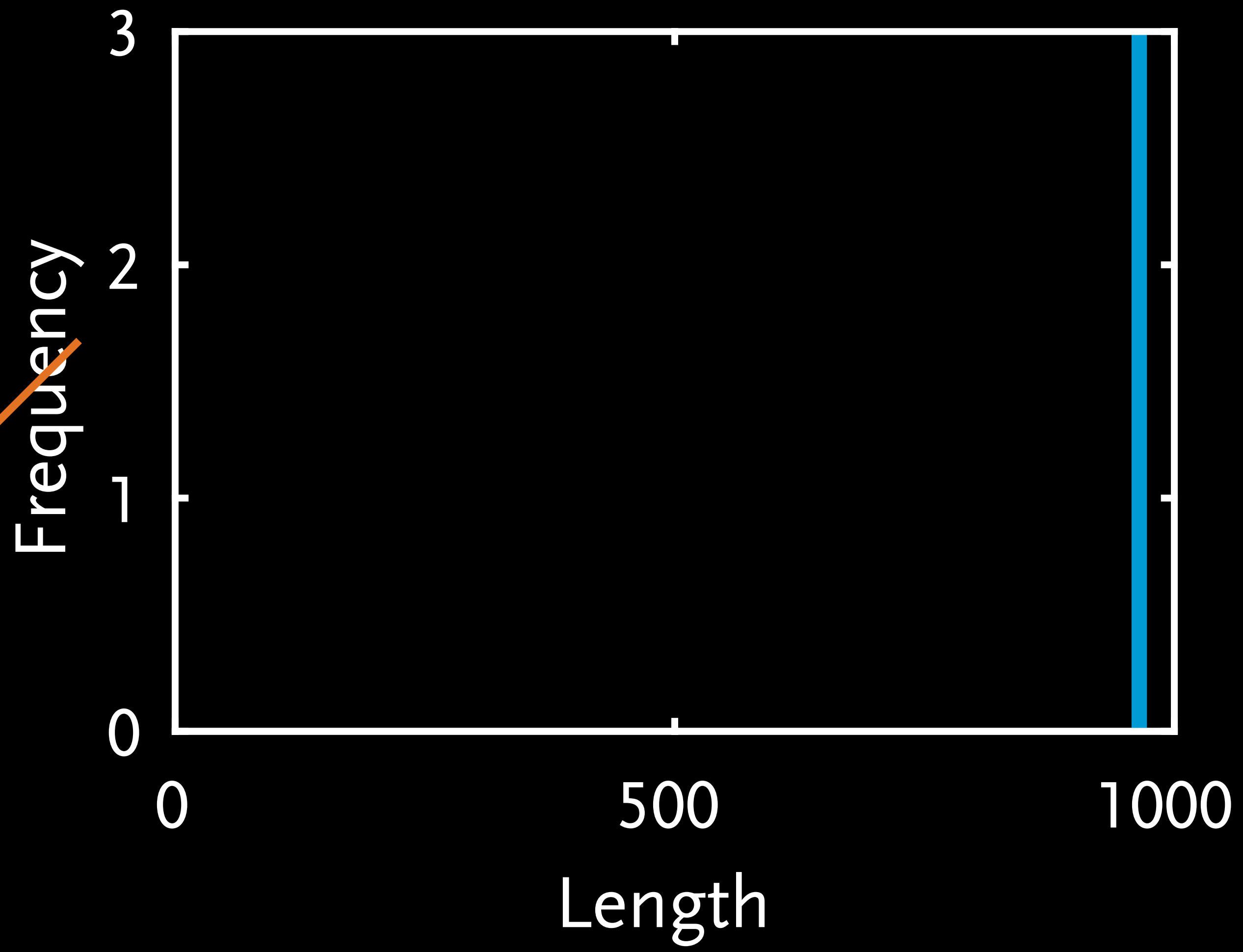




Correction of Diagonal Line Lengths

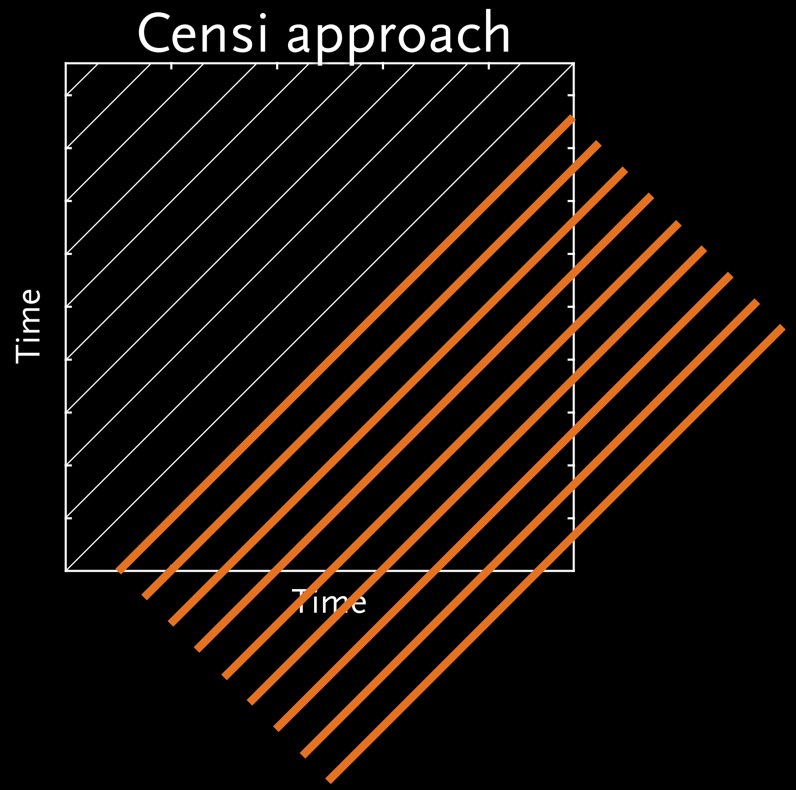


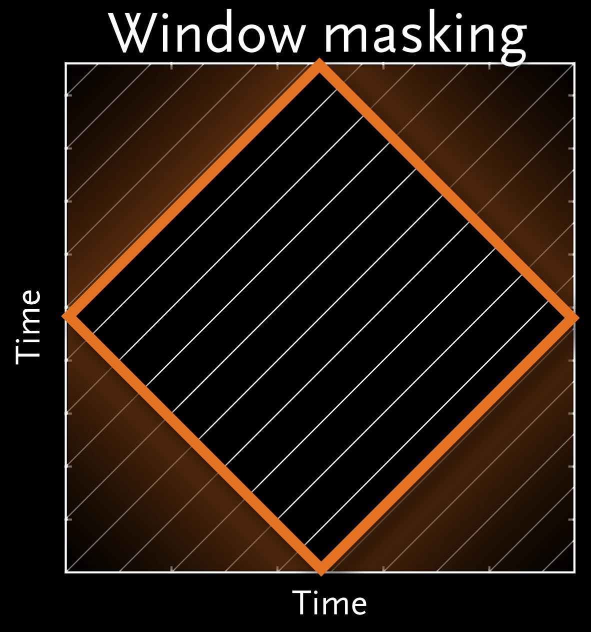
- Replace lengths of border lines by longest length

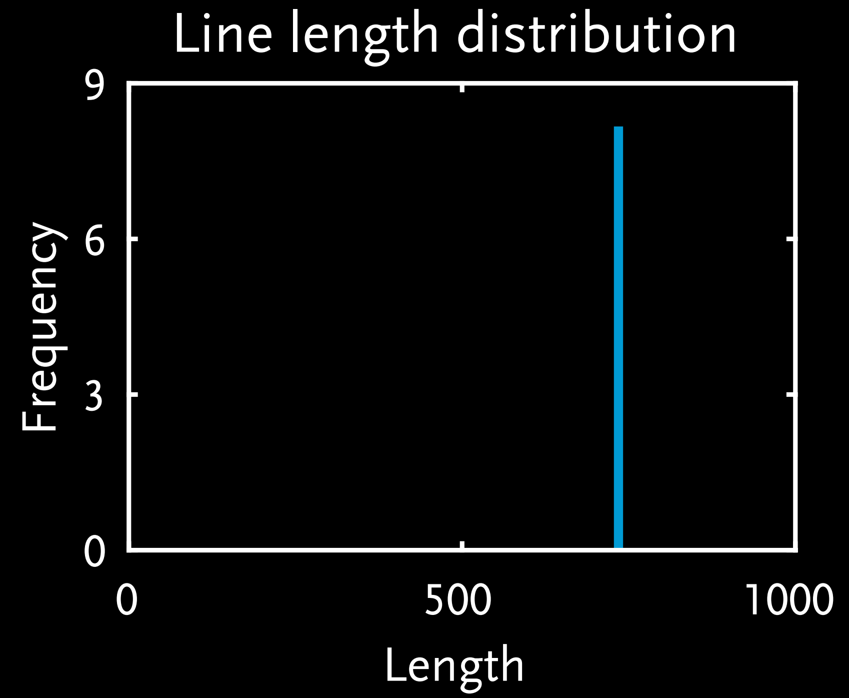


Correction of Diagonal Line Lengths

1. Discard border diagonals (dibo)
2. Keep only longest border diagonal (kelo)
3. Replace lengths of border dia-gonals by longest length (Censi)
4. Window masking



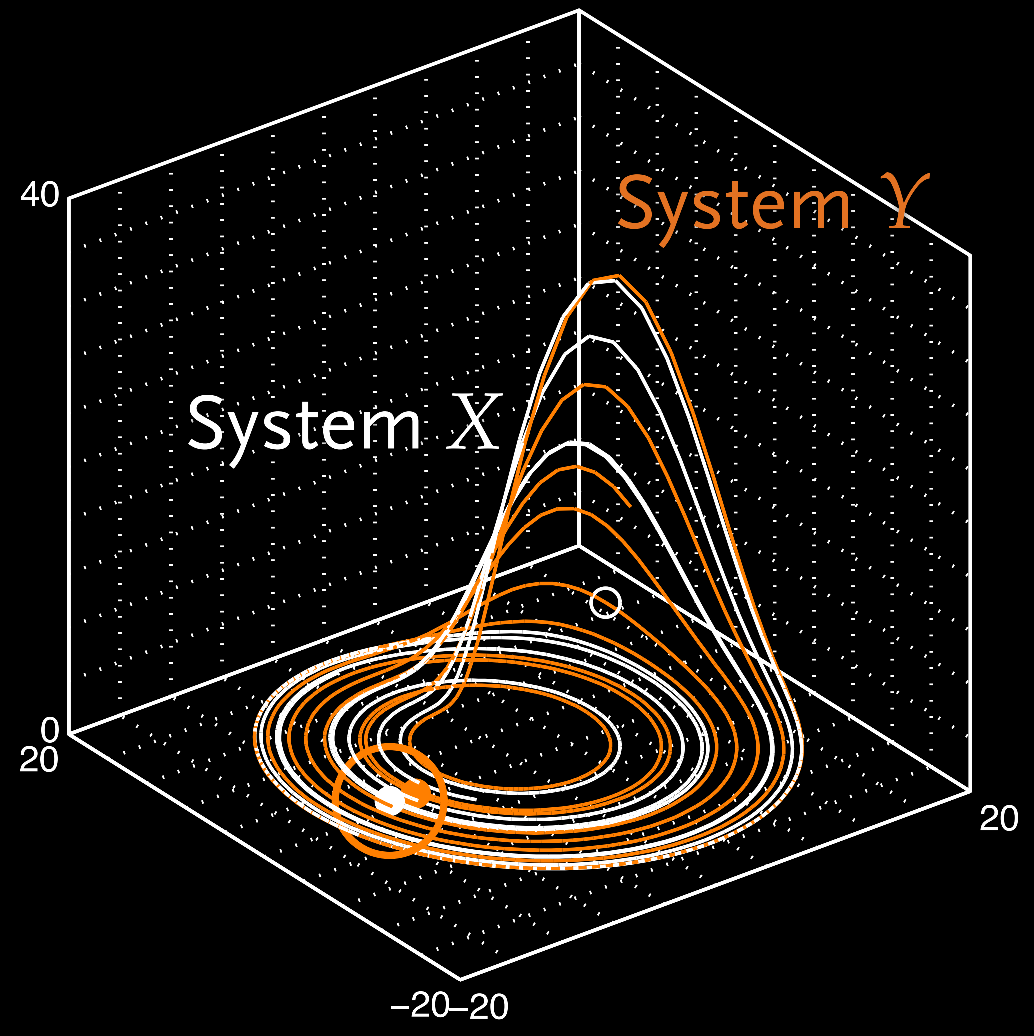


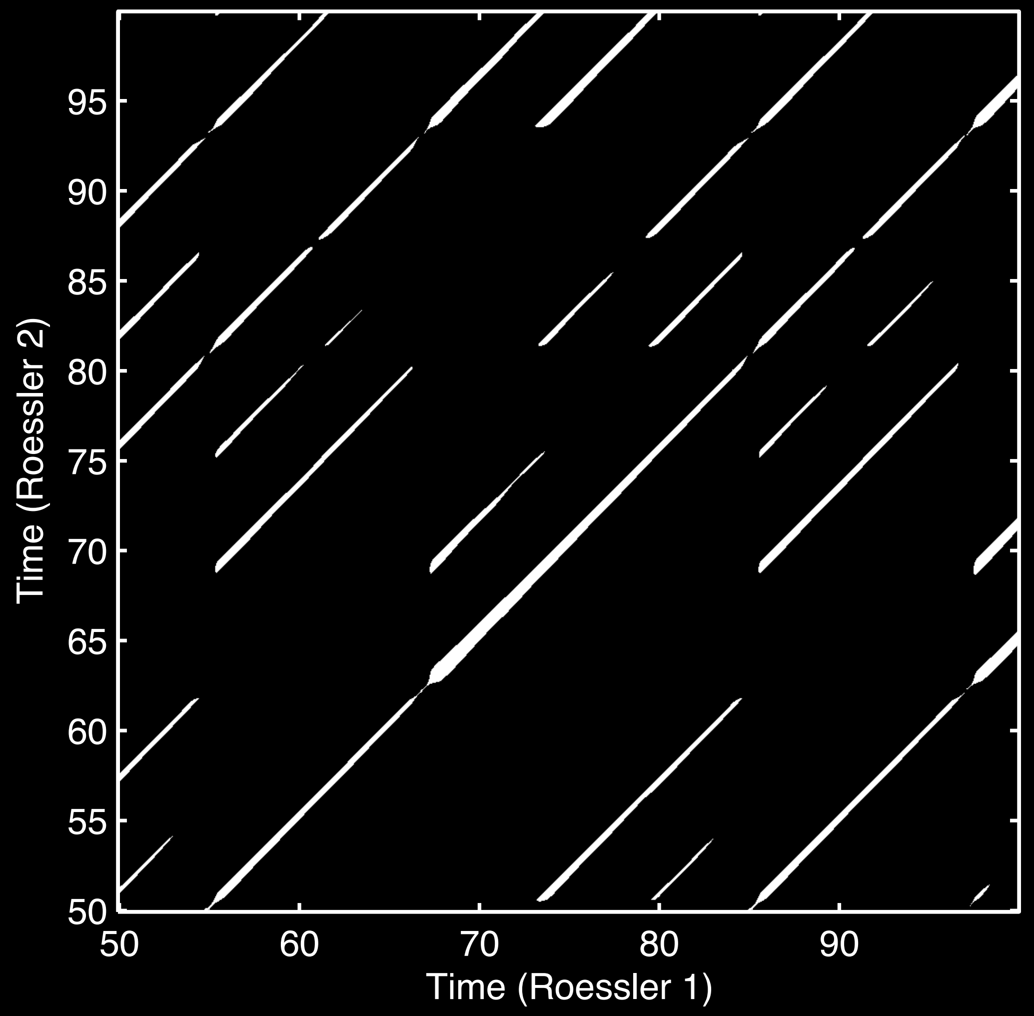


Kraemer & Marwan, Phys Lett A 383, 2019

Coupling Analysis

Cross Recurrence Plot



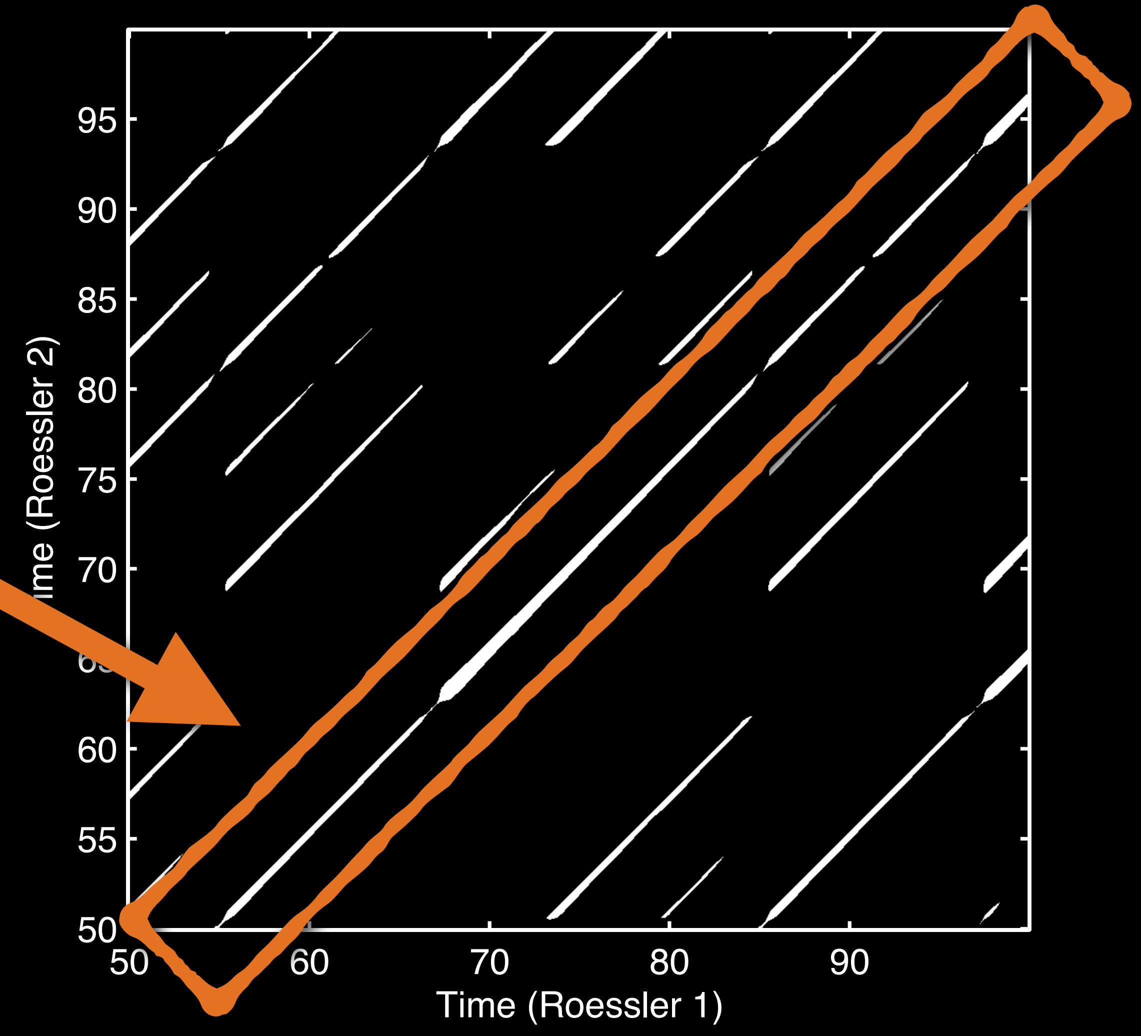


$$\mathbf{CR}_{i,j}^{\vec{x},\vec{y}} = \Theta\left(\varepsilon - ||\vec{x}_i - \vec{y}_j||\right), \quad i = 1, \ldots, N, \; j = 1, \ldots, M$$

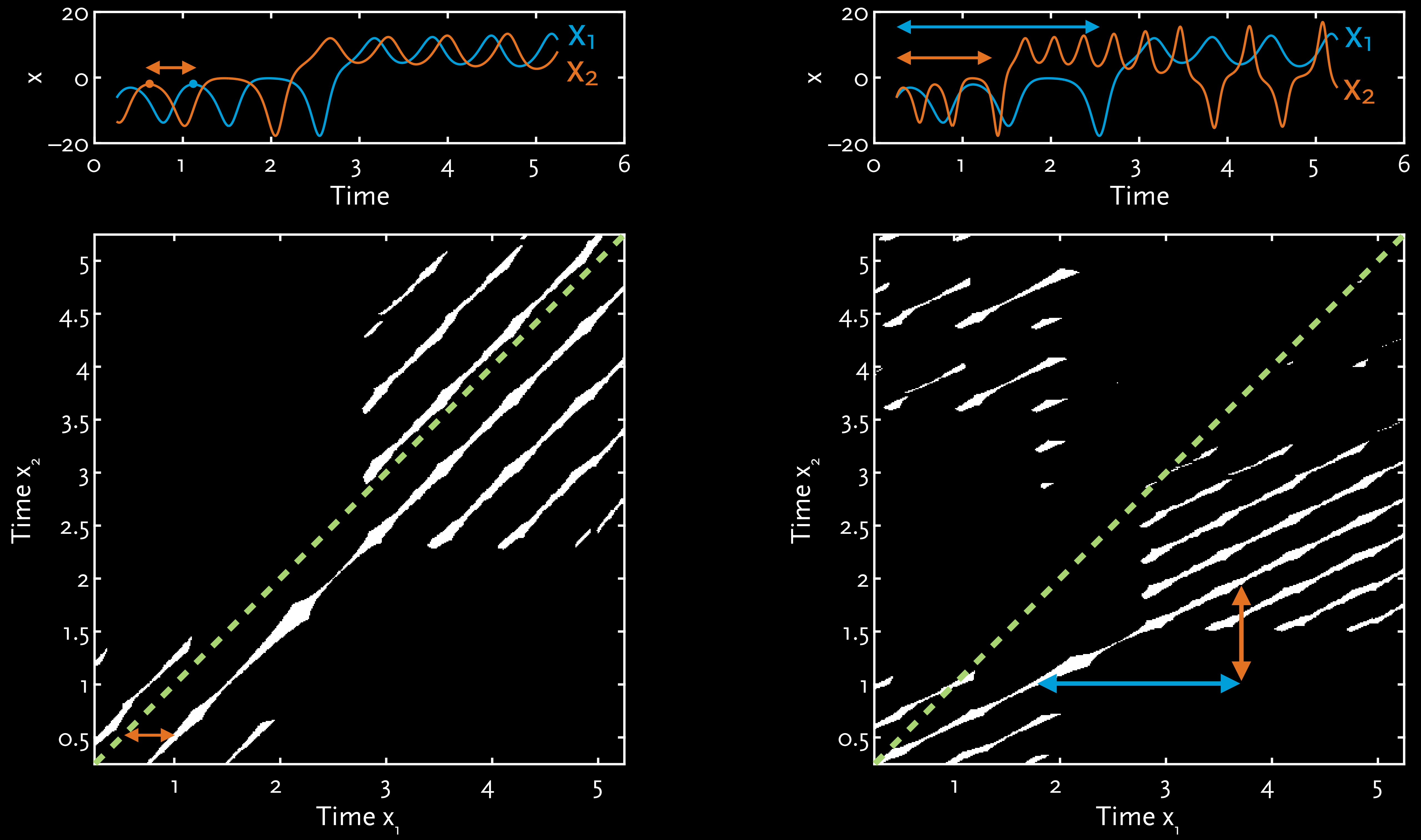
QuizZ

QuizZ

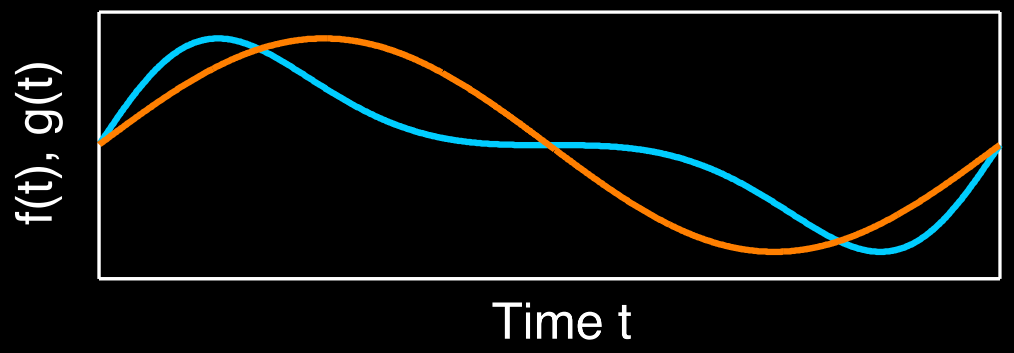
- What does this line mean/ how to interpret its shape?
  - main diagonal? (offset?)
  - straight line?
  - 45°?

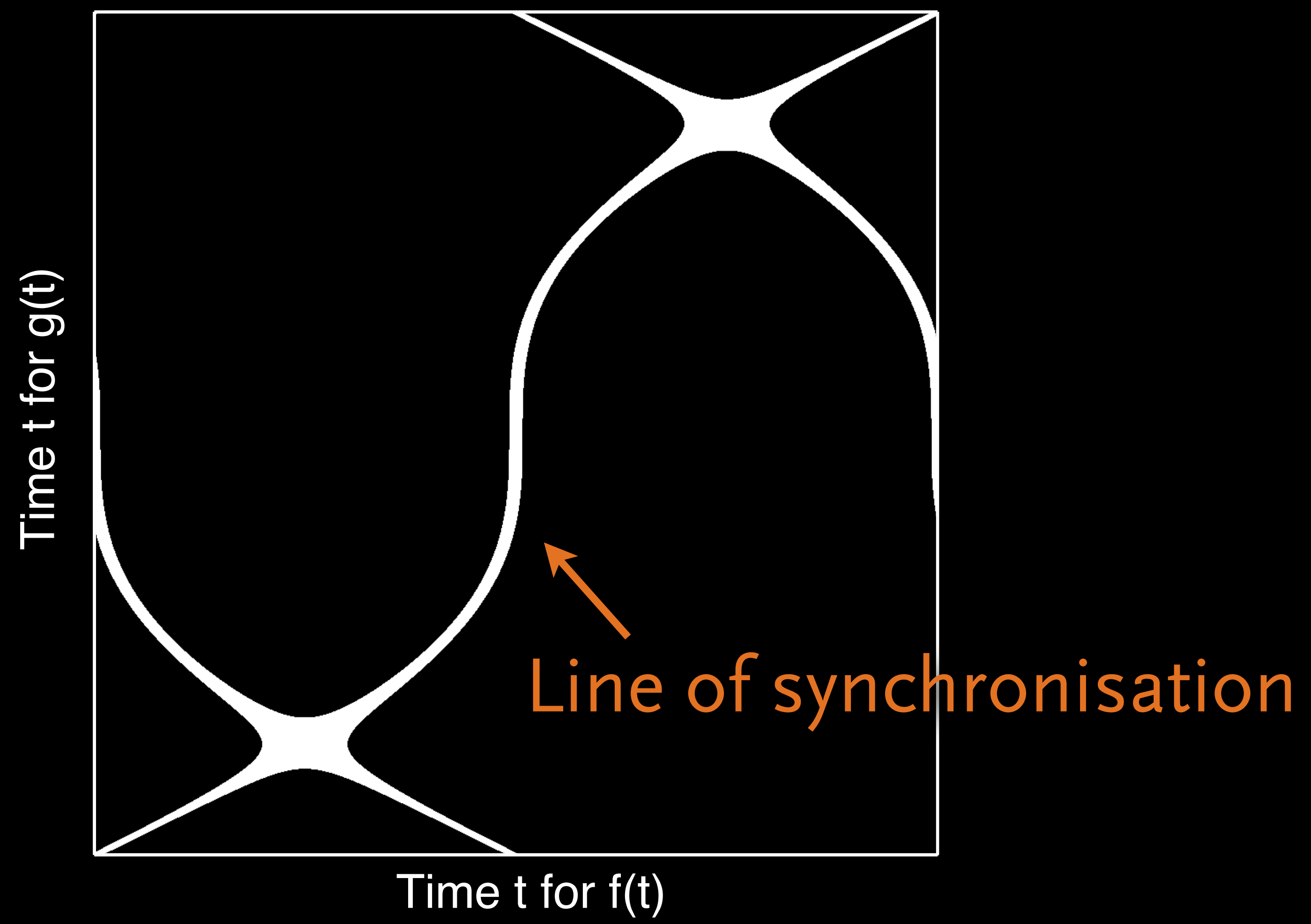


Cross Recurrence Plot



Cross Recurrence Plot



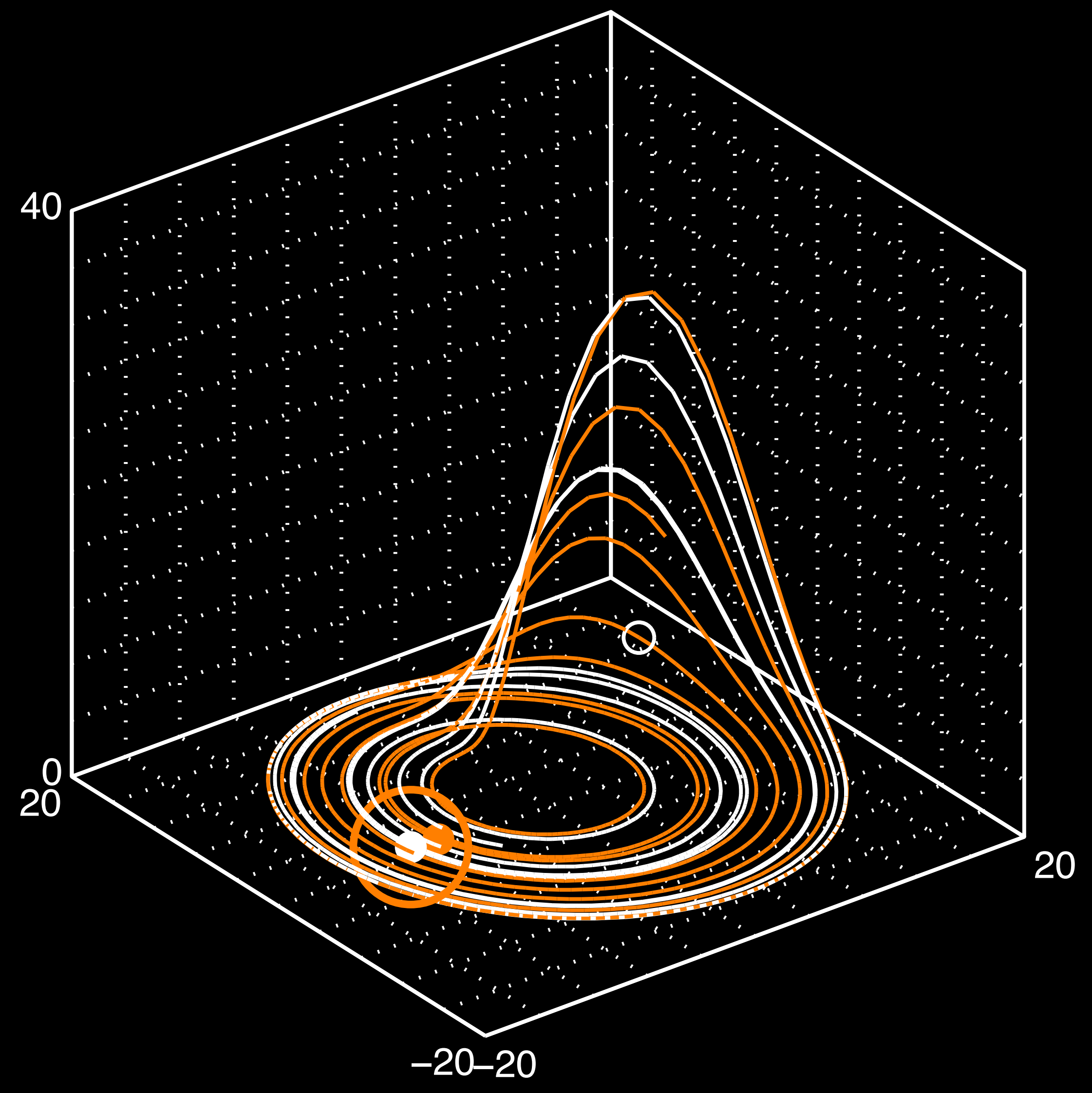


$$\mathbf{CR}_{i,j}^{\vec{x},\vec{y}} = \Theta\left(\varepsilon - ||\vec{x}_i - \vec{y}_j||\right), \quad i = 1, \ldots, N, \; j = 1, \ldots, M$$

Cross Recurrence Plot

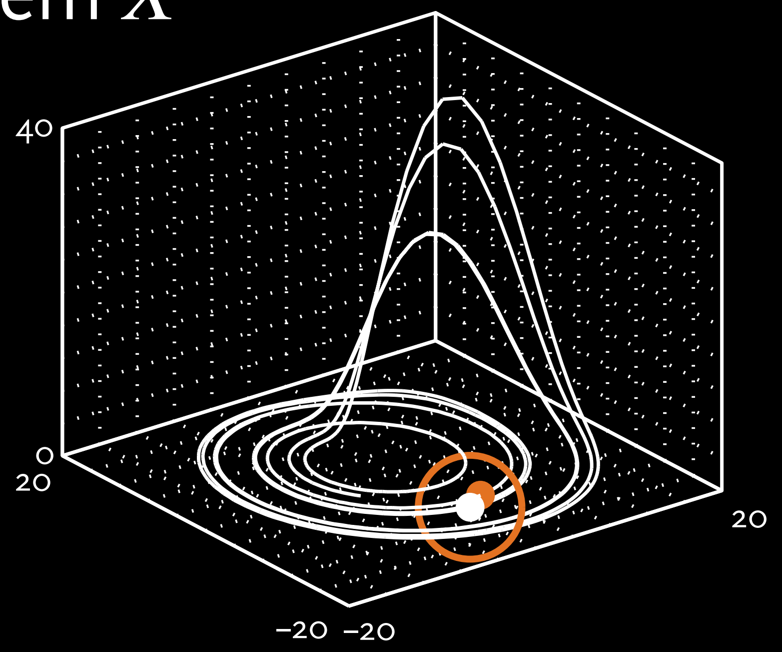
$$\mathbf{CR}_{i,j}^{\vec{x},\vec{y}} = \Theta\left(\varepsilon - ||\vec{x}_i - \vec{y}_j||\right), \quad i = 1,\ldots,N,\ j = 1,\ldots,M$$

- Occurrences of a similar state
- States should be comparable/ similar; comparable dynamics
- Normalise time series

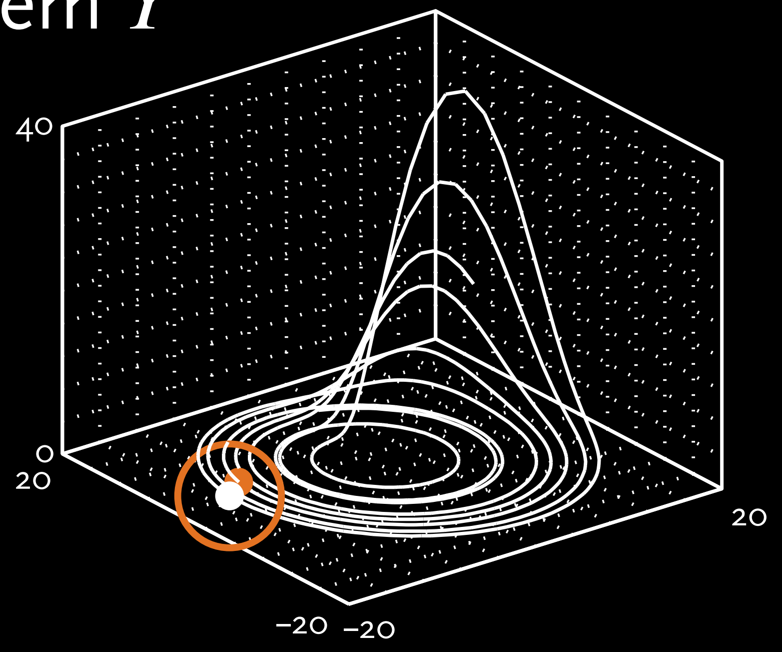


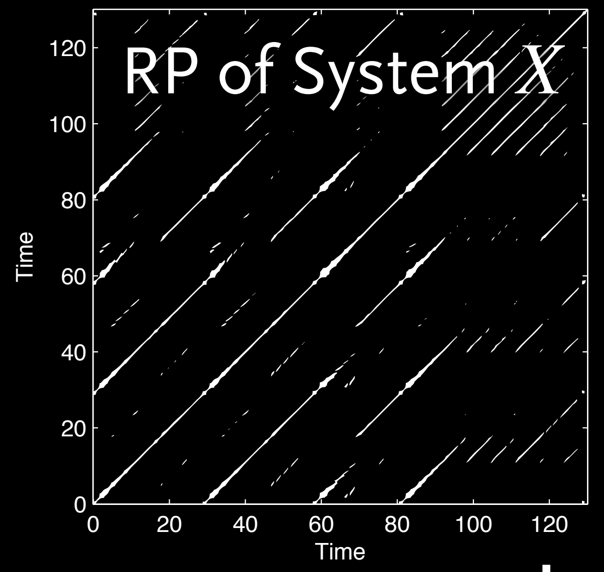
Joint Recurrence Plot

System X

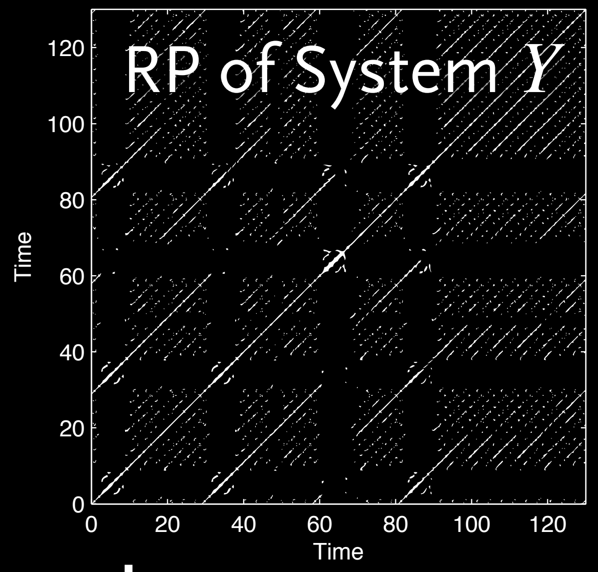


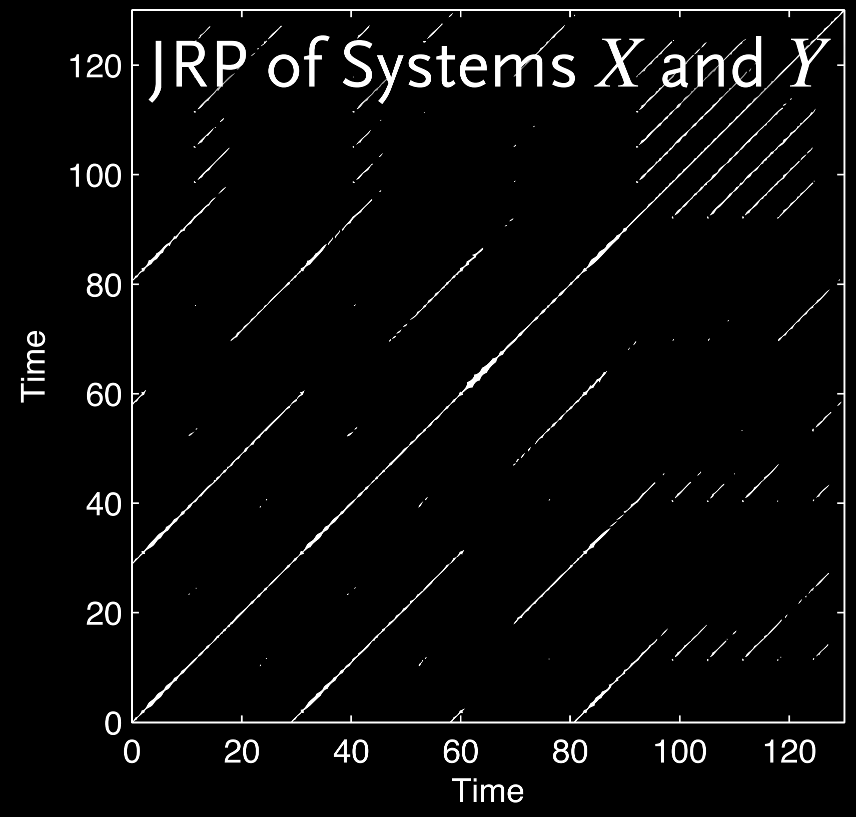
System Y





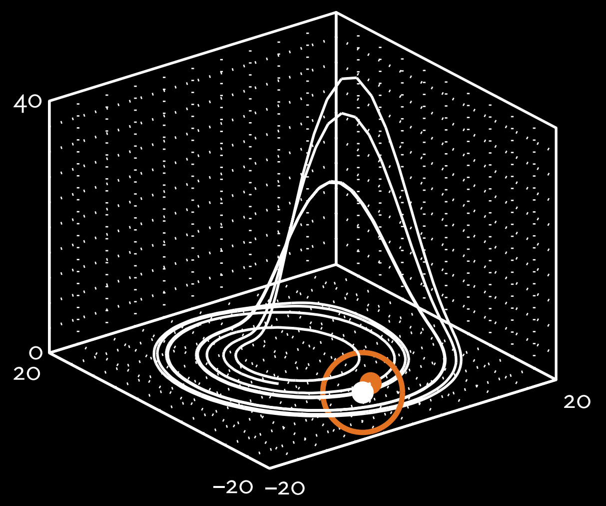
×



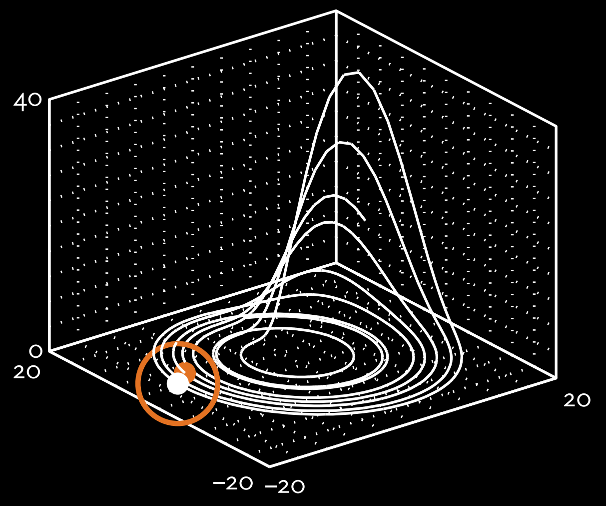


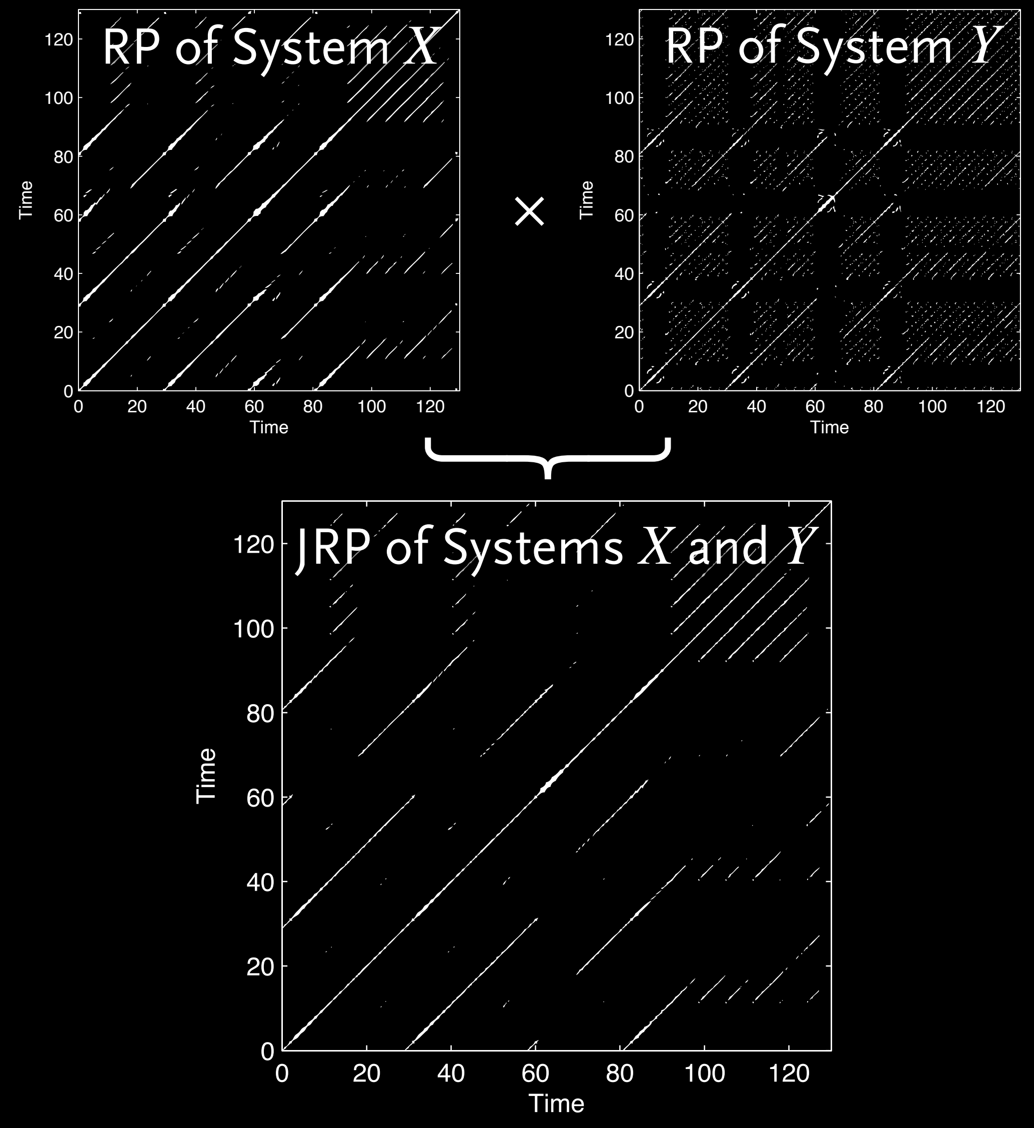
Joint Recurrence Plot

System X



System Y





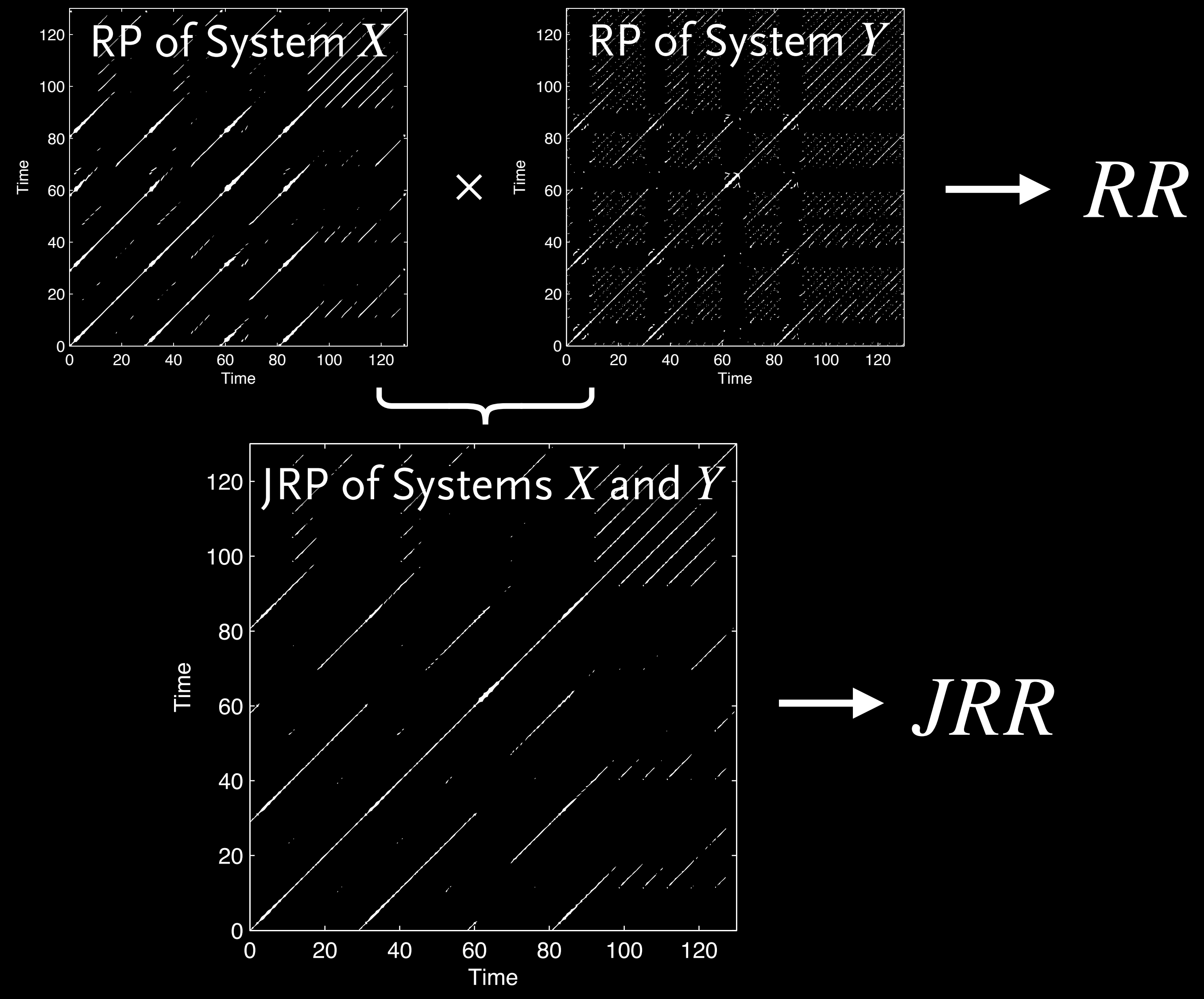
$$\mathbf{JR}_{i,j}^{\vec{x}^{(1,\dots,n)}} = \prod_{k=1}^{n} \mathbf{R}_{i,j}^{\vec{x}^{(k)}}, \quad i,j = 1,\dots,N$$

JRP based Synchronisation Measure

- Similar recurrence pattern: generalised synchronisation (GS)
- Fix RR for system X and system Y
- Synchronisation measure:

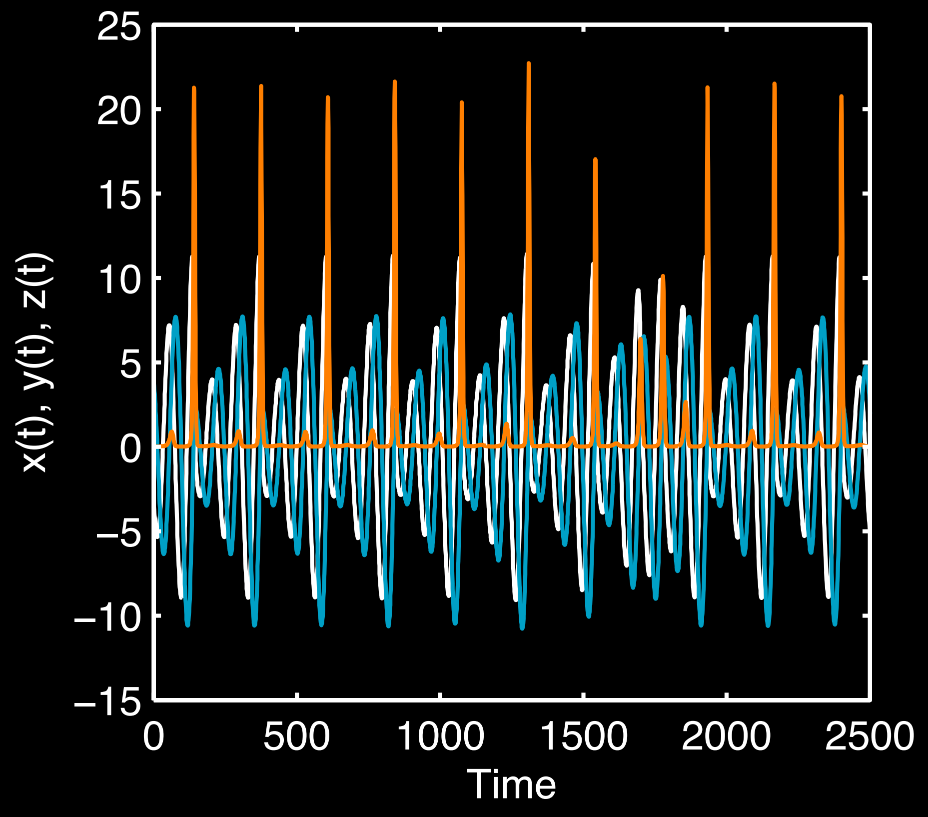
$$S = \frac{JRR}{RR}$$

(0 – independent; 1 – GS)

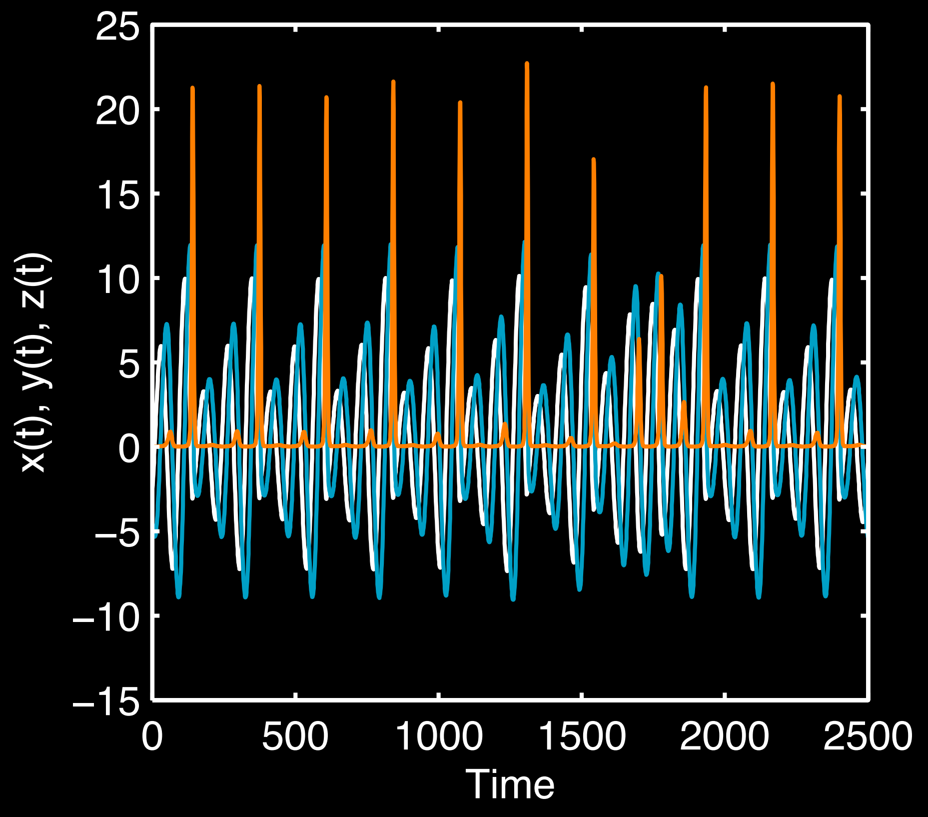


Difference CRP – JRP

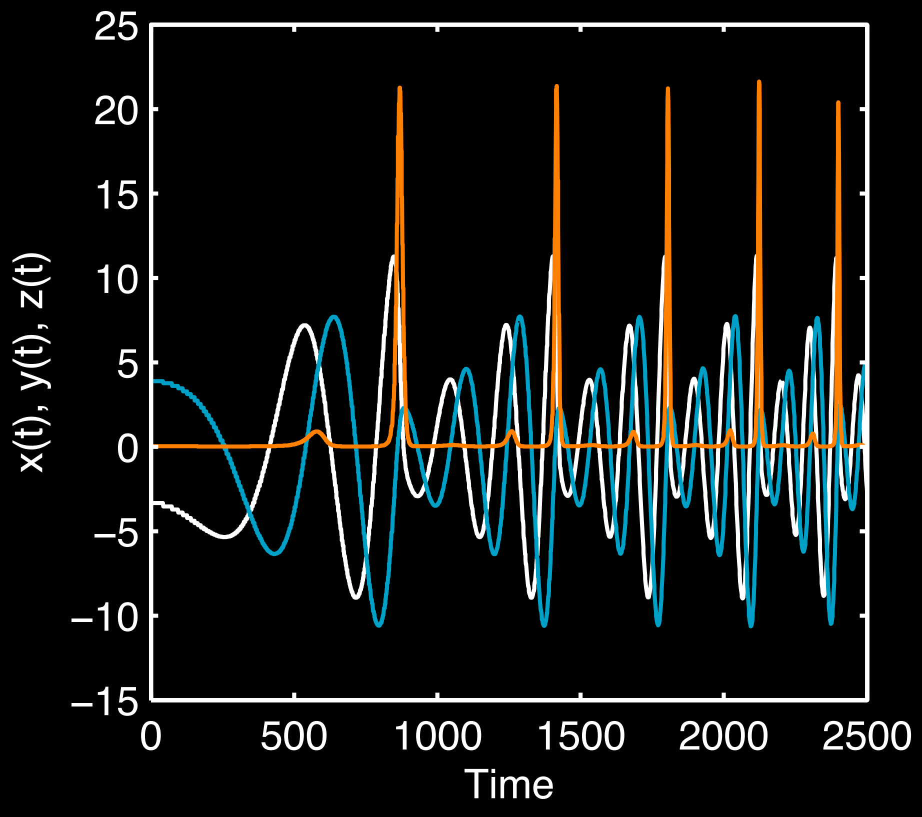
- Transformations of the Rössler system



original



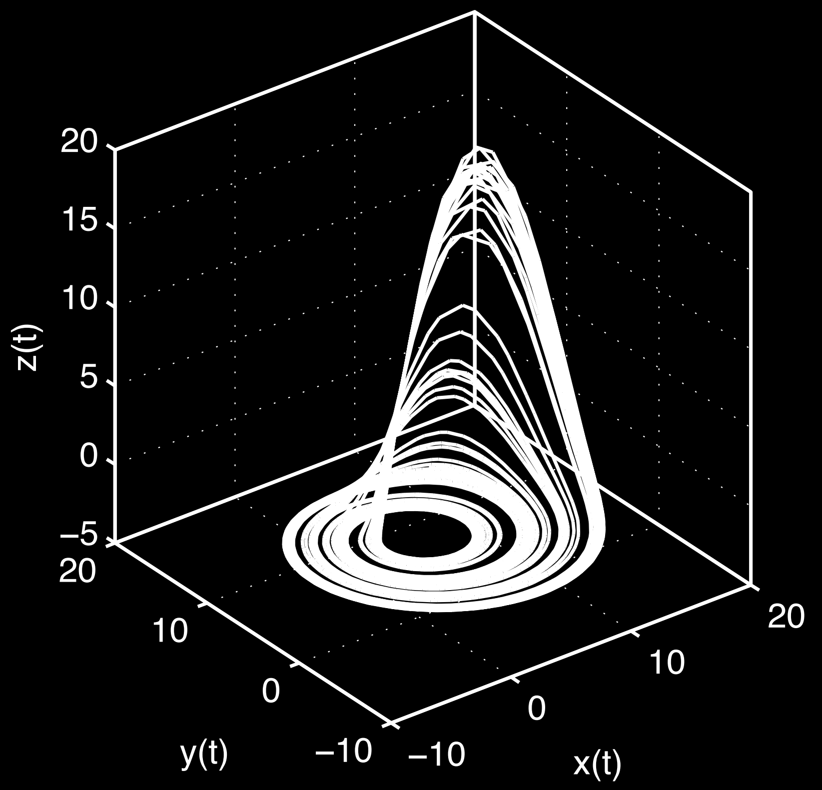
rotated



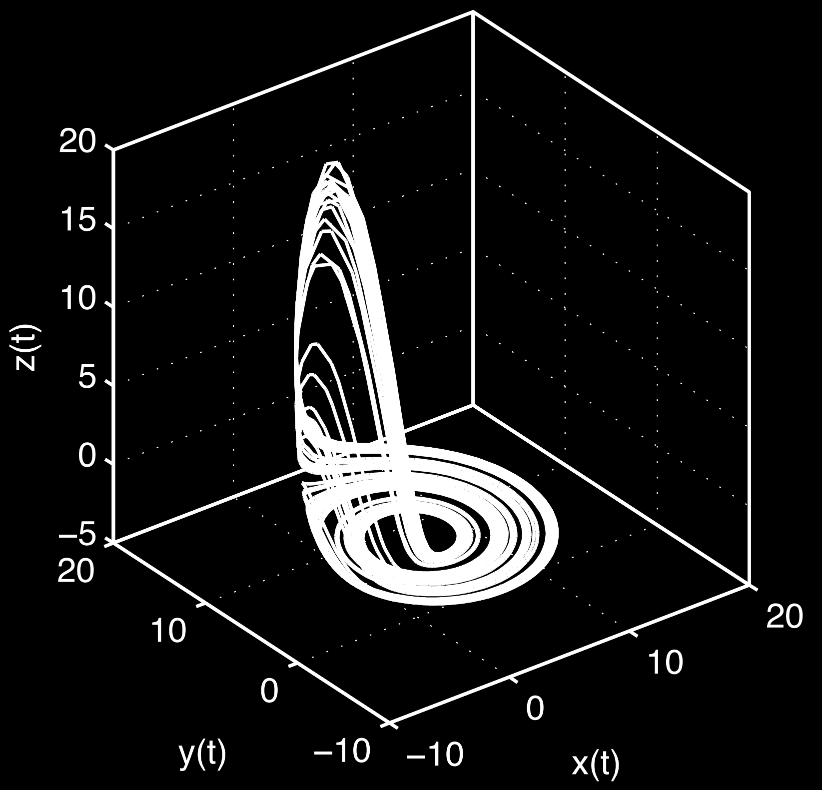
time compressed

Difference CRP – JRP

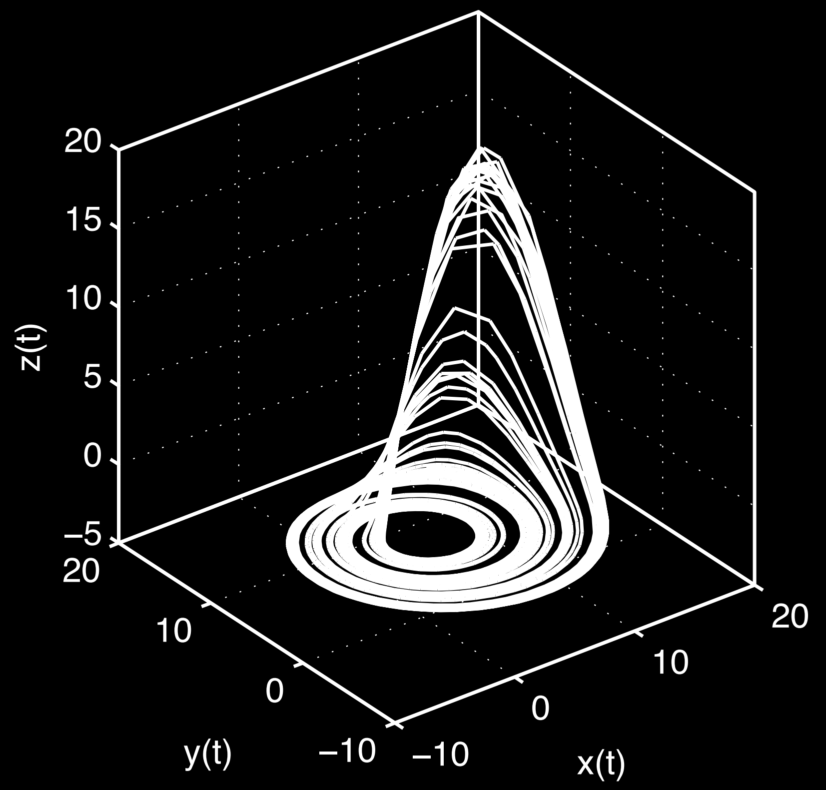
- Transformations of the Rössler system



original



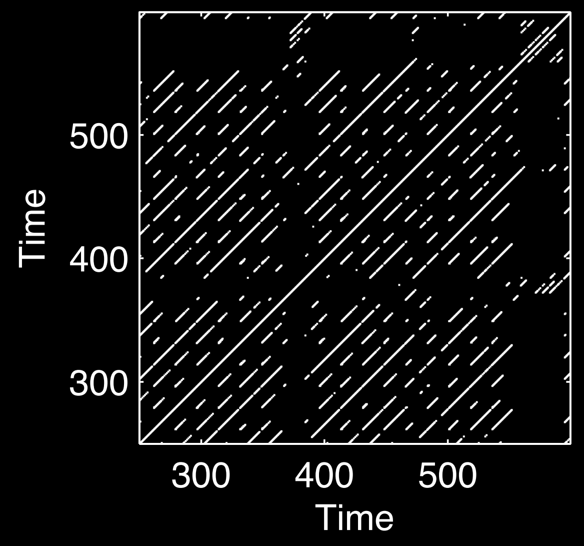
rotated



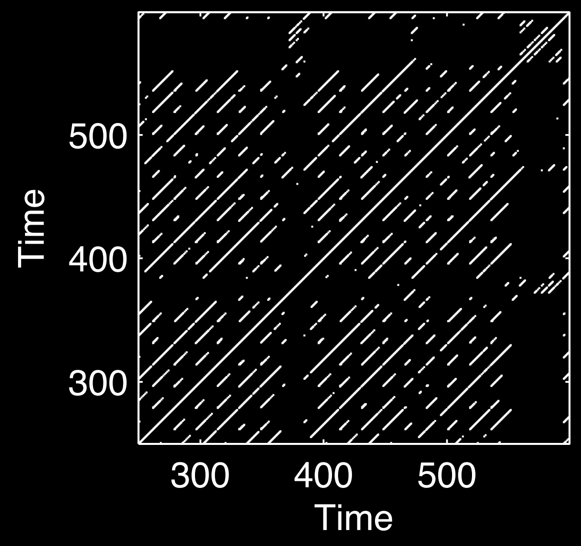
time compressed

Difference CRP – JRP

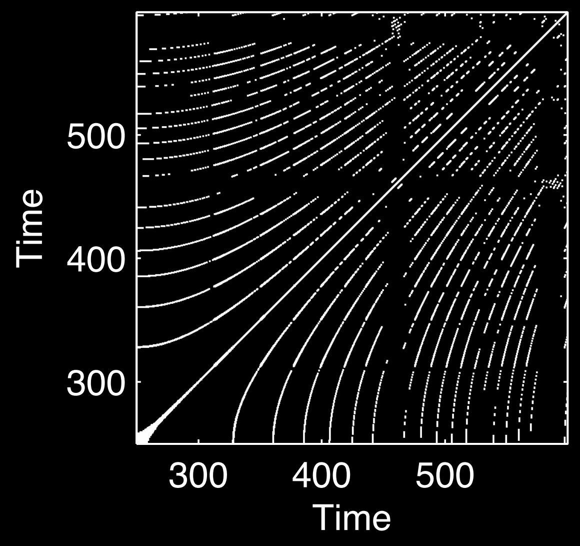
RP original

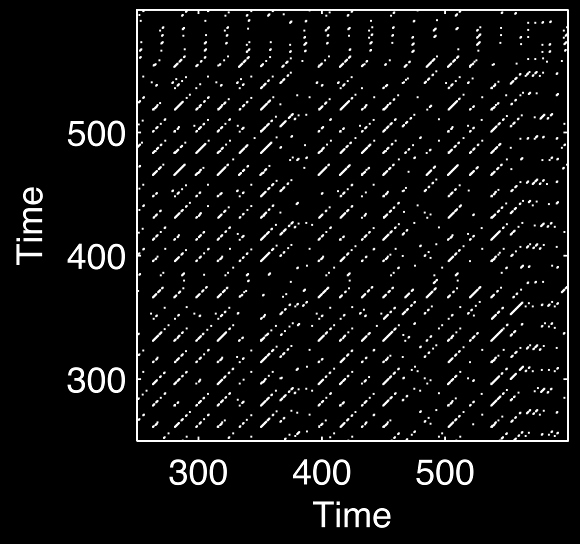


RP rotated

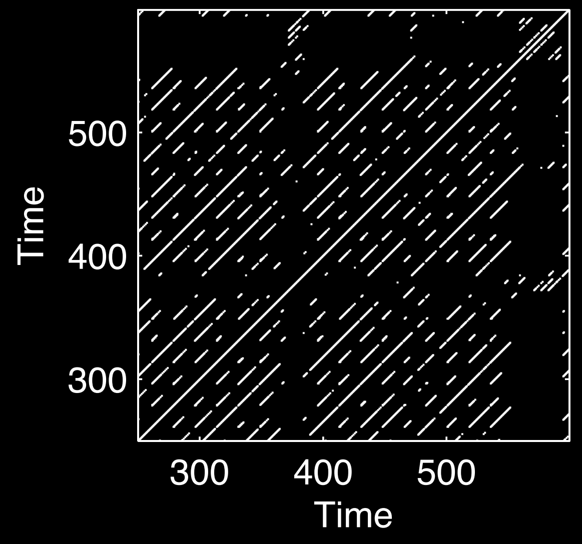


RP time compressed



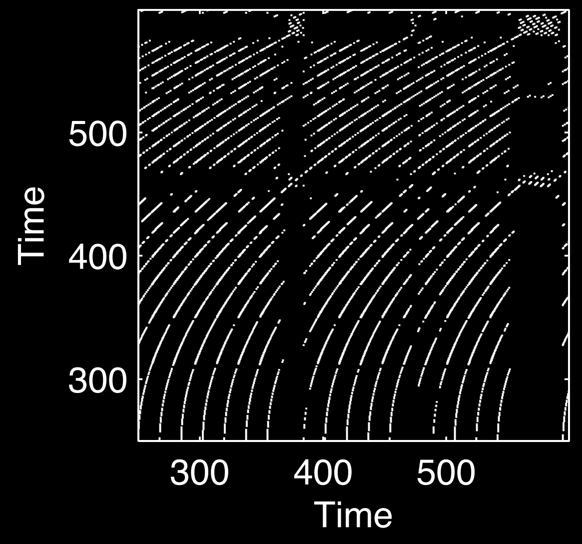


CRP

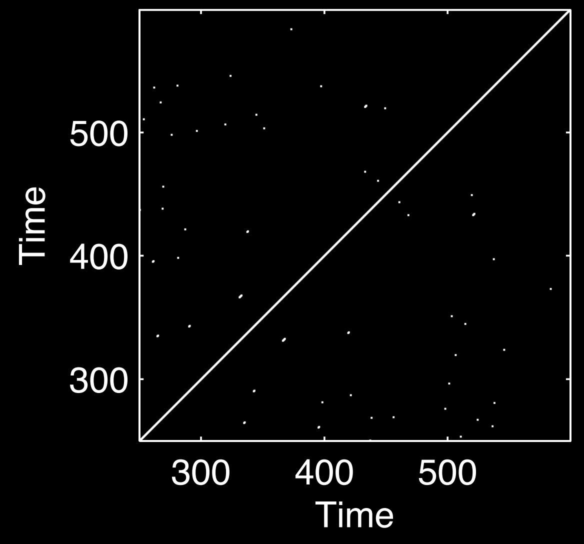


JRP

Original & rotated



CRP

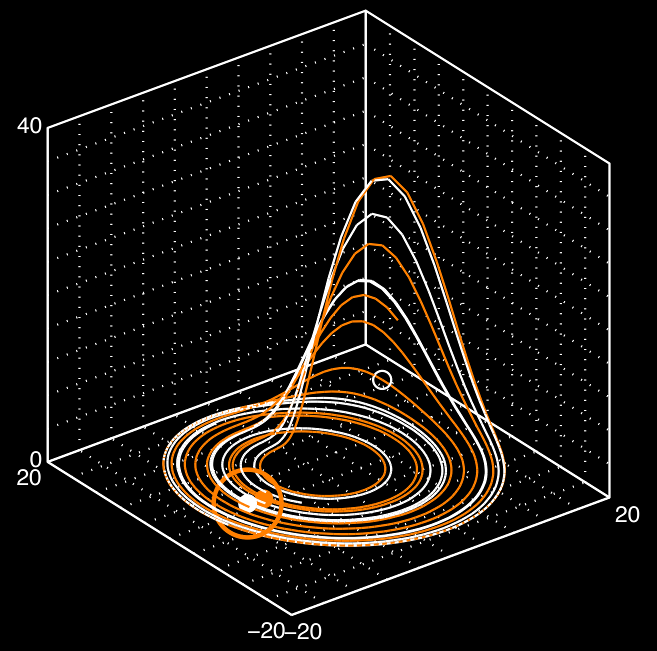


JRP

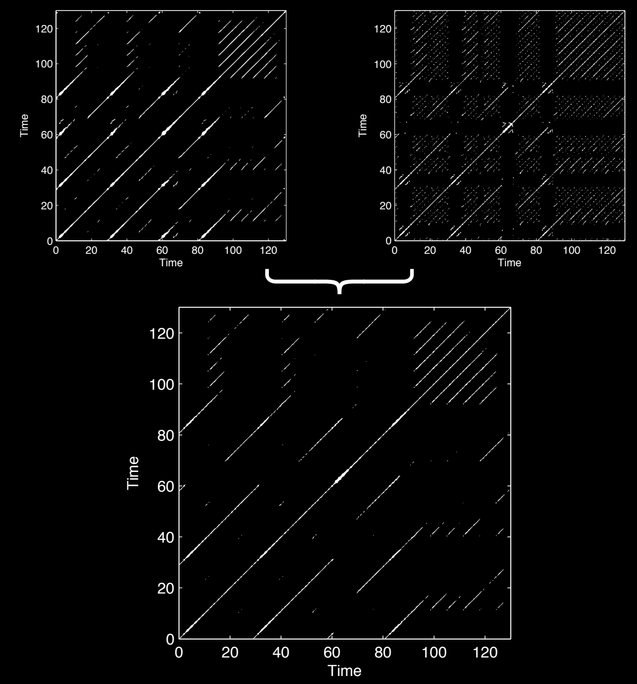
Original & time compressed

Difference CRP–JRP

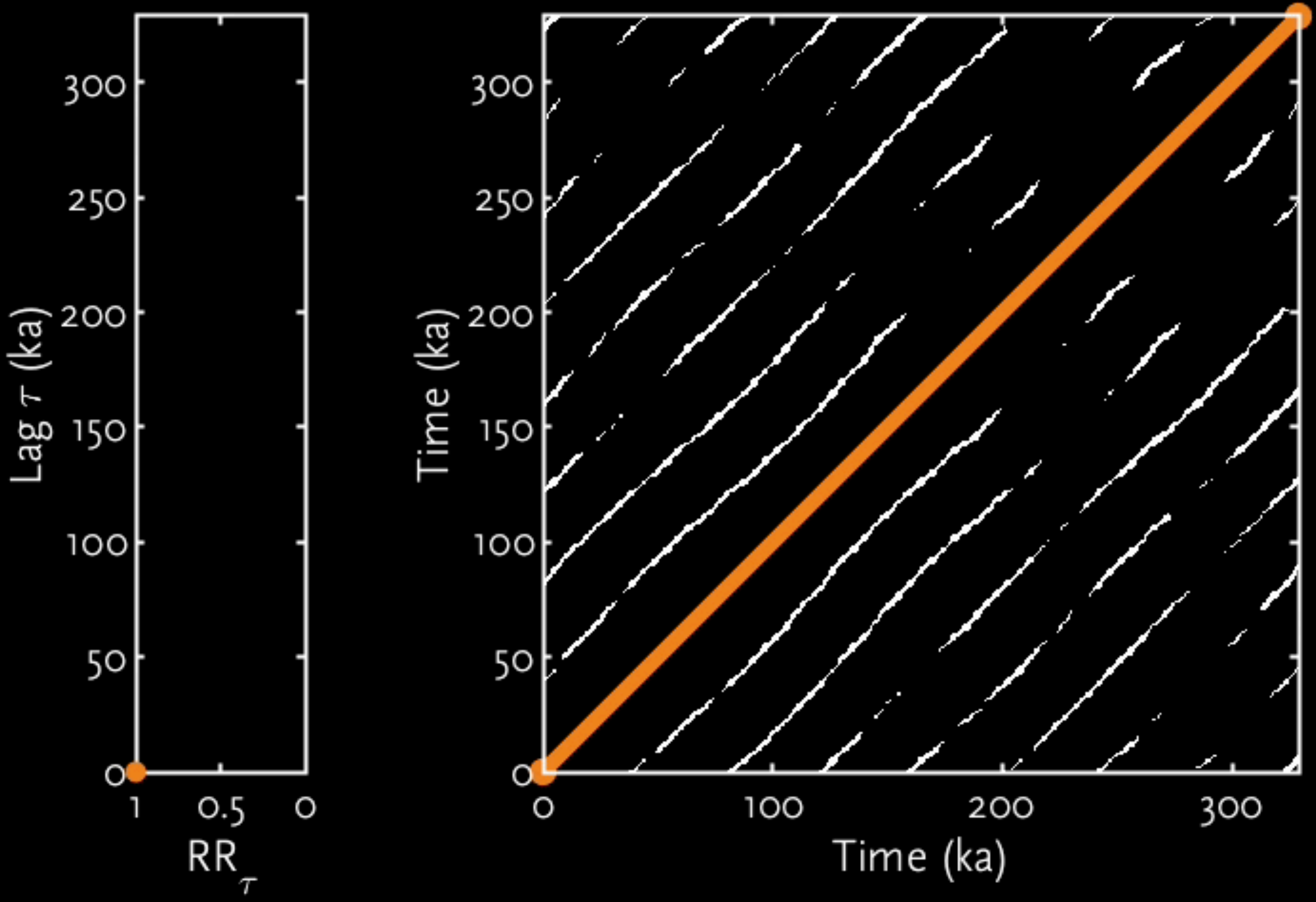
- CRP: simultaneous occurrence of a similar state
  - ➡ Comparable/ similar states
  - ➡ Differences in timing (time alignment)



- JRP: simultaneous occurrence of a recurrence
  - ➡ Different states
  - ➡ Same timing (generalised synchronisation)

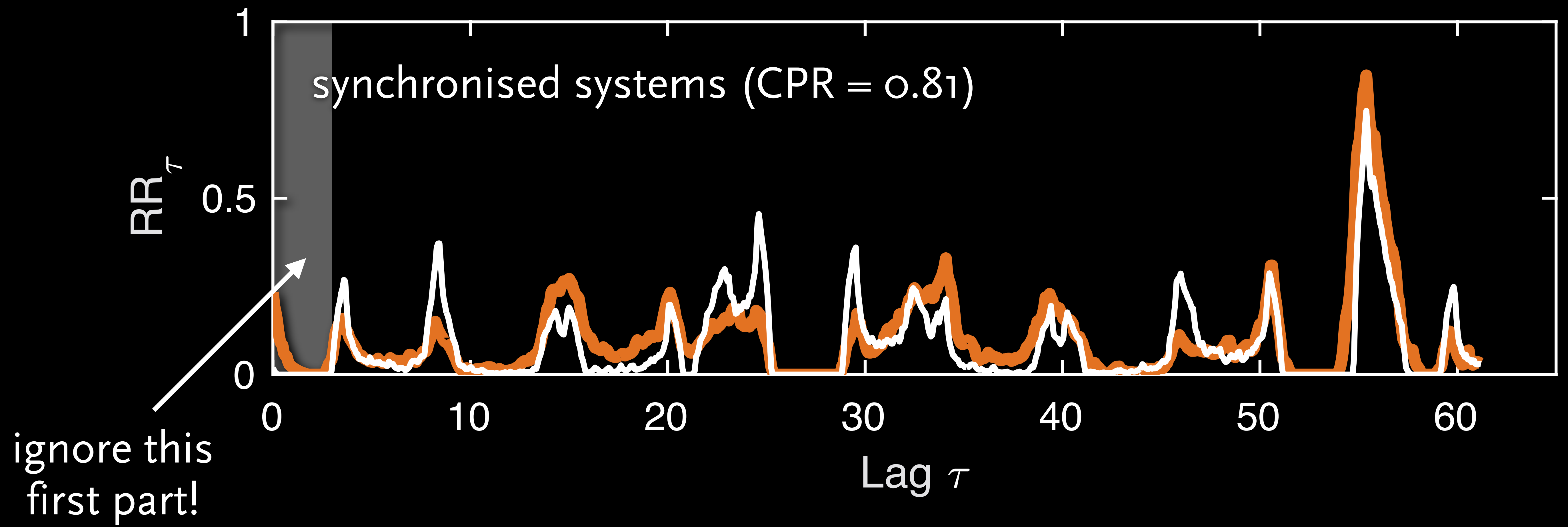


Phase Synchronisation using τ-Recurrence Rate



Phase Synchronisation

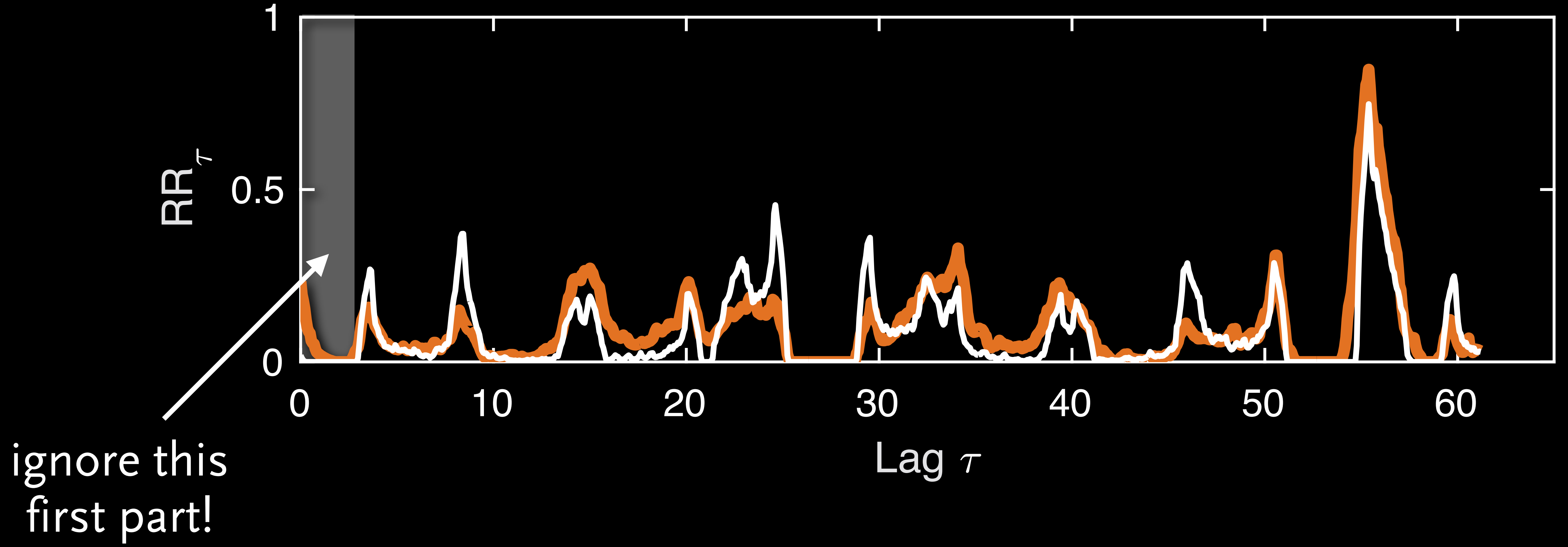
- Synchronisation index: correlation coefficient $CPR = \langle RR^x_\tau \cdot RR^y_\tau \rangle$



QuizZ

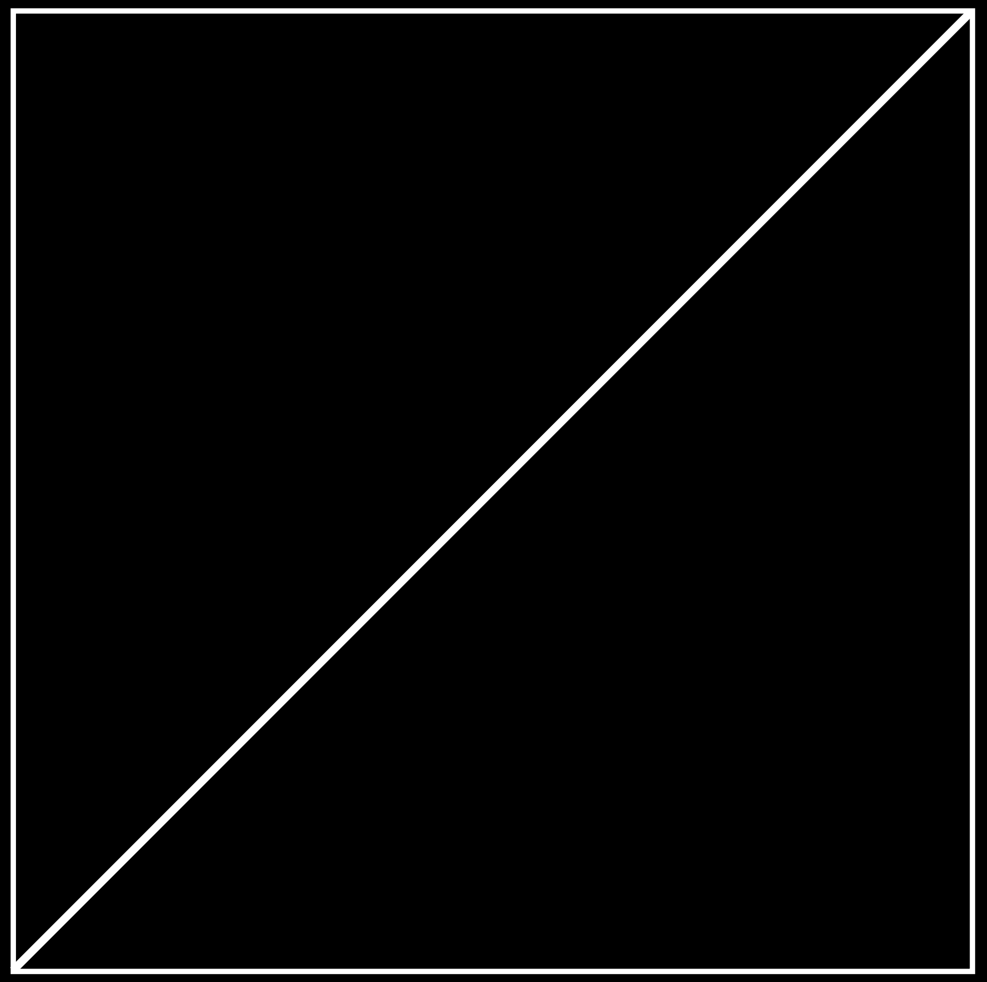
QuizZ

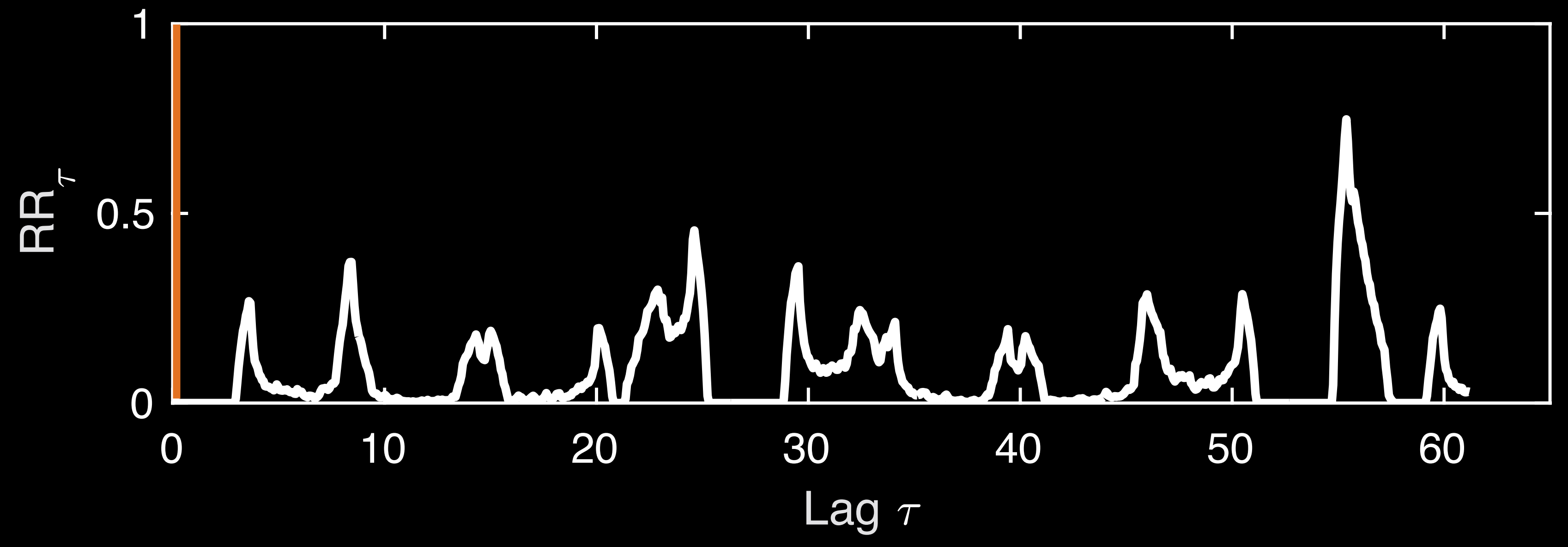
- Why should we remove this first part from analysis?



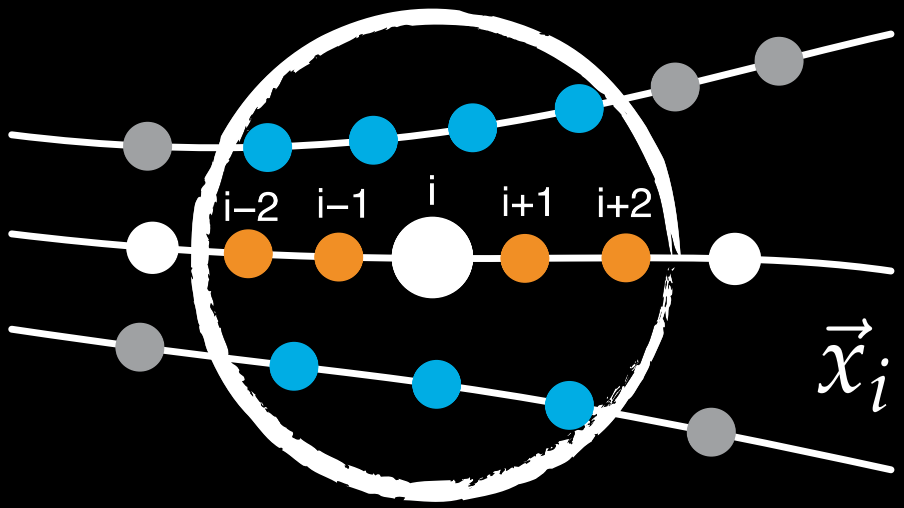
Phase Synchronisation

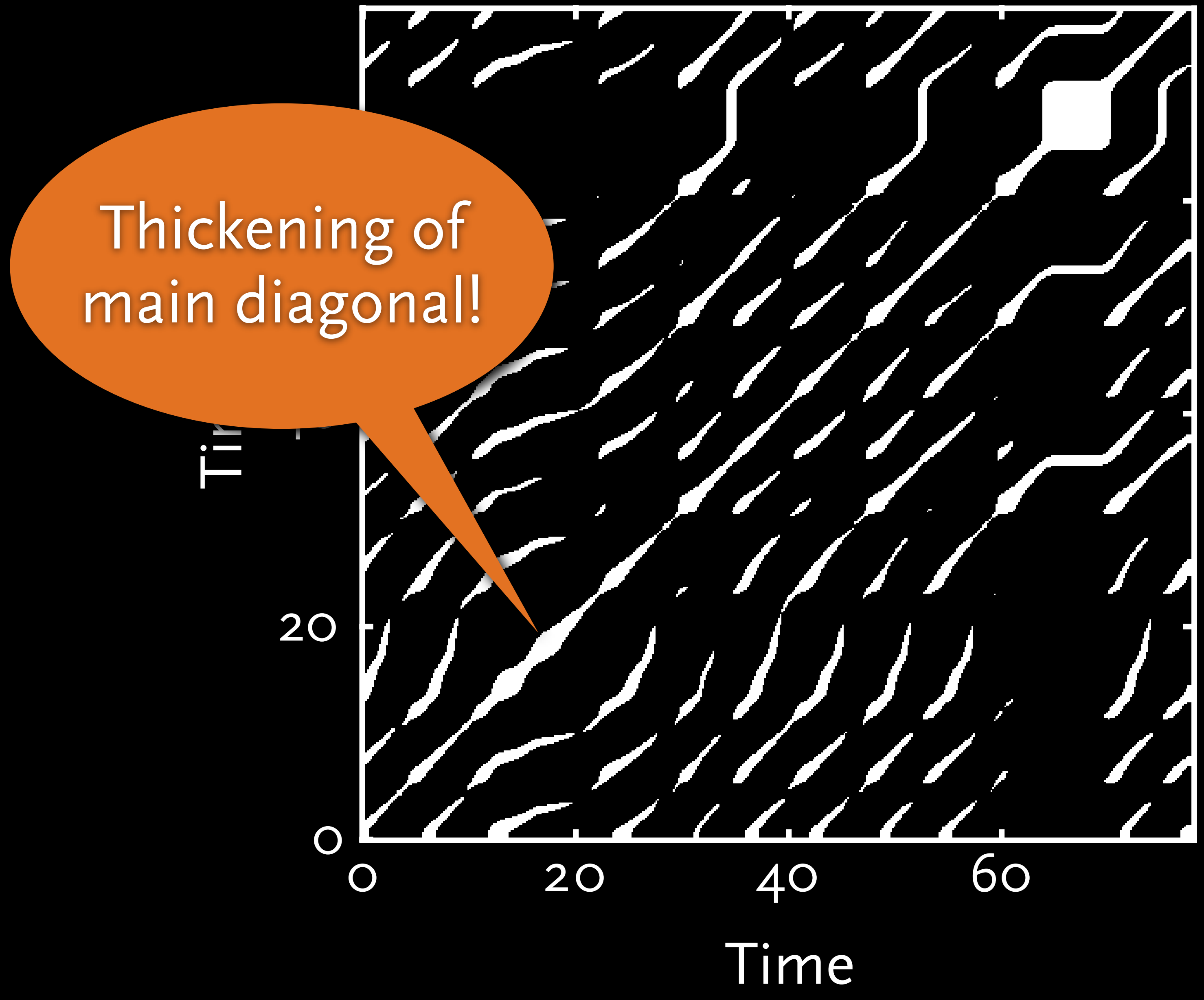
- We have the main diagonal (even when no other recurrences)!





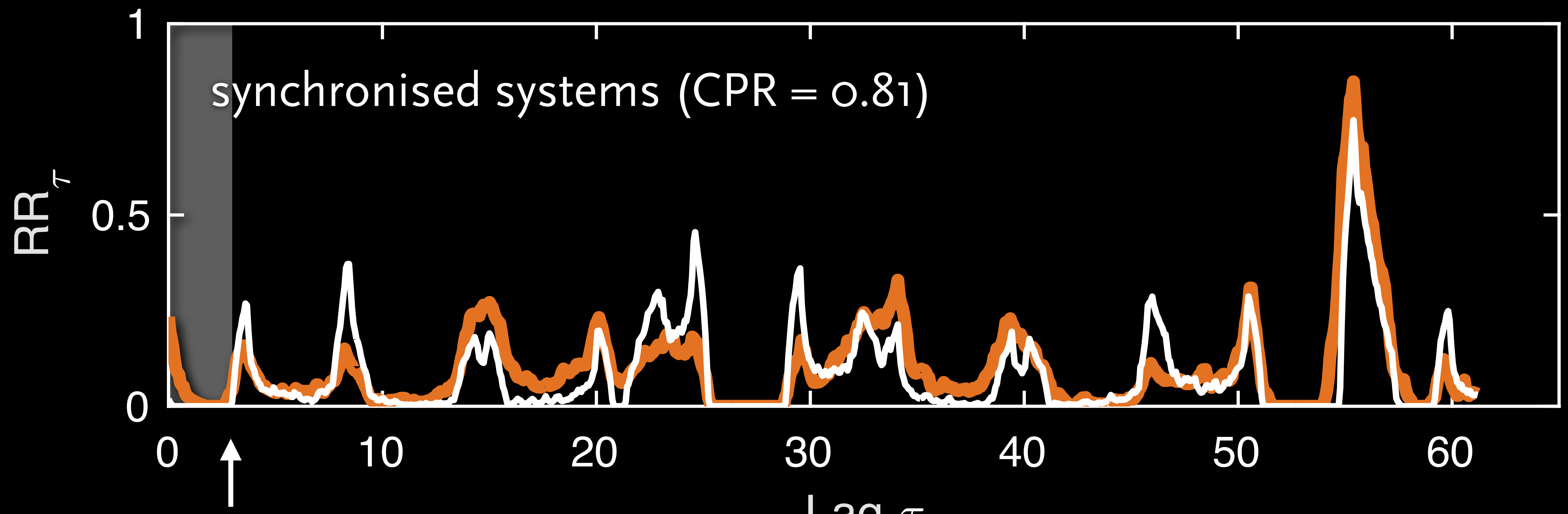
Phase Synchronisation





Phase Synchronisation

- Synchronisation index: correlation coefficient $CPR = \langle RR^x_\tau \cdot RR^y_\tau \rangle$

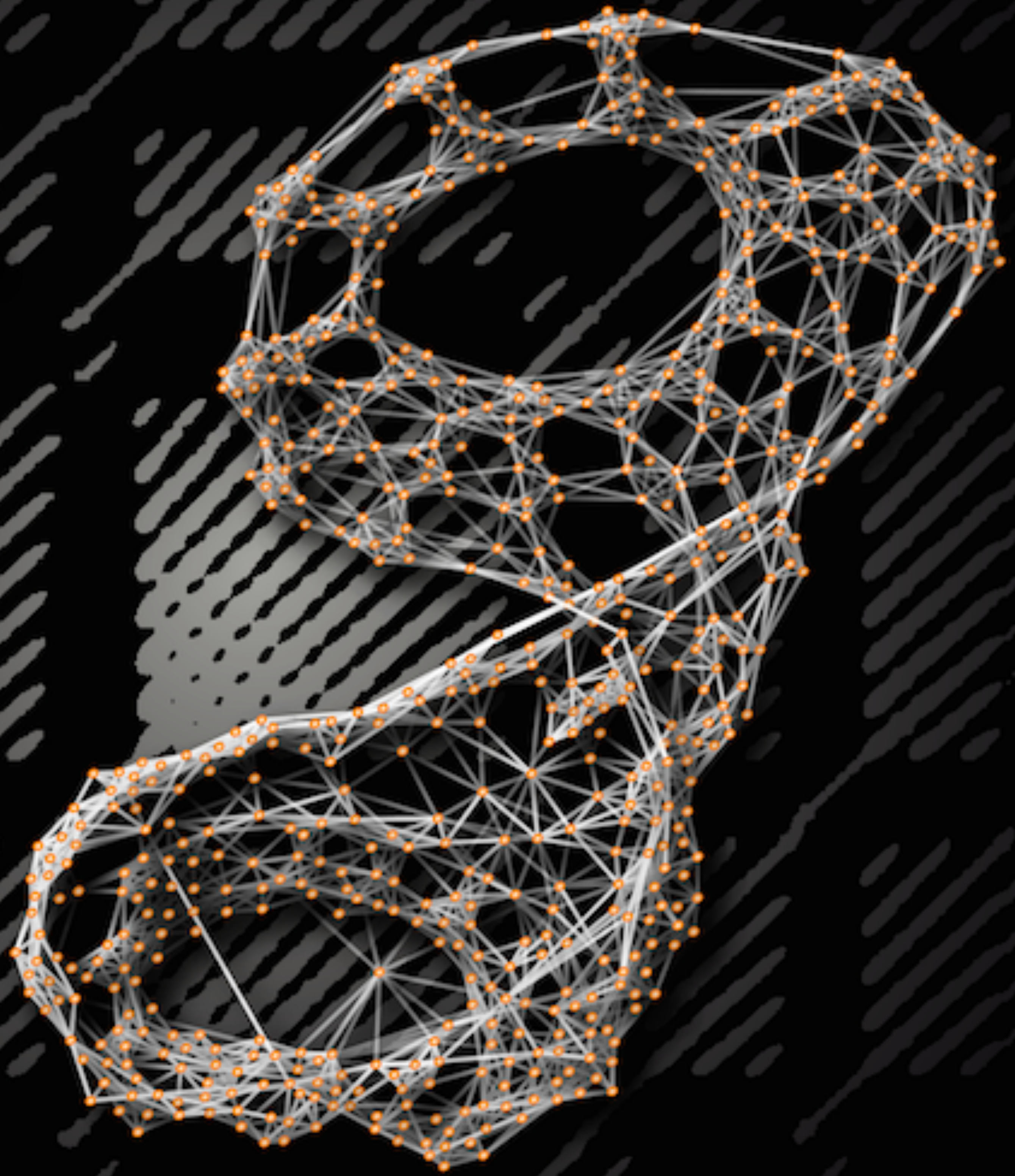


use auto-correlation time
(where ACF falls below 1/e)

Potsdam Institute for
Climate Impact Research

Recurrence Plots Workshop

1. Phase space
2. Visual interpretation
3. Sensitivity on Parameters
4. Recurrence quantification analysis
5. Coupling analysis

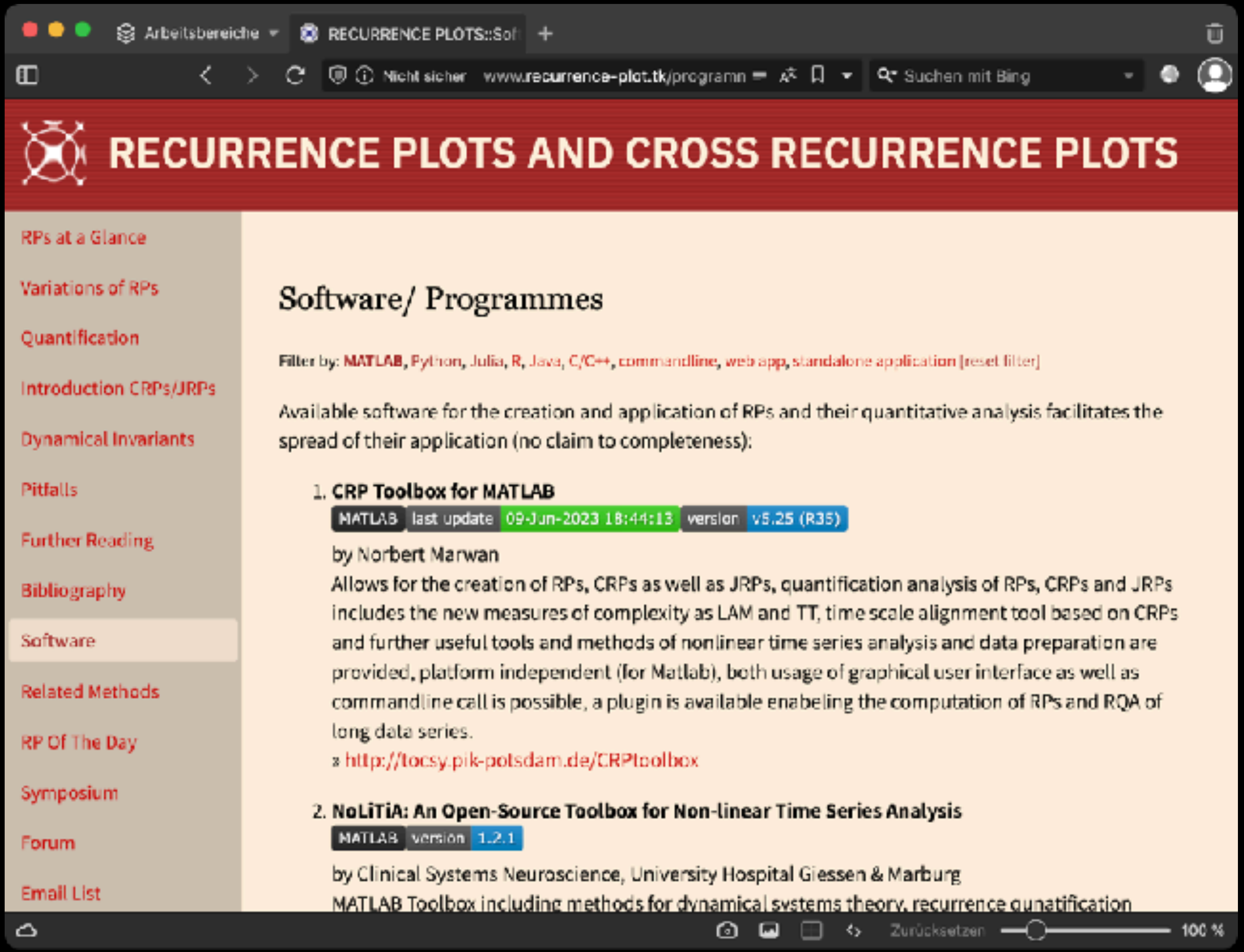


Software Tools

- MATLAB: CRP Toolbox
- Python: pyunicorn, igraph, networkx, pyRQA
- Julia: DynamicalSystems.jl

```julia
# dry run to pre-compile
x = embed(sol[1,1000:1500], 3, 6);
R = RecurrenceMatrix(x, 1.2, parallel=false);
Q = rqa(R, theiler = 1, onlydiagonal=false);

for (i,N_) in enumerate(N)
    x = embed(sol[1,1000:1000+N_], 3, 6);
    tRP_ = 0;
    tRQA_ = 0;
    for j in 1:K
        t1 = @elapsed R = RecurrenceMatrix(x, 1.2, parallel=false);
        t2 = @elapsed Q = rqa(R, theiler = 1, onlydiagonal=false);
        tRP_ = tRP_ + t1;
        tRQA_ = tRQA_ + t2;
        print("  " ,j, "\n")
    end
```



CRP Toolbox for MATLAB

- Web site:
  https://tocsy.pik-potsdam.de/CRPtoolbox (CRP Toolbox)
- Access data:
  ID: workshop
  password: potsdam
- Download installation file install.m
- Call install from the MATLAB commandline