PIK
Potsdam Institute for
Climate Impact Research

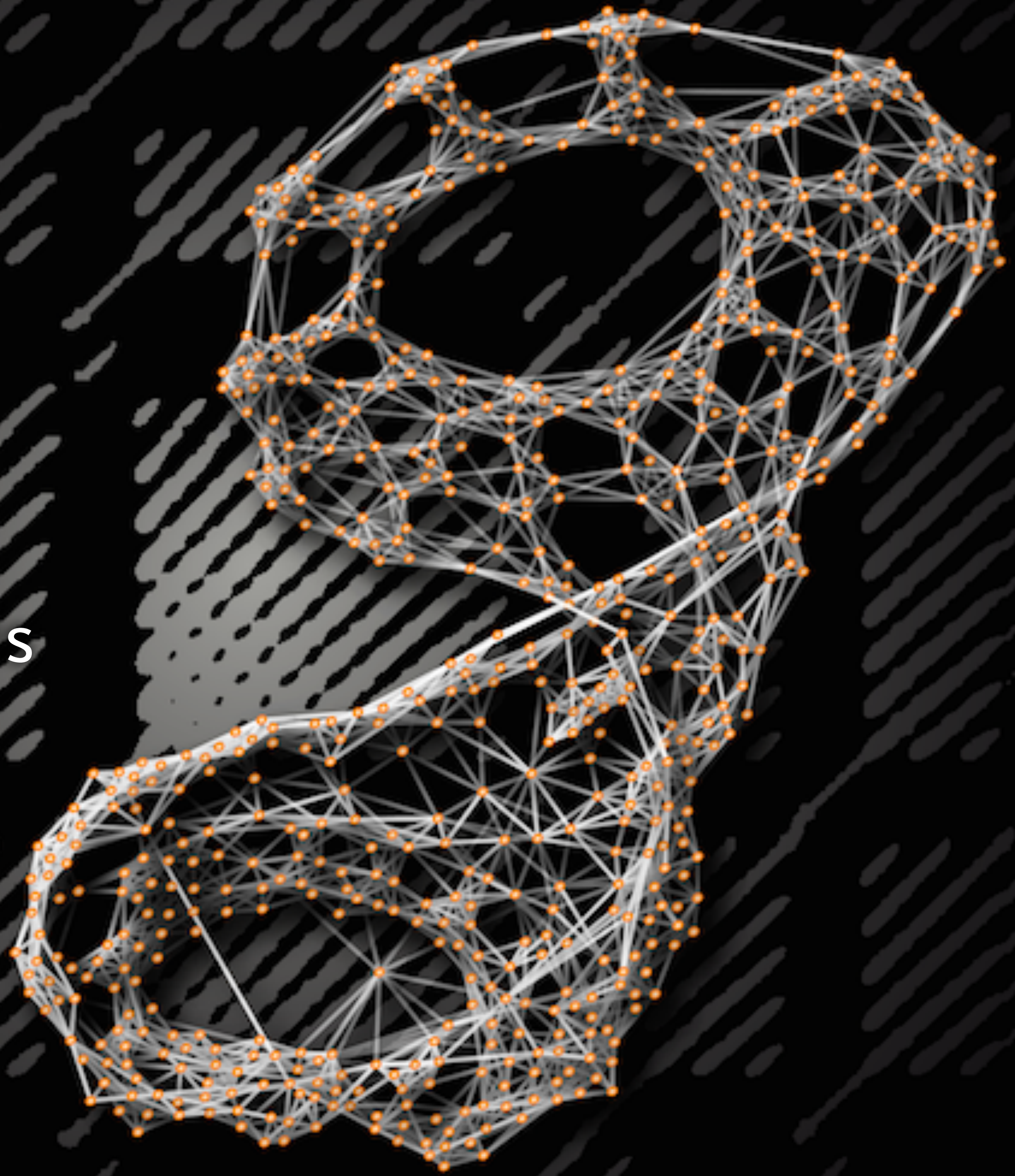
Norbert Marwan

Recurrence Analysis for Complex Systems

Leibniz-Gemeinschaft

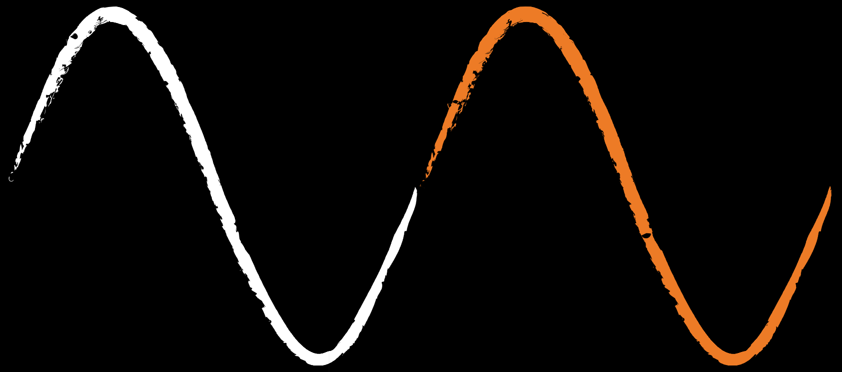
Recurrence Analysis for Complex Systems

- Phase space and recurrence plot
- Recurrence quantification
- Coupling and synchronization analysis
- Outlook



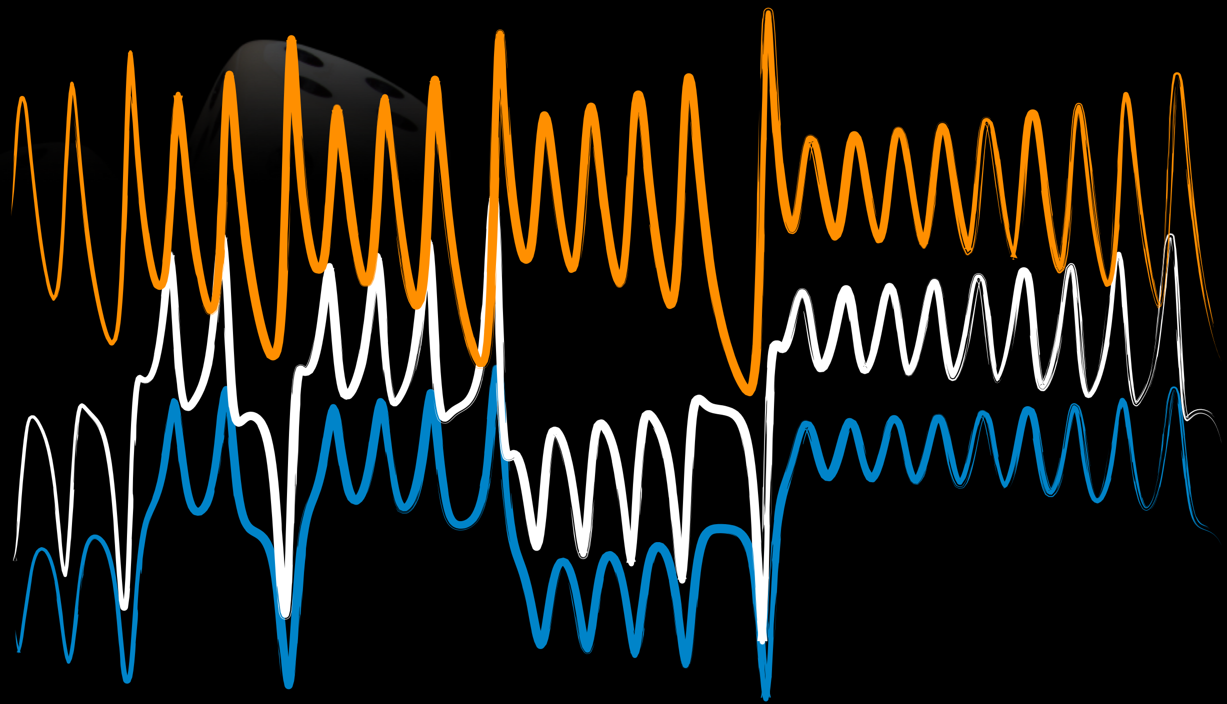
Examples of Recurrence

- Periodic process:
  return of a state after a given time (period length)



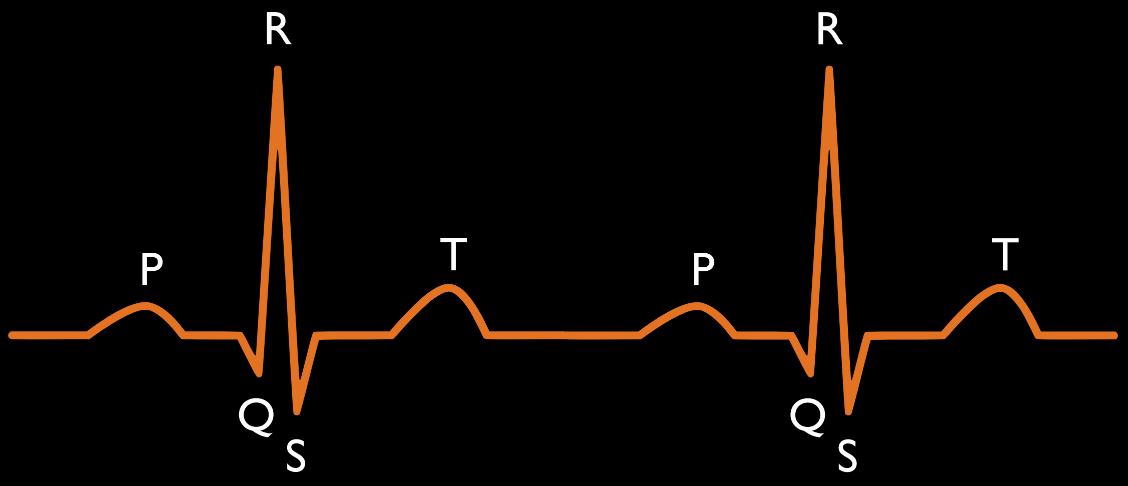
- Stochastic process:
  return of a state just by chance

- Complex dynamical process:
  return patterns contain a lot of information!



Recurrence

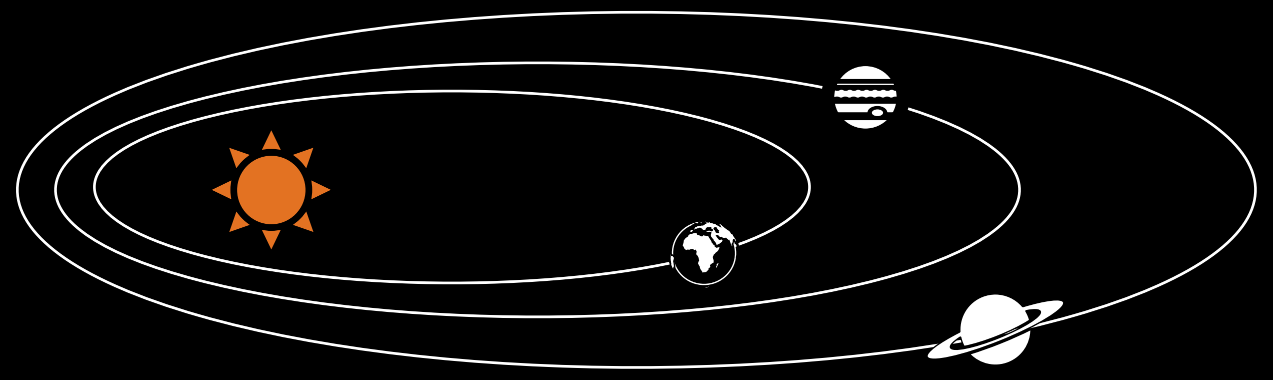
- Recurrences ubiquitous in real life:
  oscillations and resonance in engineering, heart beat after exertion, predator-prey cycles, etc.
- Fundamental characteristic of many dynamical systems



Recurrence

- H. Poincaré, 1890:
  (in the attempt to solve the 3-body problem of celestial mechanics)

  "a system recurs infinitely many times as close as one wishes to its initial state"

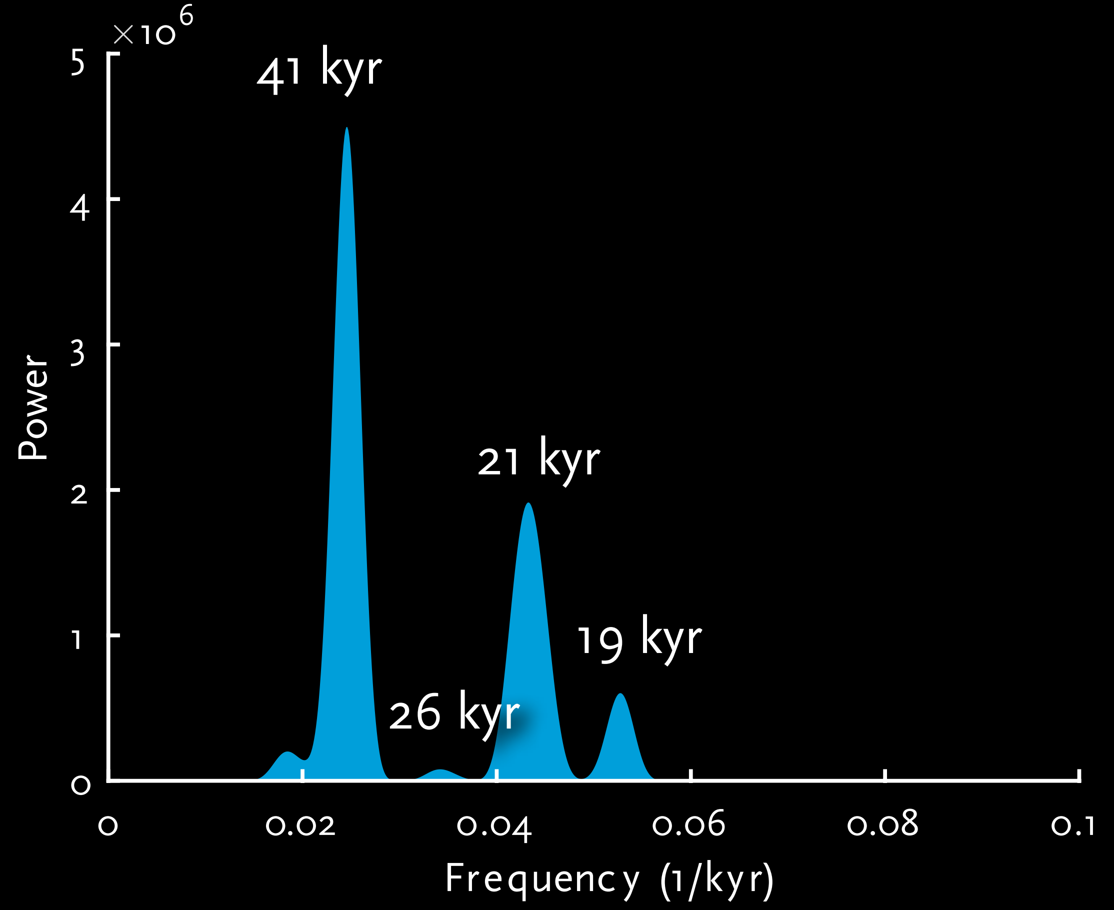


Recurrence

Courtesy: G. Wieschendahl, 2004

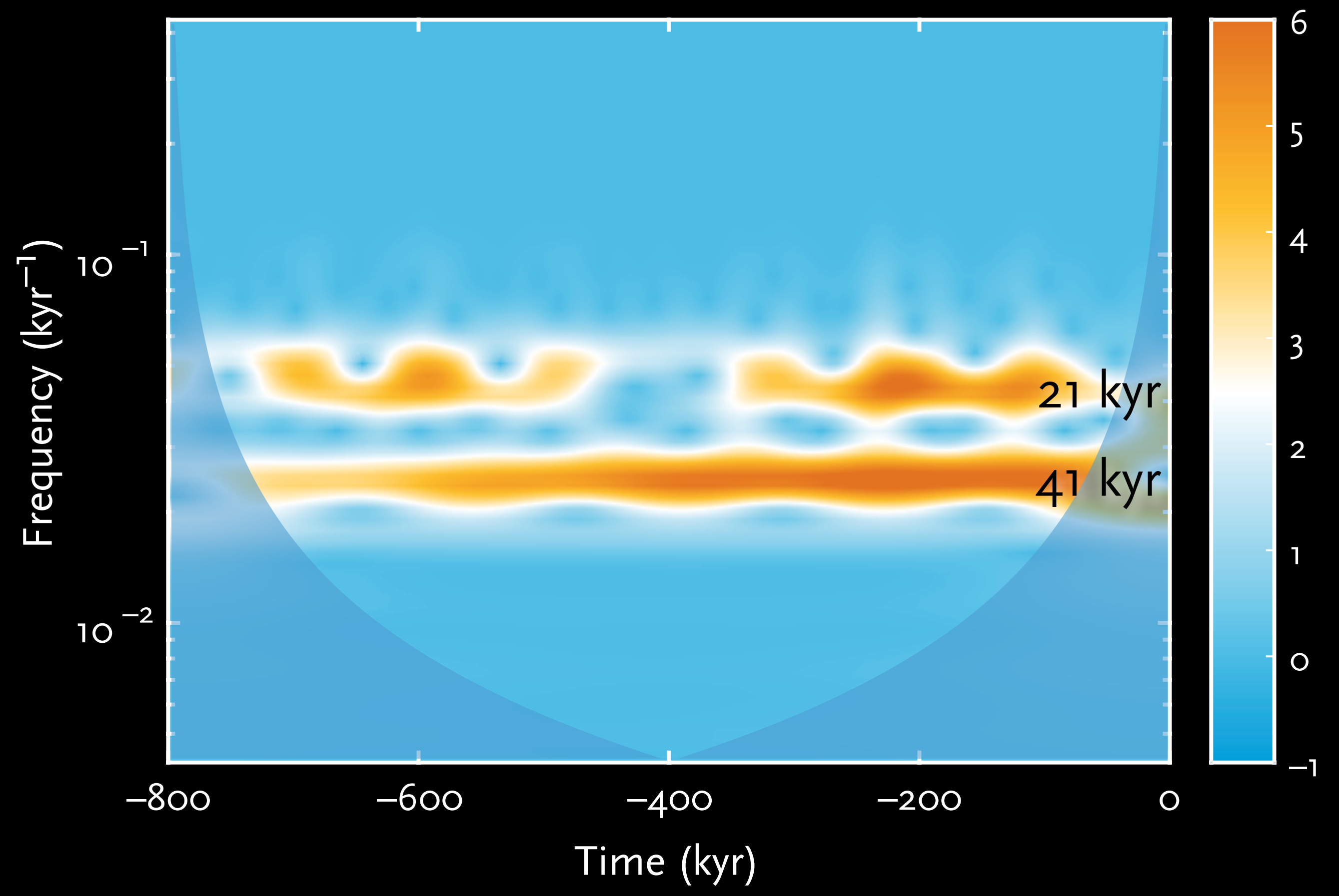
Tools for Studying Recurrences

Power spectrum
(insolation at 46°N)



$$F(f) = \left| \int_{-\infty}^{\infty} e^{-2\pi i f t} x(t)\, dt \right|^2$$

Wavelet analysis
(insolation at 46°N)



$$S_w(a,b) = \frac{1}{|a|^{1/2}} \int_{-\infty}^{\infty} x(t) \overline{\psi} \left(\frac{t-b}{a} \right) dt$$

Tools for Studying Recurrences

Power spectrum
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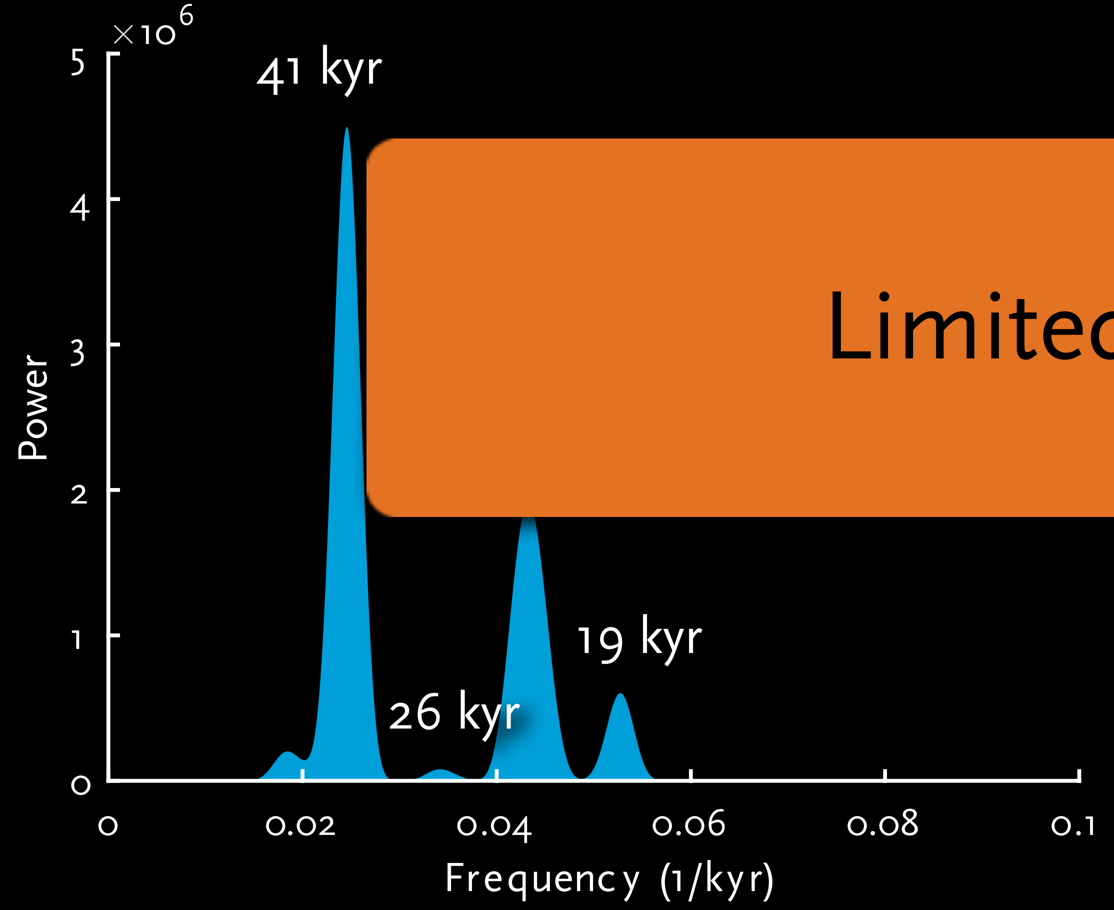

Wavelet analysis
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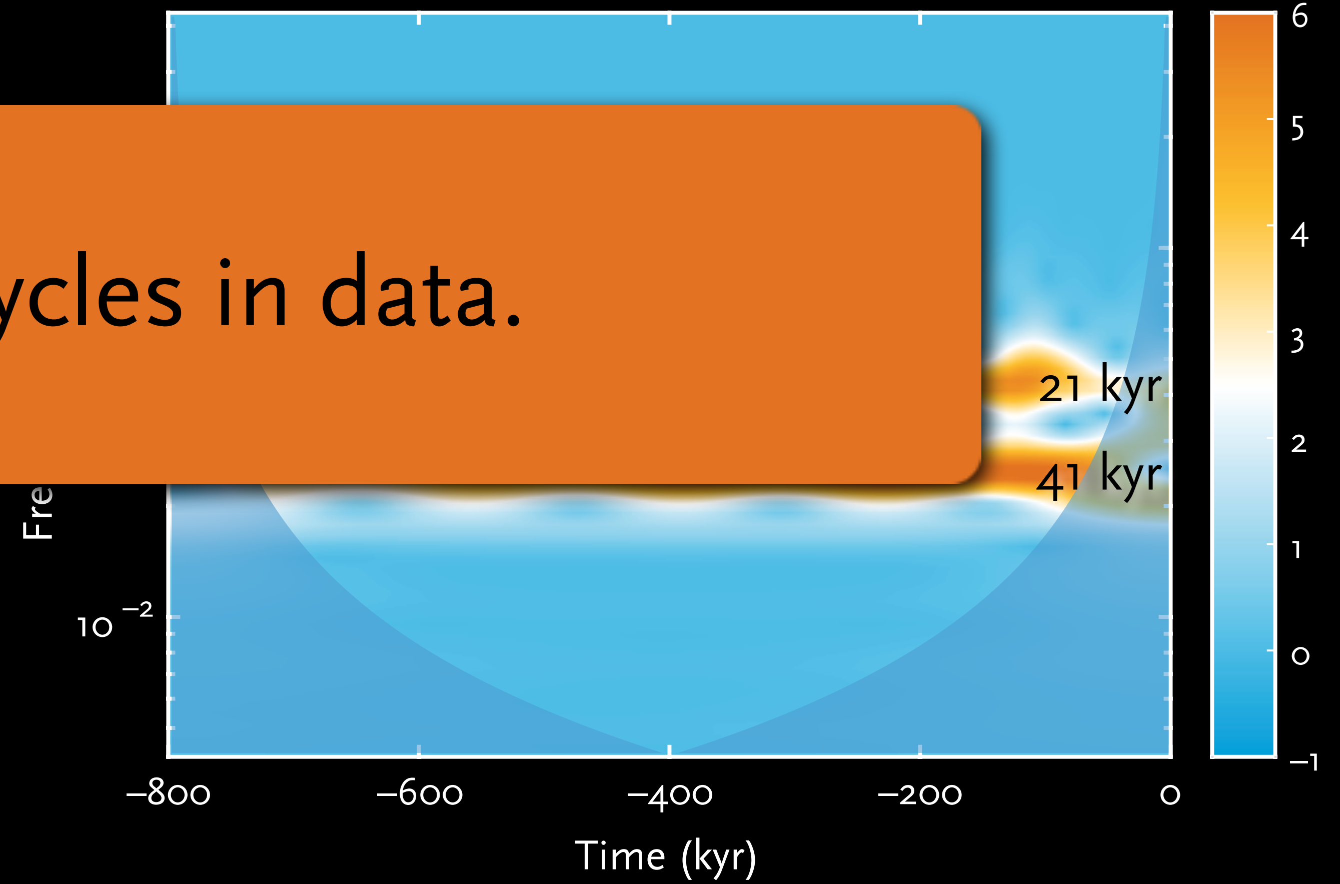

Limited to cycles in data.

$$F(f) = \left| \int_{-\infty}^{\infty} e^{-2\pi i f t} x(t)\, dt \right|^2$$

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Tools for Studying Recurrences

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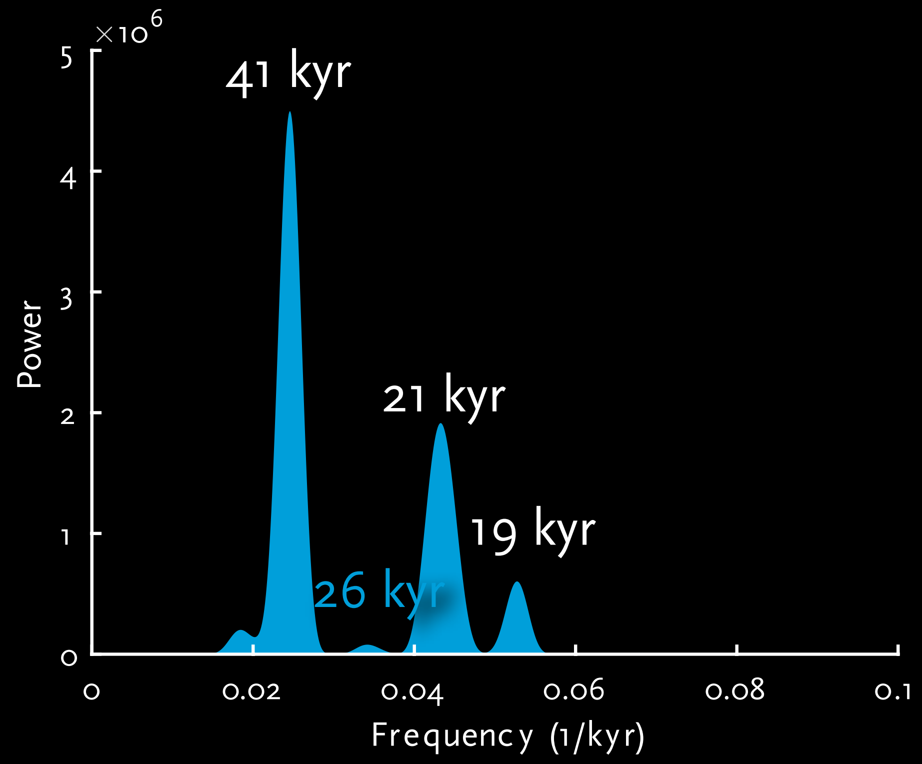

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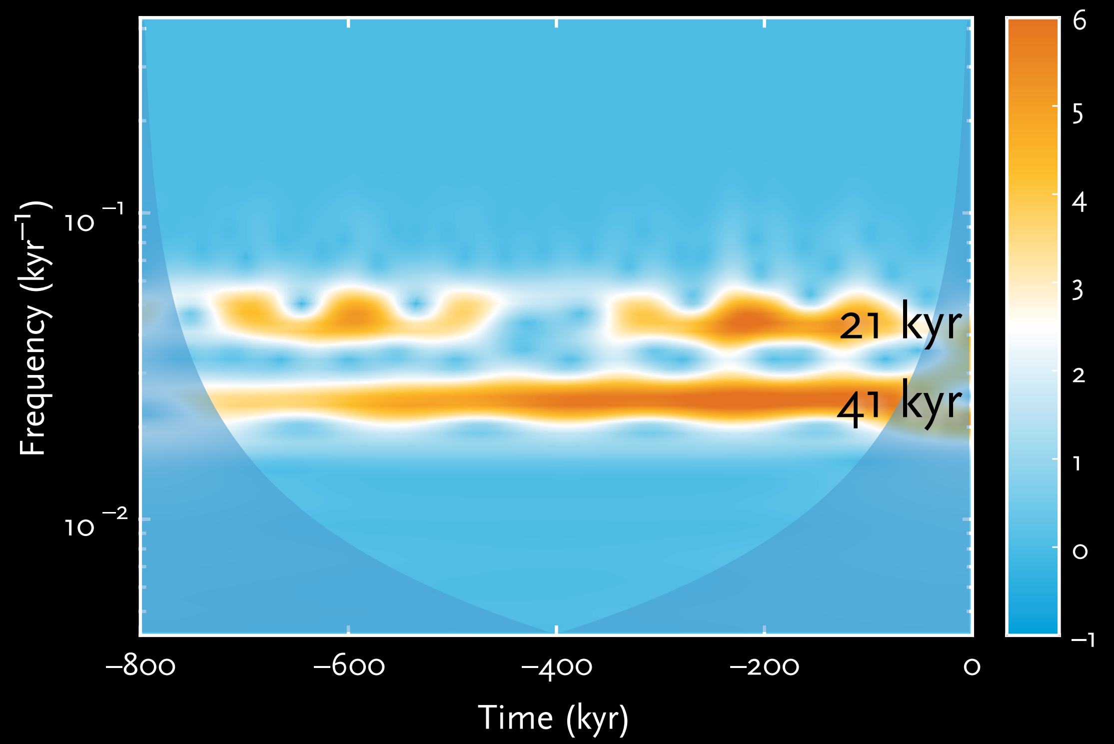
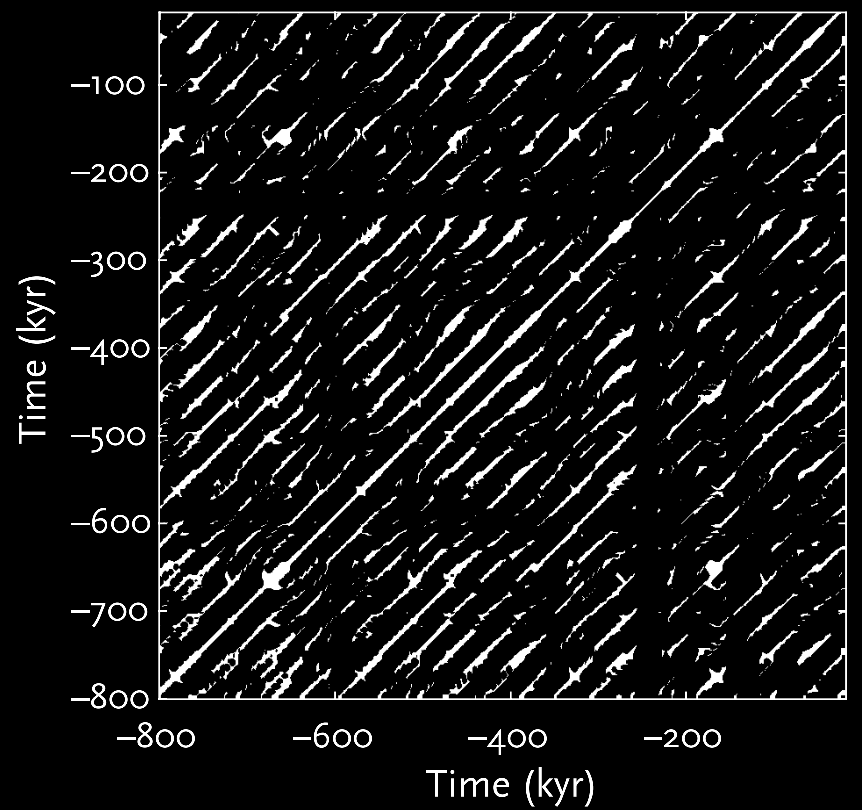


$$S_w(a,b) = \frac{1}{|a|^{1/2}} \int_{-\infty}^{\infty} x(t) \overline{\psi} \left(\frac{t-b}{a} \right) dt$$

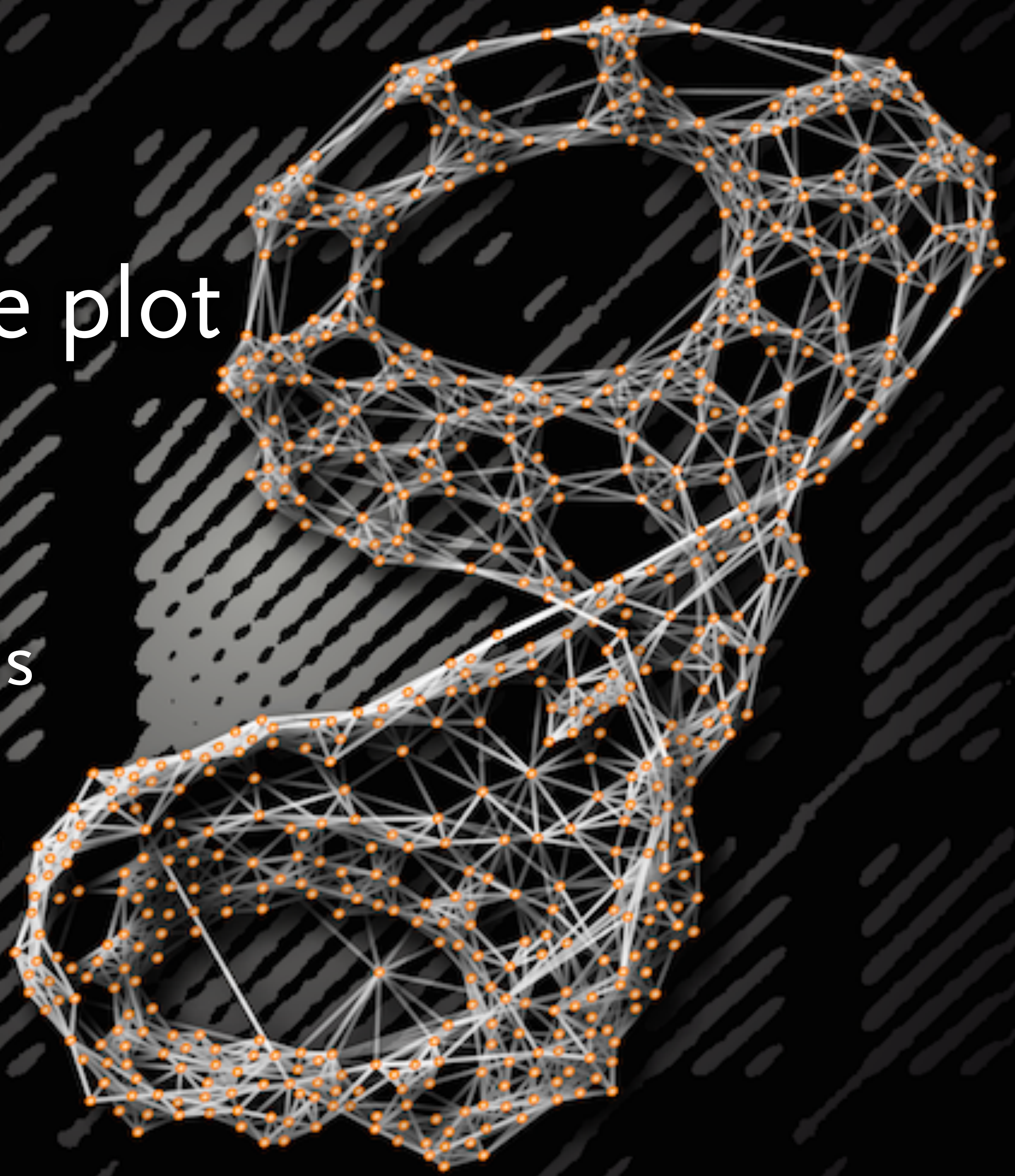
Recurrence plot (insolation at 46°N)



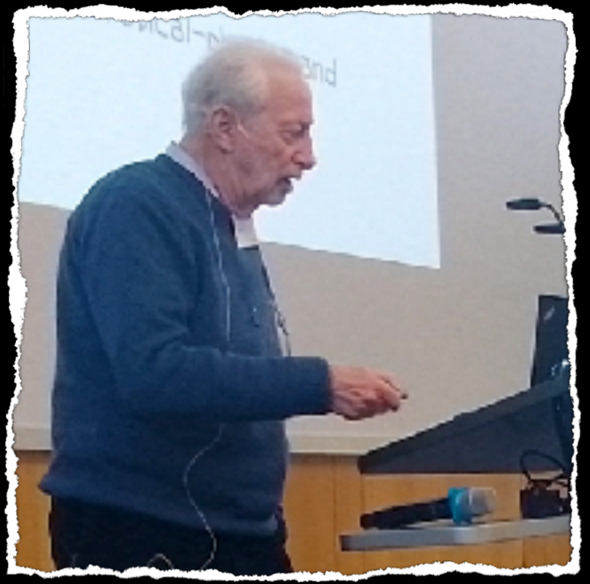
$$R(t_1, t_2) = \begin{cases} 1 : \vec{x}(t_1) \approx \vec{x}(t_2) \\ 0 : \vec{x}(t_1) \not\approx \vec{x}(t_2) \end{cases}$$

Recurrence Analysis for Complex Systems

- **Phase space and recurrence plot**
- Recurrence quantification
- Coupling and synchronization analysis
- Outlook



J. P. Eckmann, 1987



EUROPHYSICS LETTERS 1 November 1987

Europhys. Lett., 4 (9), pp. 973-977 (1987)

Recurrence Plots of Dynamical Systems.

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(*) *Département de Physique Théorique, Université de Genève*
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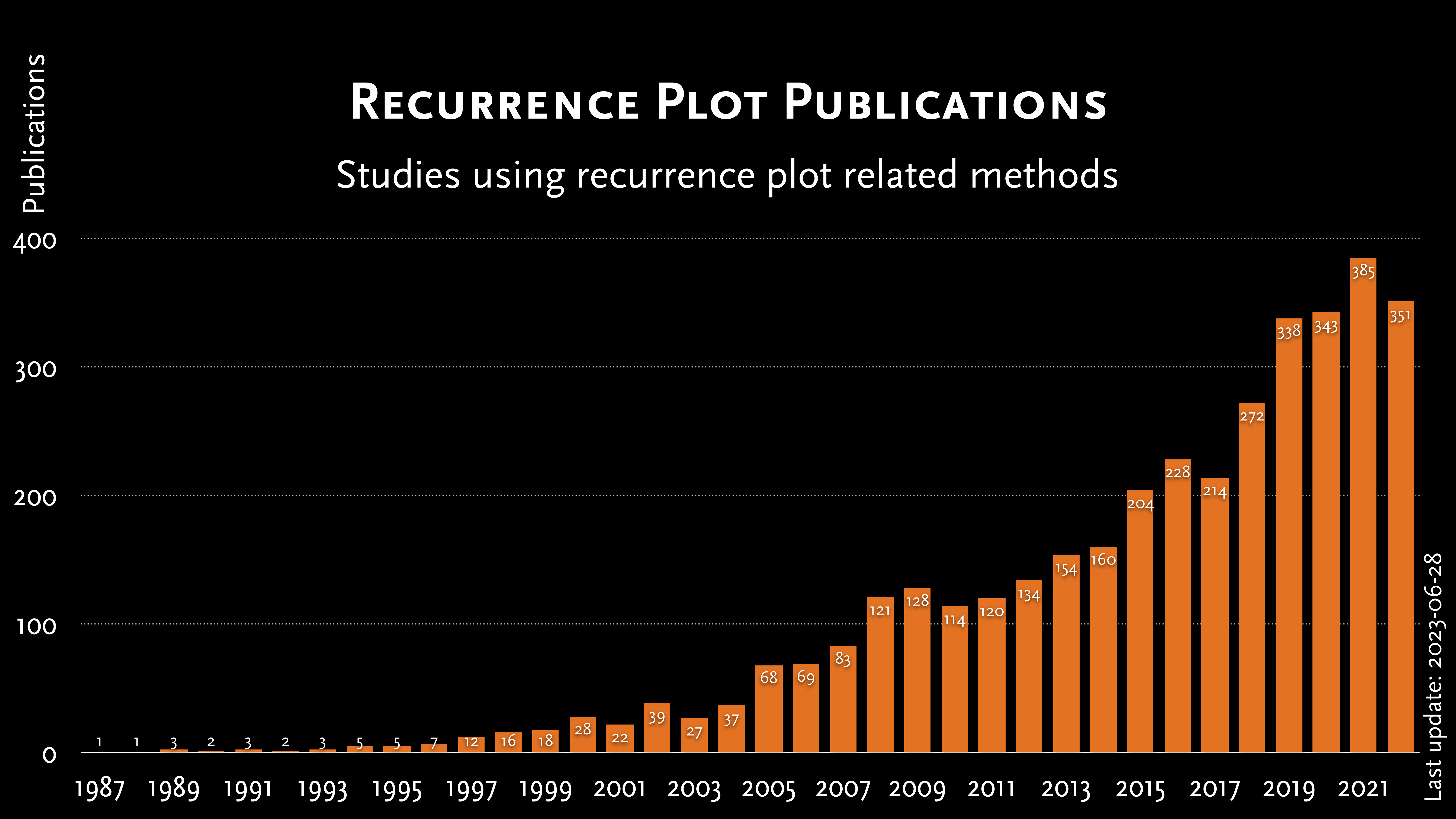
(received 23 June 1987; accepted 10 August 1987)

PACS 05.40. – Fluctuation phenomena, random processes, and Brownian motion.

Abstract. – A new graphical tool for measuring the time constancy of dynamical systems is presented and illustrated with typical examples.

In recent years a number of methods have been devised to compute dynamical parameters from time series [1]. Such parameters are the information dimension, entropy, Liapunov exponents, dimension spectrum, etc. In all cases it is assumed that the time series is obtained from an *autonomous* dynamical system, *i.e.* the evolution equations do not contain the time explicitly. It is also assumed that the time series is much longer than the characteristic times of the dynamical system. In the present letter we present a new diagnostic tool which we call *recurrence plot*; this tool tests the above assumptions, and gives useful information also when they are not satisfied. As the examples will show, the information obtained from recurrence plots is often surprising, and not easily obtainable by

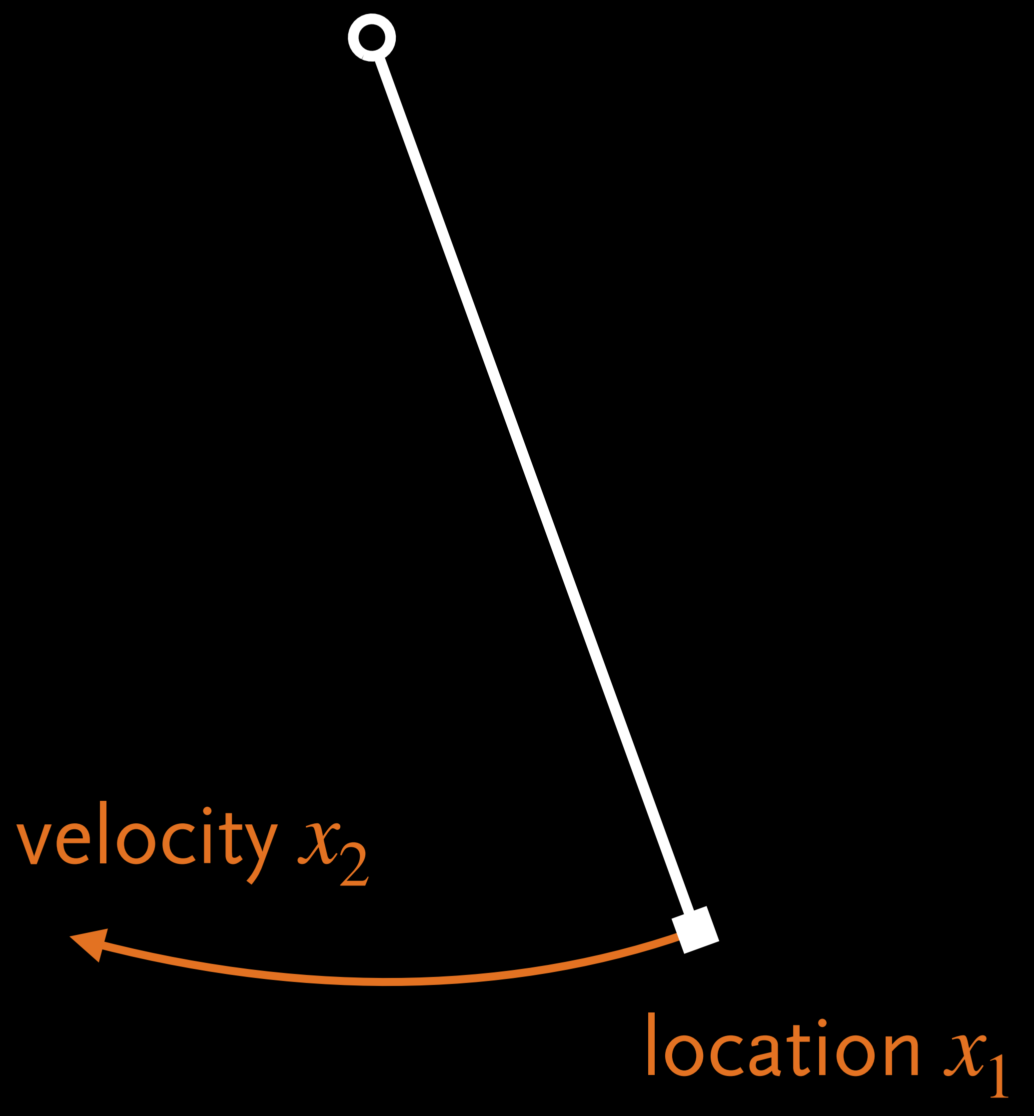

Recurrence Plot Publications
Studies using recurrence plot related methods
Publications
400
300
200
100
0
1 1 3 2 3 2 3 5 5 7 12 16 18 28 22 39 27 37 68 69 83 121 128 114 120 134 154 160 204 228 214 272 338 343 385 351
1987 1989 1991 1993 1995 1997 1999 2001 2003 2005 2007 2009 2011 2013 2015 2017 2019 2021
Last update: 2023-06-28

Phase Space

- State of a physical system:
  state variables

- Example: pendulum
  ➡ 2 state variables:
  location $x_1(t)$, velocity $x_2(t)$

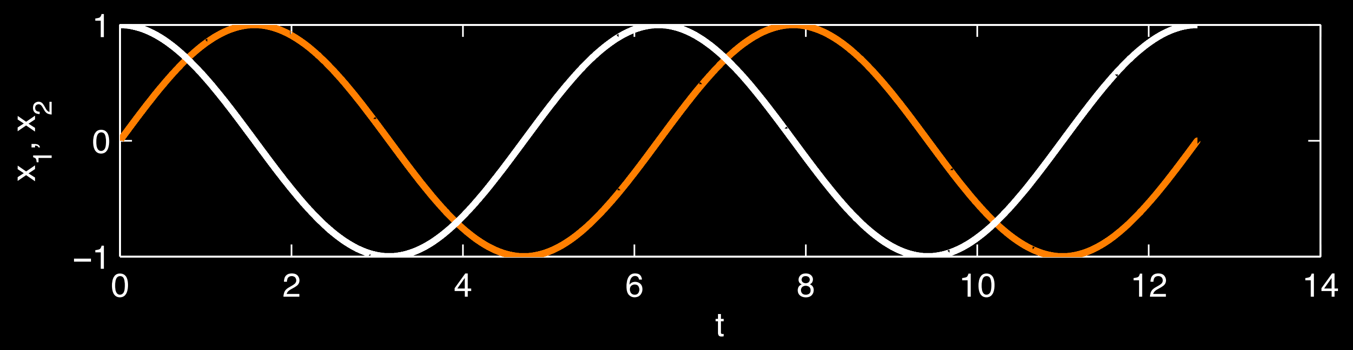
- State variables = vector $\overrightarrow{x}(t) \in \mathbb{R}^d$

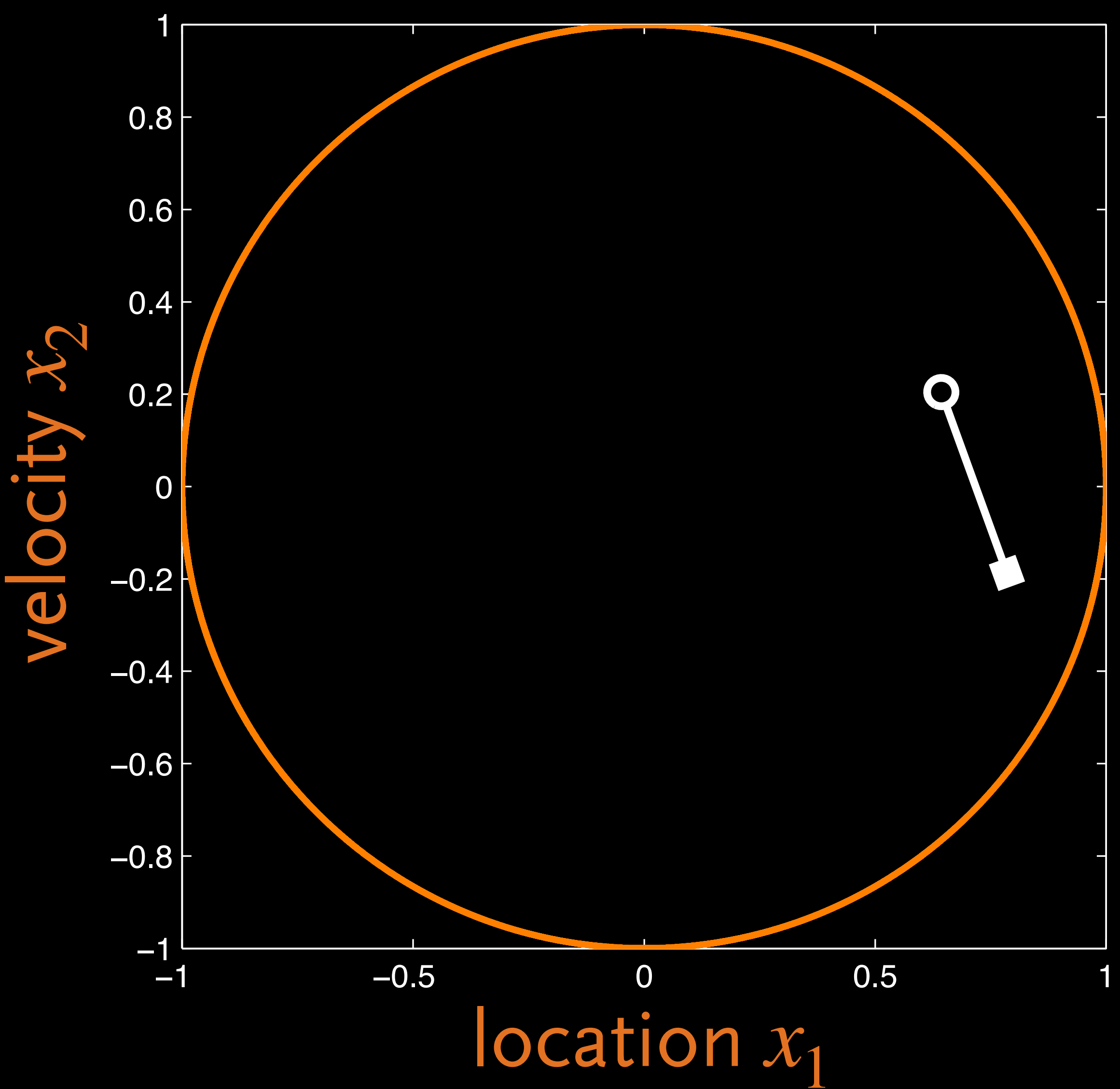


Phase Space

- Dynamics:
  differential equations
  $\dot{\vec{x}}(t) = \vec{F}\left(\vec{x}(t)\right), \qquad F:\ \mathbb{R}^d \to \mathbb{R}^d$

- Temporal order of the state vectors:
  phase space trajectory





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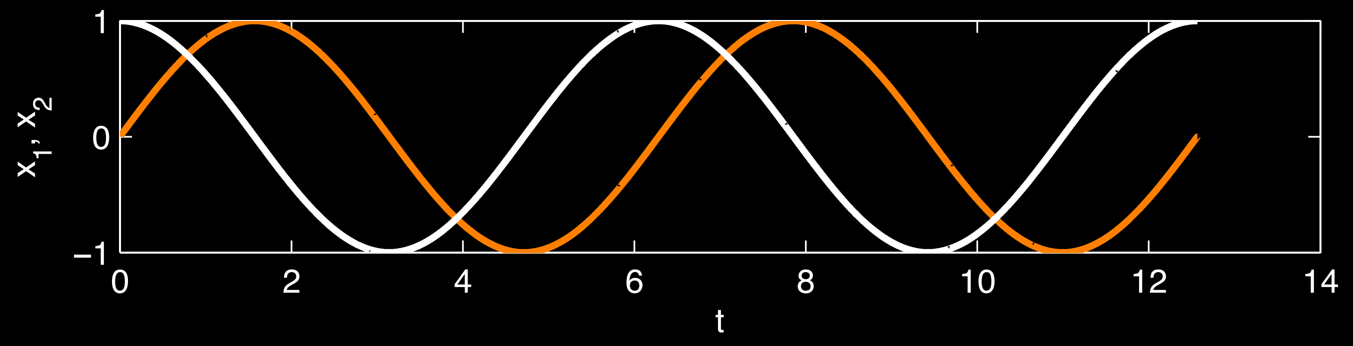

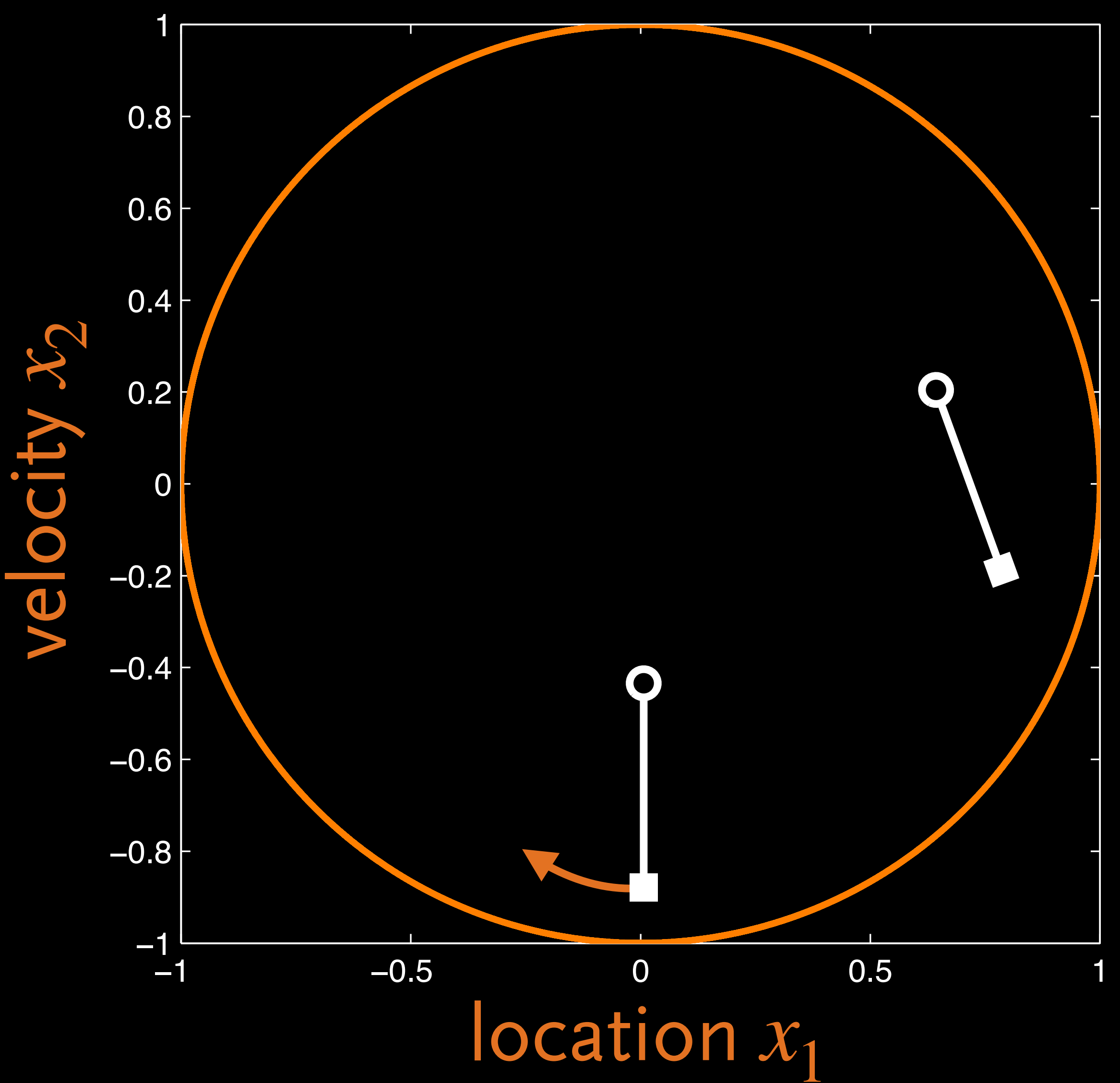

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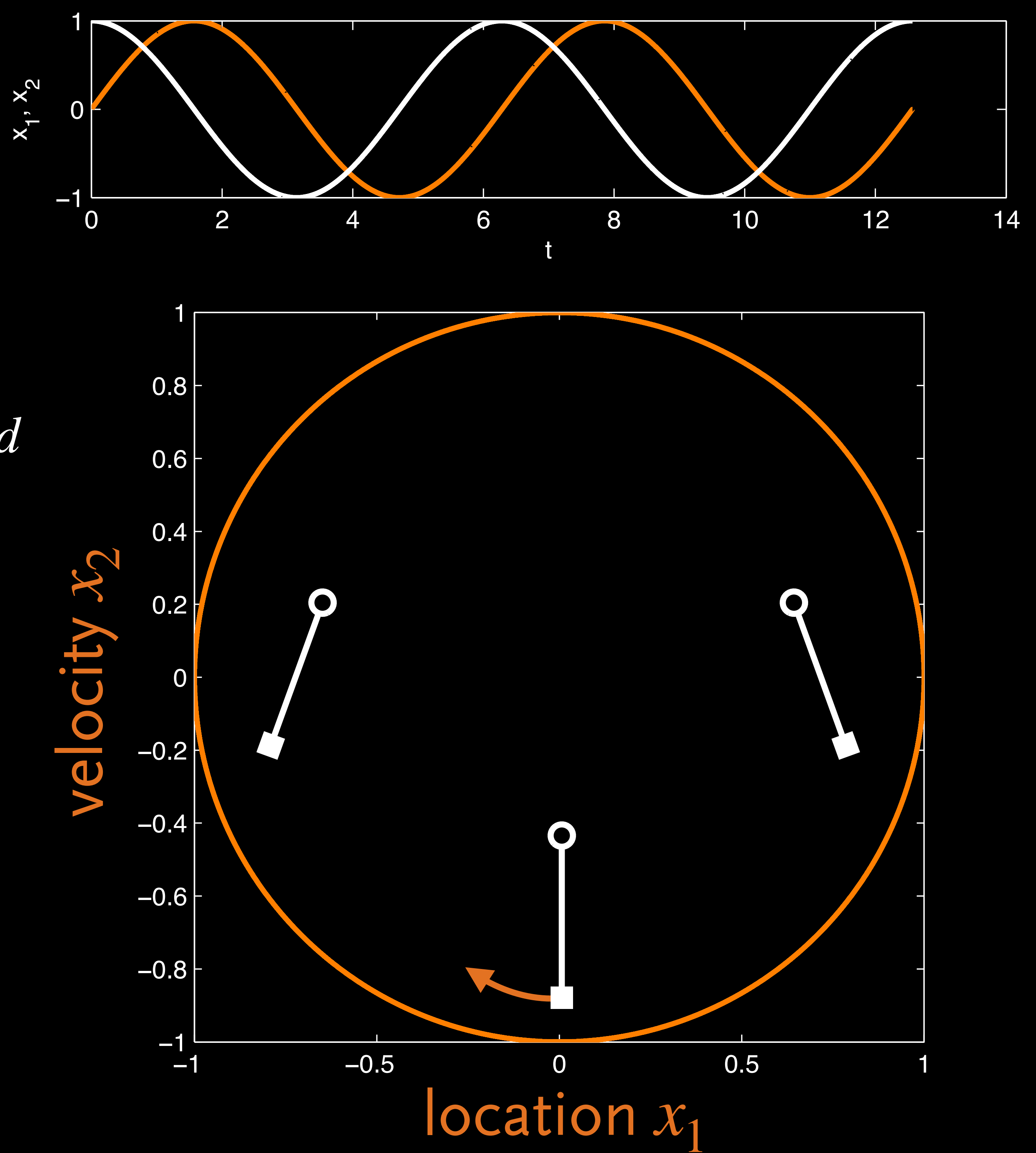

Phase Space

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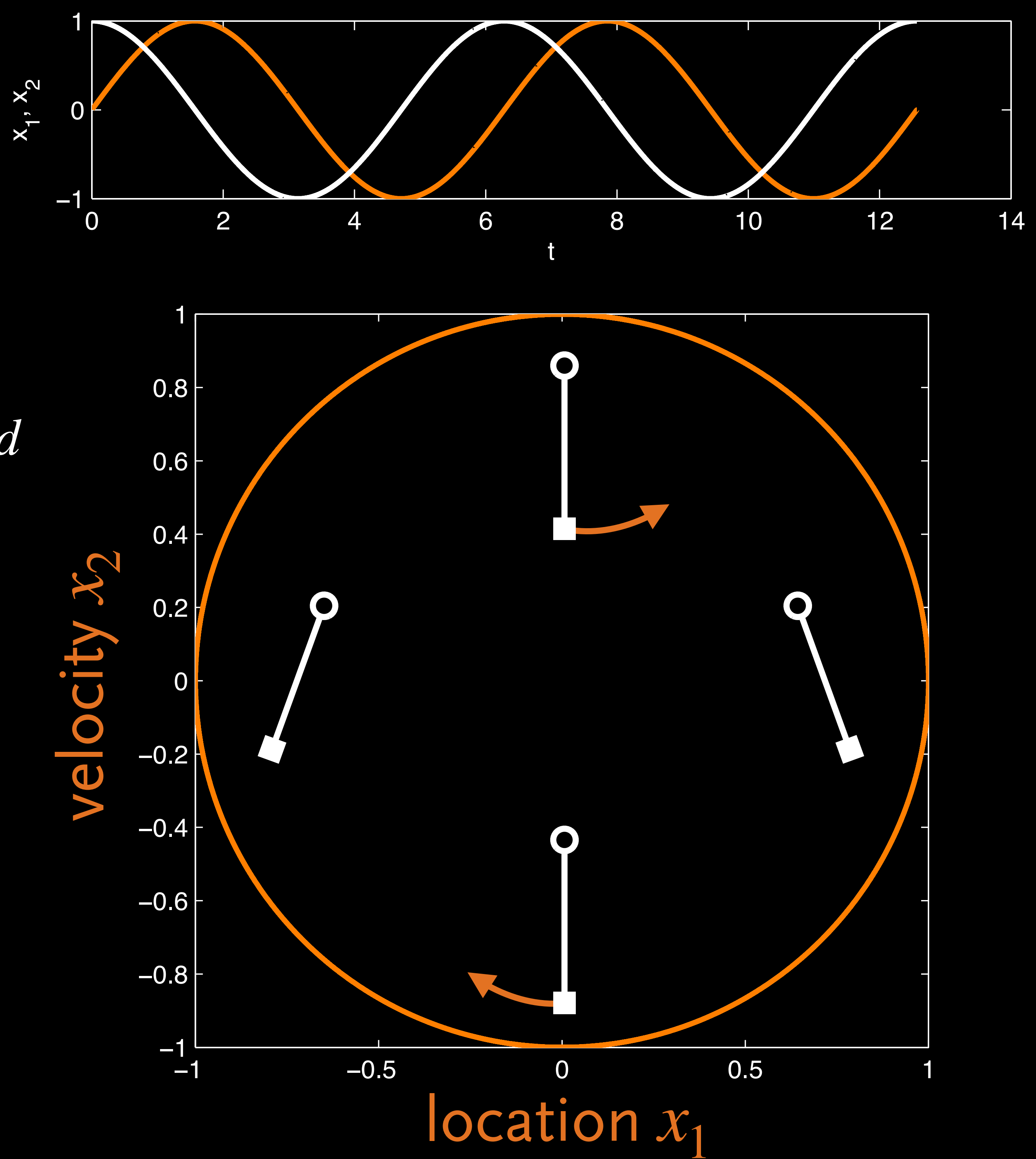
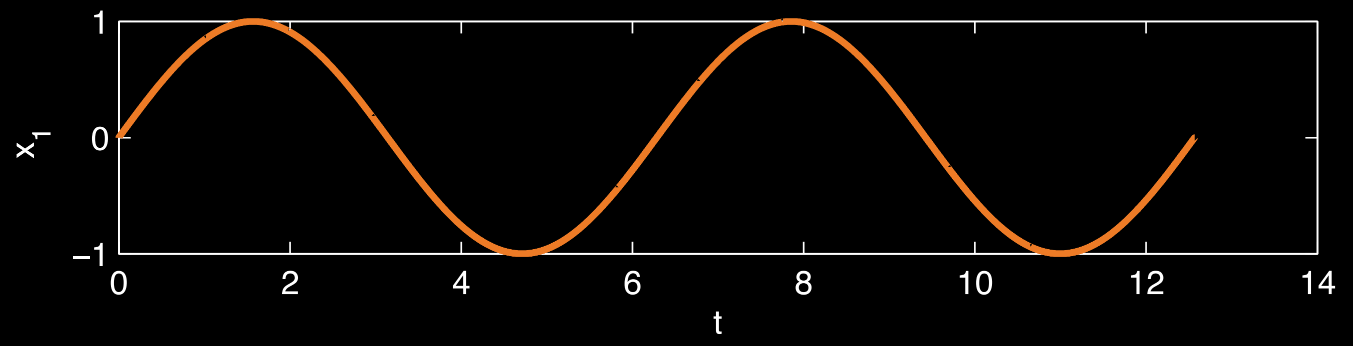


Phase Space

- Measurement:
  only one variable measured,
  e.g., $u(t)$ (or u_i with $t = i\Delta t$)



➡ phase space reconstruction

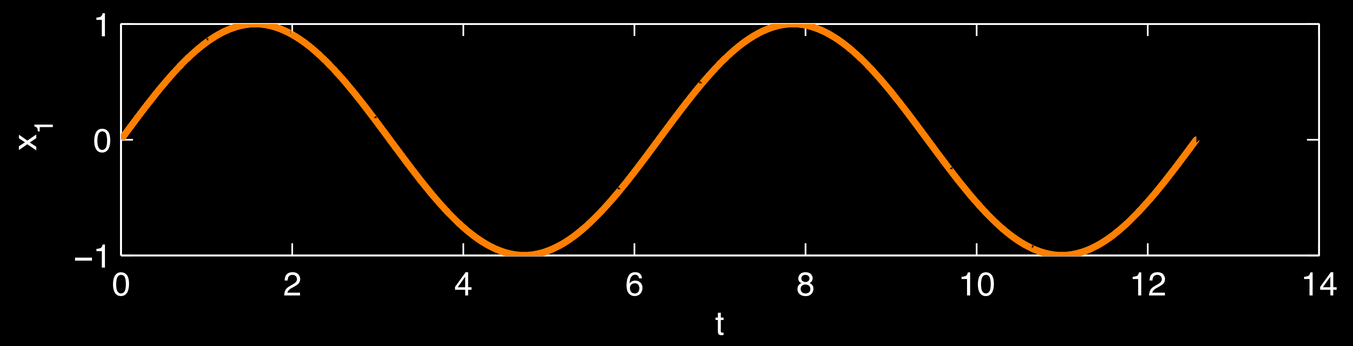
Phase Space Reconstruction

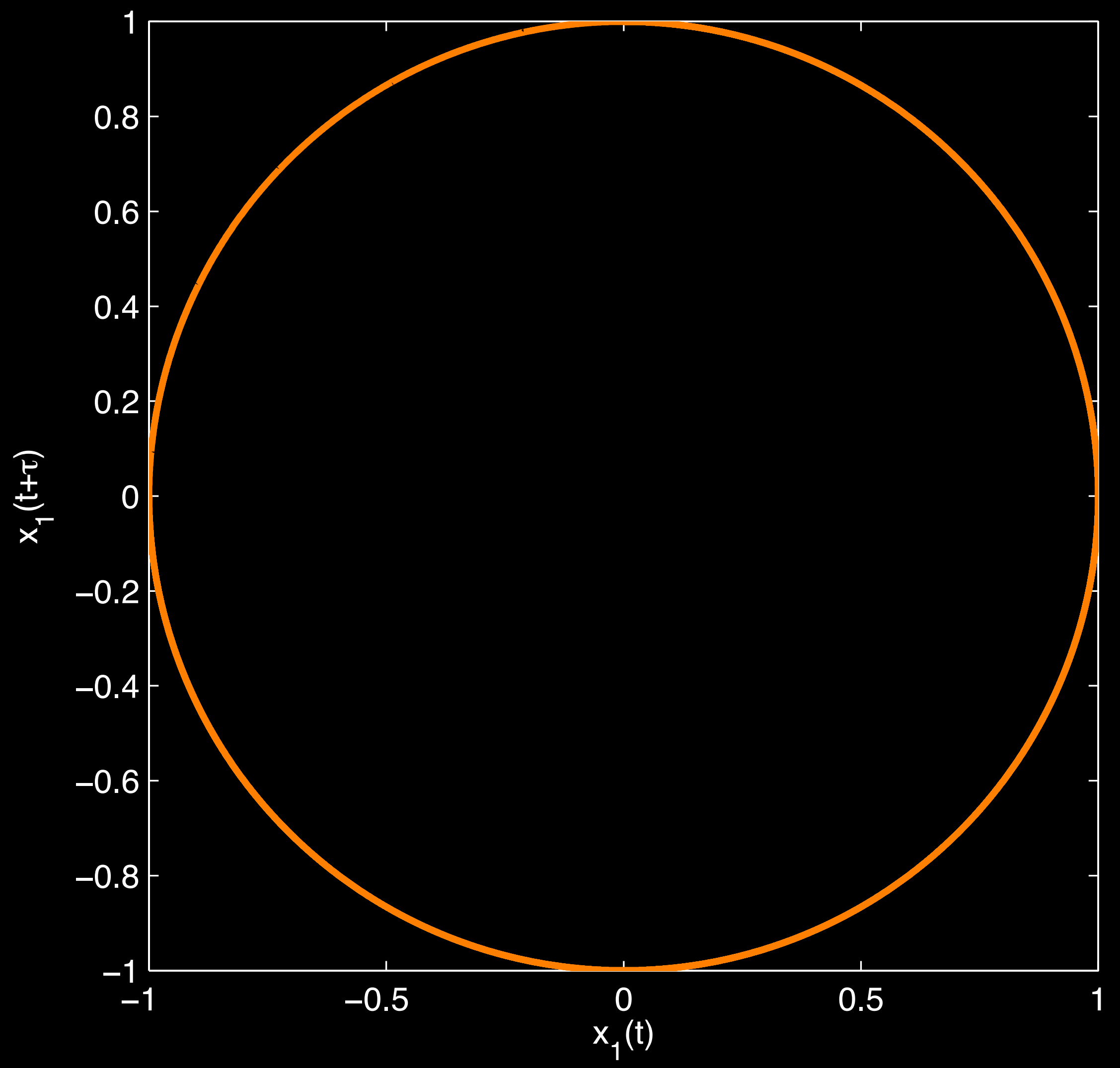
- Measurement:
  only one variable measured,
  e.g., $u(t)$ (or u_i with $t = i\Delta t$)

- Time delay embedding

$$\overrightarrow{x}(i) = \left(u_i, u_{i+\tau}, u_{i+2\tau} \ldots, u_{i+(m-1)\tau} \right)$$

time delay embedding dimension

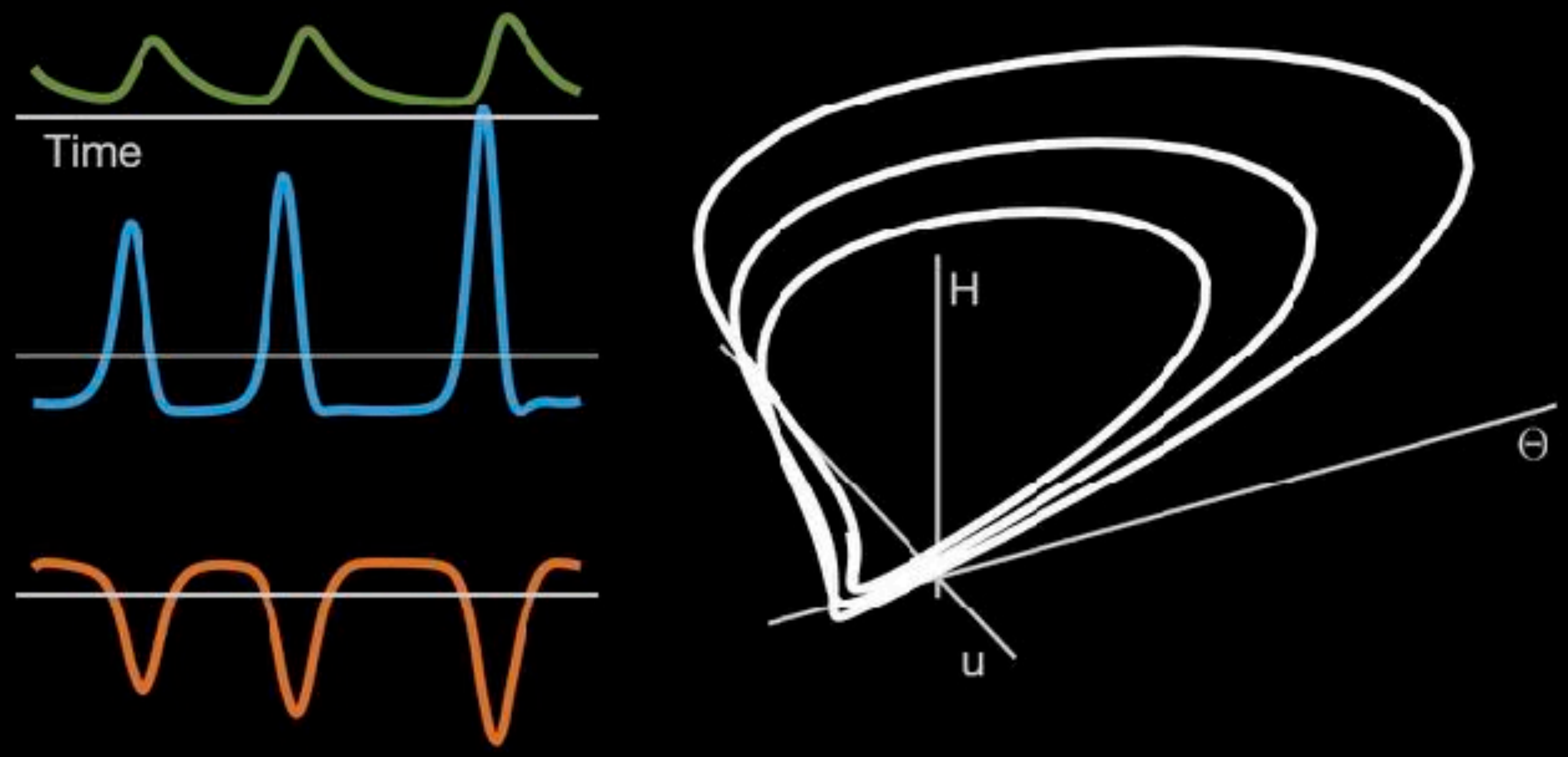




Packard et al, Phys Rev Lett 45, 1980
Takens, Lect Notes Math 898, 1981

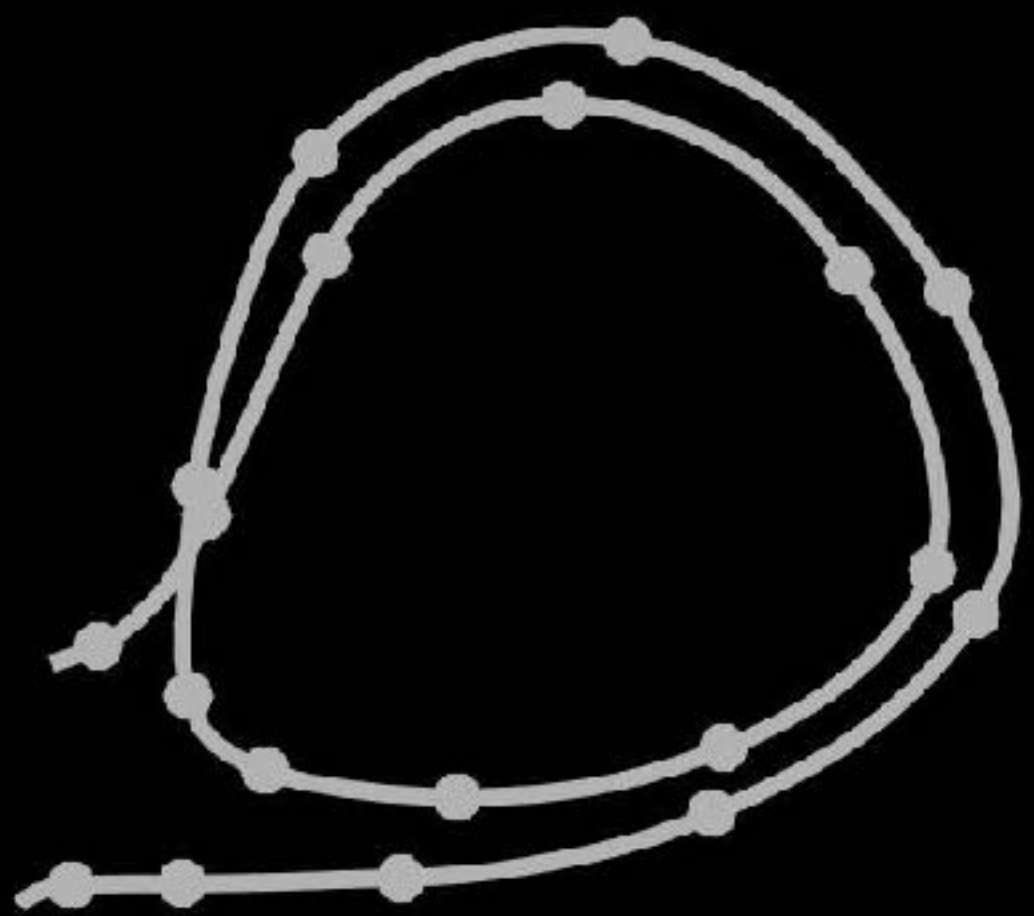
Remark on Phase Space Reconstruction

- Not part of recurrence plots
- Recurrence plots just check recurrences in phase space – how to get it is another question!
- “Multidimensional recurrence plot” – nothing special, just using the standard definition of a phase space

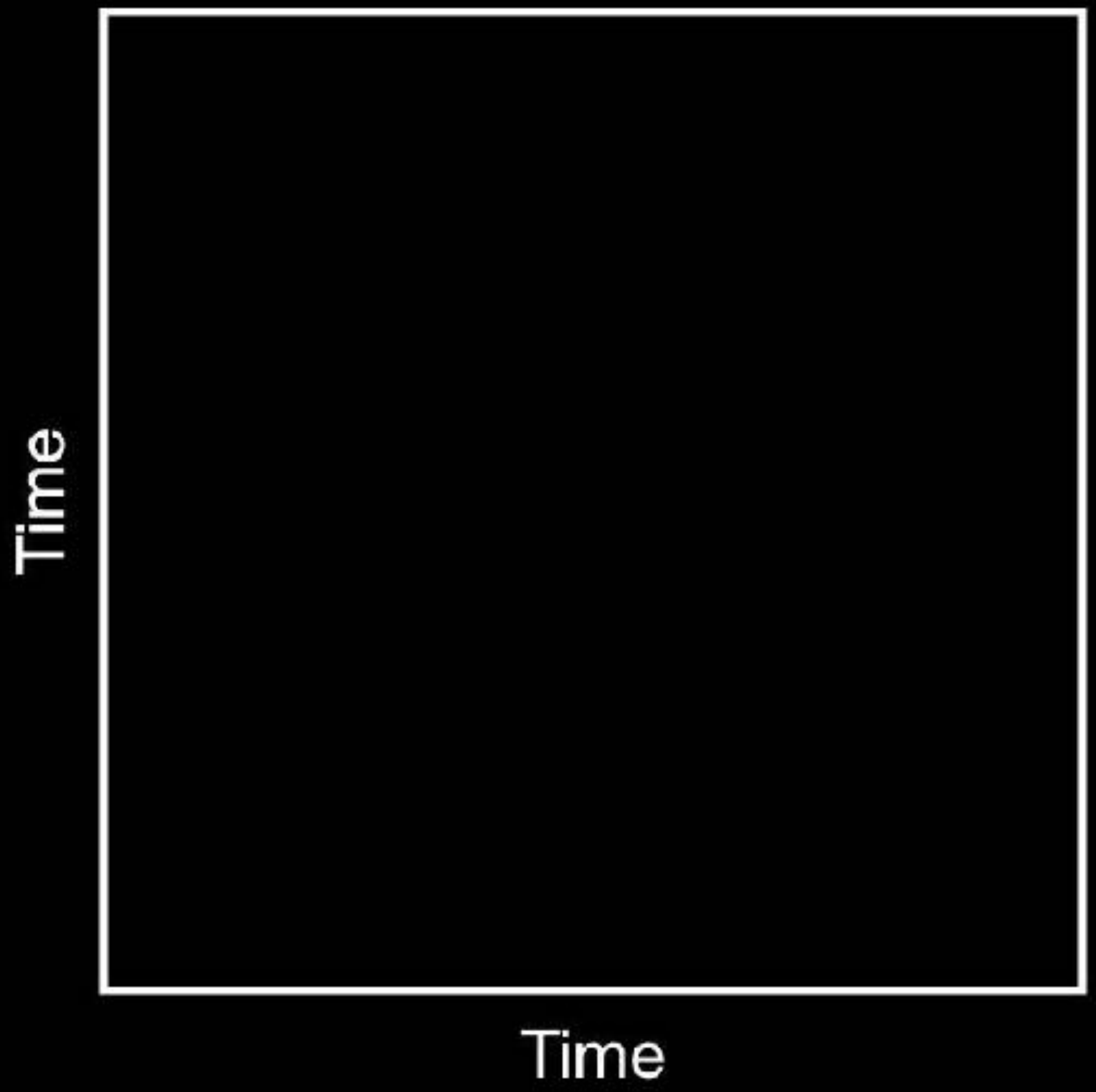


Recurrence Plot

Phase space trajectory

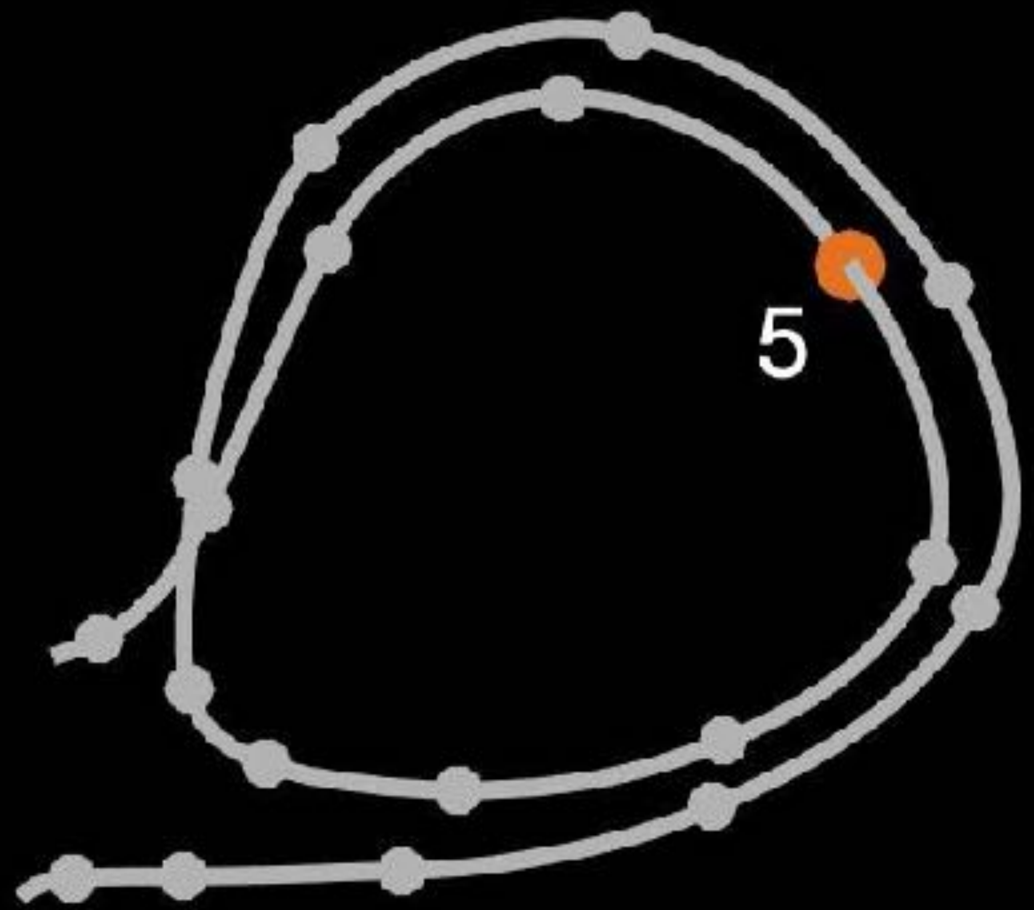


Recurrence matrix

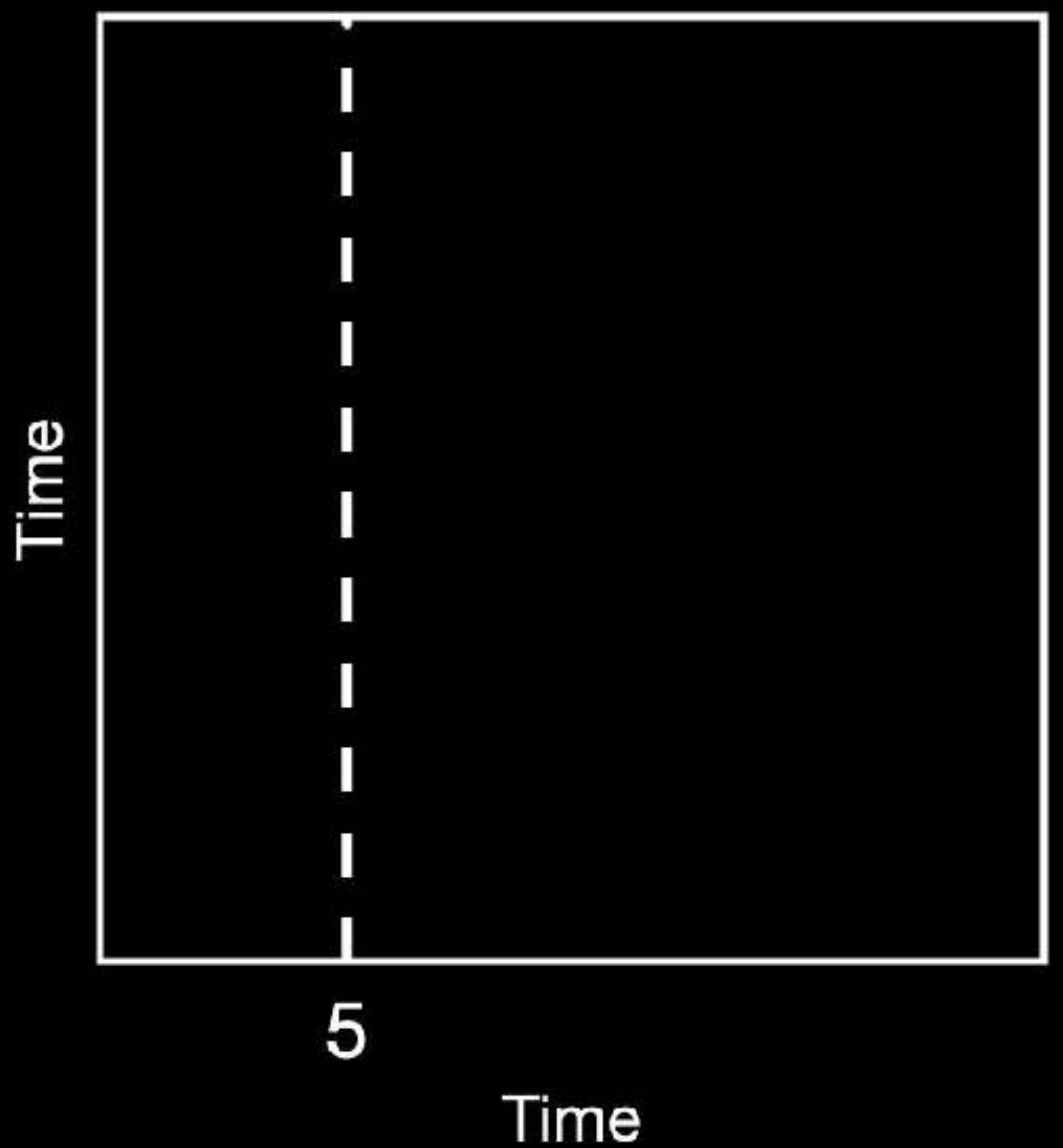


Recurrence Plot

Phase space trajectory

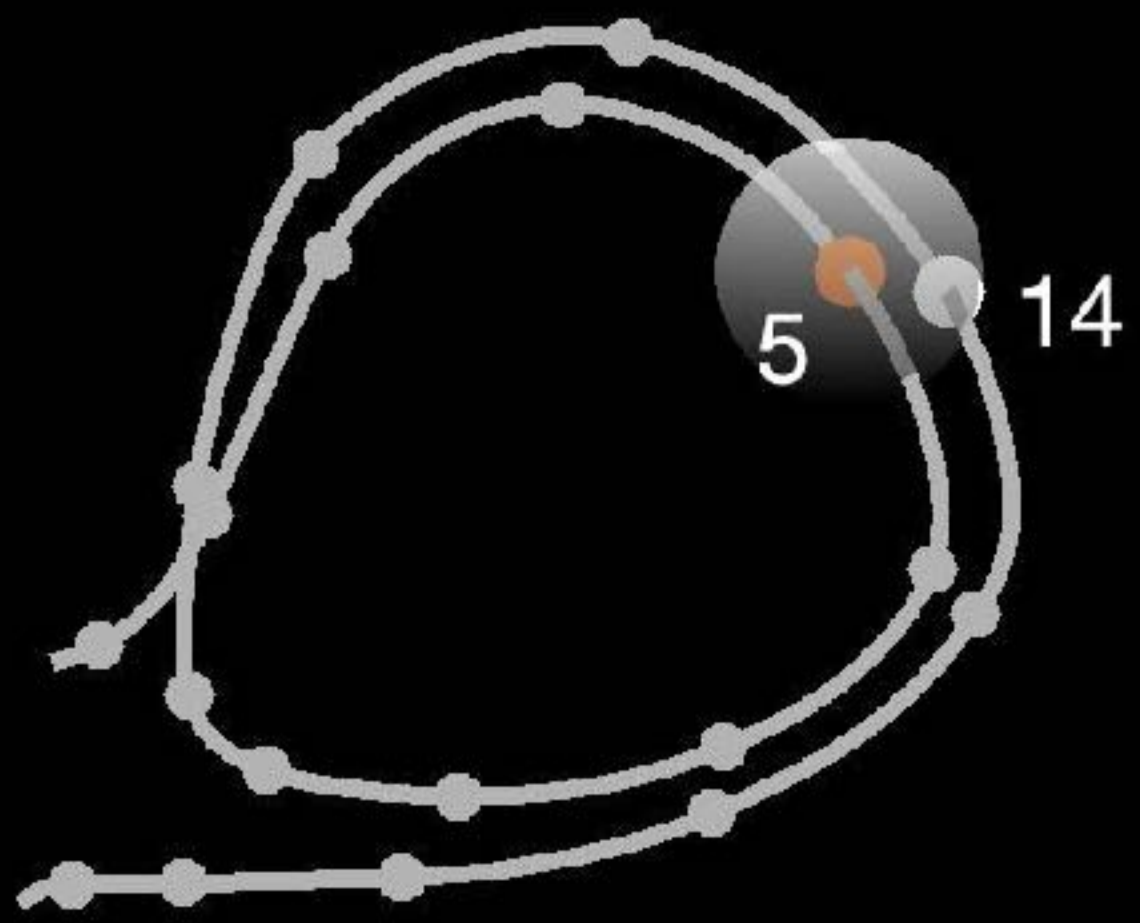


Recurrence matrix



Recurrence Plot

Phase space trajectory



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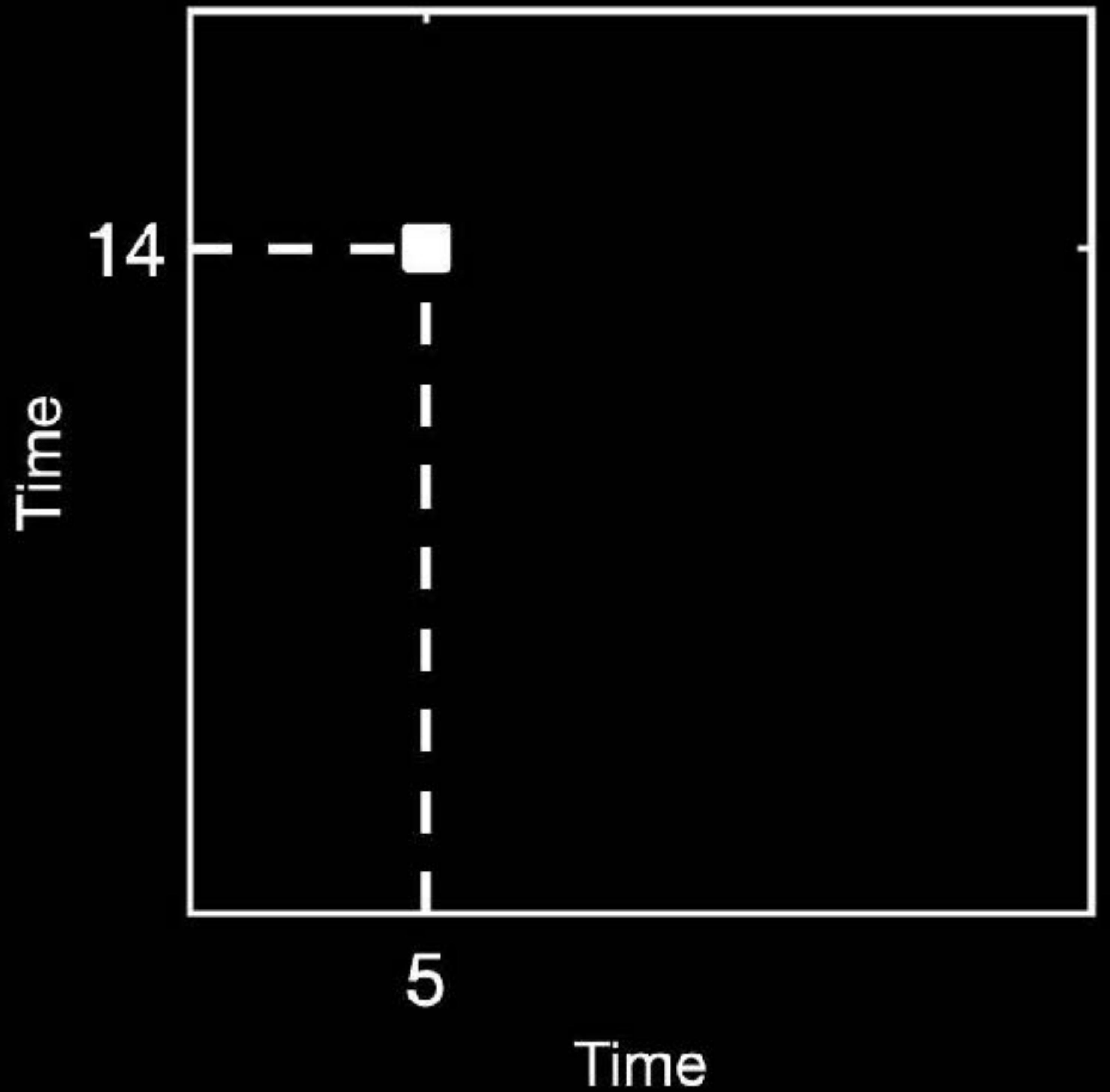
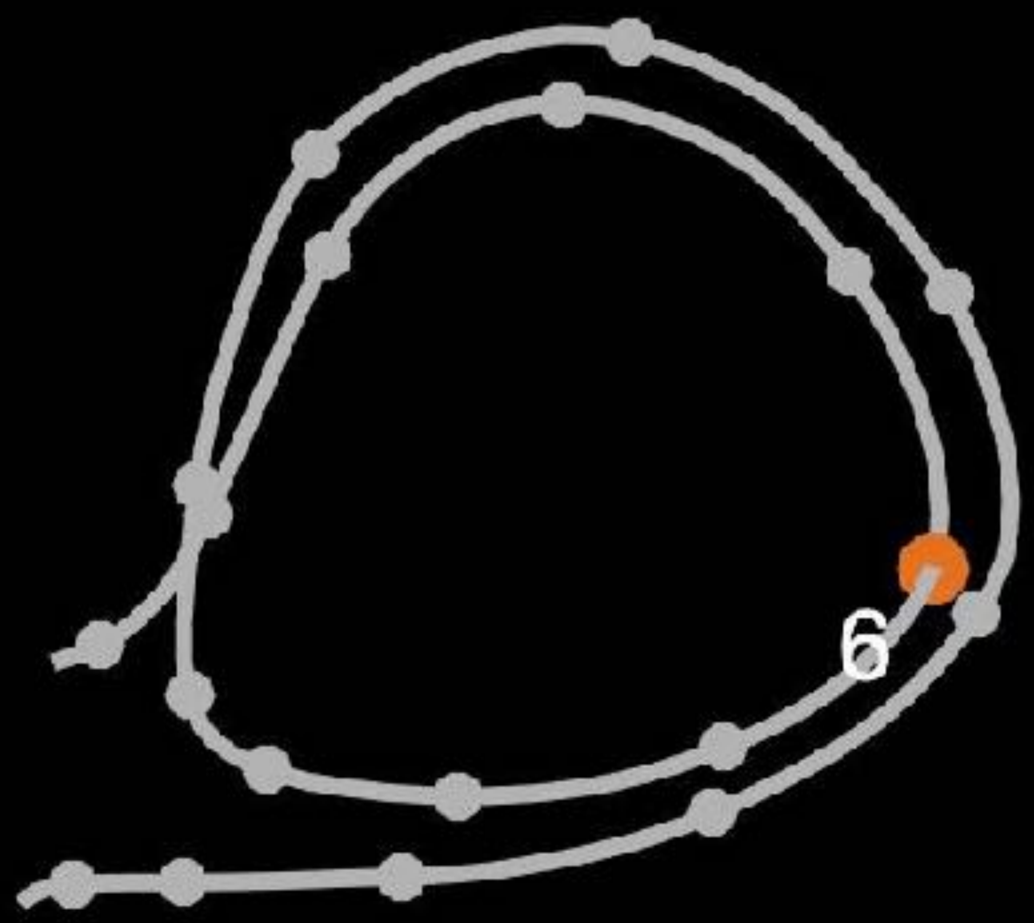
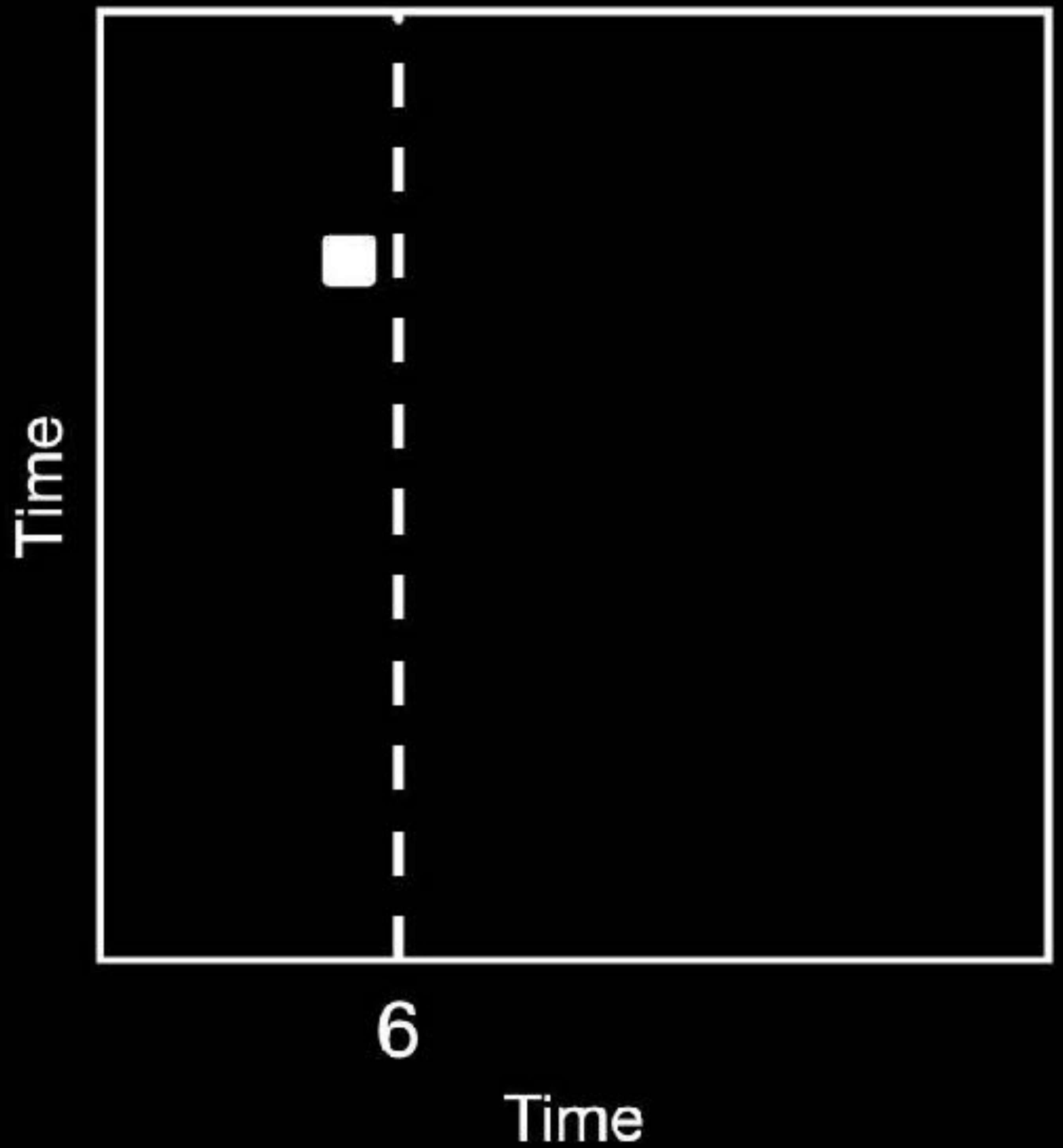


Recurrence Plot

Phase space trajectory

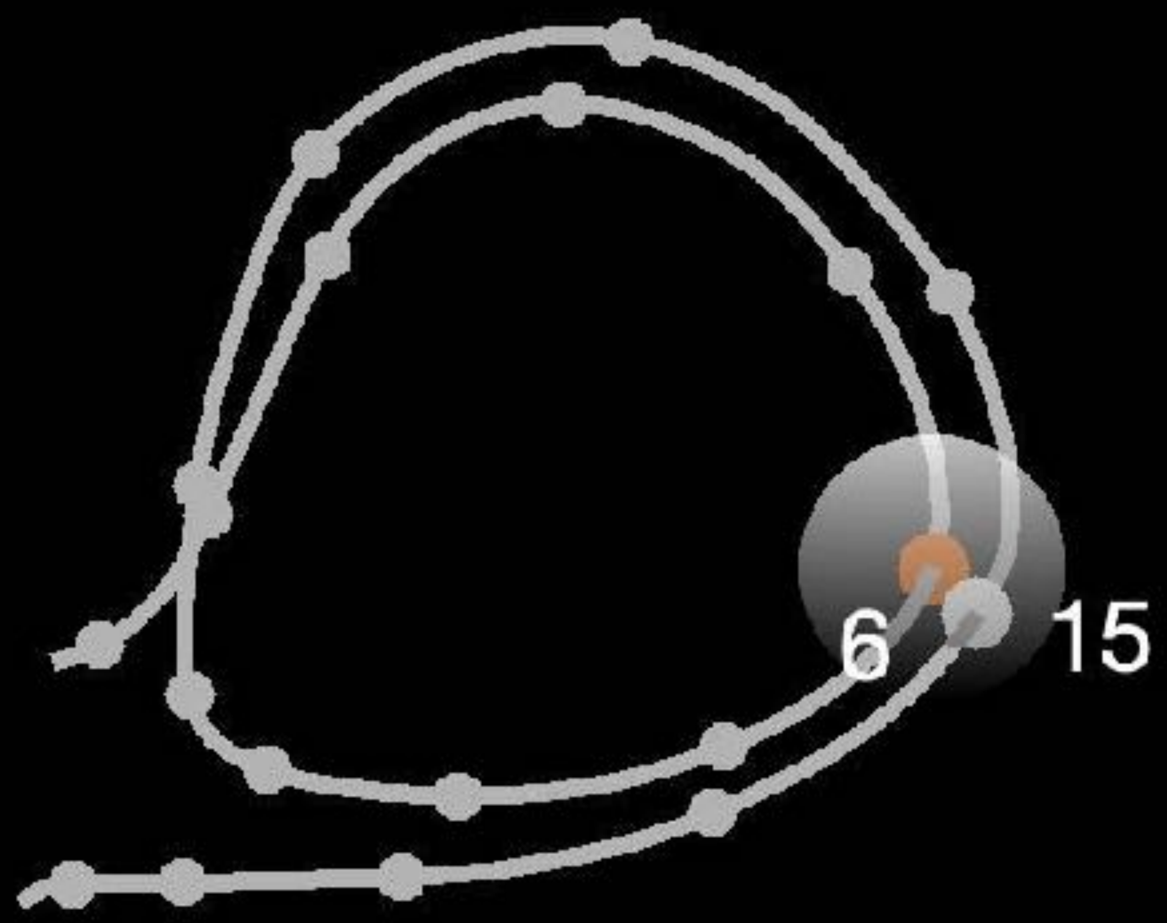


Recurrence matrix



Recurrence Plot

Phase space trajectory



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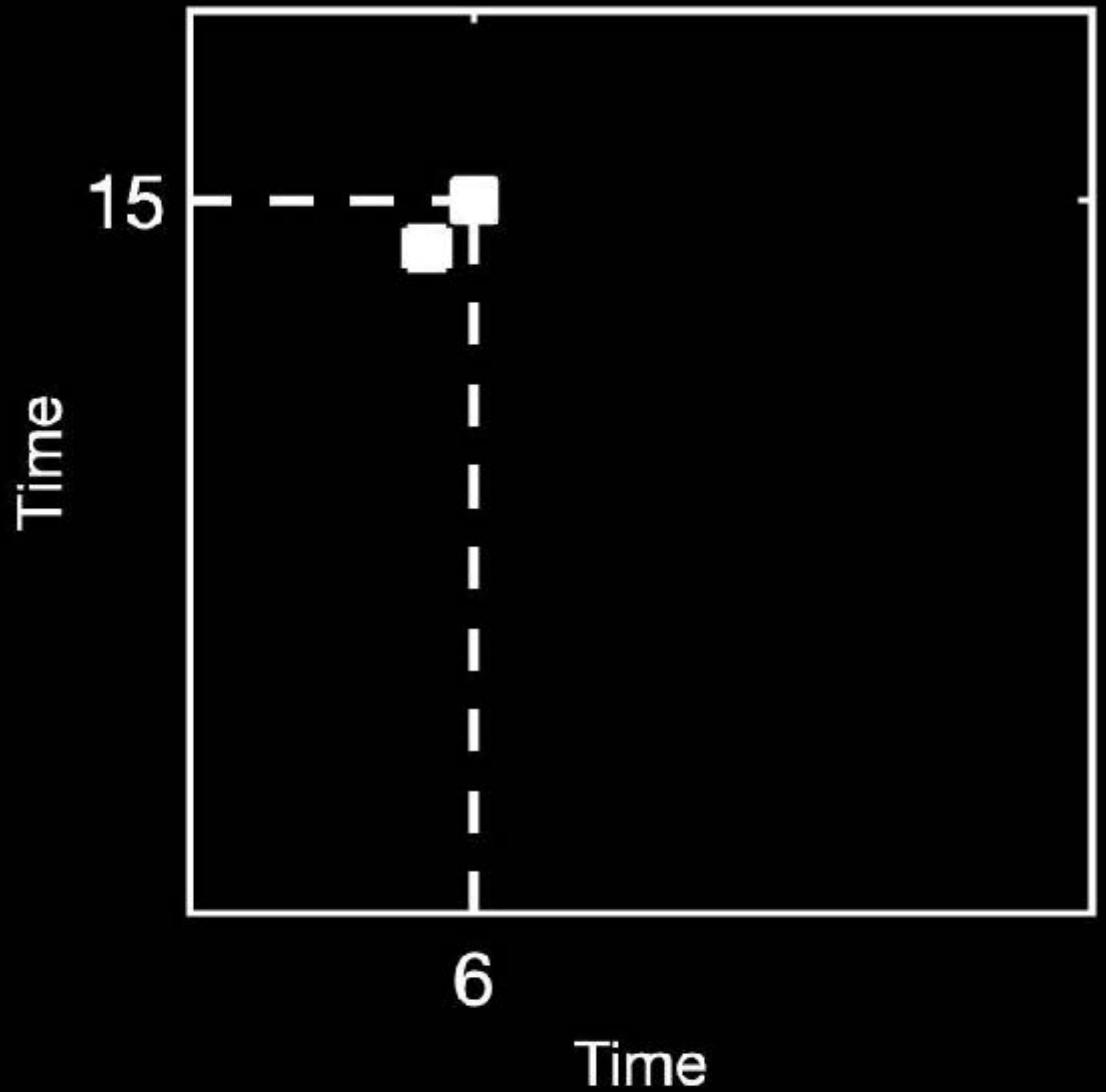
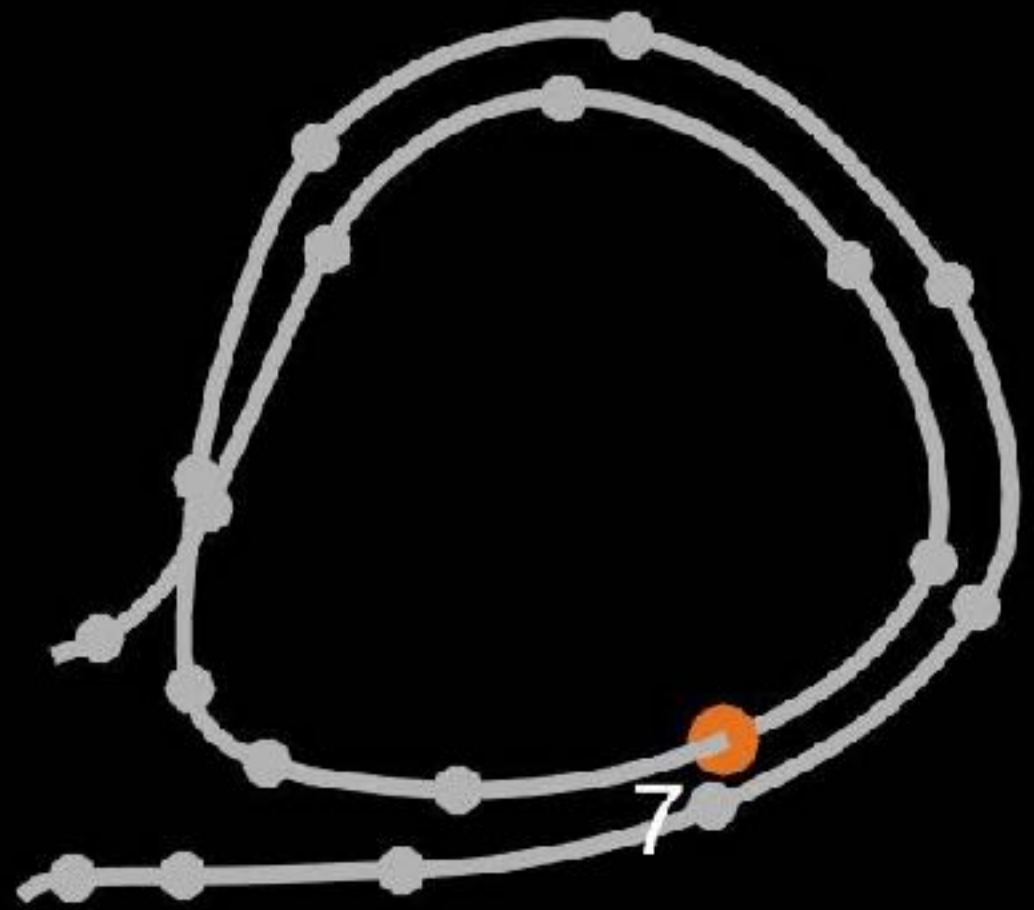
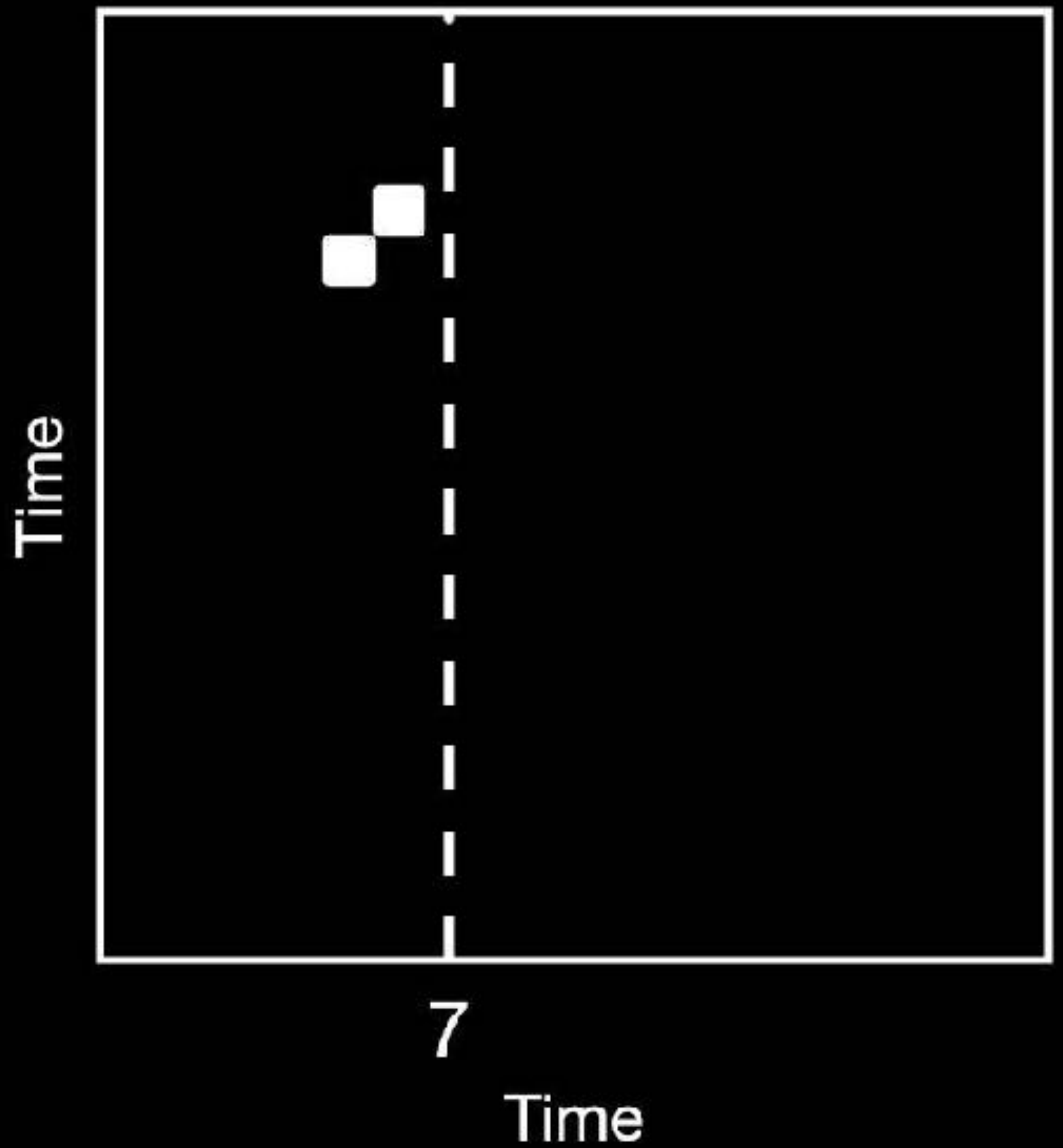


Recurrence Plot

Phase space trajectory

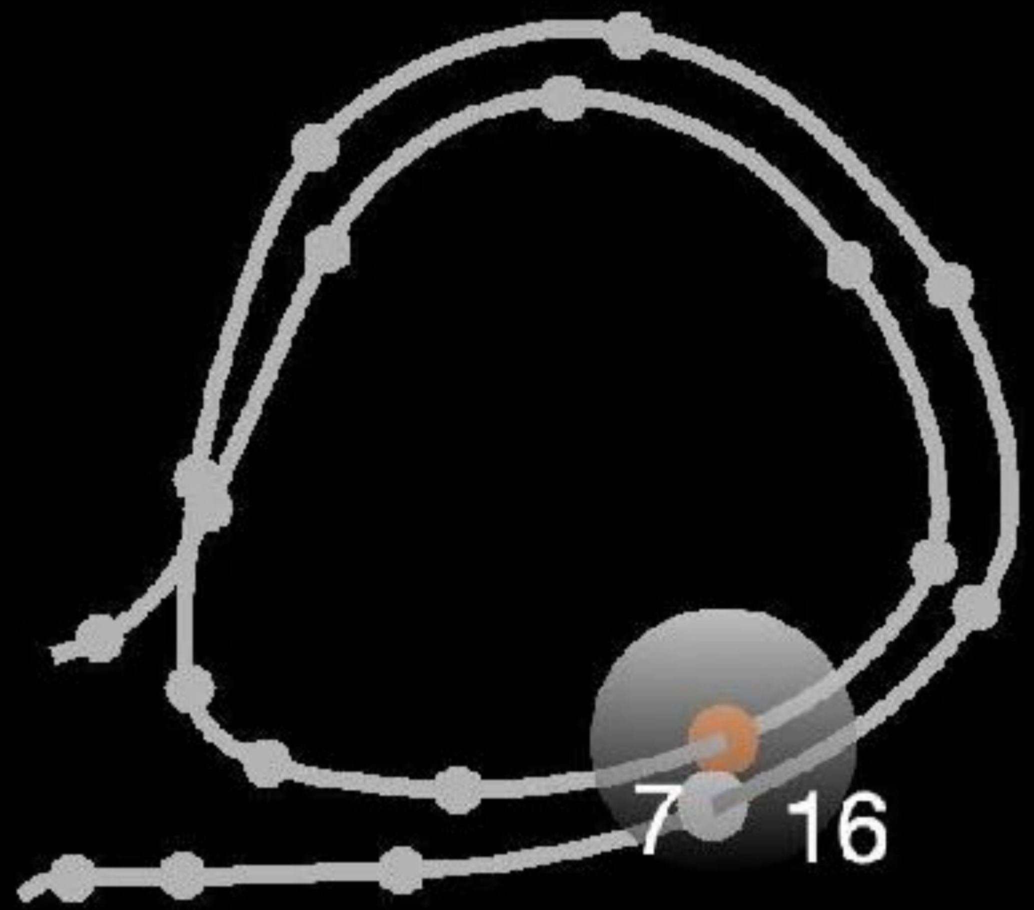


Recurrence matrix



Recurrence Plot

Phase space trajectory



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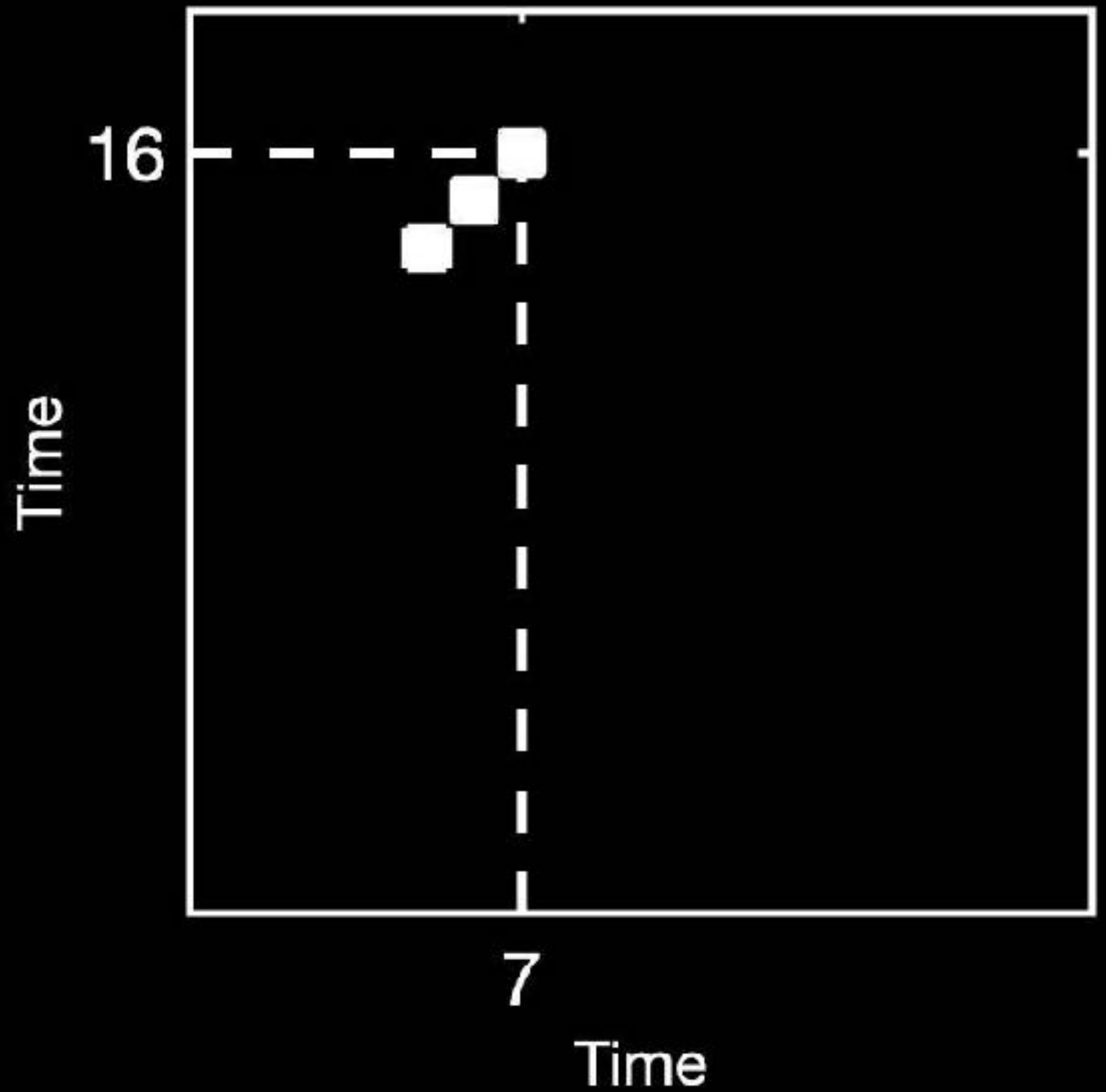

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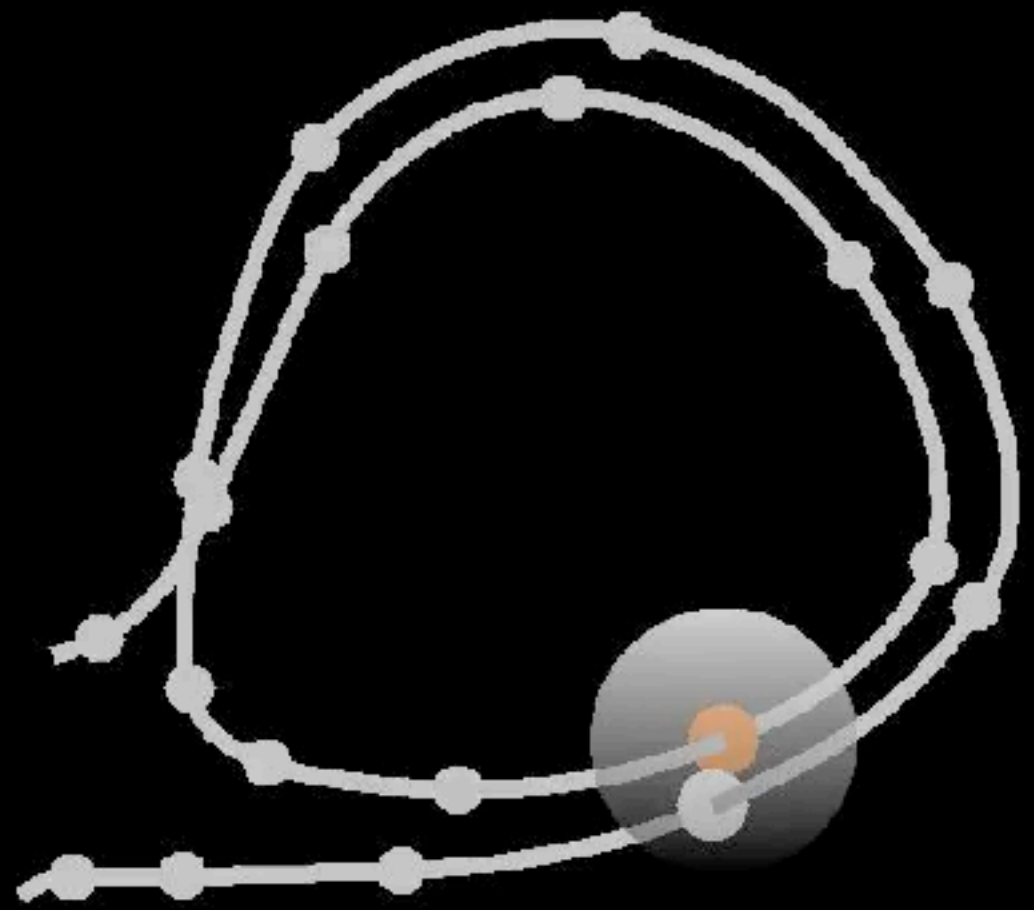

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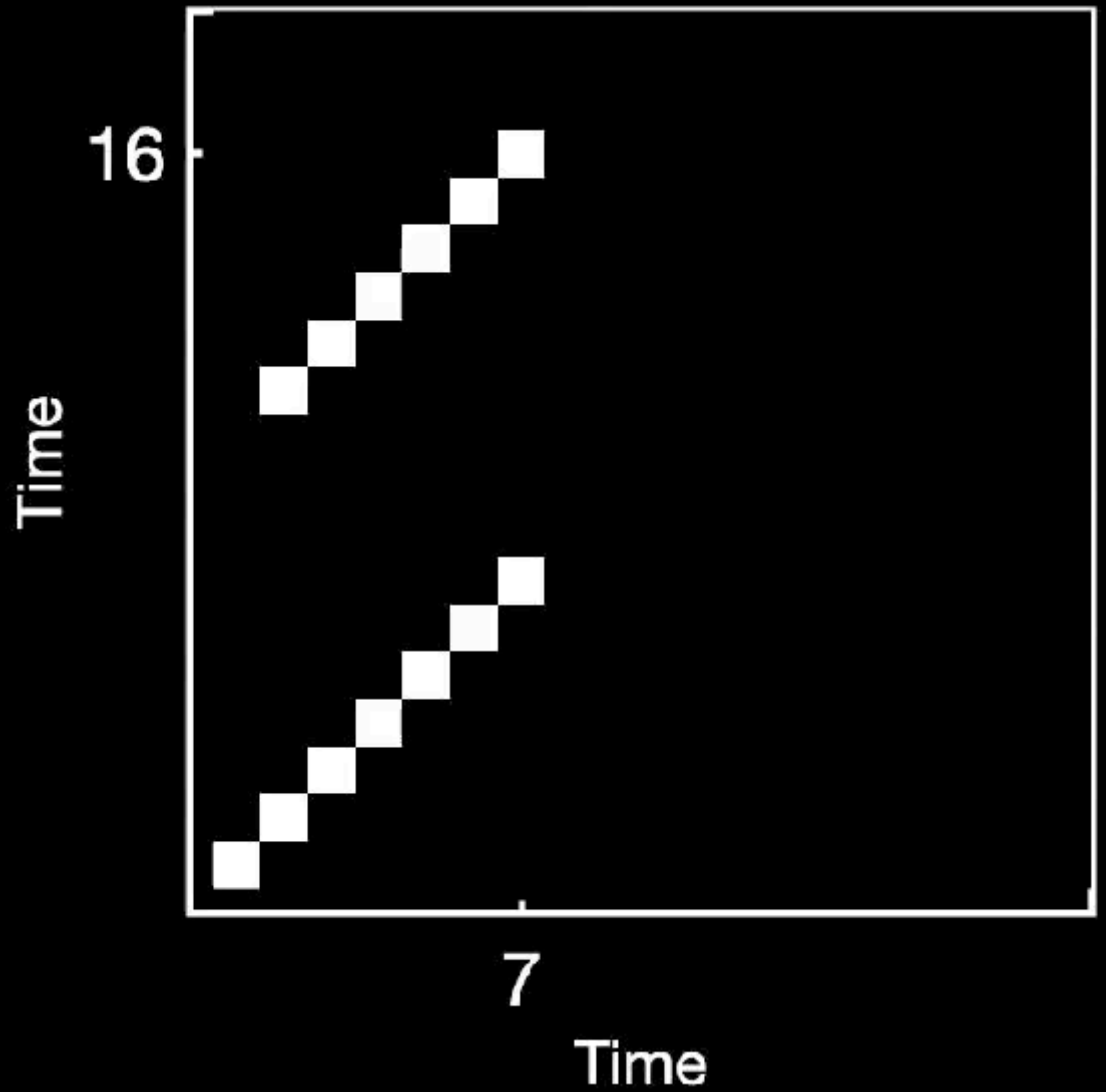


Recurrence Plot

- To visualise the phase space trajectory by its recurrences
- Recurrence matrix:
  - binary
  - symmetric

$$R_{i,j} = \begin{cases} 1 : \overrightarrow{x}_i \approx \overrightarrow{x}_j \\ 0 : \overrightarrow{x}_i \not\approx \overrightarrow{x}_j \end{cases}$$

$$R_{i,j}(\varepsilon) = \Theta\left(\varepsilon - \|\overrightarrow{x}_i - \overrightarrow{x}_j\|\right)$$

$\Theta(x \geq 0) = 1;\ \Theta(x < 0) = 0$

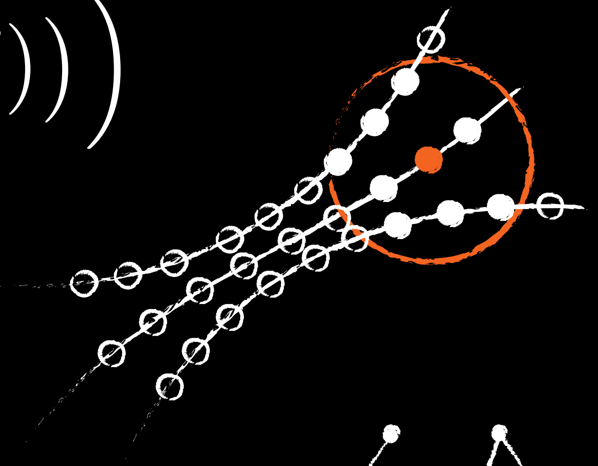
$R_{i,j} =$

1	1	0	0	1
1	1	1	0	1
0	1	1	0	0
0	0	0	1	1
1	1	0	1	1

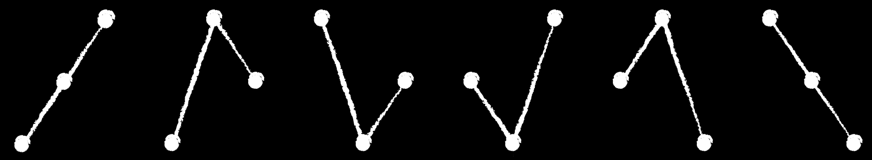
Eckmann et al, EPL 4, 1987

Variants of Recurrence Definitions

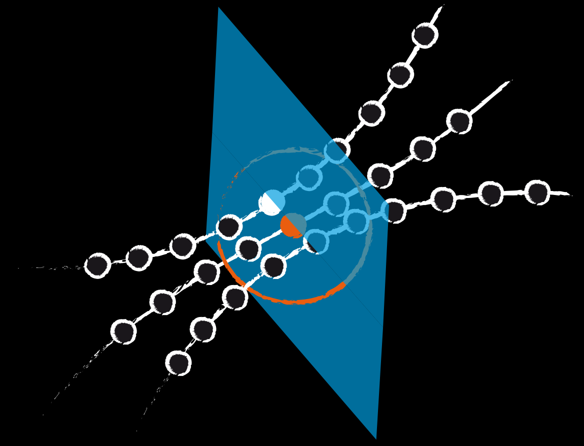
$$R_{i,j}(\varepsilon) = \Theta\Big(\varepsilon - d\big(\vec{x}(i), \vec{x}(j)\big)\Big)$$



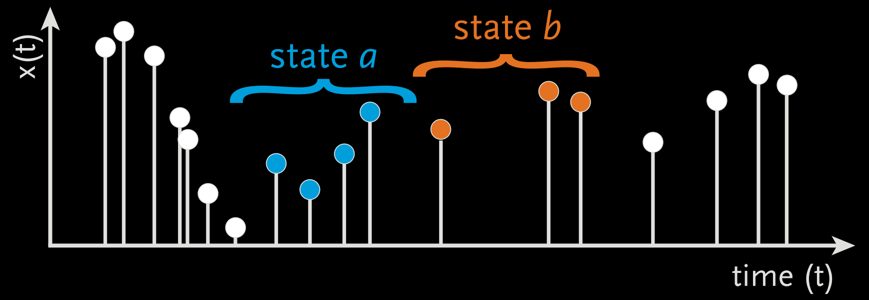
- Spatial distance (metric in phase space)
- Fixed amount of nearest neighbours



- Order patterns
- Iso-directional RP, perpendicular RP, windowed RP, meta-RP



- RPs for events, spike-trains, point processes

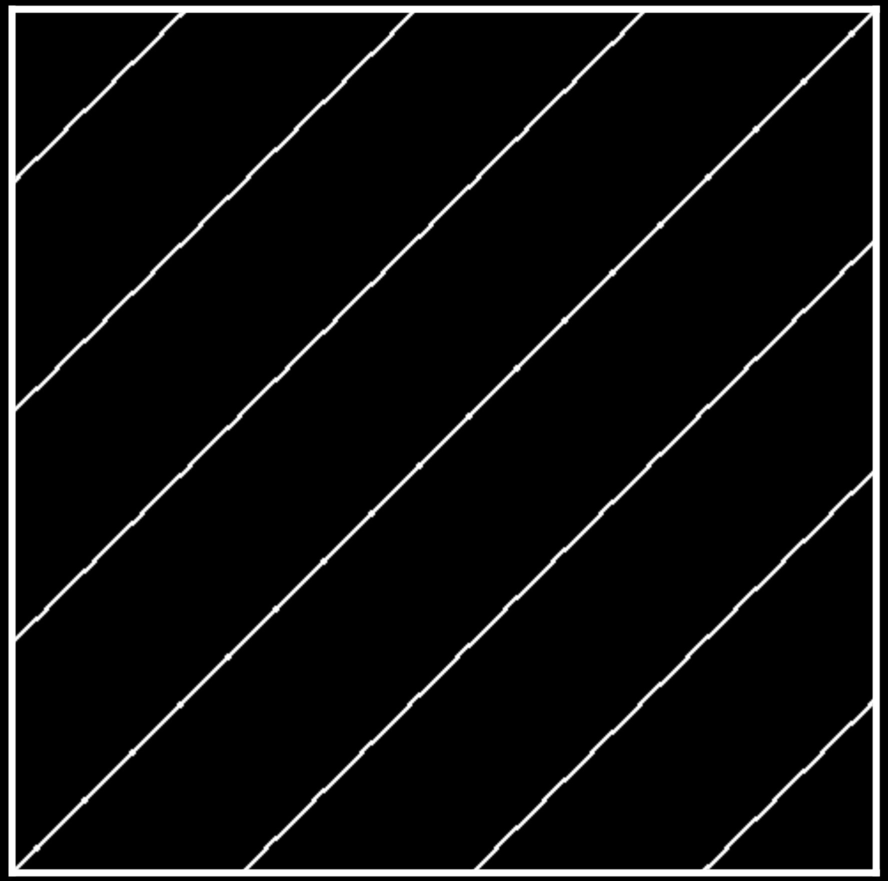


- Spatio-temporal RPs

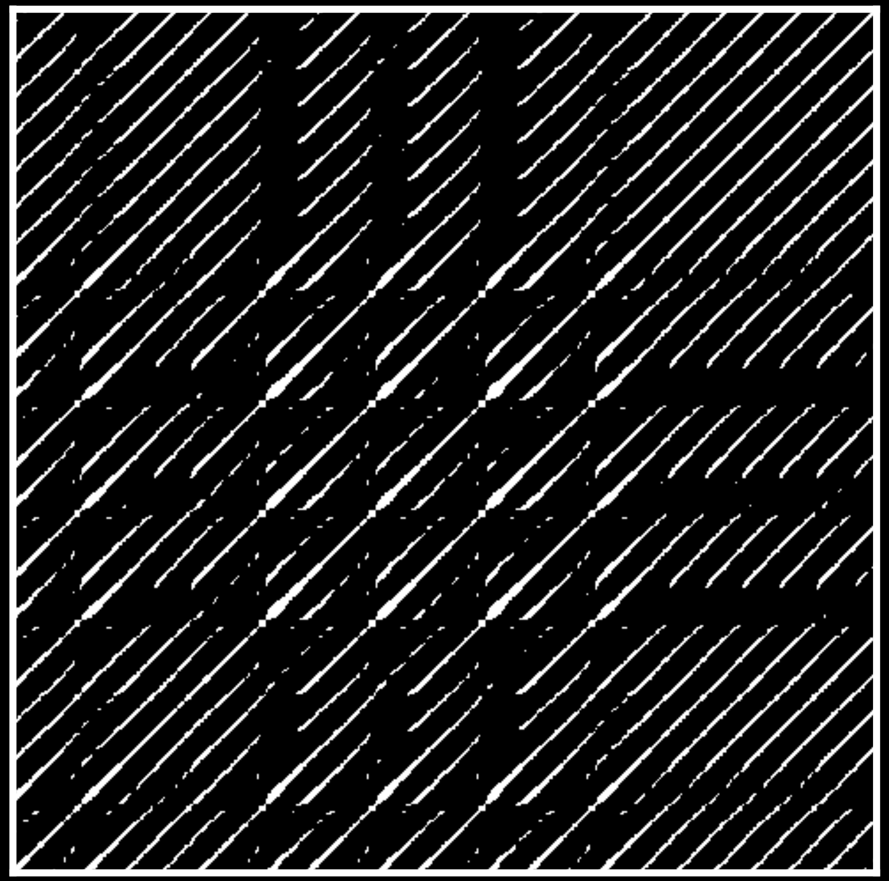
Marwan et al, Phys Rep 438, 2007

Different Dynamics: Different Patterns

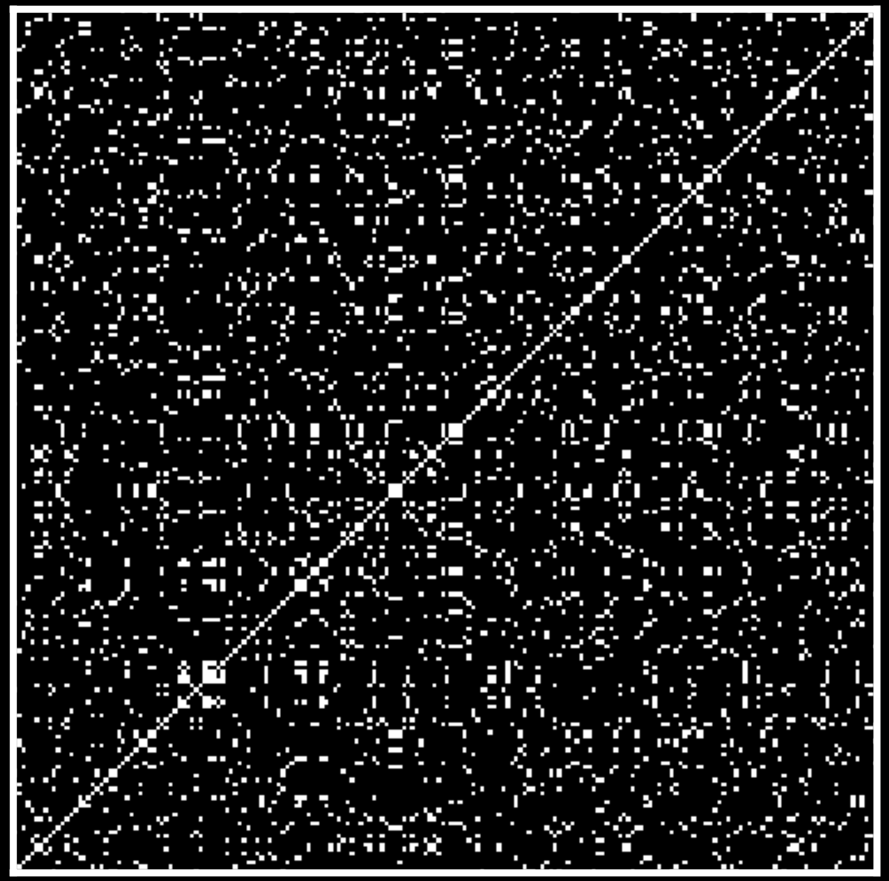
Periodic



Chaotic



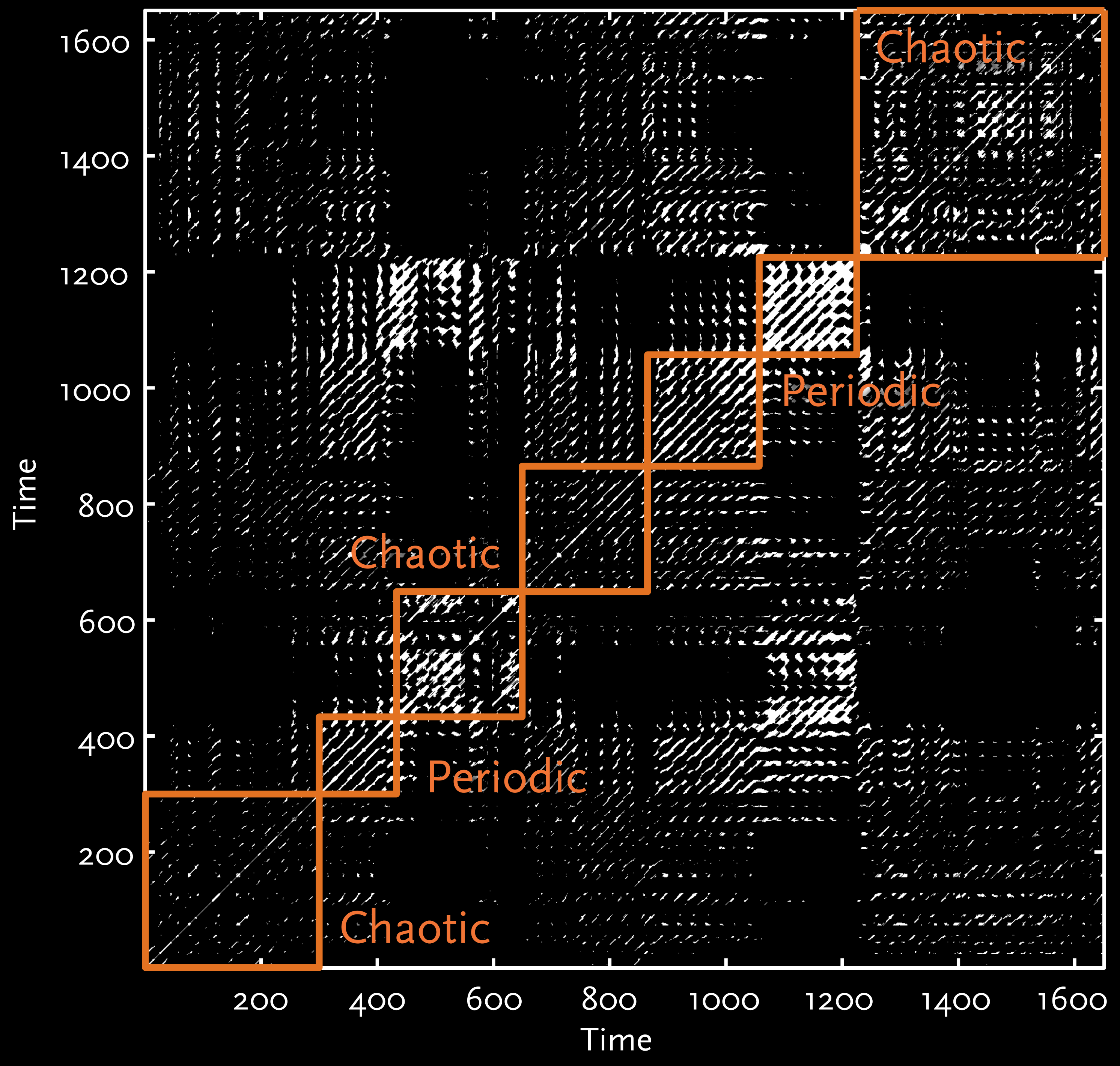
Random



Marwan et al, Phys Rep 438, 2007

Dynamics of Oxygen Crises in Lakes

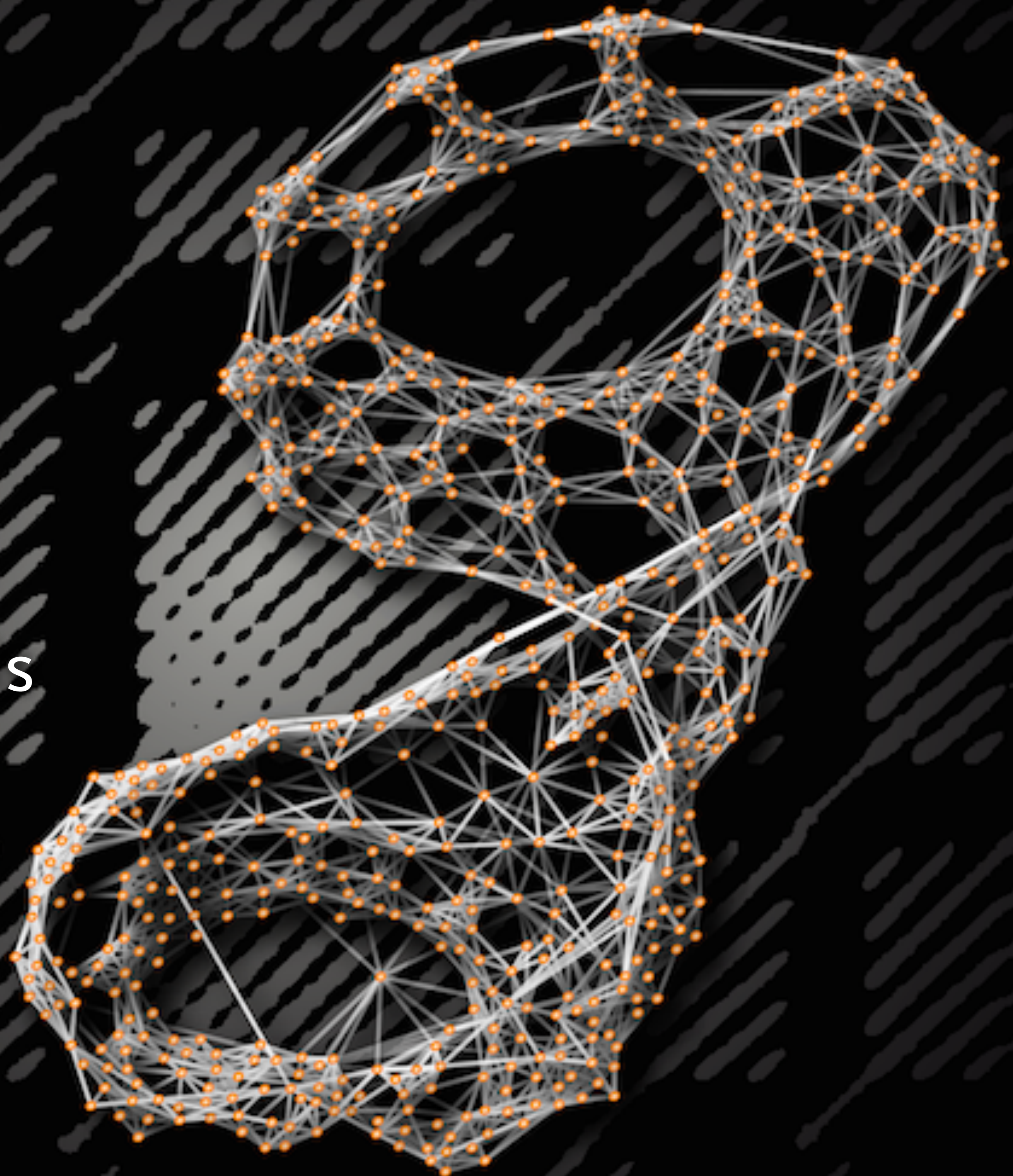
- Episodes with different dynamics



Facchini et al., Ecol Mod 203, 2007

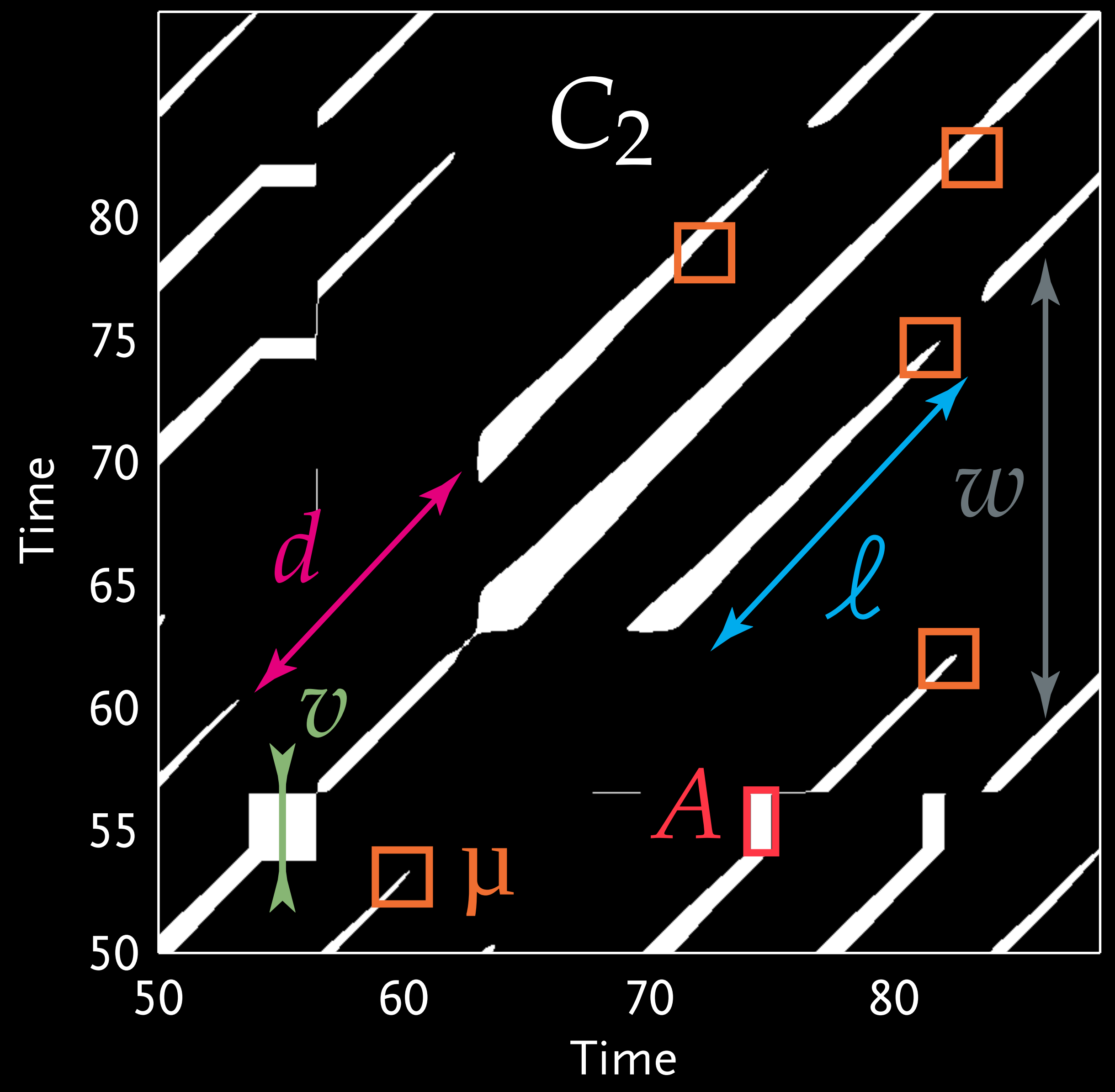
Recurrence Analysis for Complex Systems

- Phase space and recurrence plot
- **Recurrence quantification**
- Coupling and synchronization analysis
- Outlook



Structures in Recurrence Plots

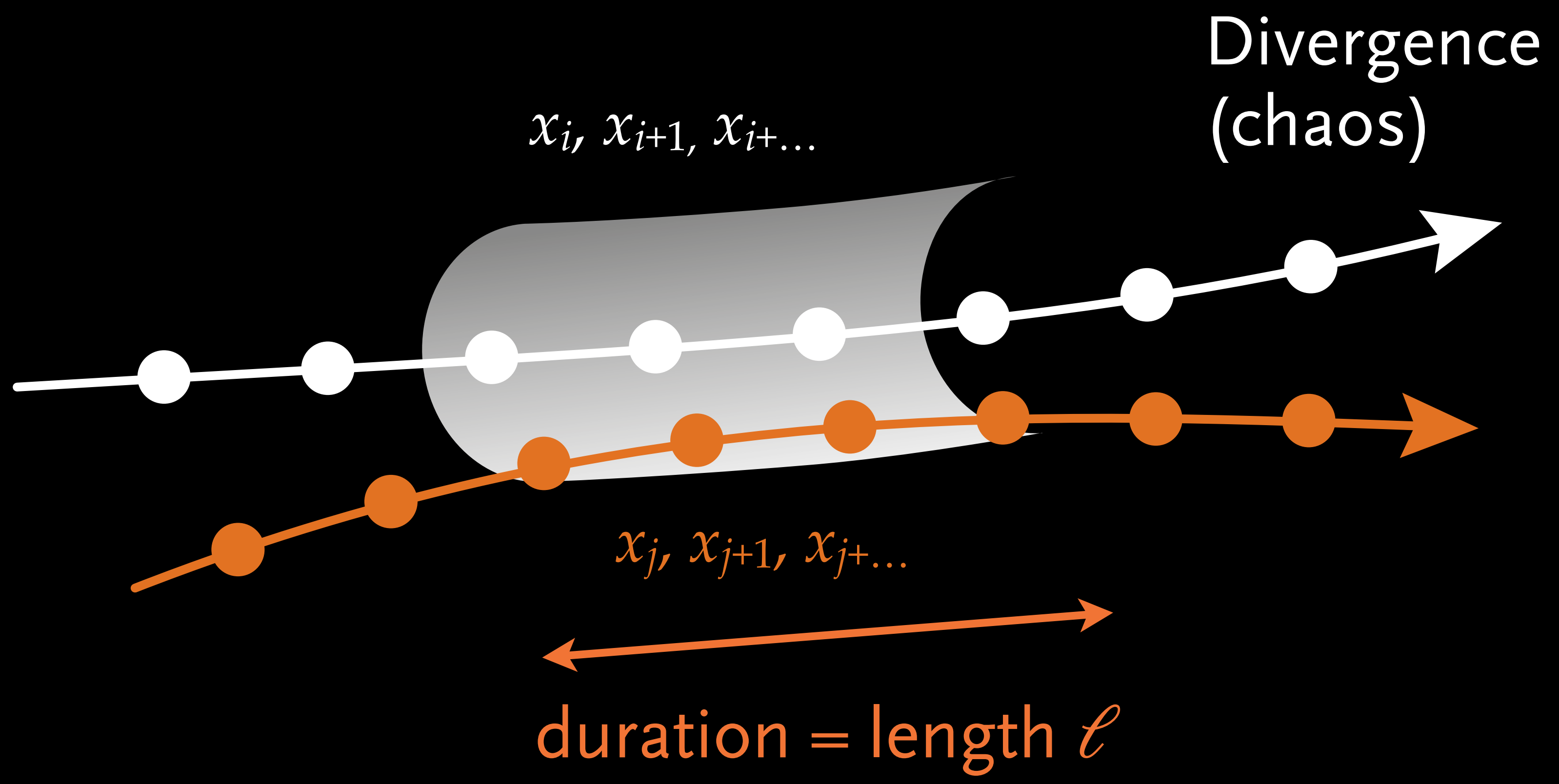
- Diagonal lines
- Diagonal empty lines
- Vertical lines
- Vertical empty lines
- Areas
- Microstates
- Point density (correlation sum)

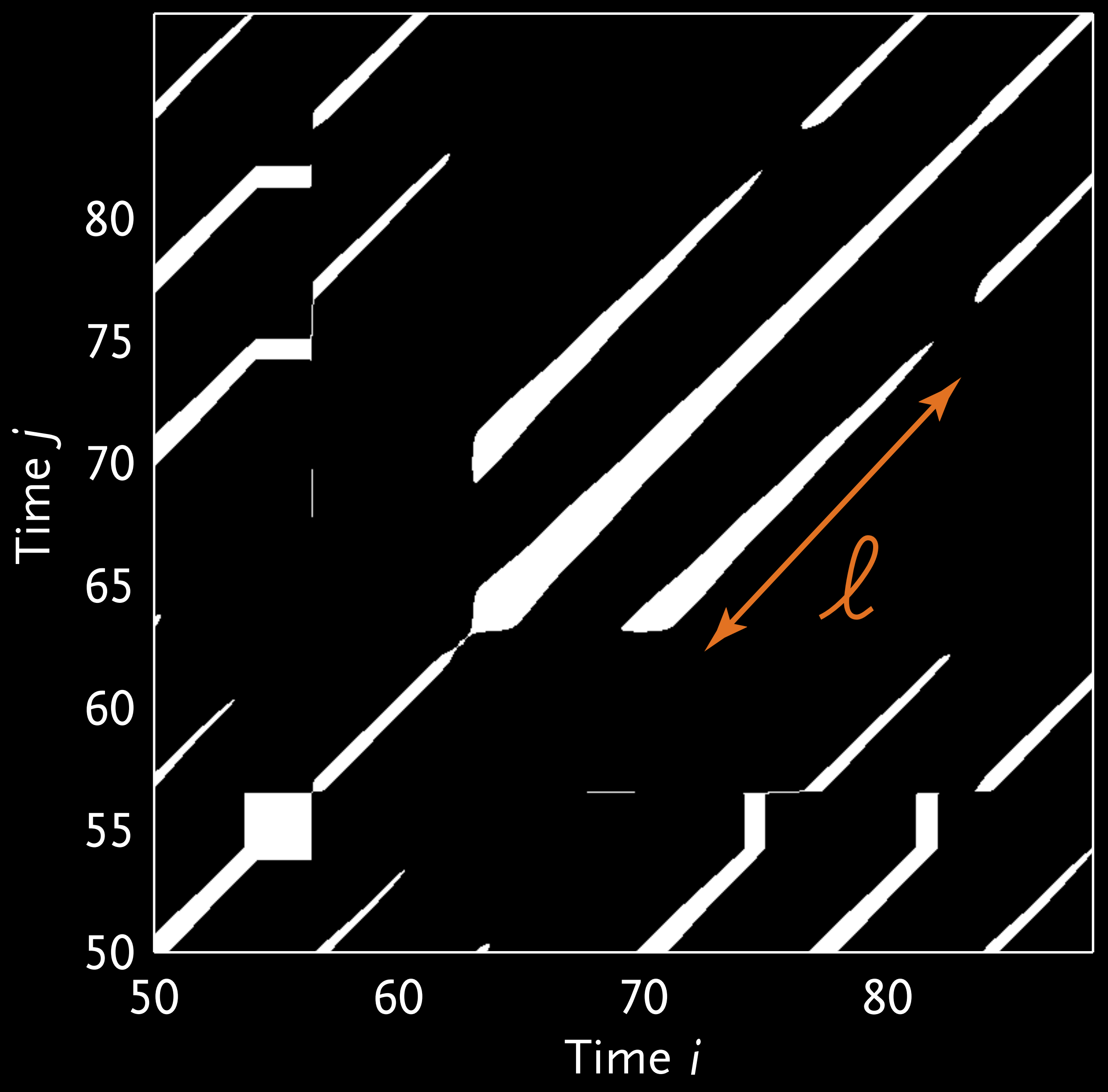


examples of microstates μ:

Zbilut & Webber
1992, 1994

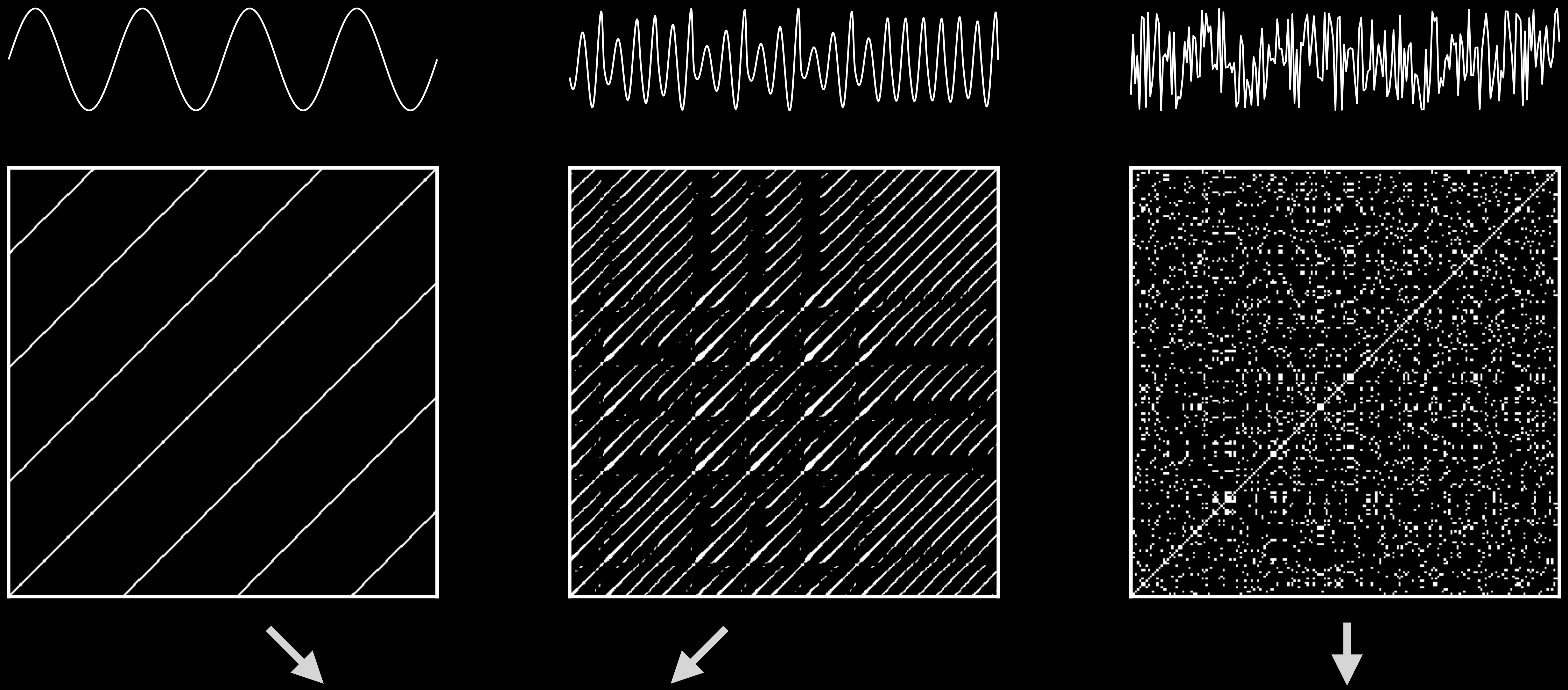
Diagonal Lines





Zbilut & Webber Jr., Phys Lett A 171, 1992

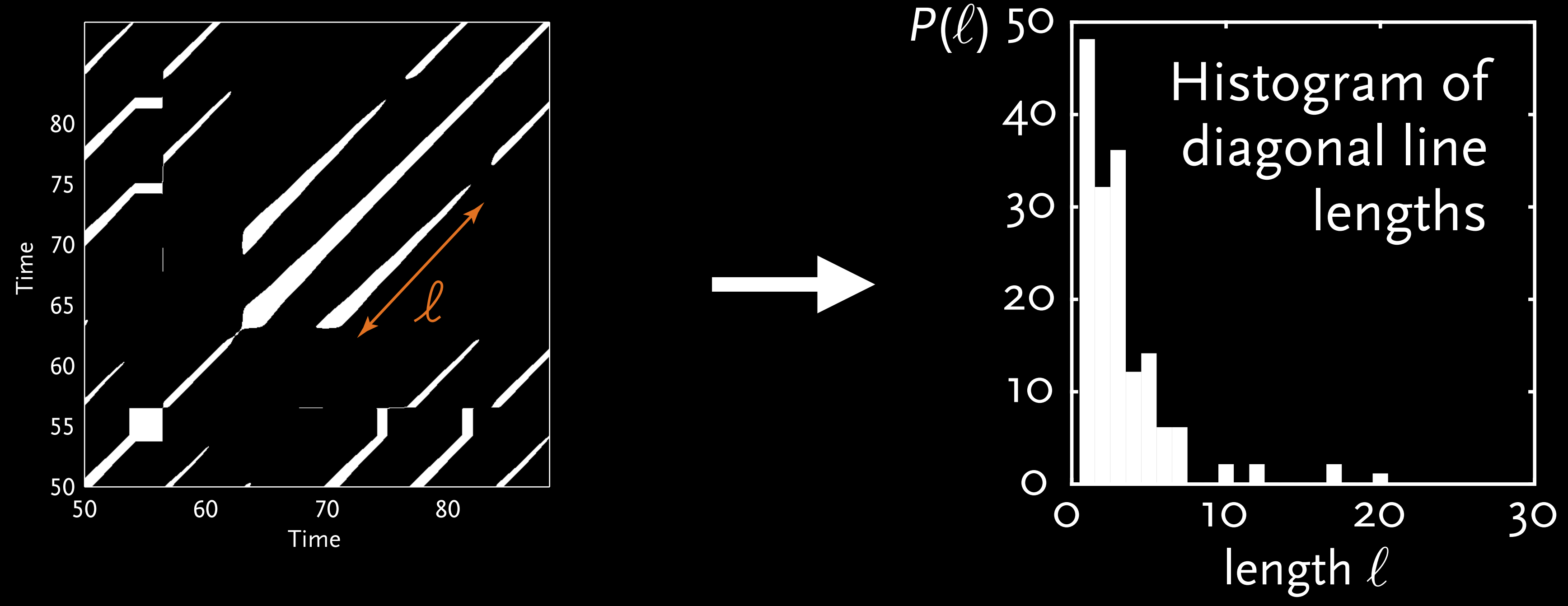
Determinism



Deterministic dynamics:
mainly diagonal lines

Stochastic dynamics:
mainly single points

Determinism



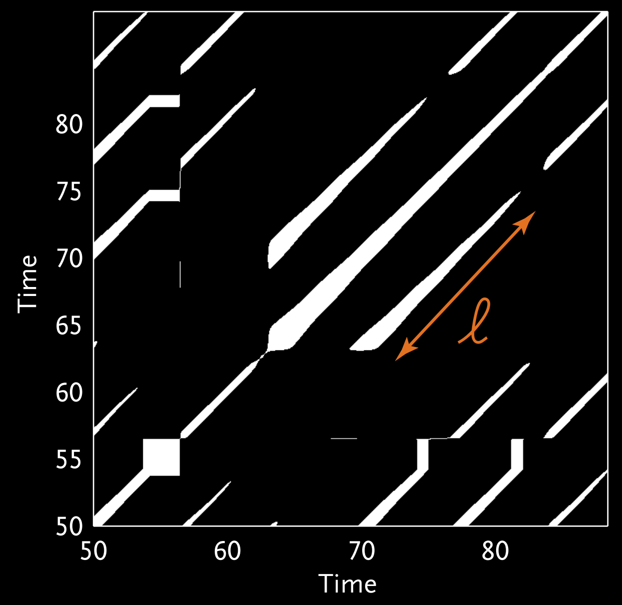
- Fraction of points forming diagonal lines

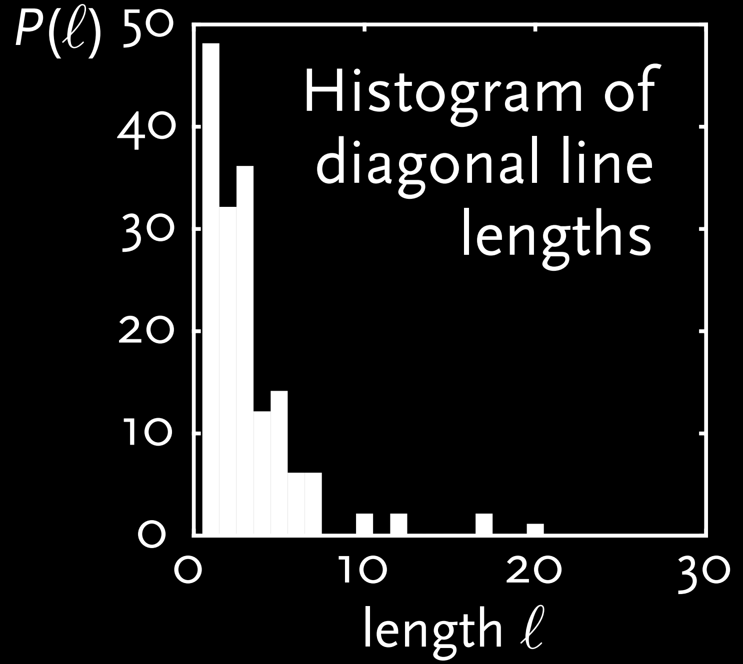
$$DET = \frac{\sum_{\ell=\ell_{\min}}^{N_m} \ell P(\ell)}{\sum_{\ell=1}^{N_m} \ell P(\ell)}$$

Probability that dynamics will evolve in a similar way as at another point of time

Zbilut & Webber Jr., Phys Lett A 171, 1992

Determinism





- Fraction of points forming diagonal lines

$$DET = \frac{\sum_{\ell=\ell_{\min}}^{N_m} \ell P(\ell)}{\sum_{\ell=1}^{N_m} \ell P(\ell)}$$

Not a mathematical definition of determinism!
(heuristic measure)

Zbilut & Webber Jr., Phys Lett A 171, 1992

Recurrence Quantification (RQA)

Measure	Structure	Meaning
Recurrence rate	recurrence points	mean recurrence probability, C_2
Determinism	diagonal lines	predictability/ determinism
Mean diagonal line length		predictable time, divergence
Max. diagonal line length		divergence, Lyapunov exponent
Entropy		complexity of the dynamics
Rényi entropy K_2		Lyapunov exponent
Laminarity	vertical lines	intermittency
Trapping time		duration of laminar phases
Recurrence time	(empty) vertical lines	mean waiting time until recurrence
Recurrence time entropy		complexity of the time scale of recurrence

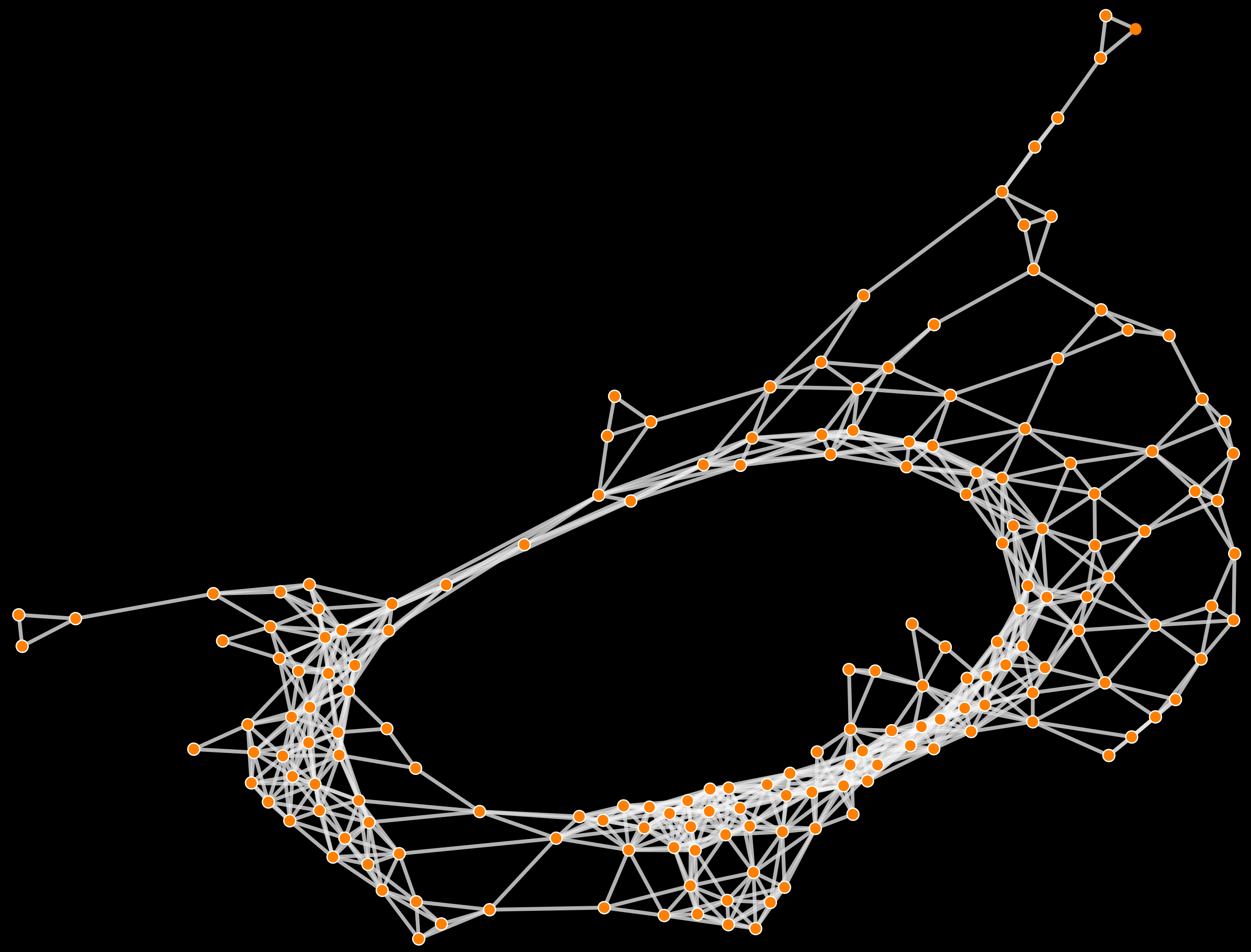
Marwan et al, Phys Rep 438, 2007

Recurrence Networks

$$R_{i,j} = \begin{array}{|c|c|c|c|c} \hline 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 1 & 1 \end{array}$$

nodes $\longrightarrow$ time points

links $\longrightarrow$ recurrences

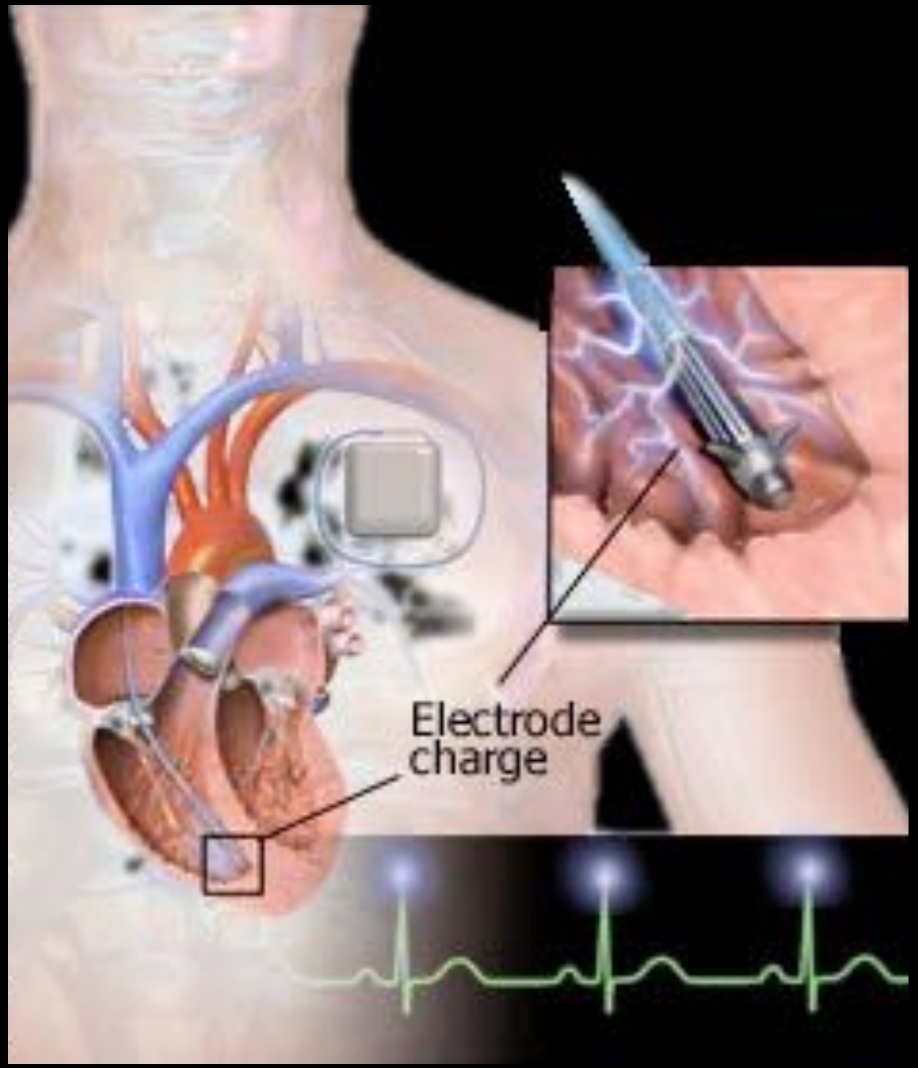


complex network measures
explaining topological properties

Marwan et al, Phys Lett A 373, 2009

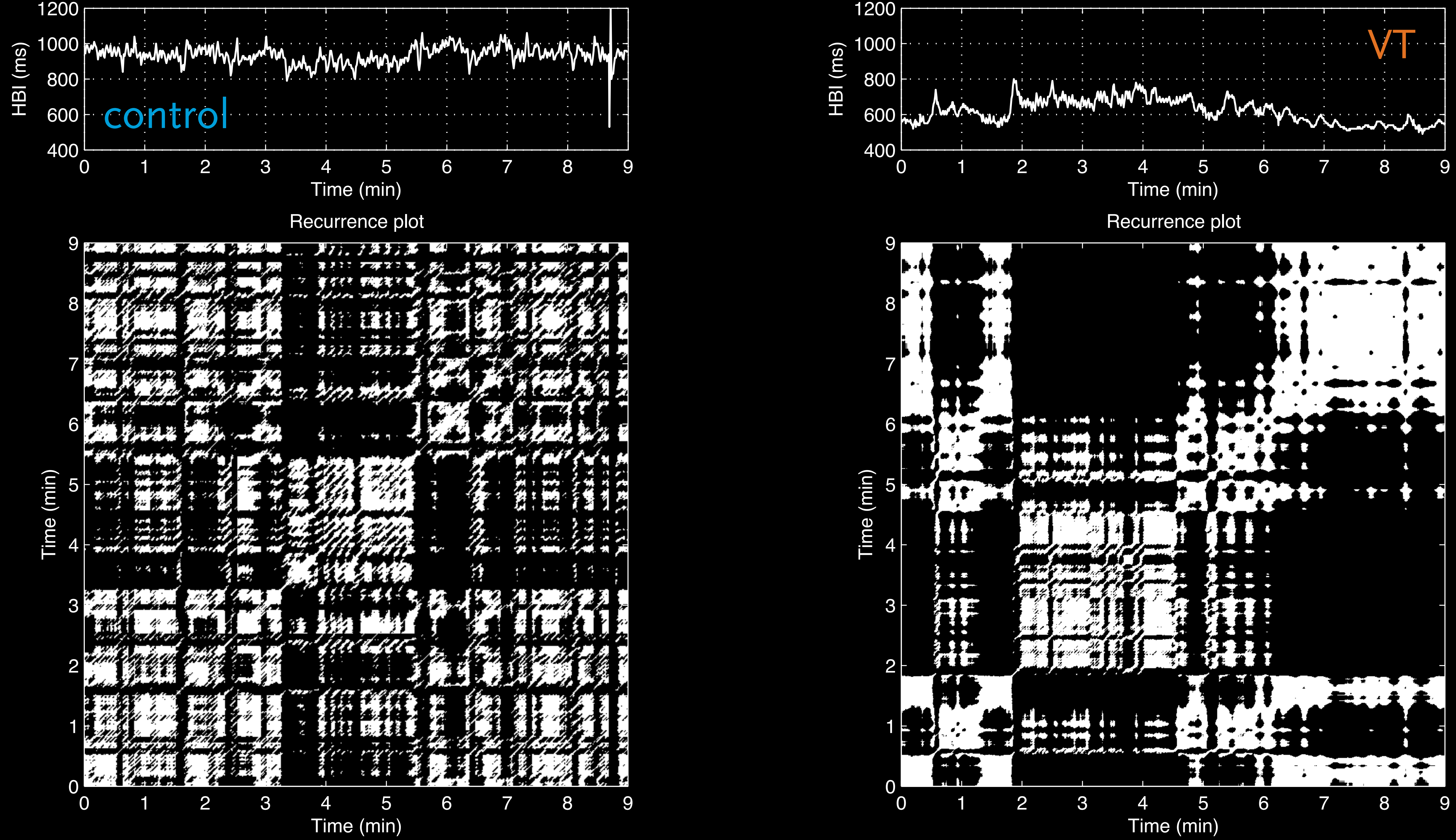
Detecting Cardiovascular States

- Predicting ventricular tachycardia (VT)
- Implanted cardioverter defibrillator (ICD)
- 1000 heart beat intervalls
- 17 VT patients, 24 control subjects, 24 VT episodes

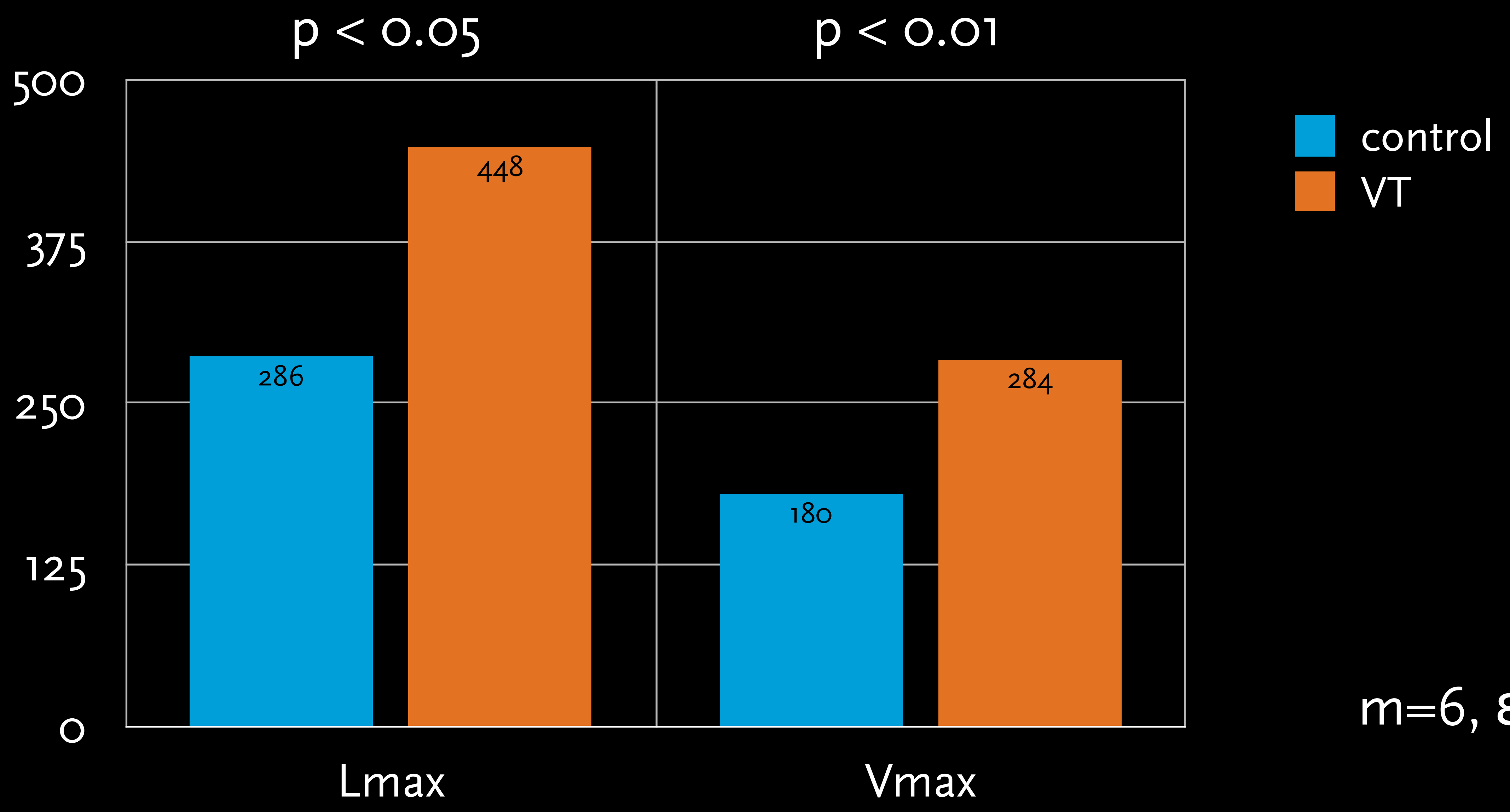


Marwan et al, Phys Rev E 66, 2002

Detecting Cardiovascular States



Detecting Cardiovascular States

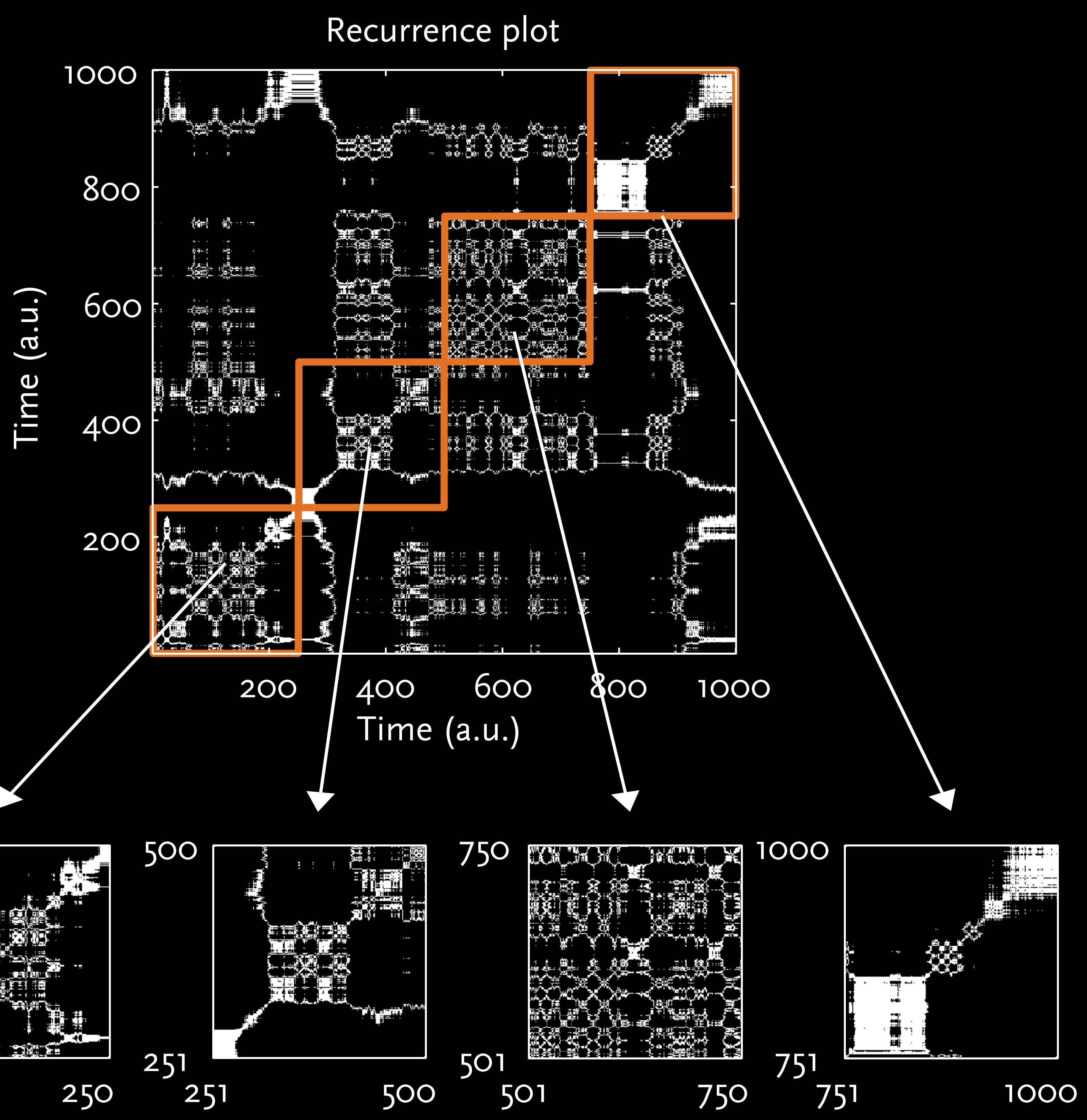


m=6, ε=110

- significant differences between VT and control

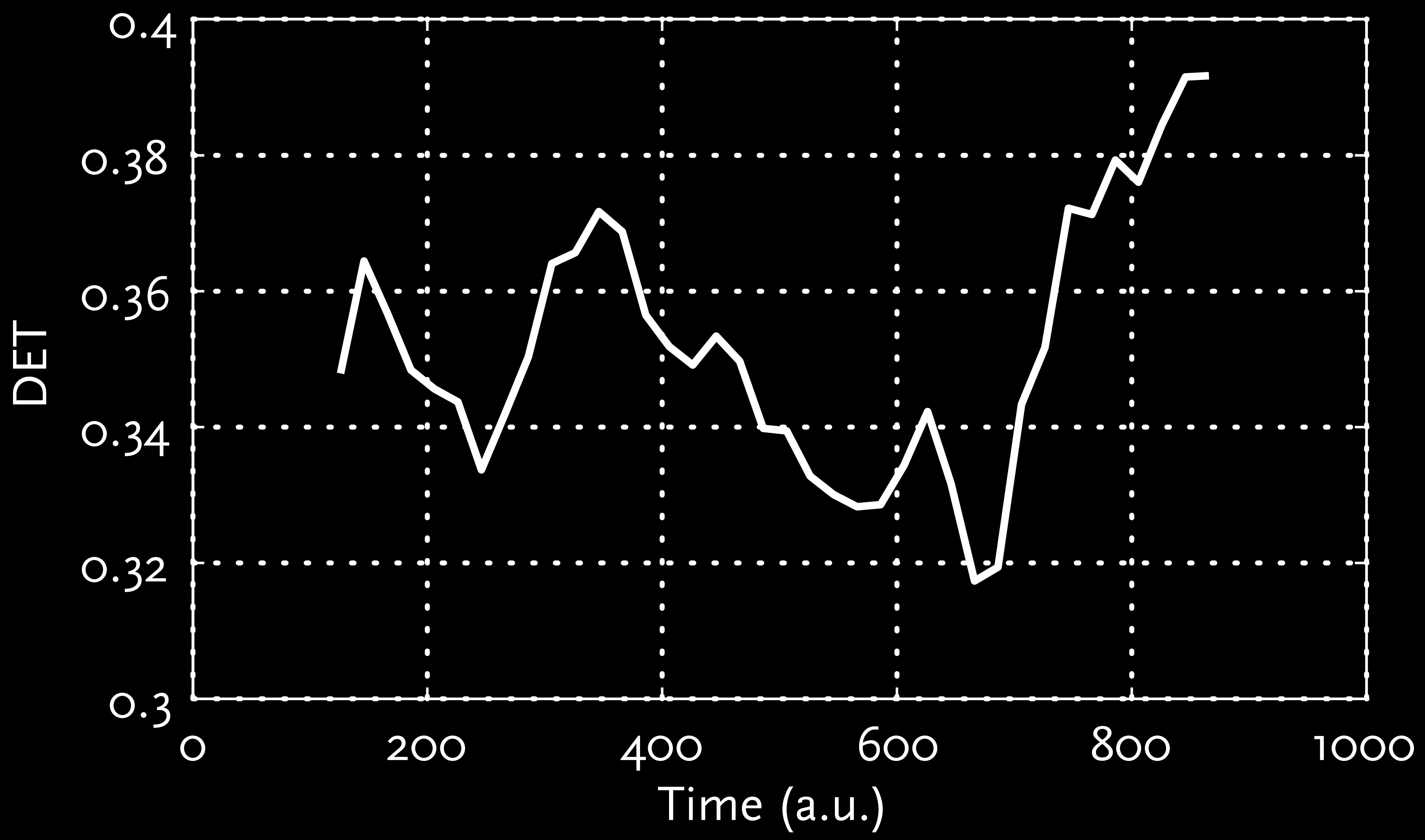
Marwan et al, Phys Rev E 66, 2002

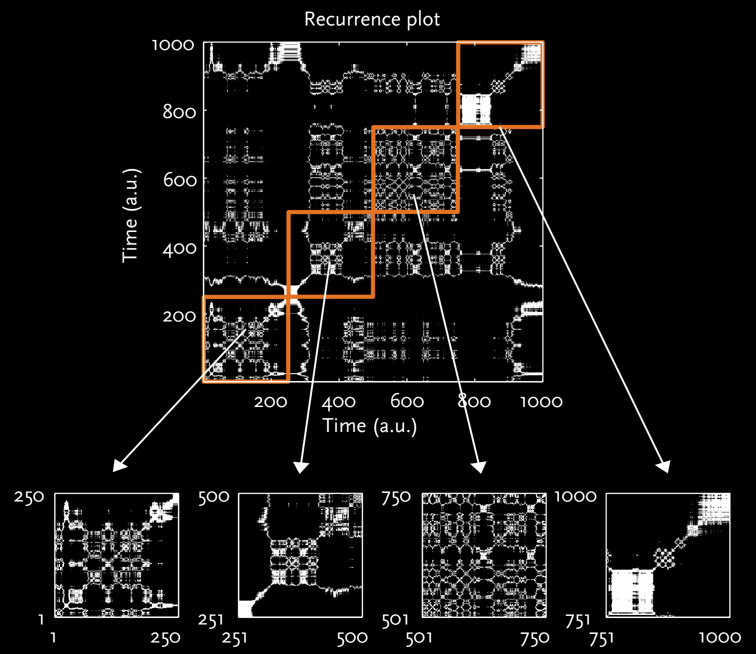
Indicator for Transitions



Trulla et al, Phys Lett A 223, 1996

Indicator for Transitions

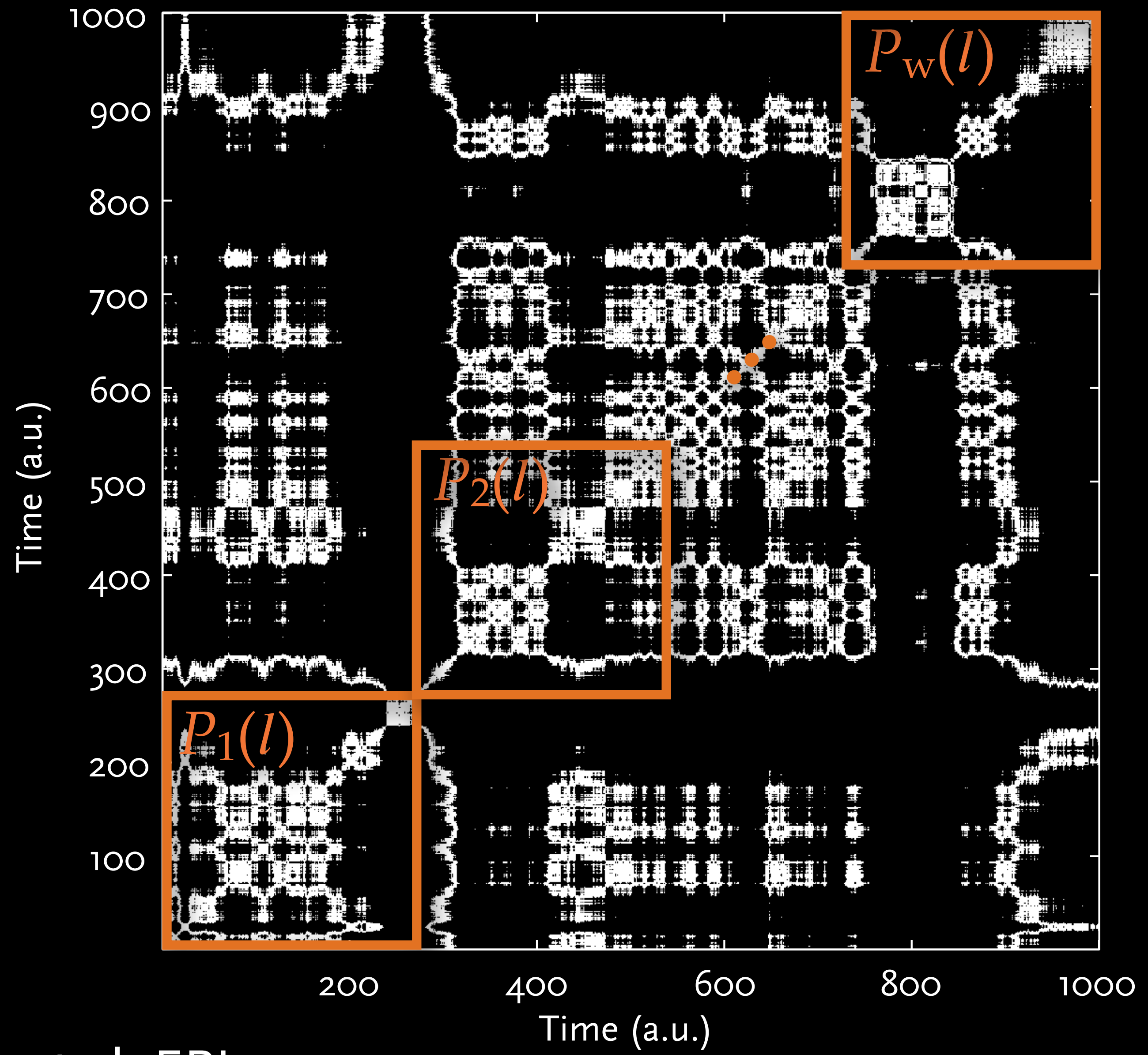




- RQA measures: transitions/ nonstationarity?

Significance Test

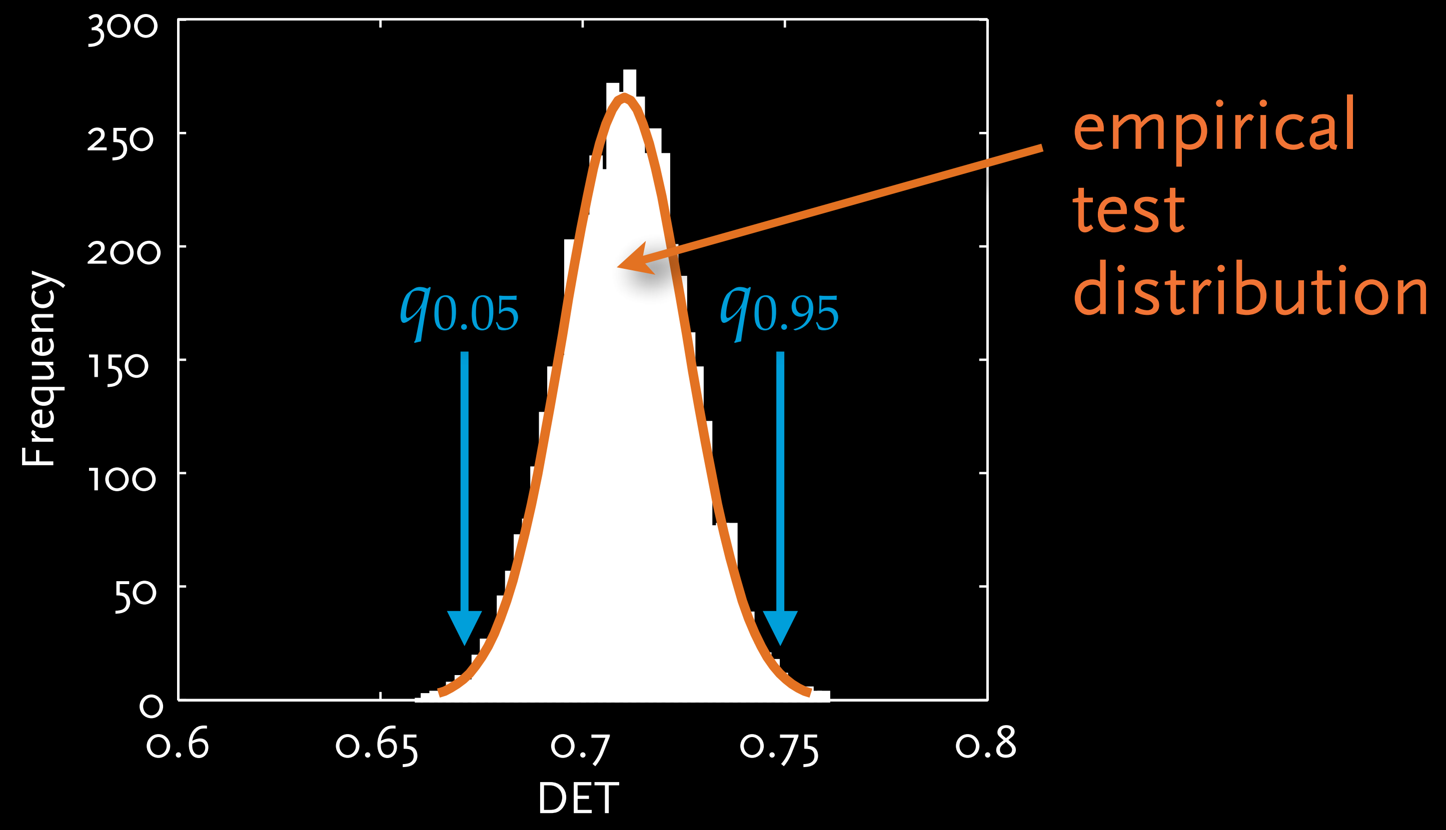
- Bootstrapping line structures from all windows



Merged distribution of line lengths

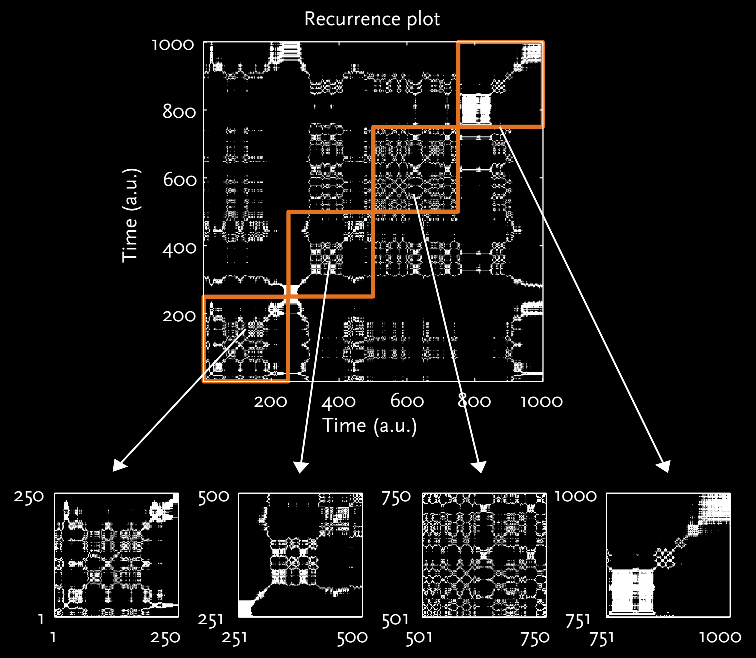
$$\hat{P}(l) = \sum_{w} P_w(l)$$

(Calculate DET_{surr} by bootstrapping lines from P_{w})

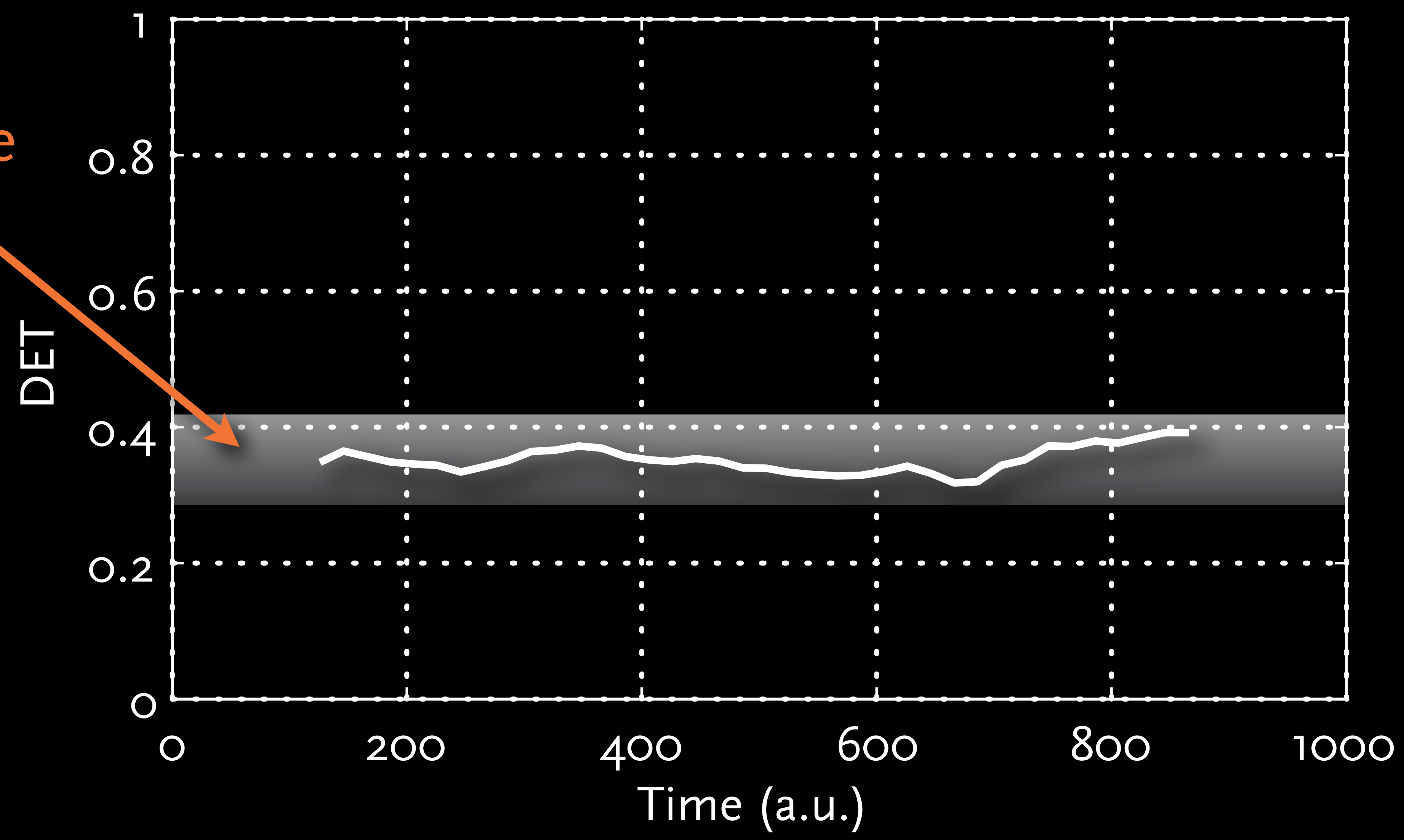


Marwan et al, EPL 101, 2013

Indicator for Transitions

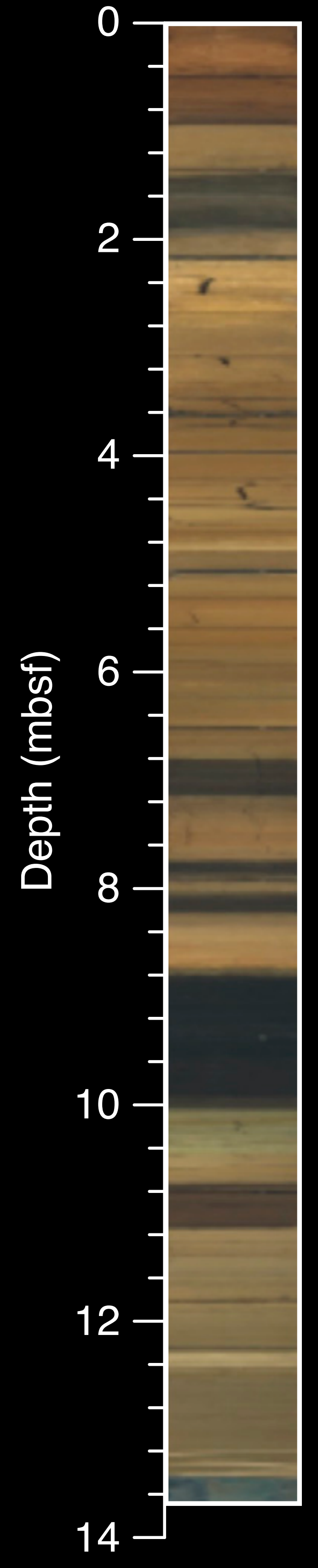


95% confidence interval



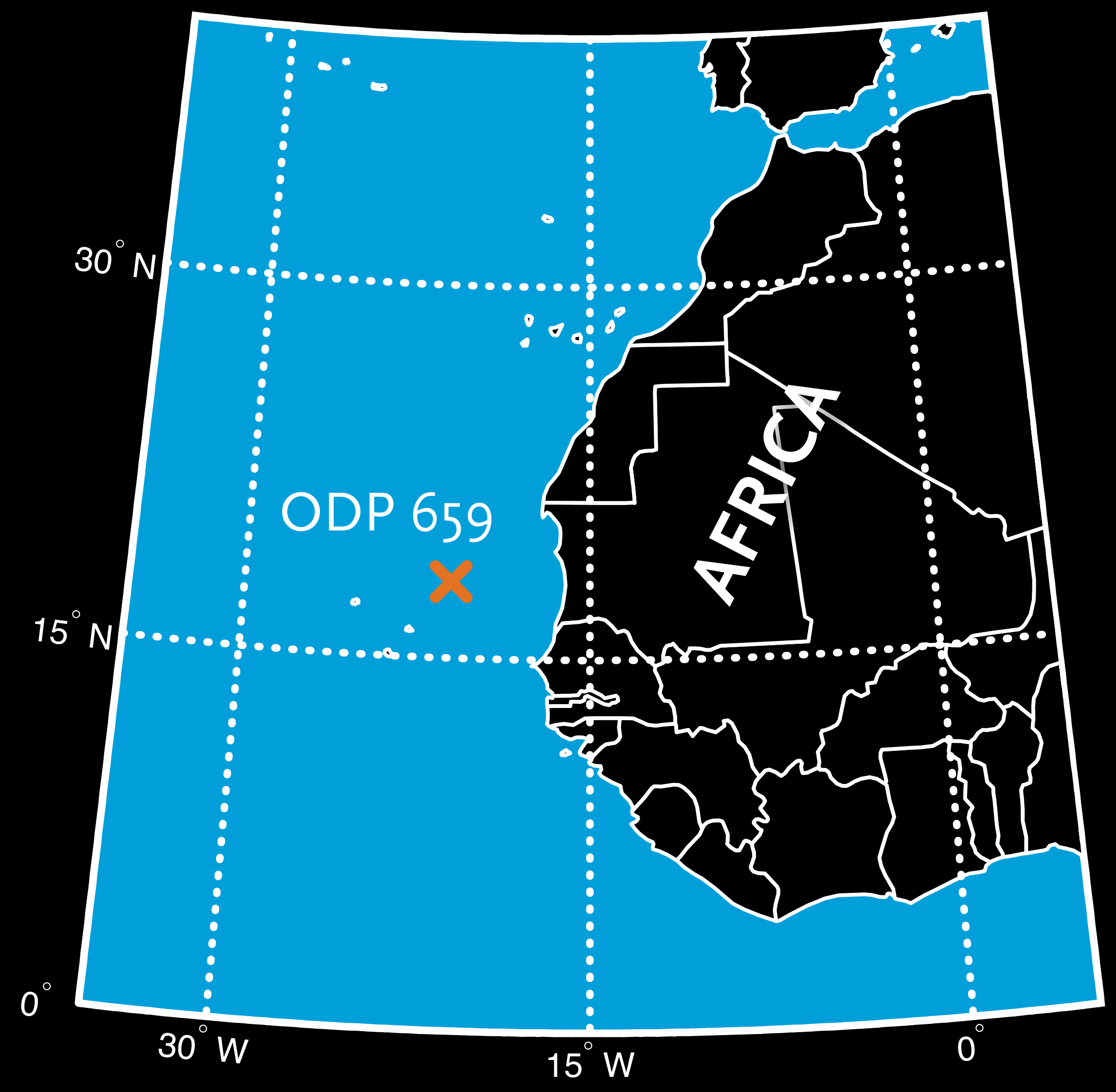
➡ no significant transition

Marwan et al, EPL 101, 2013



Palaeoclimate Regime Changes

- Marine dust flux record
- Proxy for aridity in Africa
- ODP site 659



Palaeoclimate Regime Changes

- Marine dust flux record
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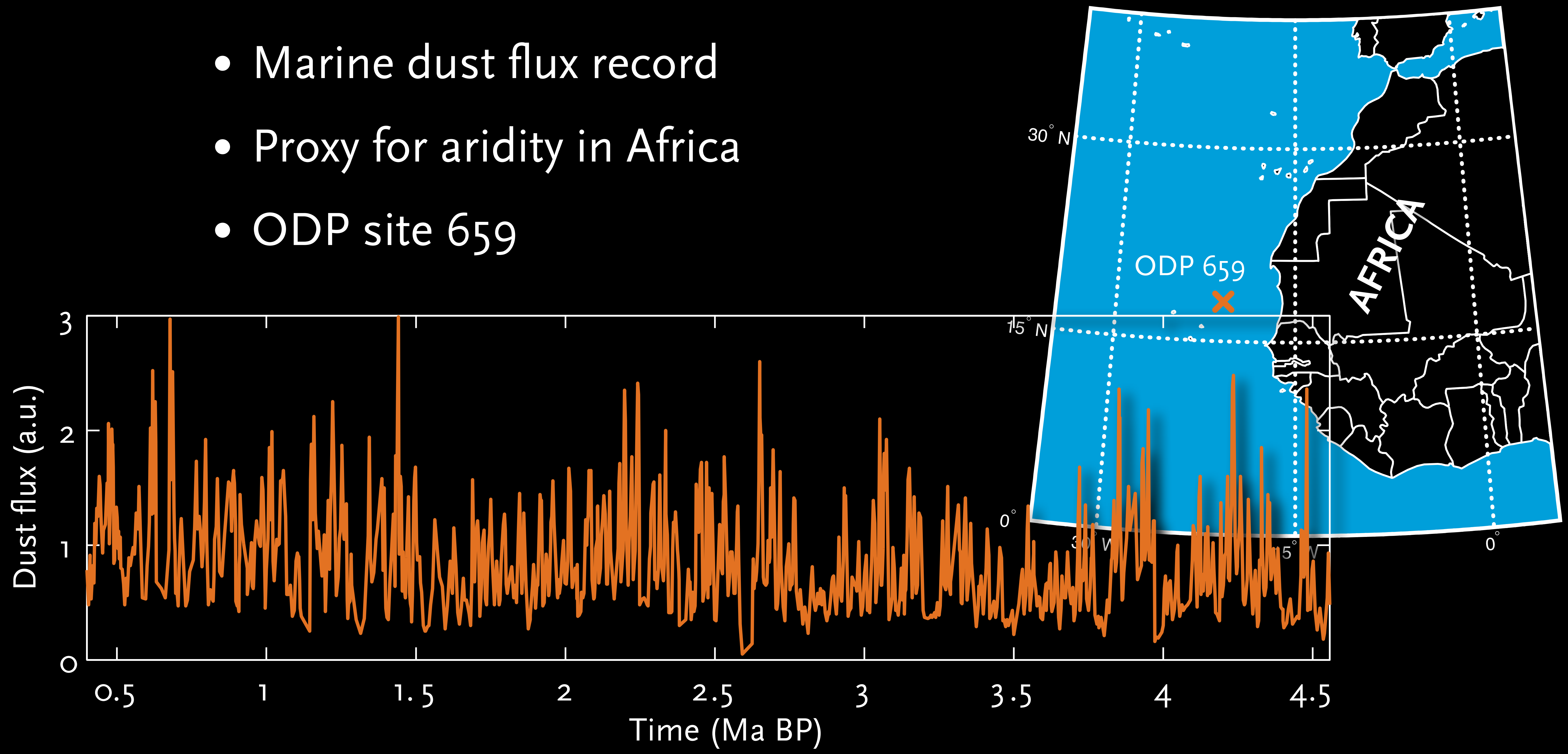
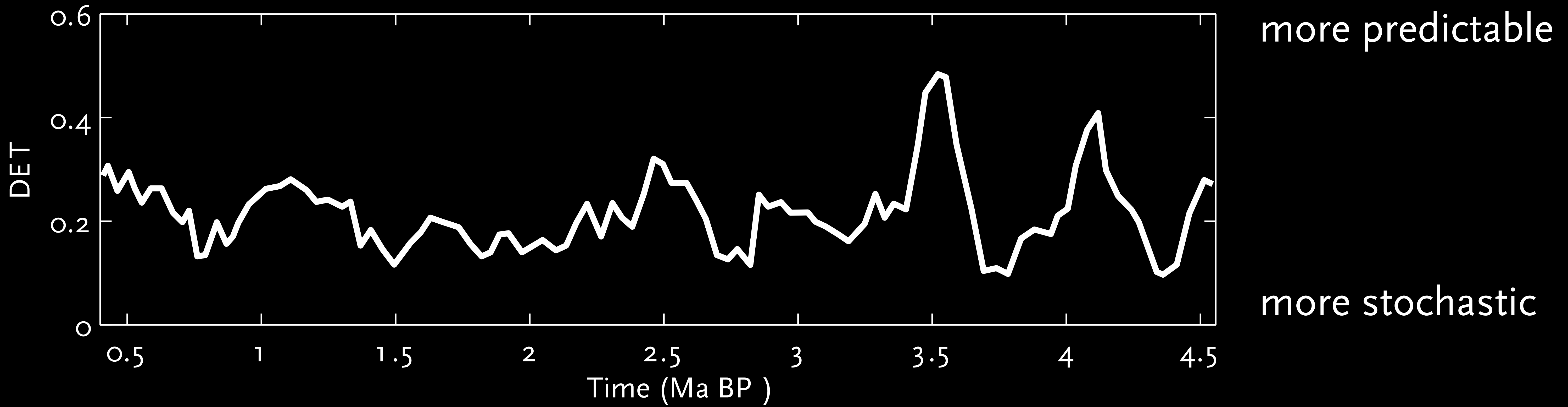
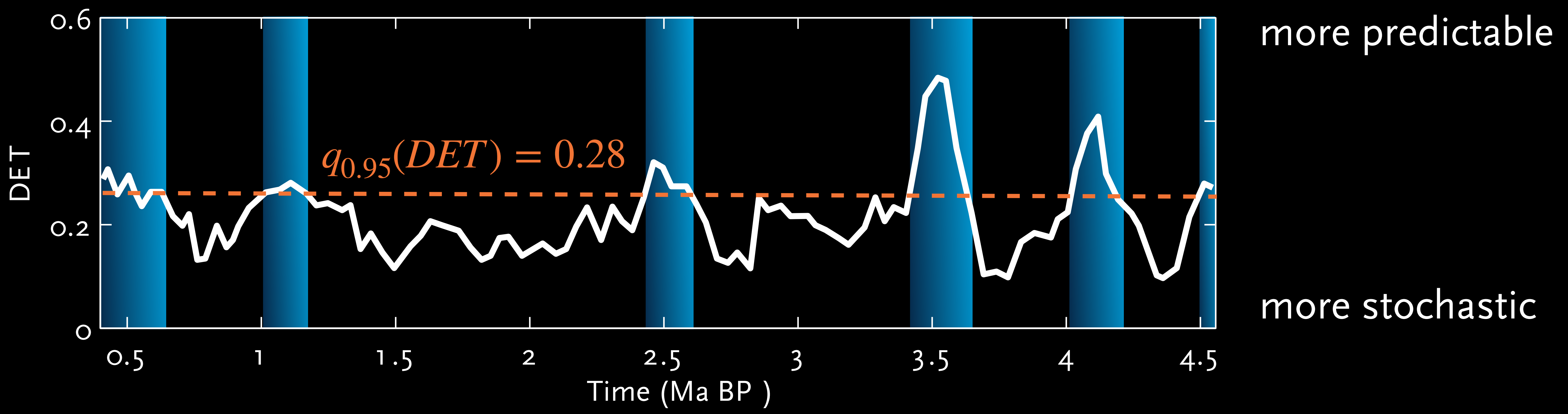


Palaeoclimate Regime Changes

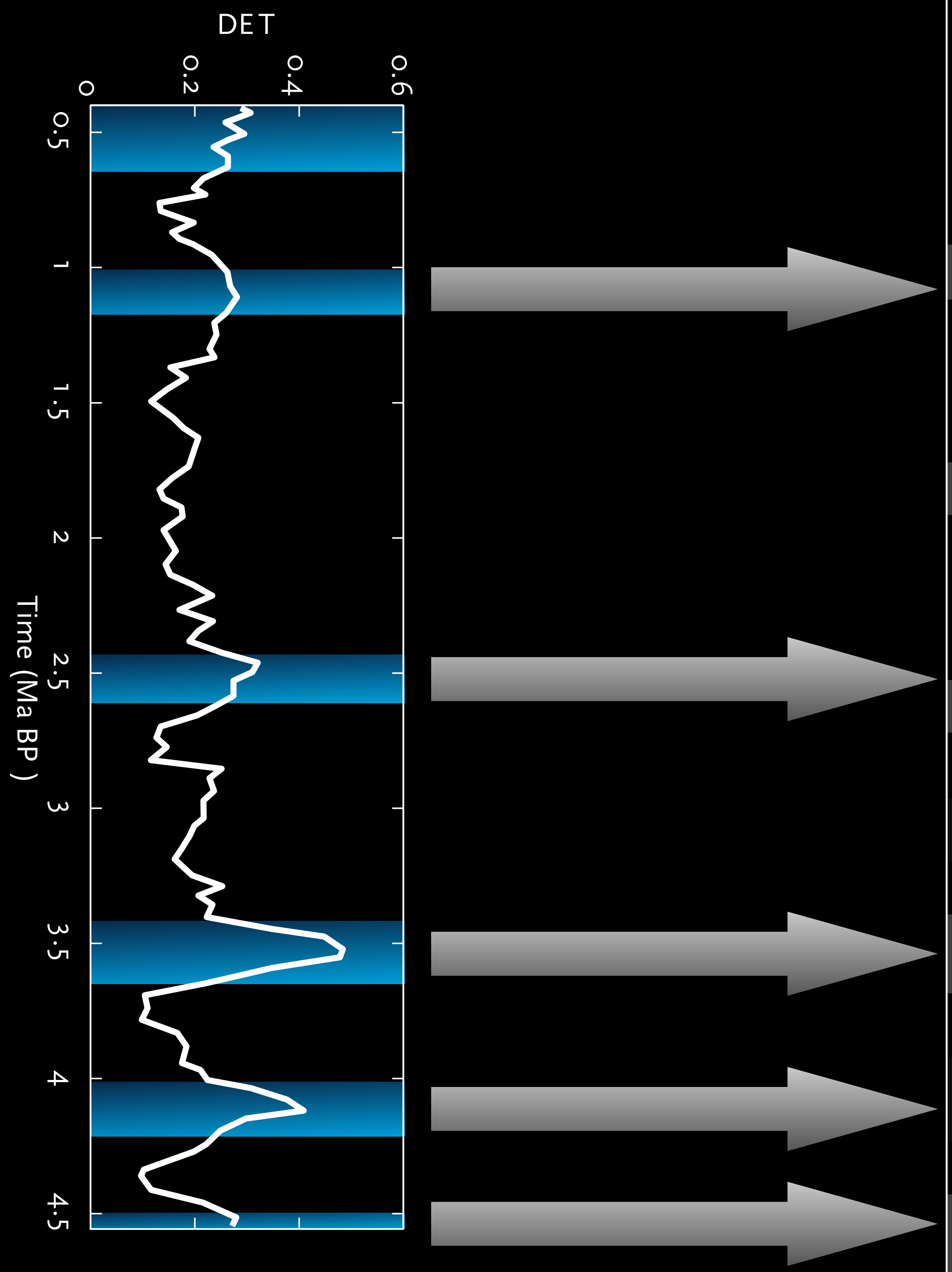


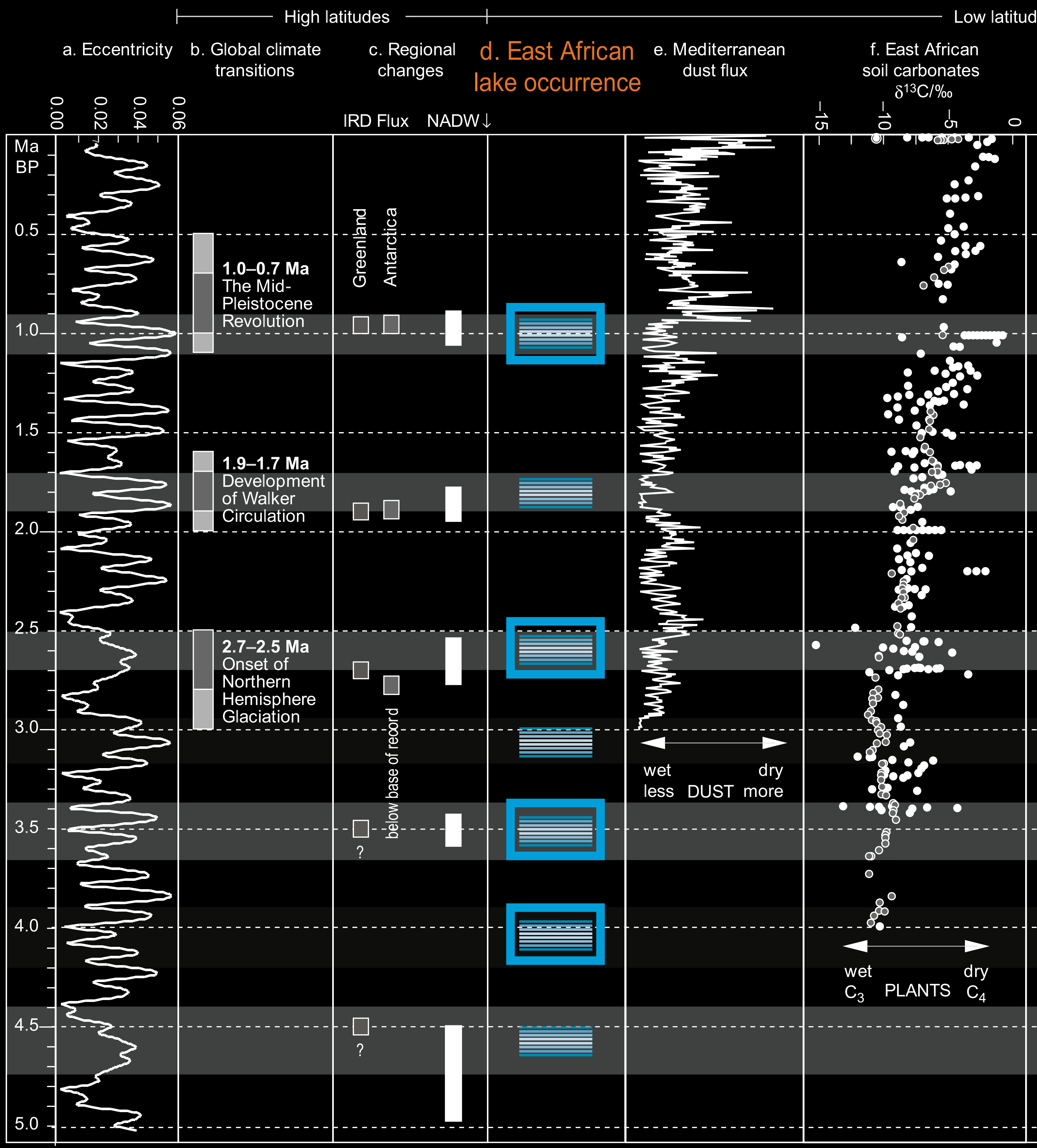
moving window: 112 ka

Palaeoclimate Regime Changes



moving window: 112 ka

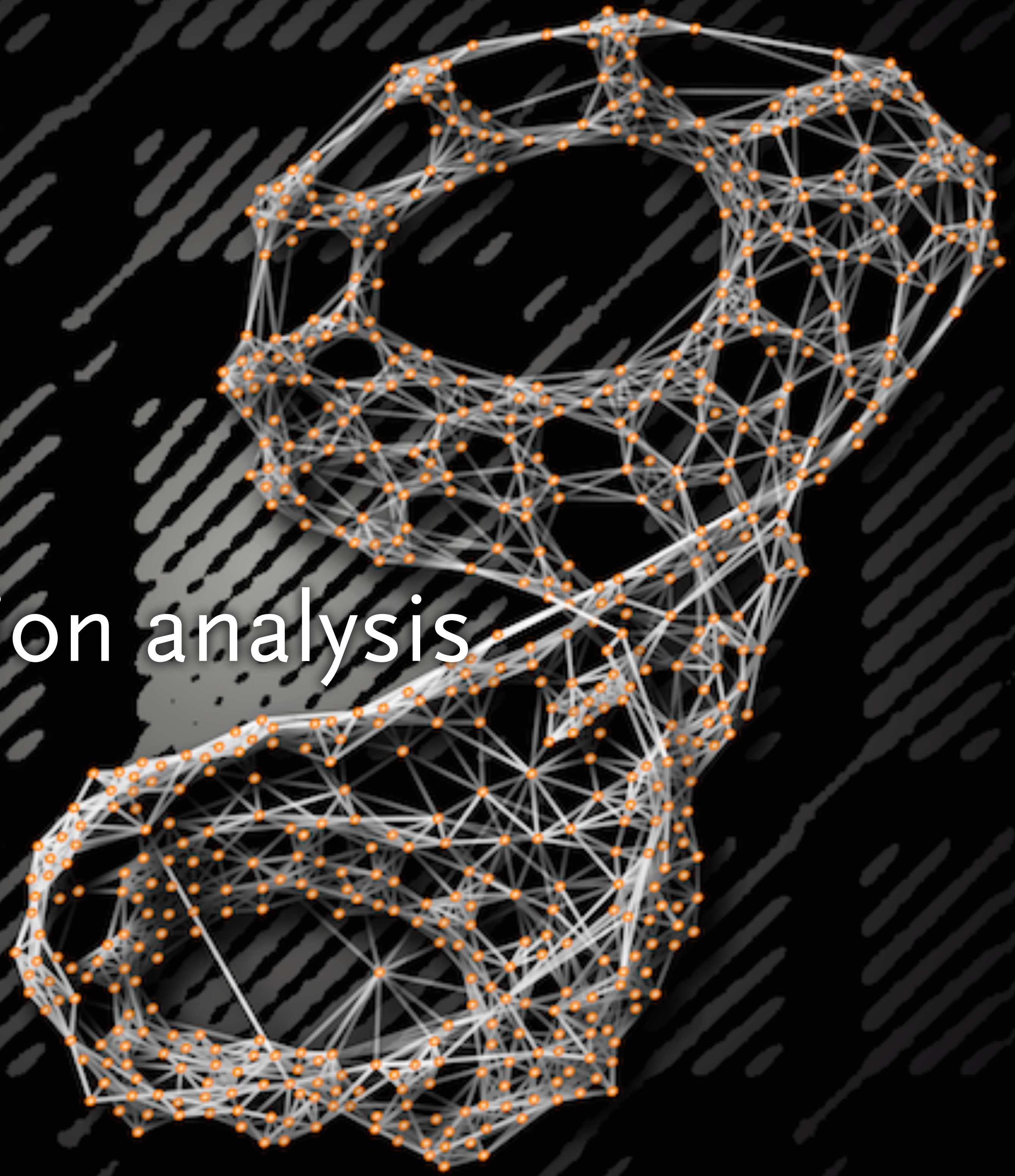




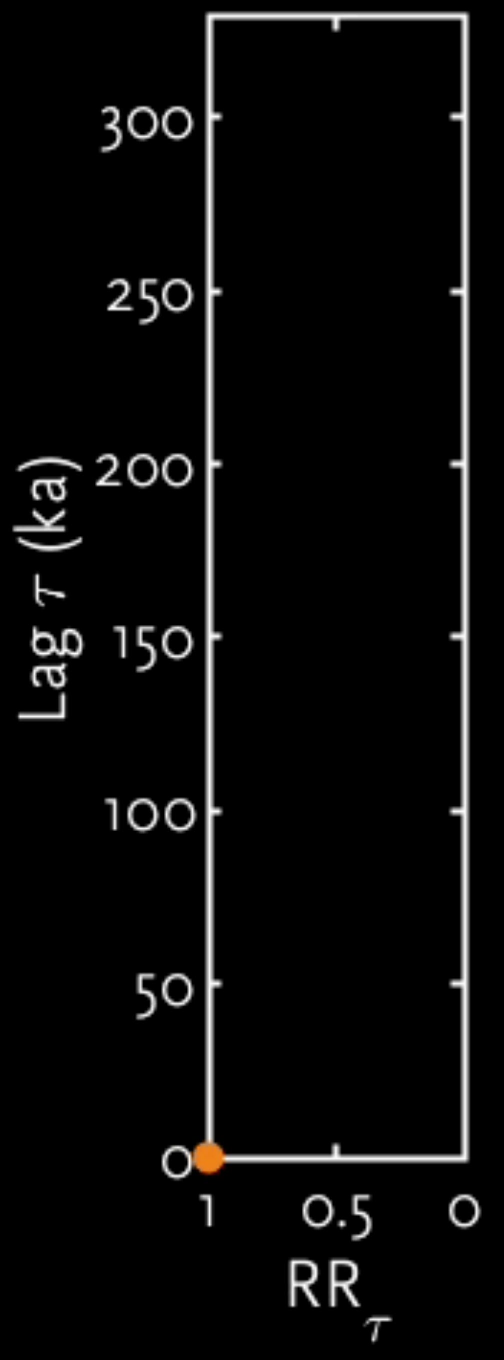
Trauth et al, J Hum Evol 53, 2007
Marwan et al, Quat Sc Rev 274, 2021

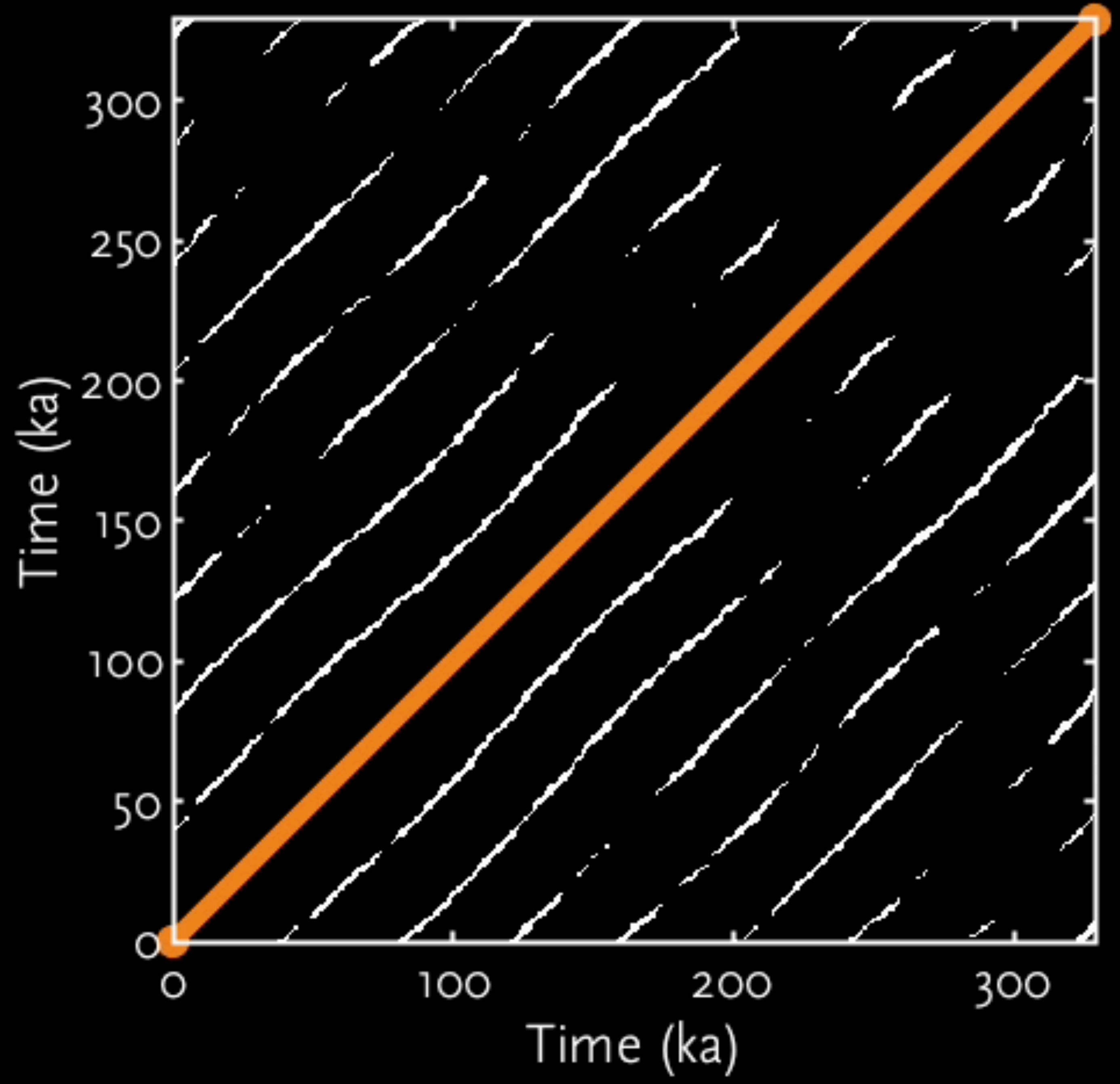
Recurrence Analysis for Complex Systems

- Phase space and recurrence plot
- Recurrence quantification
- **Coupling and synchronization analysis**
- Outlook



τ-Recurrence Rate





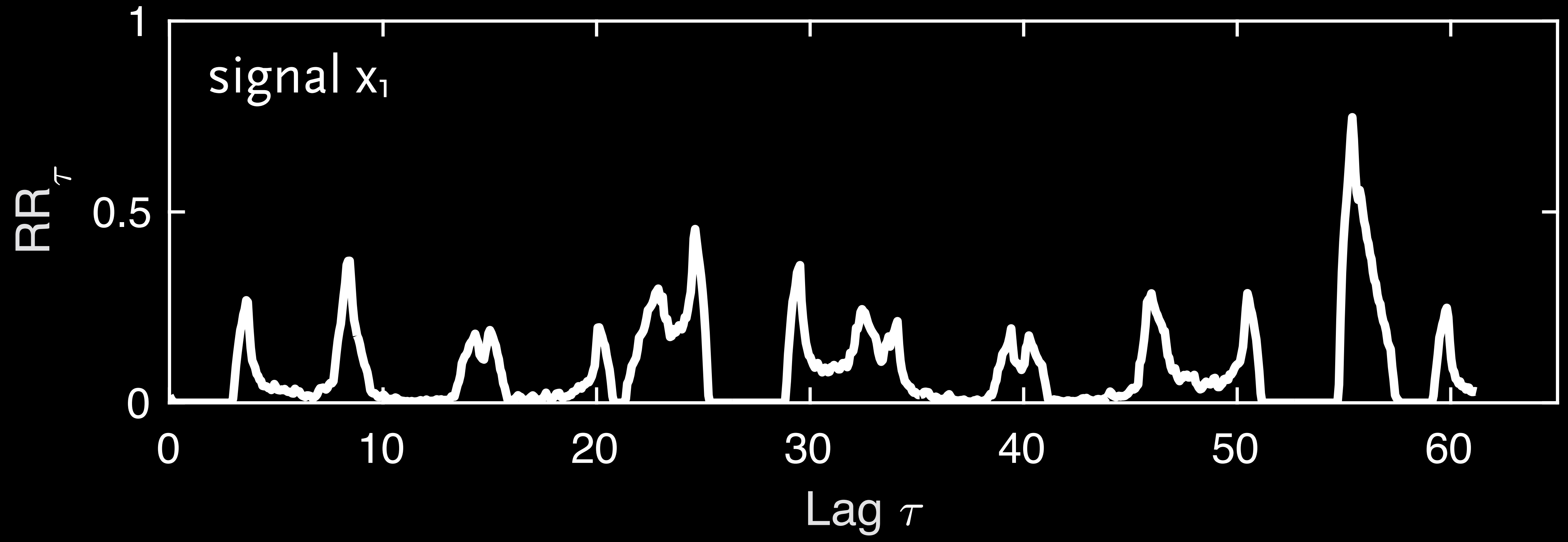
- Density of recurrence points along a diagonal

$$\tau RR(\tau) = \frac{1}{N-\tau}\sum_{i=1}^{N-\tau} R_{i,i+\tau}$$

- Probability that the system will return to a former state after lag τ

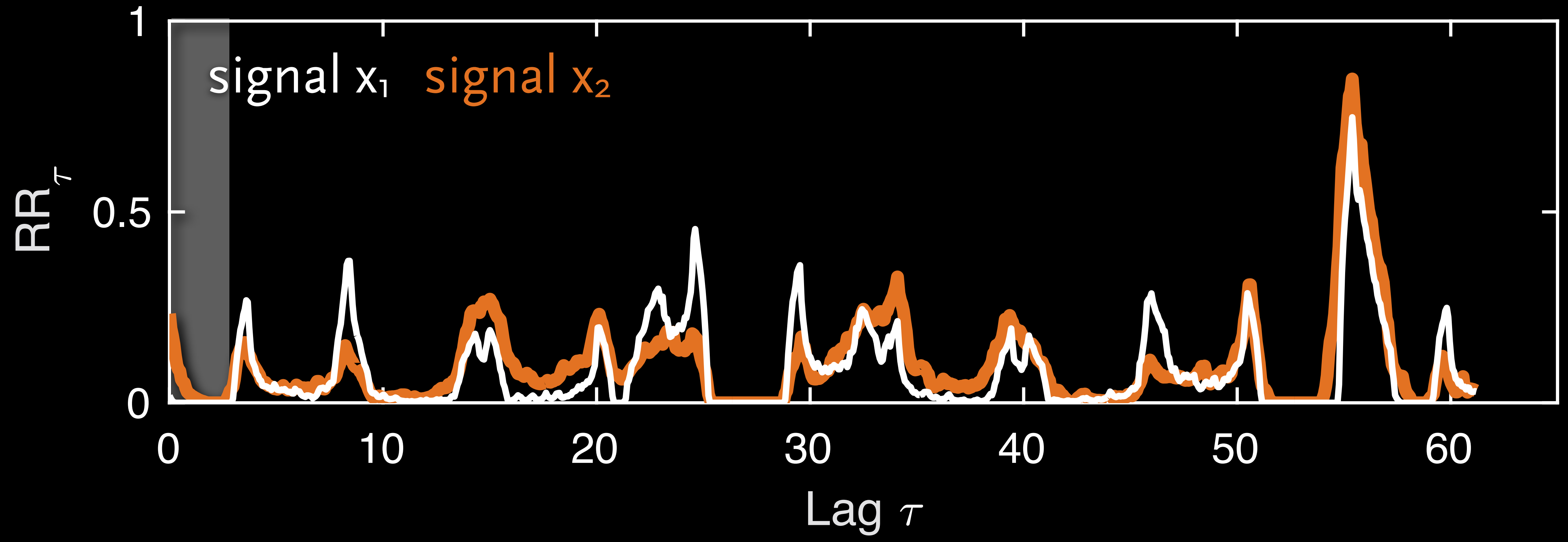
Phase Synchronisation

- Probability RR_τ that system recurs after time τ



Phase Synchronisation

➡ High coincidence for synchronised systems

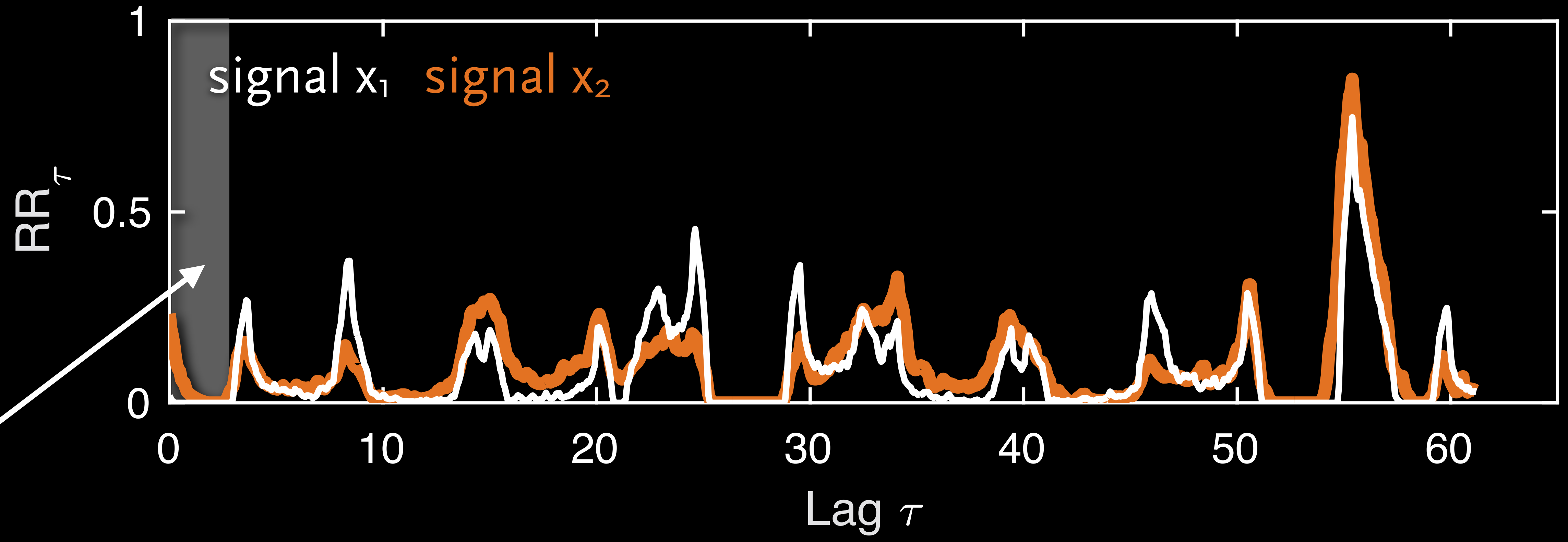


- Synchronisation index: correlation coefficient $CPR = \langle RR_\tau^x \cdot RR_\tau^y \rangle$

Romano et al, EPL 71, 2005

Phase Synchronisation

➡ High coincidence for synchronised systems

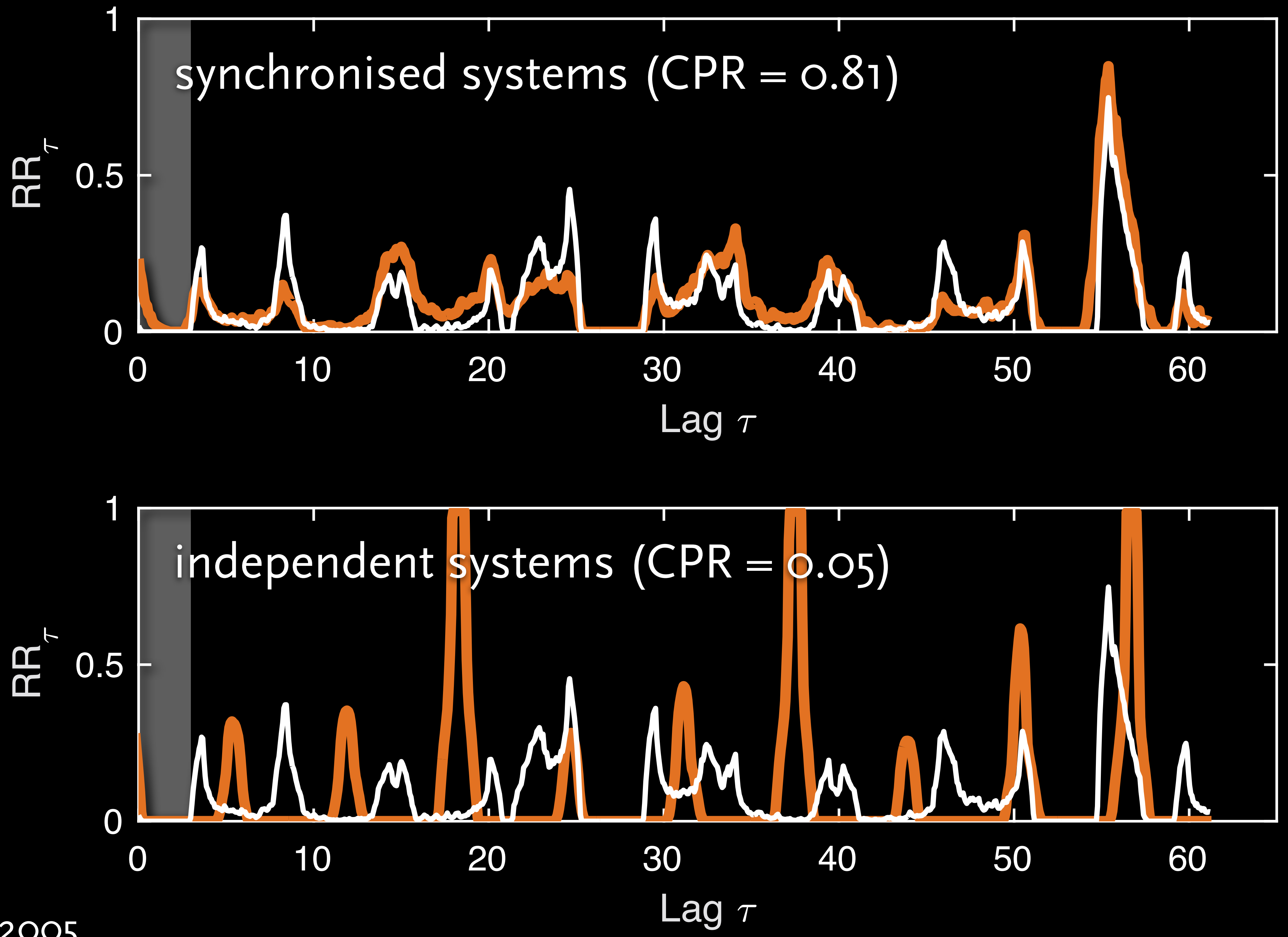


ignore this first part!

- Synchronisation index: correlation coefficient $CPR = \langle RR_\tau^x \cdot RR_\tau^y \rangle$

Romano et al, EPL 71, 2005

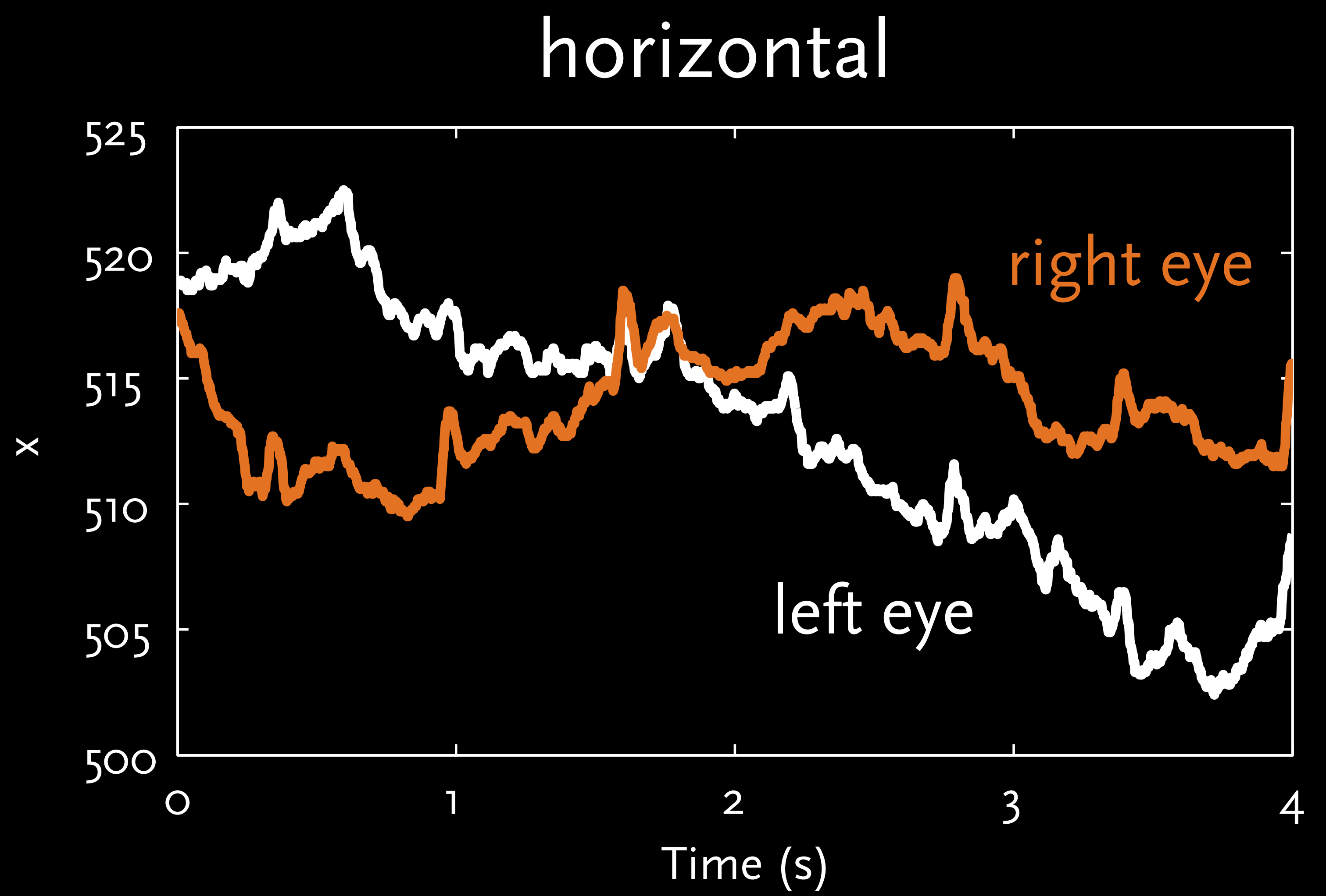
Phase Synchronisation

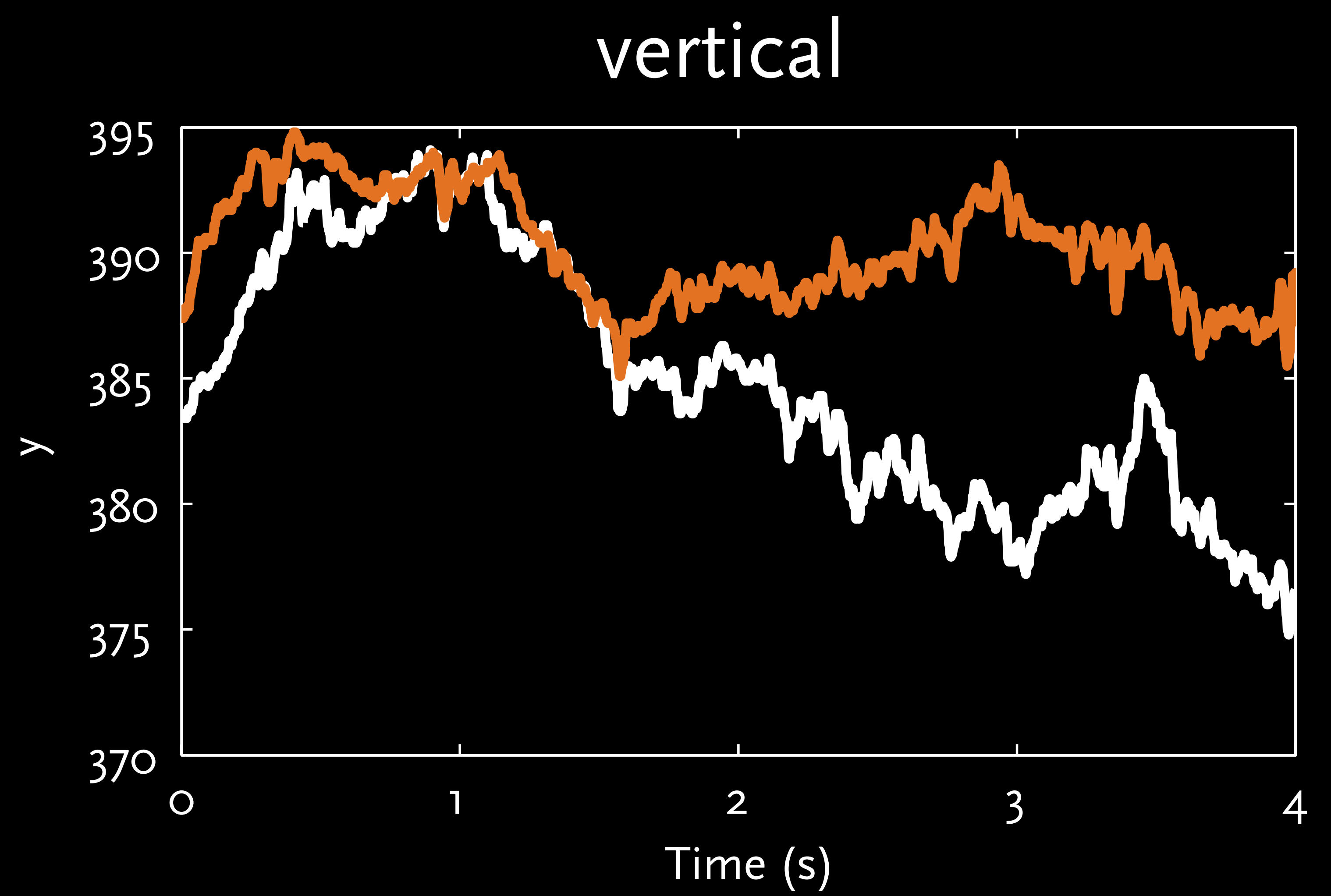


Romano et al, EPL 71, 2005

Fixational Eye Movements

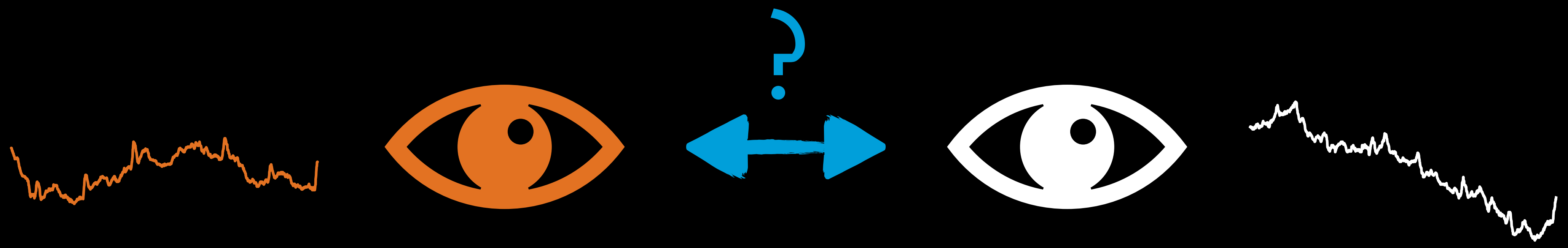
Fixational Eye Movements



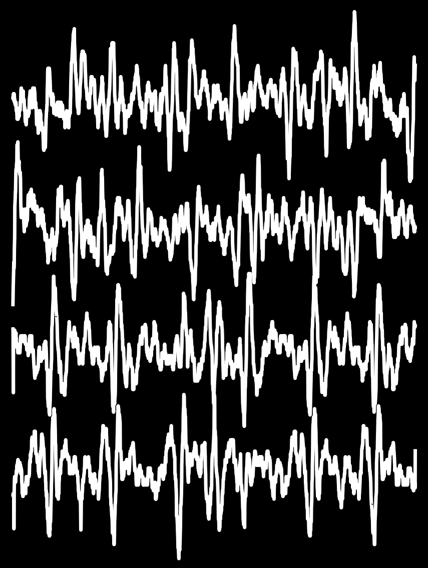


Romano et al, Handbook of Time Series Analysis, 2006

Fixational Eye Movements

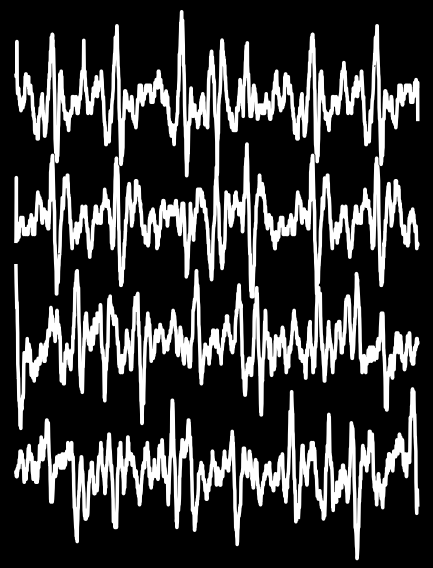


$$CPR = \langle \bar{p}_{\vec{x}}(\tau)\bar{p}_{\vec{y}}(\tau)\rangle$$



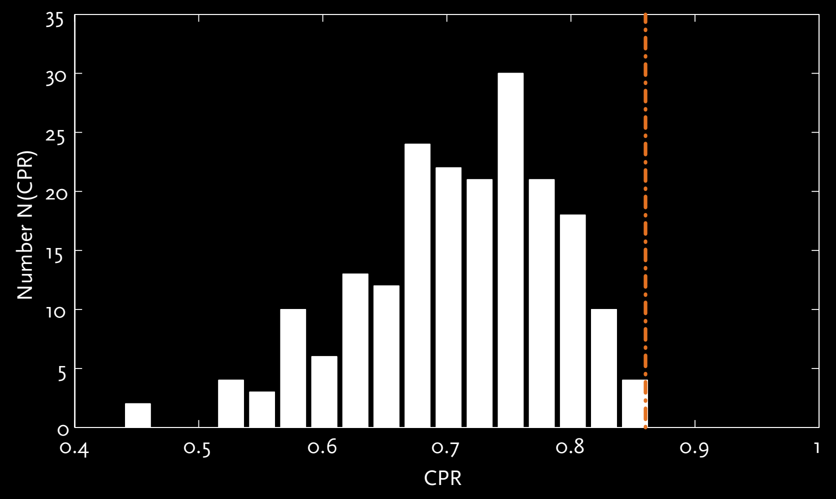
Surrogates

Nullmodel:
independence



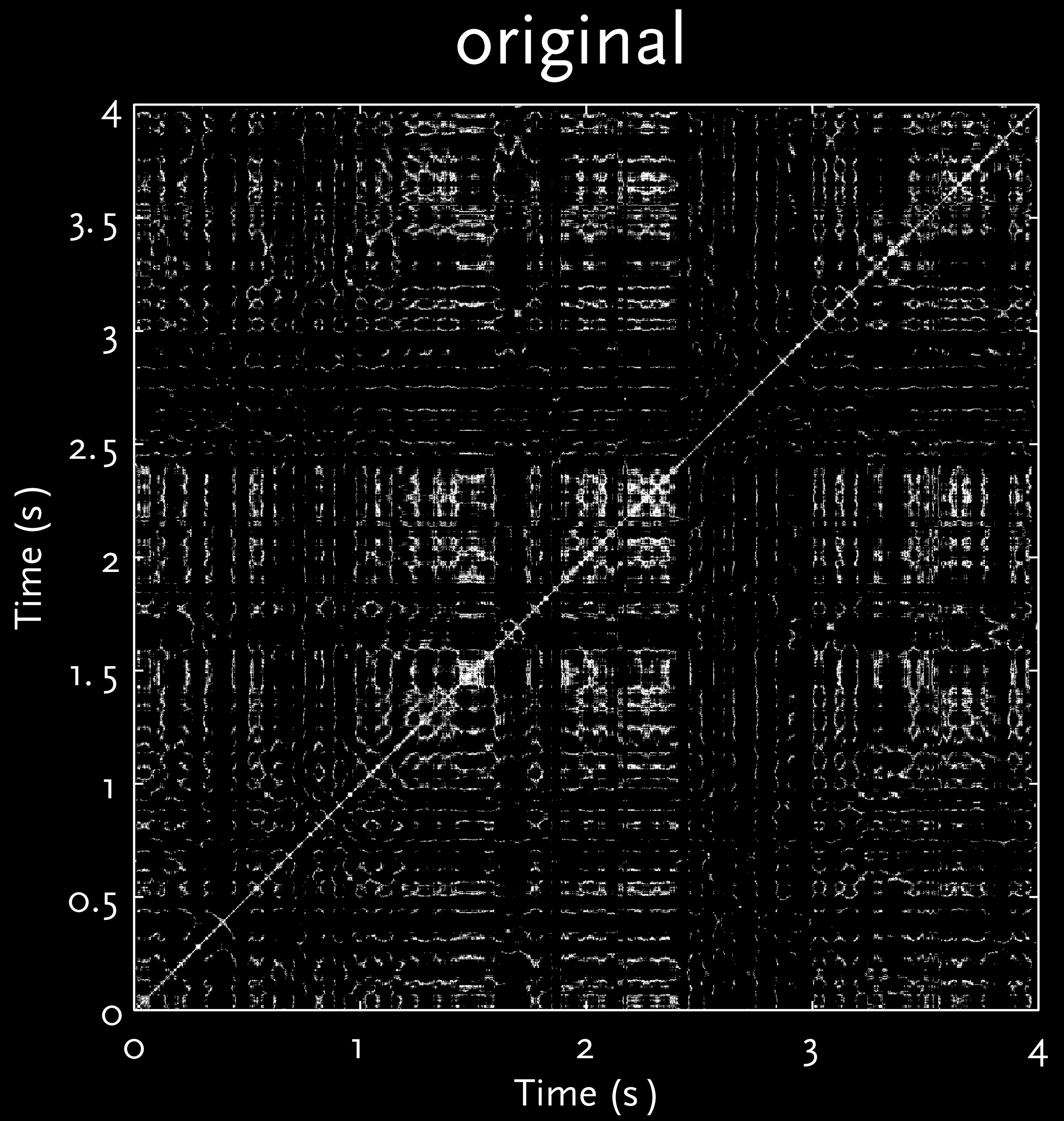
Surrogates

Empirical test distribution for CPR

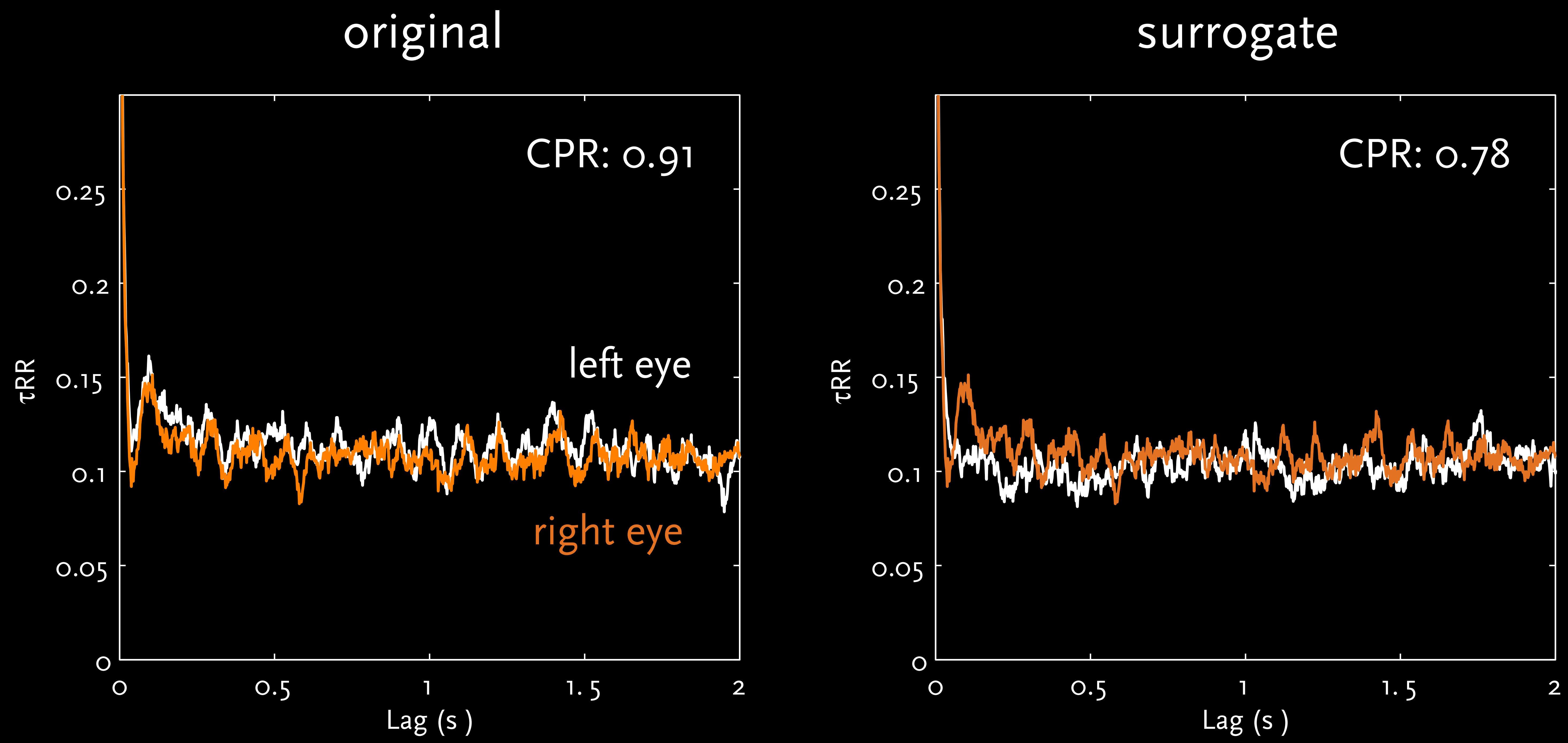


Romano et al, Handbook of Time Series Analysis, 2006

Fixational Eye Movements

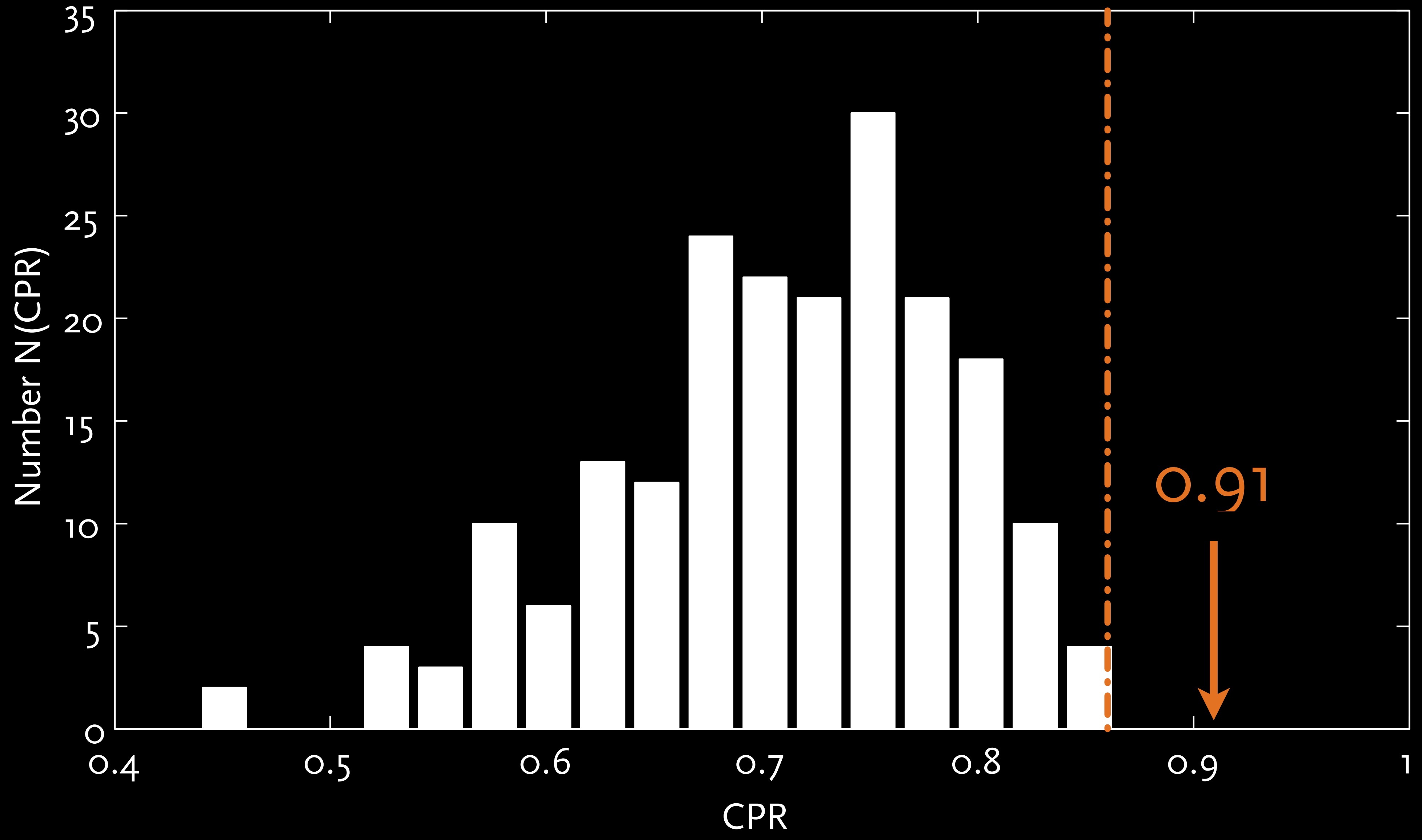


Fixational Eye Movements



Romano et al, Handbook of Time Series Analysis, 2006

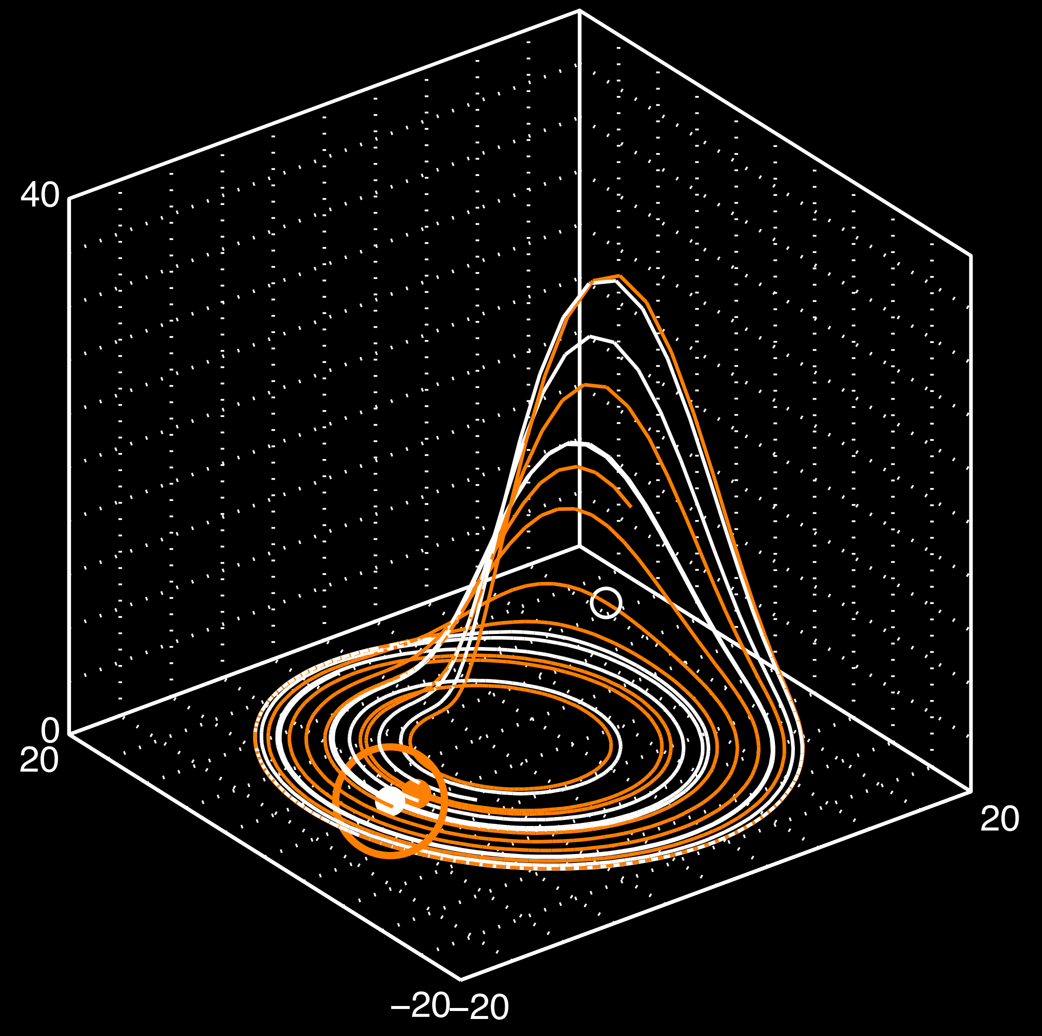
Fixational Eye Movements

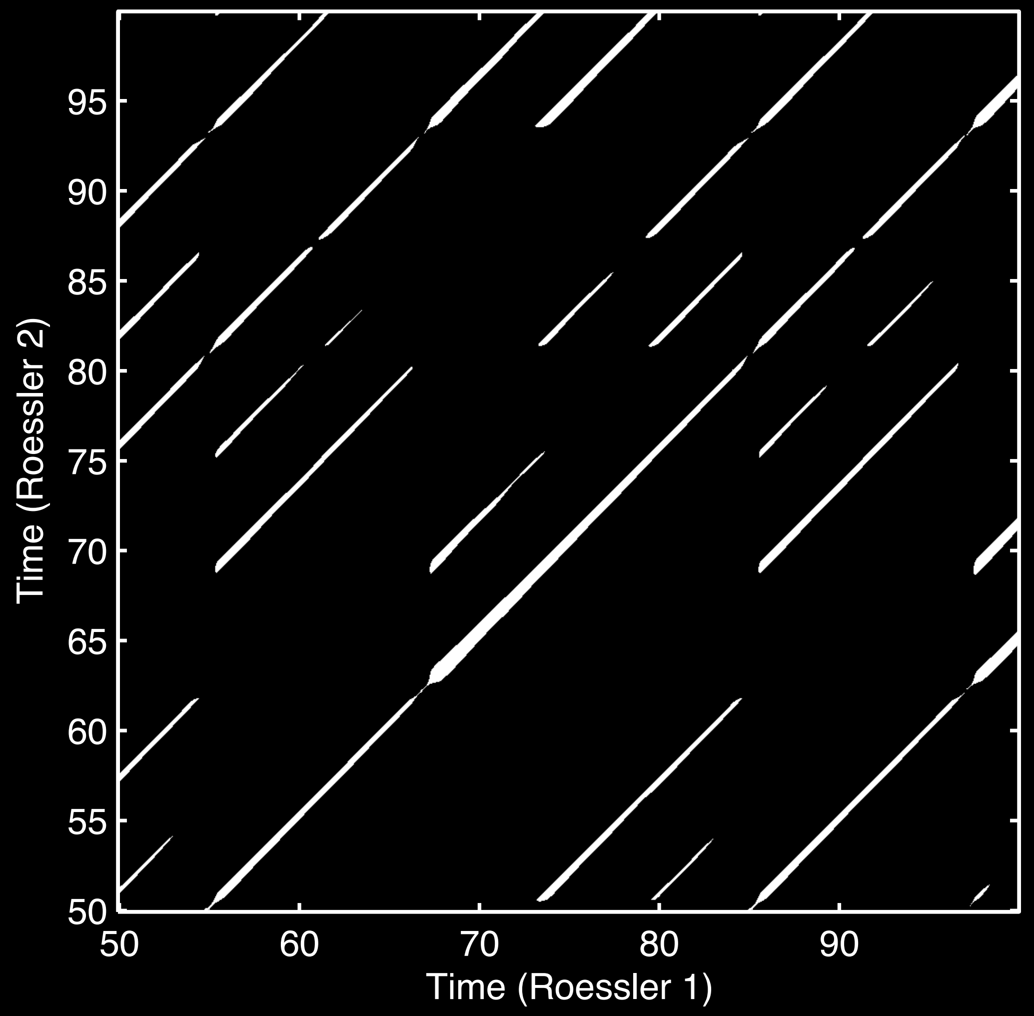


- phase synchronisation of left and right eye movement
- one centre in the brain controlling the eyes‘ movement

Romano et al, Handbook of Time Series Analysis, 2006

Cross Recurrence Plot

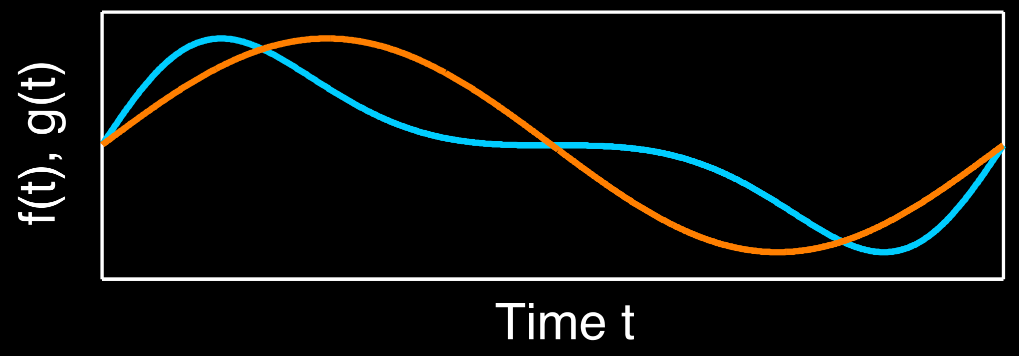


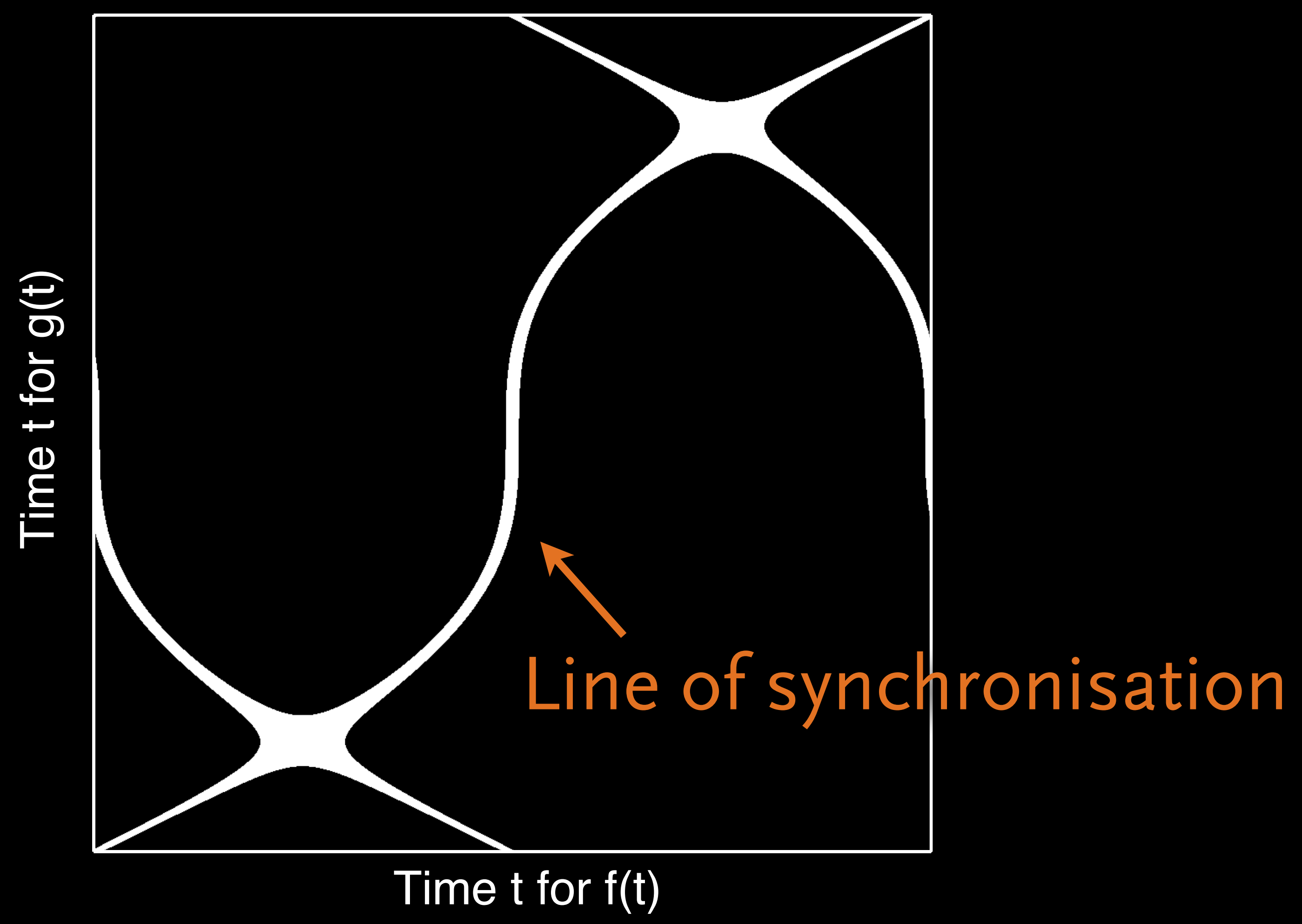


$$\mathbf{CR}_{i,j}^{\vec{x},\vec{y}} = \Theta\left(\varepsilon - ||\vec{x}_i - \vec{y}_j||\right), \quad i = 1, \dots, N,\ j = 1, \dots, M$$

Marwan et al, Nonlin Proc Geophys 9, 2002

Cross Recurrence Plot



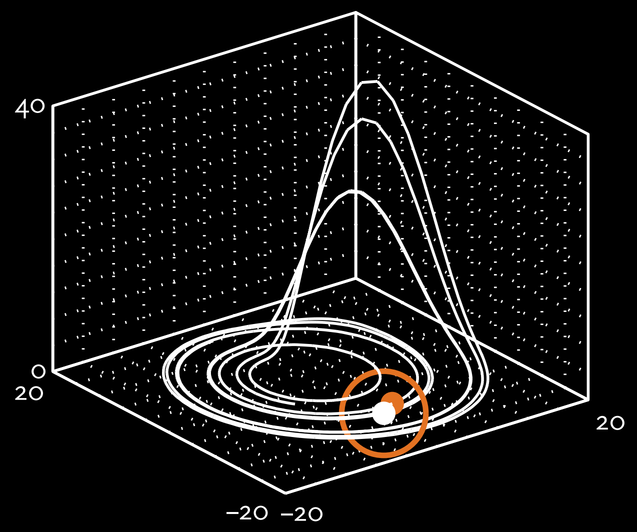


$$\mathbf{CR}_{i,j}^{\vec{x},\vec{y}} = \Theta\left(\varepsilon - ||\vec{x}_i - \vec{y}_j||\right), \quad i = 1, \ldots, N, \; j = 1, \ldots, M$$

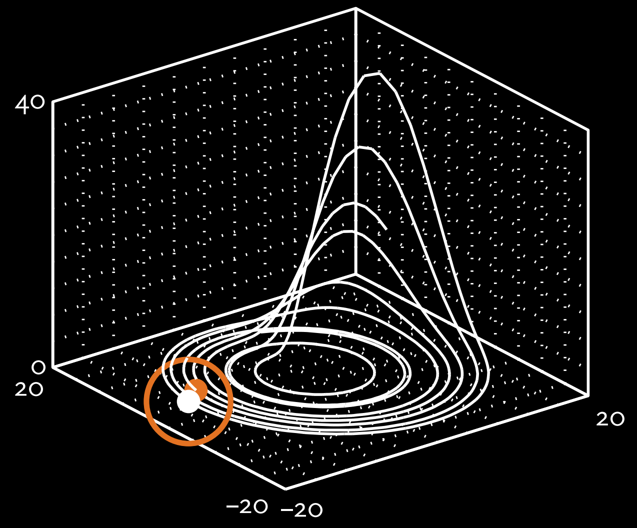
Marwan et al, Nonlin Proc Geophys 9, 2002

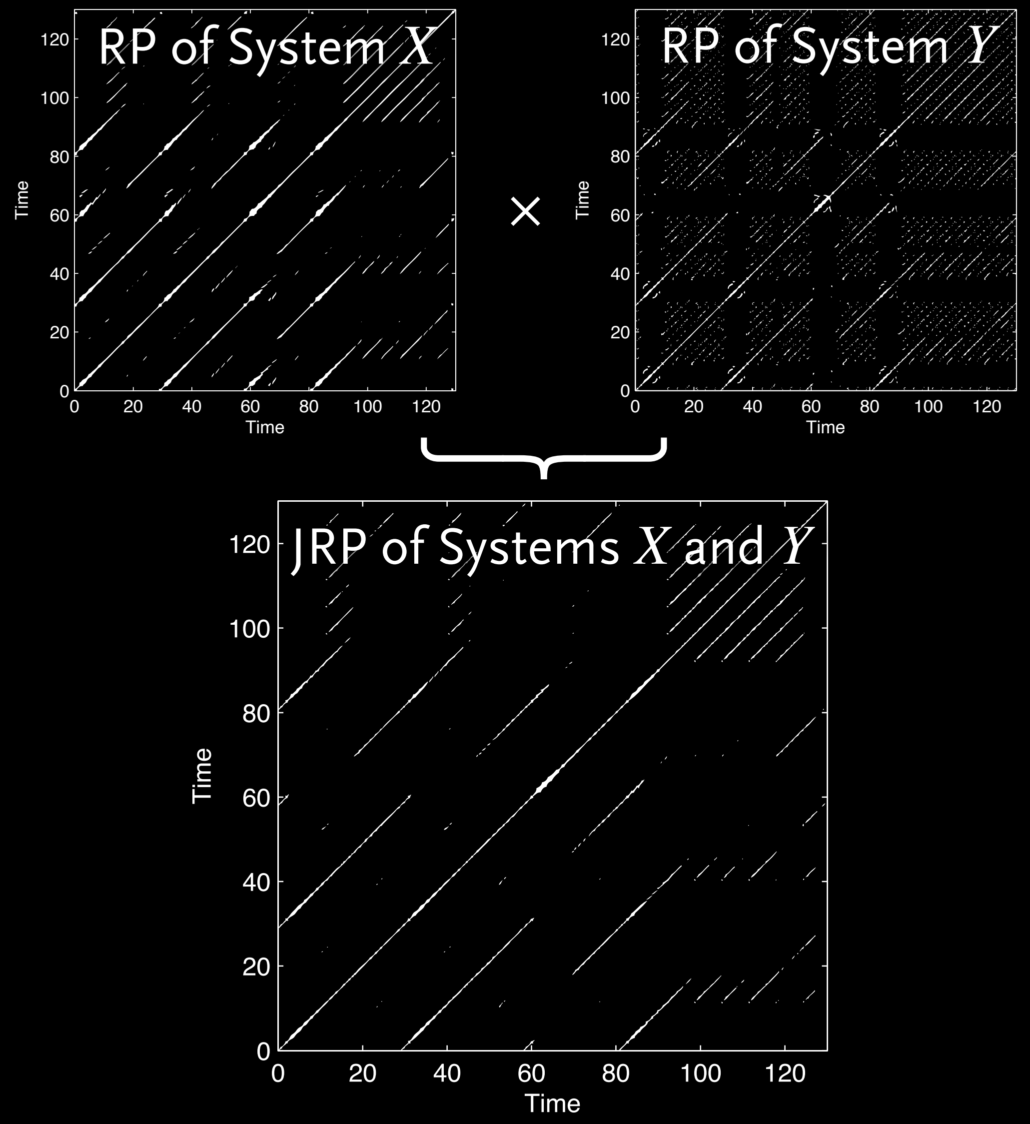
Joint Recurrence Plot

System X



System Y





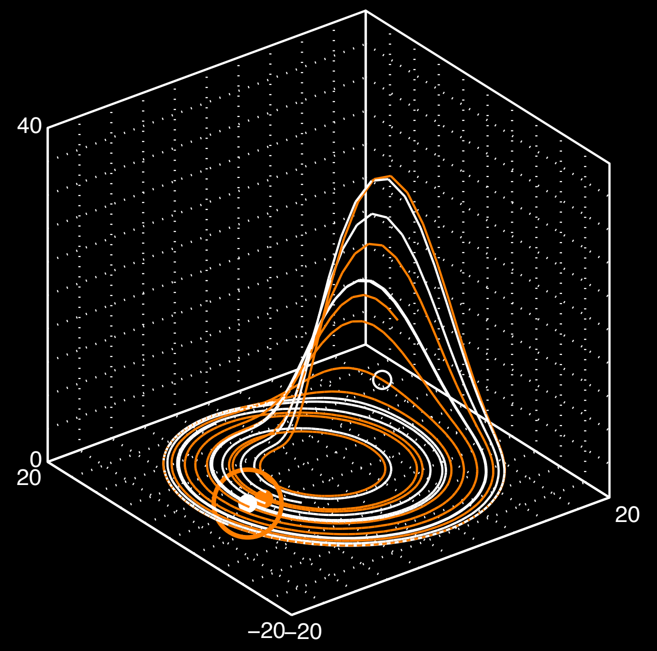
$$\mathbf{JR}_{i,j}^{\vec{x}(1,\ldots,n)} = \prod_{k=1}^{n} \mathbf{R}_{i,j}^{\vec{x}(k)}, \quad i,j = 1,\ldots,N$$

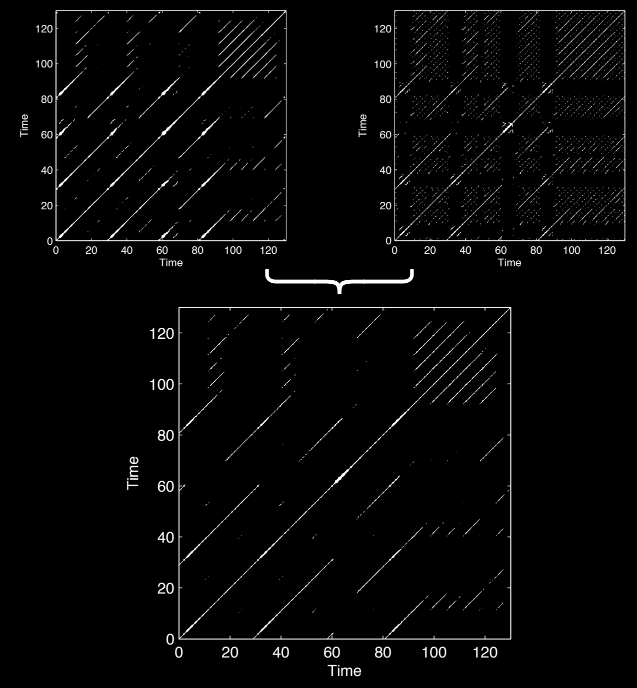
Romano et al, Phys Lett A 330, 2004

Difference CRP–JRP

- CRP: simultaneous occurrence of a similar state
  - ➡ Comparable/ similar states
  - ➡ Differences in timing (time alignment)

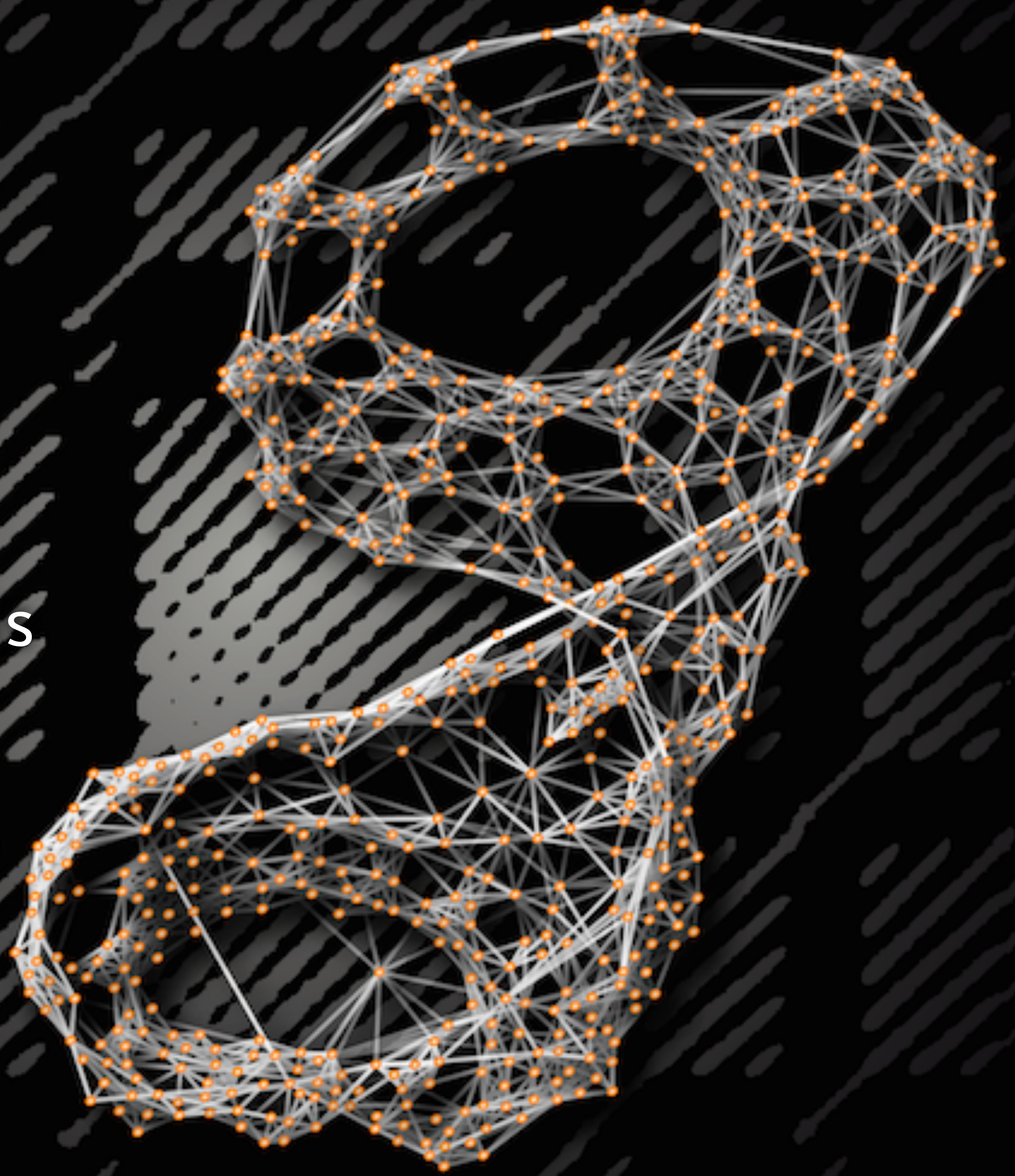
- JRP: simultaneous occurrence of a recurrence
  - ➡ Different states
  - ➡ Same timing (generalised synchronisation)



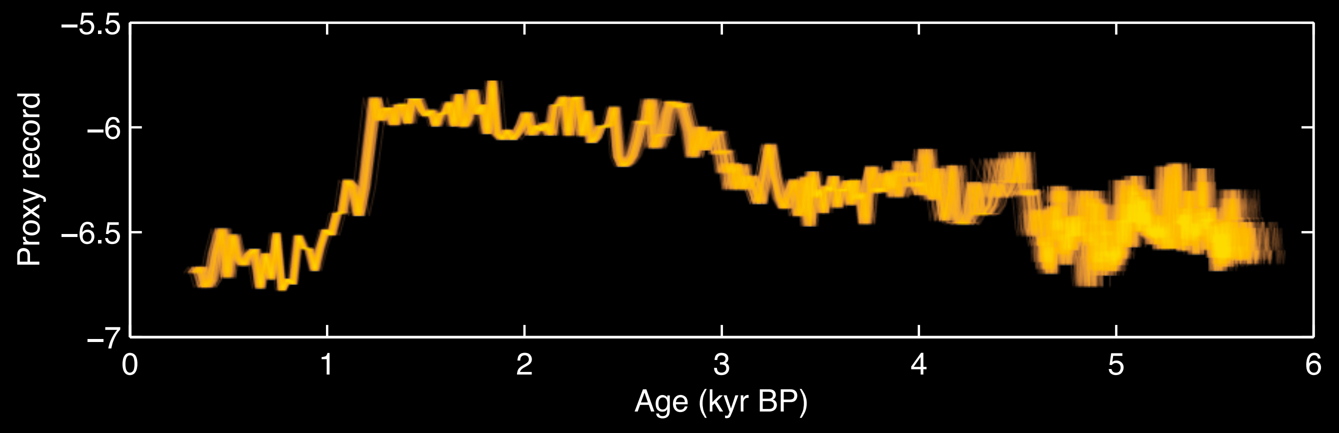


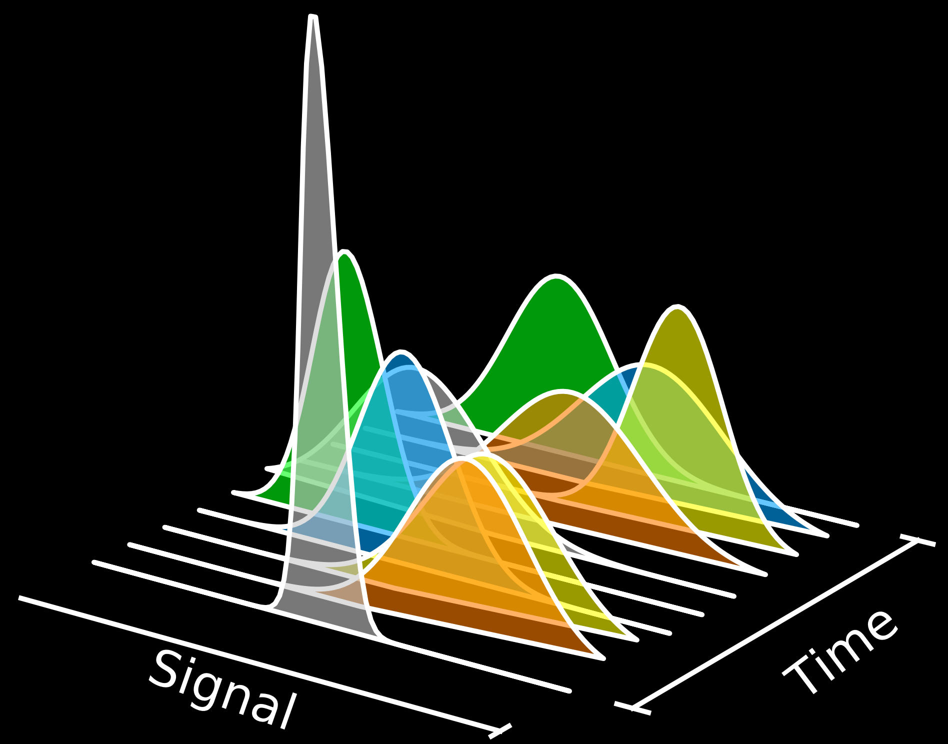
Recurrence Analysis for Complex Systems

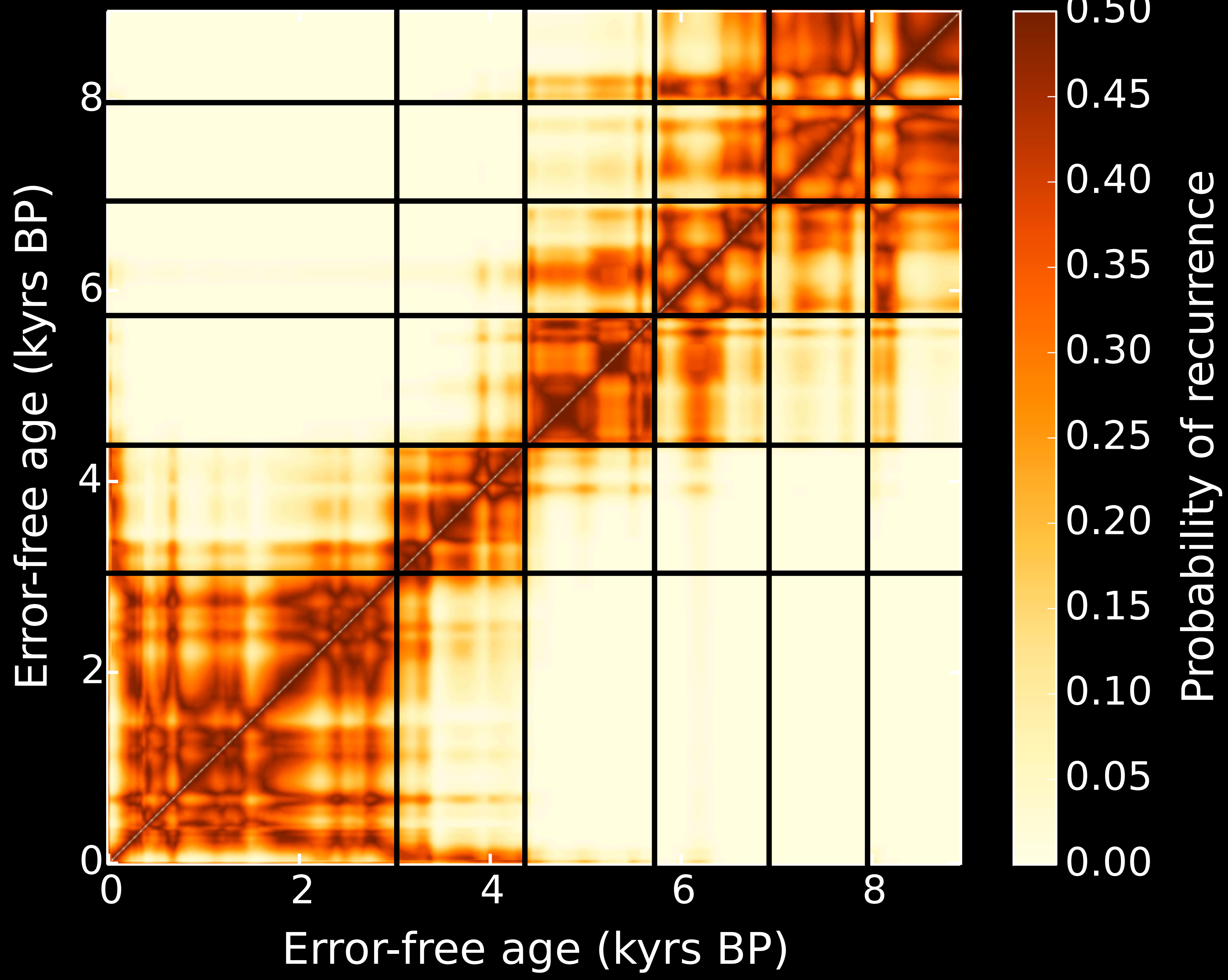
- Phase space and recurrence plot
- Recurrence quantification
- Coupling and synchronization analysis
- **Outlook**



Data with Uncertainties

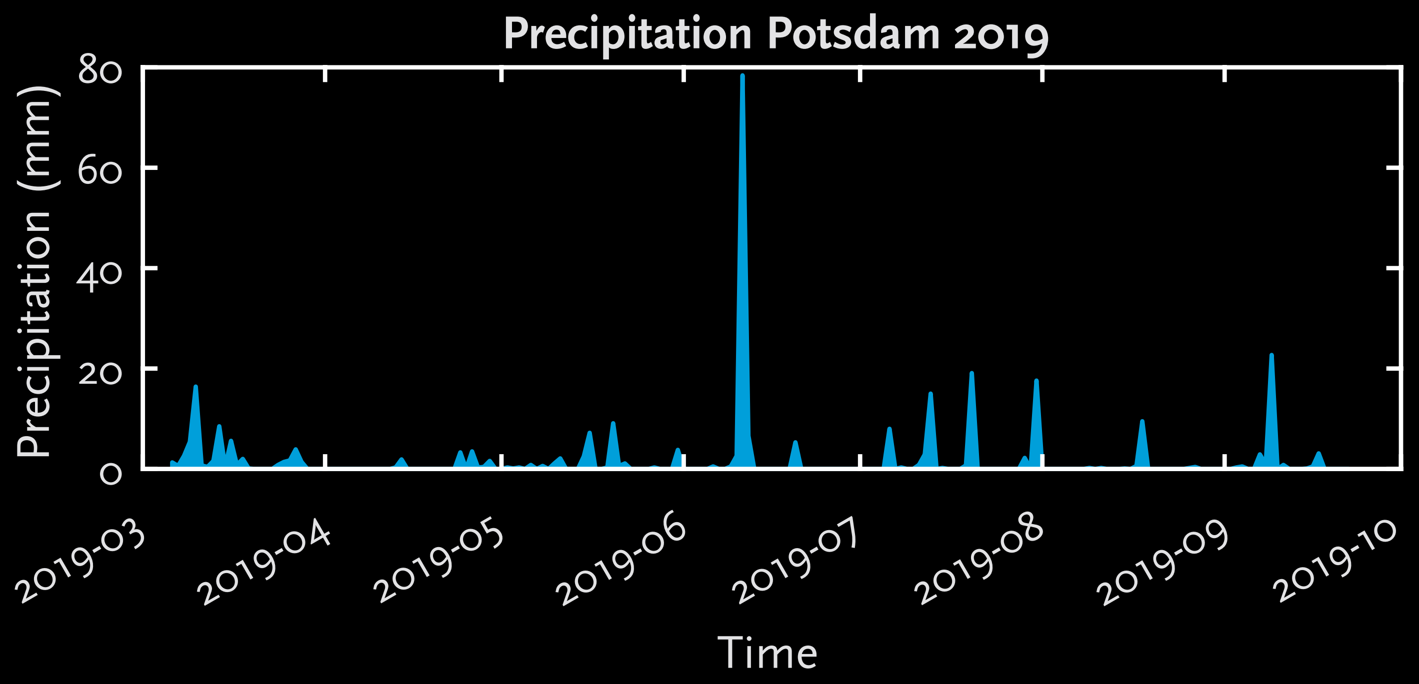


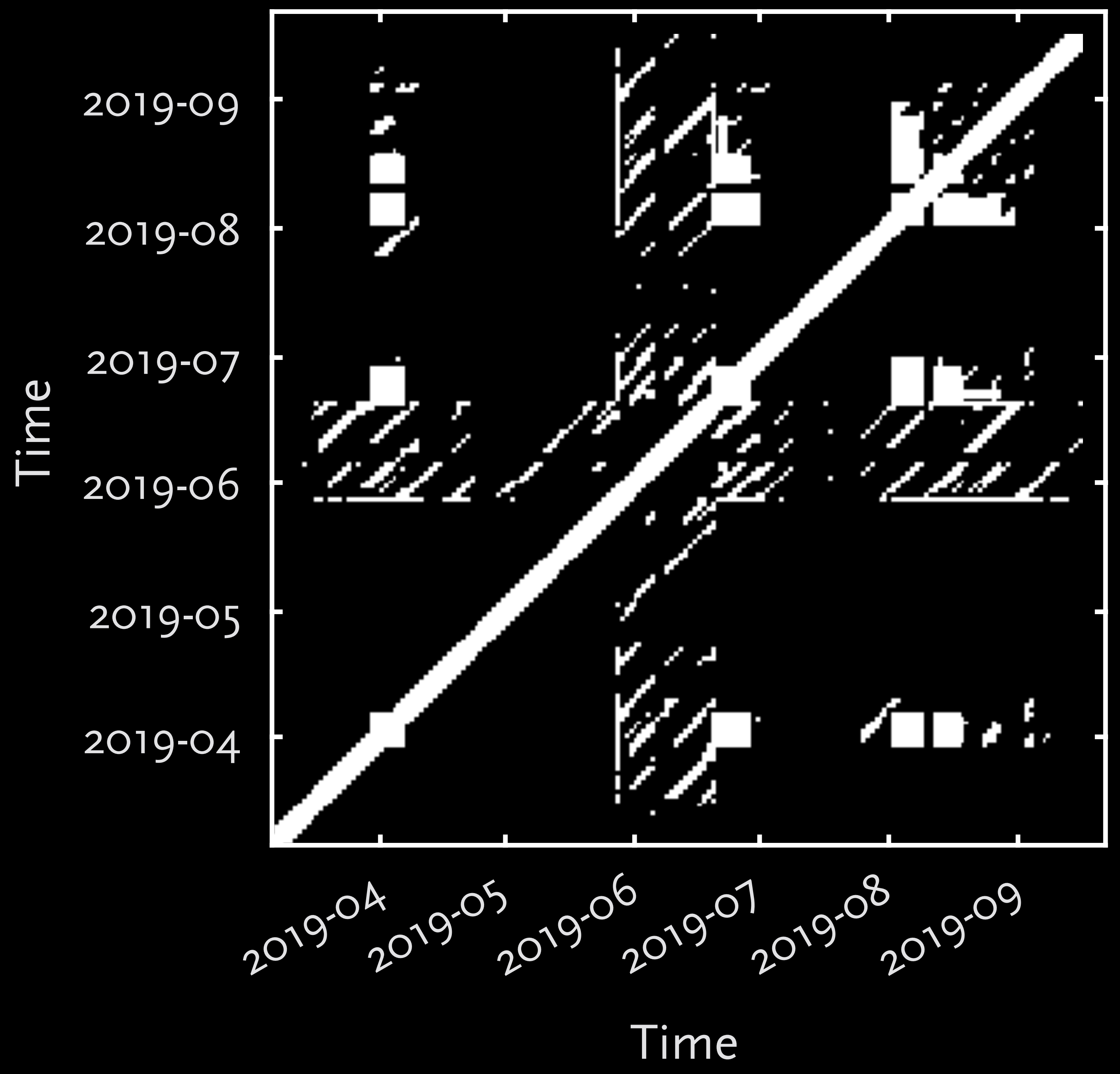




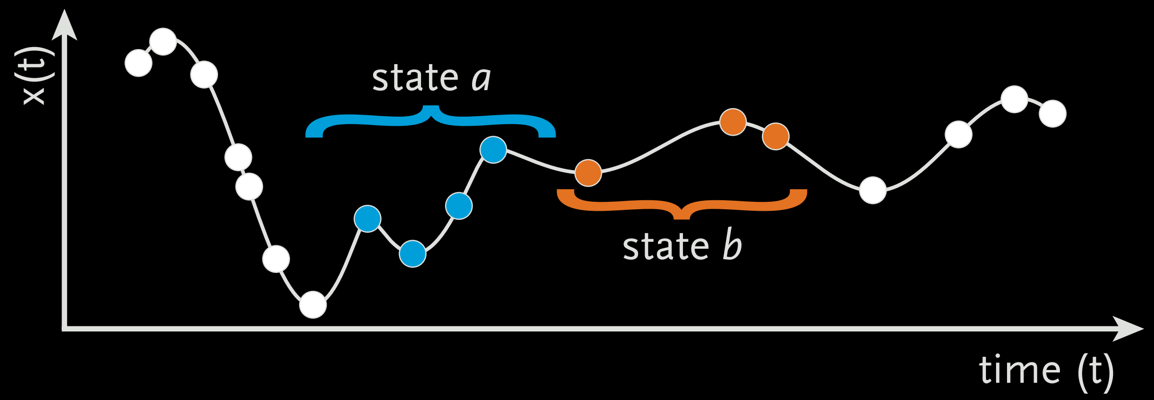
Goswami et al, Nat Comm 9, 2018

Edit Distance Recurrence Plot





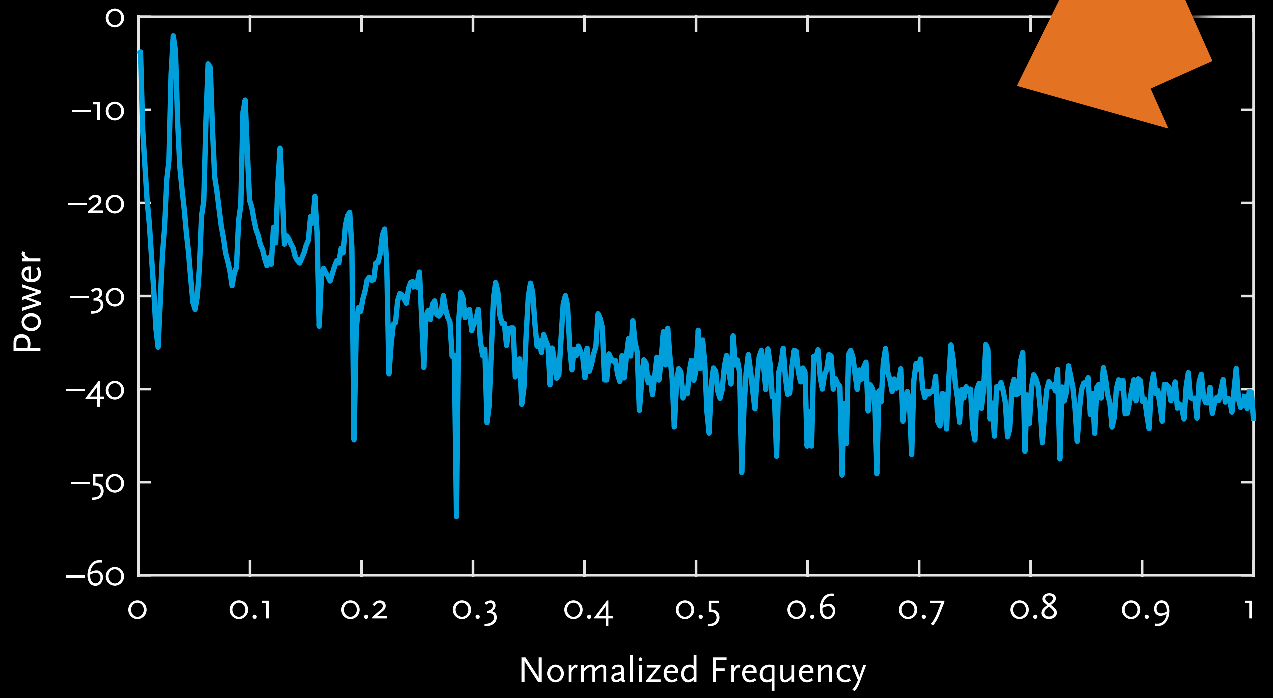
Transformation cost based distance metric:



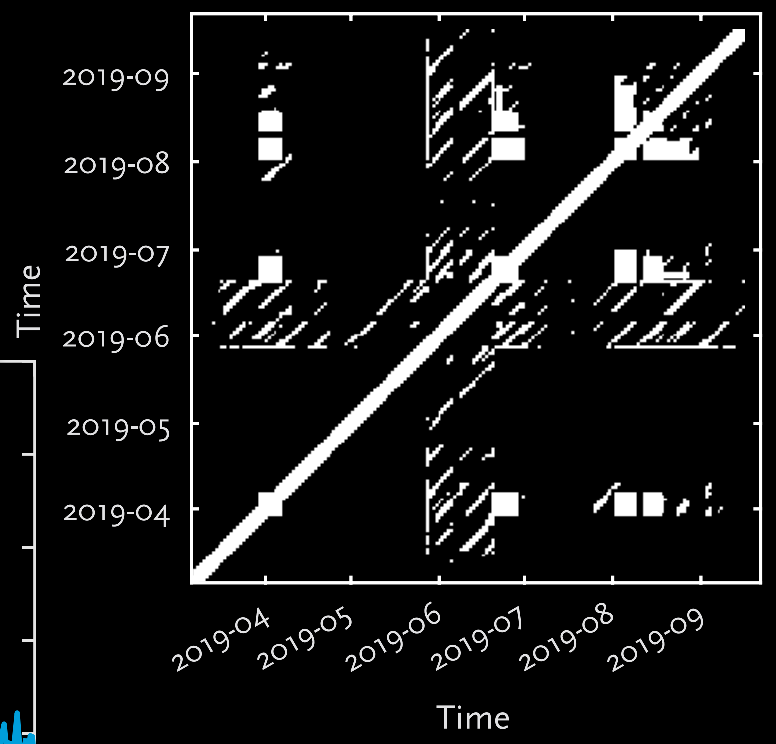
Ozken et al., Phys. Rev. E 91, 2015

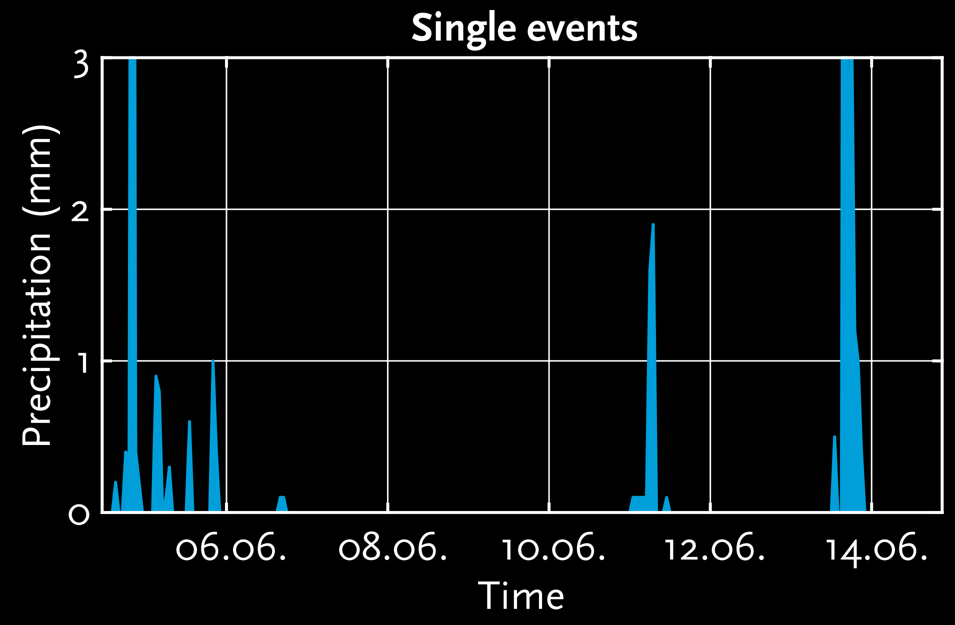
Edit Distance Recurrence Plot

Spectrogram

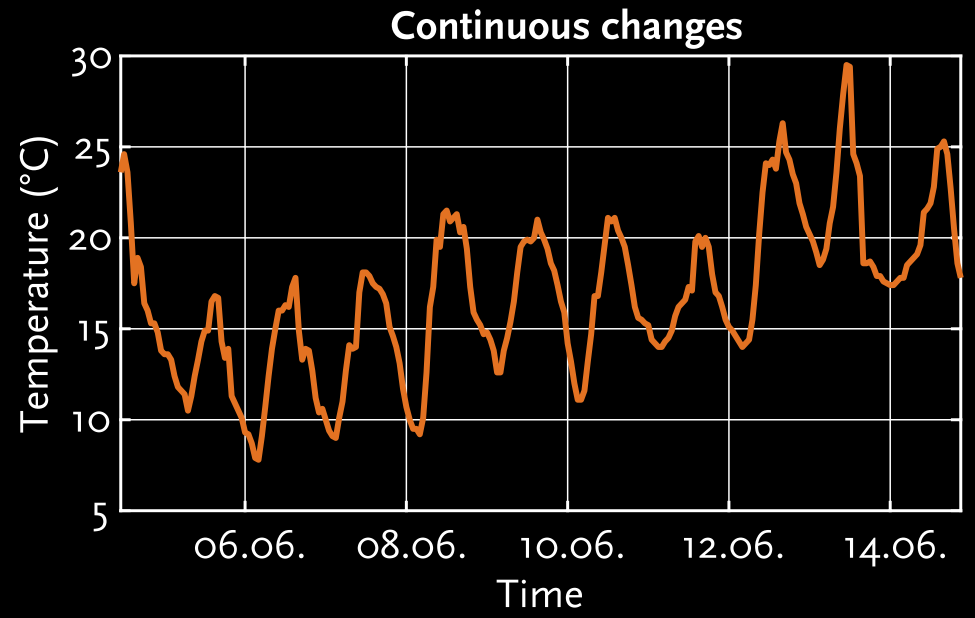


Applications





Correlation tests



Kraemer et al, Entropy 24, 2022
Marwan, Front Appl Math Stat 9, 2023

Recurrence and Machine Learning

Hatami et al., Patt Anal Appl 22, 2019
Nam & Kang, Appl Sc 11, 2021

- Classification (time series imaging)

Uribe et al., Transl Recur, 2014
Abid & Lefebvre, J Loc Based Serv 15, 2021

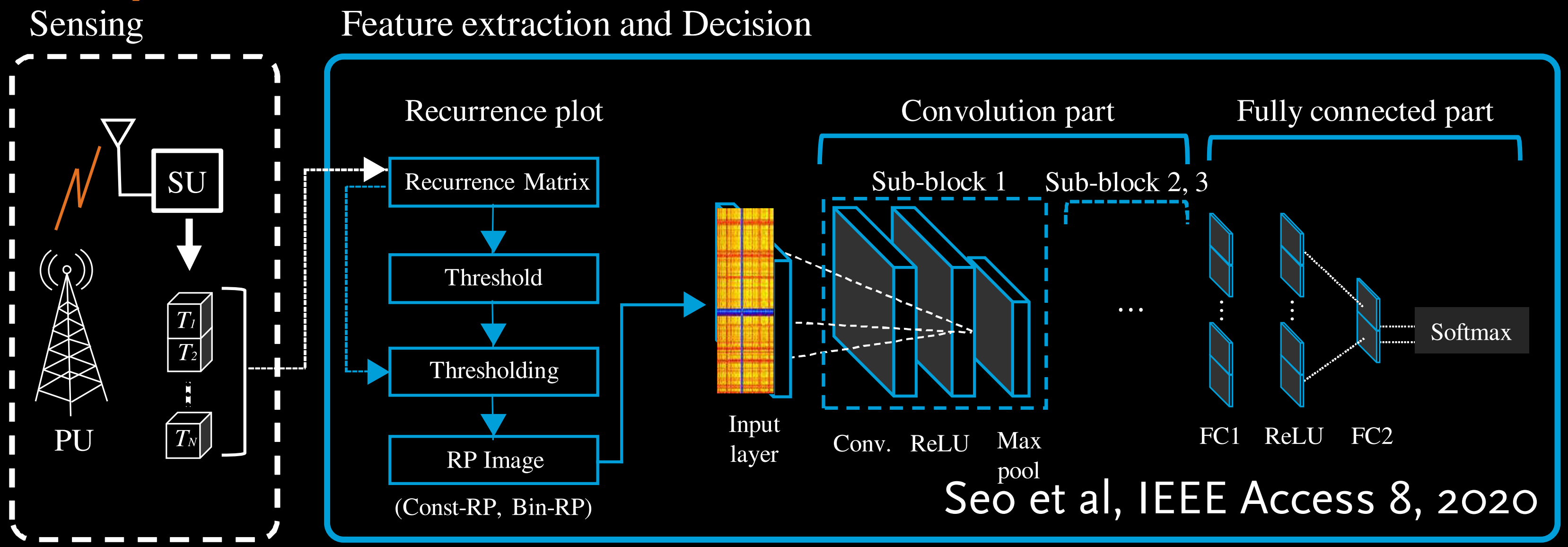
- Pattern recognition, feature extraction (SVM)

Cui et al., IEEE PESGM, 2019
Chen & Yang, Int J Bif Chaos 30, 2020

- Transition detection, monitoring, anomaly detection

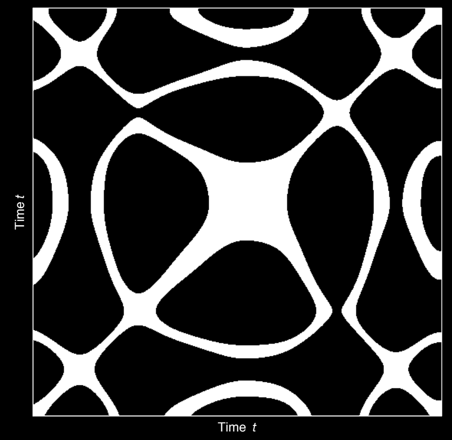
Estebsari & Rajabi, Electronics 9, 2020
Ojeda et al., ICAIIC, 2020

- Modelling (reservoir computing), prediction

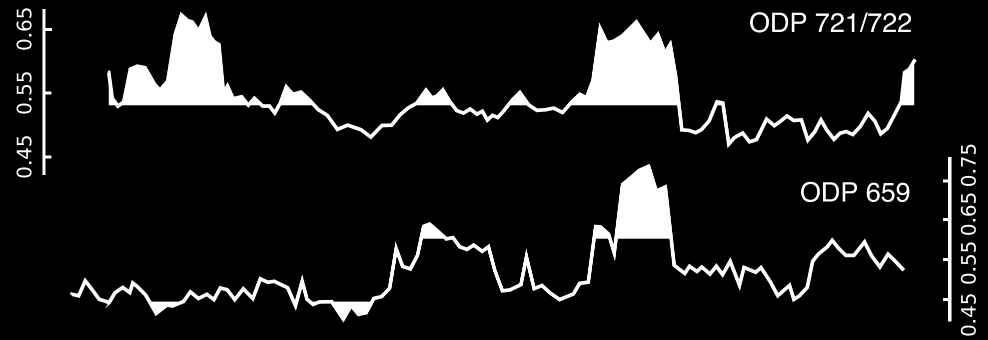


Seo et al, IEEE Access 8, 2020

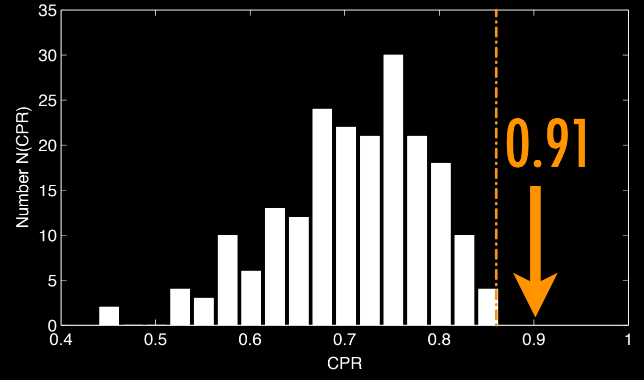
Meaning of structures



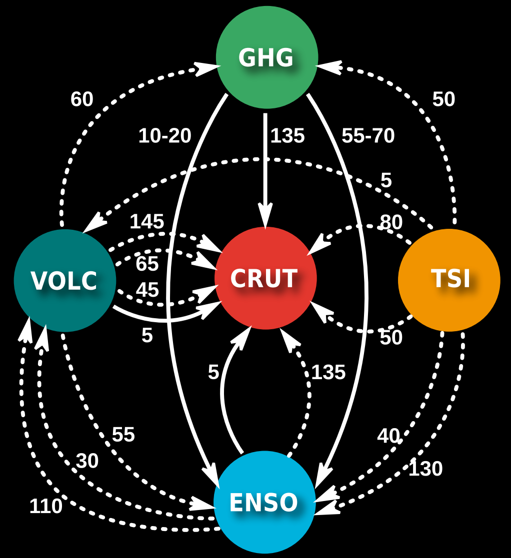
Transition detection



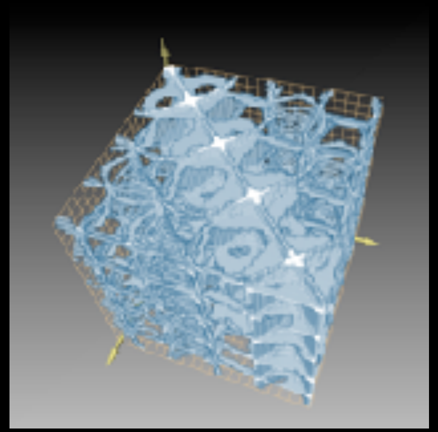
Statistical test

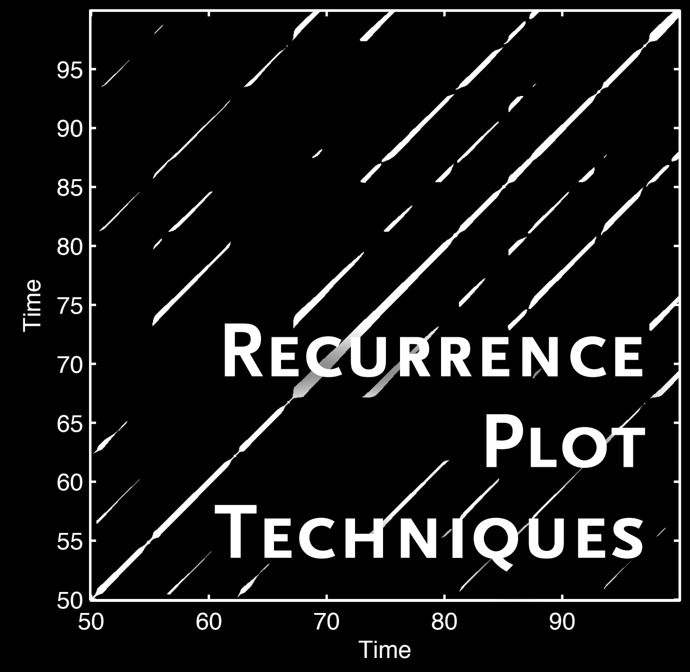


Synchronisation and coupling

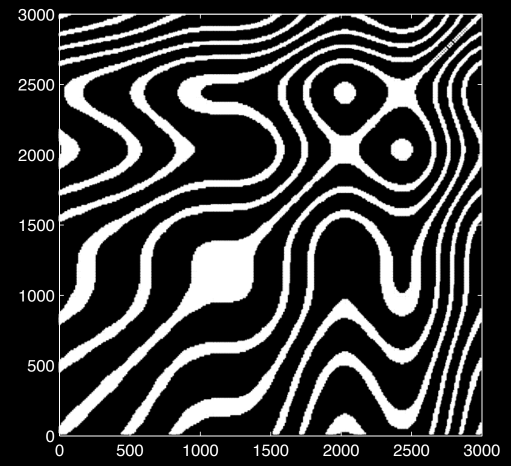


Spatial recurrences

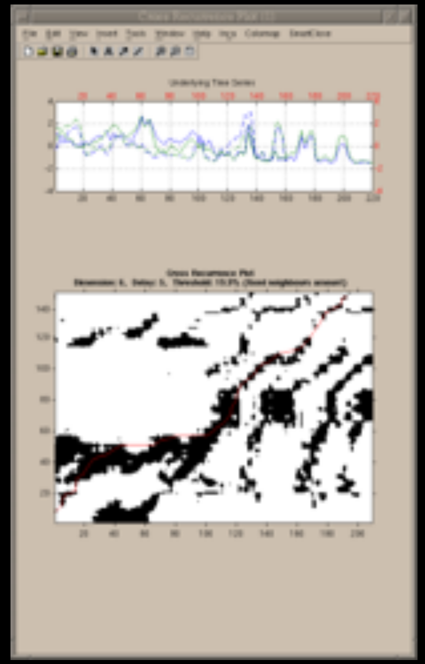




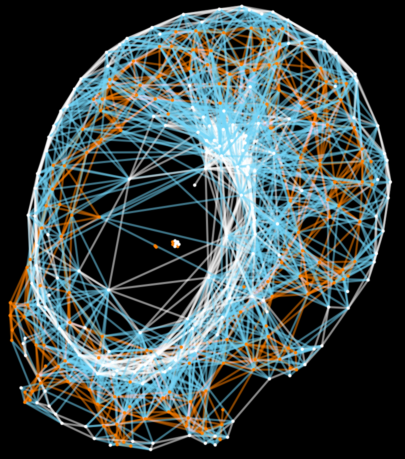
Pitfalls



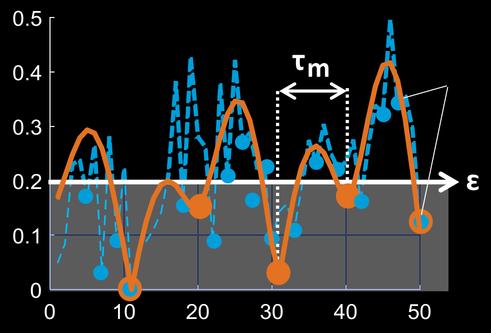
Software



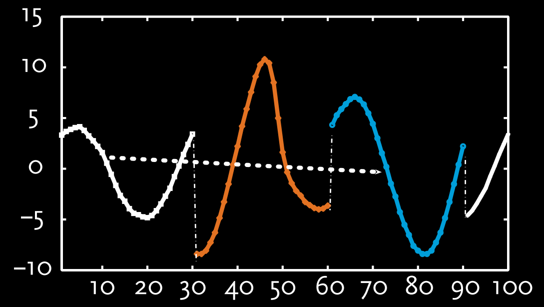
Recurrence networks



Alternative recurrence definitions



Gap filling



Challenging data