Lab 1 Report

March 24, 2025

1 Lab 1: Implementing FEM for 1D Diffusion Equation

We implement the finite elements method (FEM) with piecewise linear ($\mathcal{P}_1$) basis functions to solve the one-dimensional diffusion equation:

$$\text{(D):} \qquad \begin{cases} -(D(x)u'(x))' = f(x) & \text{for } x \in (a,b) \\ u(0) = u(1) = 0 \end{cases} .$$

In the equation (D), $f : [a,b] \to \mathbf{R}$ is the *forcing* or *source* function, $D : [a,b] \to \mathbf{R}$ is the diffusion (typically constant, in which case the equation reads $-Du'' = f$), and we search for a suitable $u : [a,b] \to \mathbf{R}$ in the *trial* space

$$\mathcal{V}([a,b]) = \{v : [a,b] \to \mathbf{R} : v \in C^0([a,b]),\ v' \text{ is piecewise continuous and bounded on } [a,b]\}$$

The constraints $u(0) = u(1) = 0$ are so-called *Dirichlet* boundary conditions.

1.0.1 Importing the code

Please refer to https://github.com/labounad/comp-diffeq/blob/main/oneD/oneD_galerkin.py for the full source code.

[1]:
```
import os
import sys
module_path = os.path.abspath(os.path.join('..'))
if module_path not in sys.path:
    sys.path.append(module_path)

from oneD.oneD_galerkin import *
```

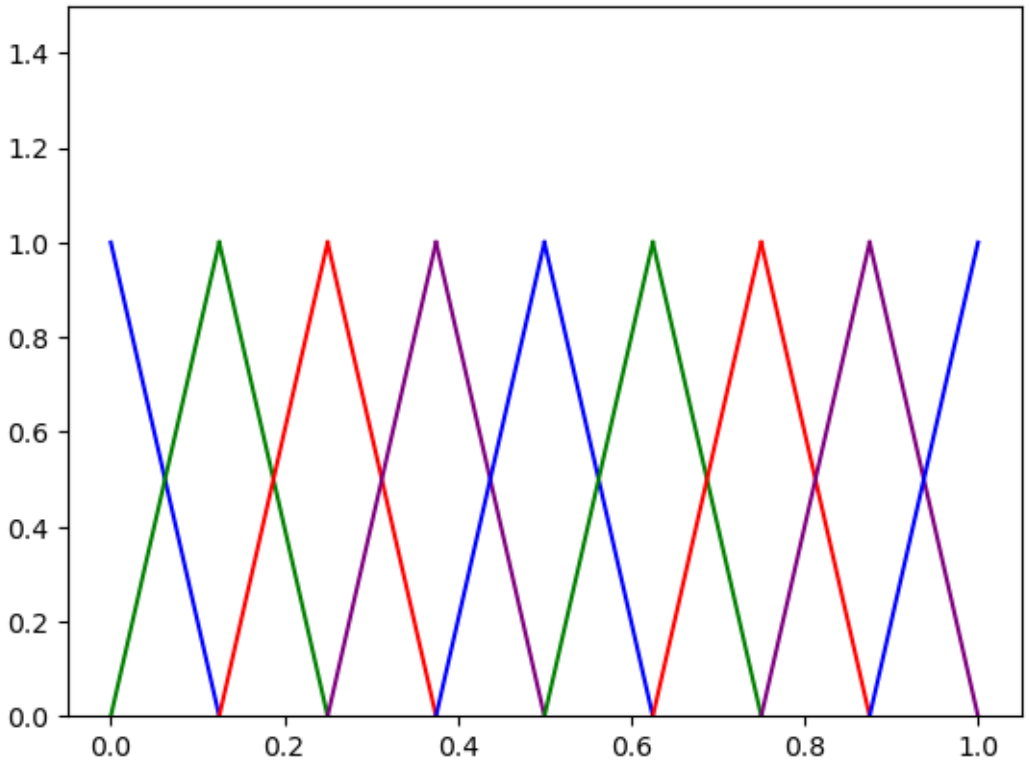
1.1 Theory

The idea is to discretize the interval $[a,b]$ into pieces (elements) of width h, and approximate u by a sequence u_h converging to u, where each u_h lives in the finite-dimensional space $\mathcal{V}_h = \text{span}\{\phi_0, \ldots, \phi_n\}$. The basis functions ϕ_i are chosen to be a piecewise linear partition of unity, such that, with x_j, $j = [0:n]$ denoting the j-th node, $\phi_i(x_j) = \delta_{ij}$. Explicitly:

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{else} \end{cases}$$

An example with $n = 8$ is depicted below:

[2]:
```
# plot basis functions:
n_elems = 8
nodes = n_elems + 1
x_start = TestParams.X_START
x_end = TestParams.X_END

x_coords = np.linspace(x_start, x_end, nodes)
colors = cycle(['blue', 'green', 'red', 'purple'])
for n in range(nodes):
    col = next(colors)
    if n > 0:
        plt.plot([x_coords[n-1], x_coords[n]], [0,1], color=col)
    if n < n_elems:
        plt.plot([x_coords[n], x_coords[n+1]], [1,0], color=col)
plt.ylim((0,1.5))
plt.show()
```



Then, one solves the *weak* formulation in $\mathcal{V}_h$: Find $u_h \in \mathcal{V}_h$ such that

$$(\mathrm{W_h}): \qquad (u_h', v') = (f, v) \quad \text{for all } v \in \mathcal{V}_h.$$

Since $u_h \in \mathcal{V}_h$ we have $u_h = \sum_j u_j \phi_j$ and since $v \in \mathcal{V}_h$ must be arbitrary, we must at least have

($\mathrm{W_h}$) hold for the ϕ_j, hence for all j:

$$(u_h', \phi_j') = \left(\sum_i u_i \phi_i', \phi_j' \right) = \sum_i u_i (\phi_i', \phi_j') = (f, \phi_j).$$

By defining $\mathbf{A} :=: \{a_{ij}\} := \{(\phi_i', \phi_j')\}$ and $\mathbf{b} :=: \{b_j\} := \{(f, \phi_j)\}$, we must solve $\mathbf{Au} = \mathbf{b}$. Such a solution $\mathbf{u} = \{u_i\}$ will then by linearity solve ($\mathrm{W_h}$) for all $v \in \mathcal{V}_h$.

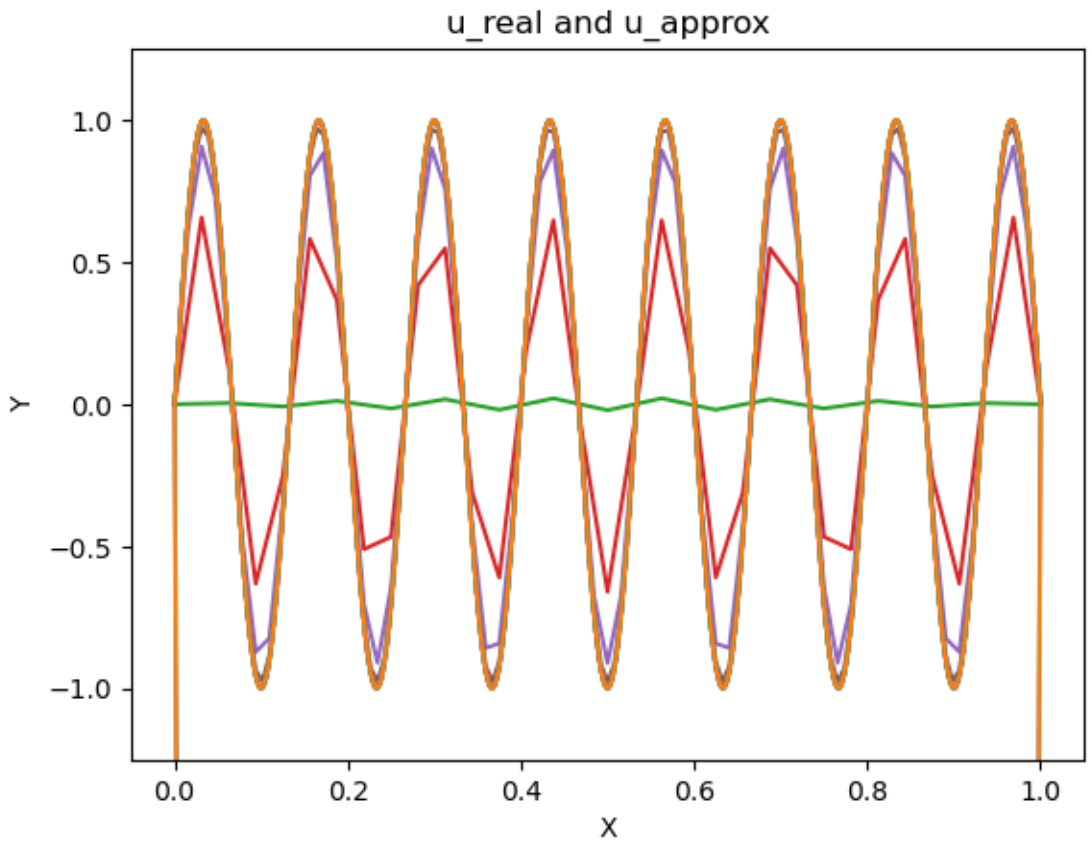
The matrix $\mathbf{A}$ is called the *stiffness* matrix and the vector $\mathbf{b}$ the *load* vector.

1.2 Implementation

To demonstrate the implementation, we take $[a, b] = [0, 1]$, constant diffusion $D = 1$, and pick a test function $u_{\text{real}}(x) = \sin(k\pi x)$ and source $f(x) = -u_{\text{real}}''(x) = (k\pi)^2 \sin(k\pi x)$, with $k = 15$, so that u_{real} is a solution of (D).

We then approximate u_{real} by functions u_h, where $h = n^{-1}$ for $n = 2^2, 2^3, \ldots, 2^{12}$.

[7]:
```
TestParams.K_CONST = 15

def constant_diffusion(x):
    return np.ones(len(x))

TestParams.diffusion_function = constant_diffusion

n_elems = [2**i for i in range(2, 13)]
x_start = 0
x_end = 1

residuals = []

for n_elem in n_elems:
    x_coords, u_approx, u_real = galerkin(n_elem, x_start, x_end)
    plt.plot(x_coords, u_approx)

    residuals.append(calc_l2err(x_coords, u_approx, u_real))

x_fine = np.linspace(x_start, x_end, 10 ** 4)
plt.plot(x_fine, TestParams.u_real(x_fine))
plt.ylim((-1.25, 1.25))

plt.xlabel("X")
plt.ylabel("Y")
plt.title("u_real and u_approx")
plt.show()
```



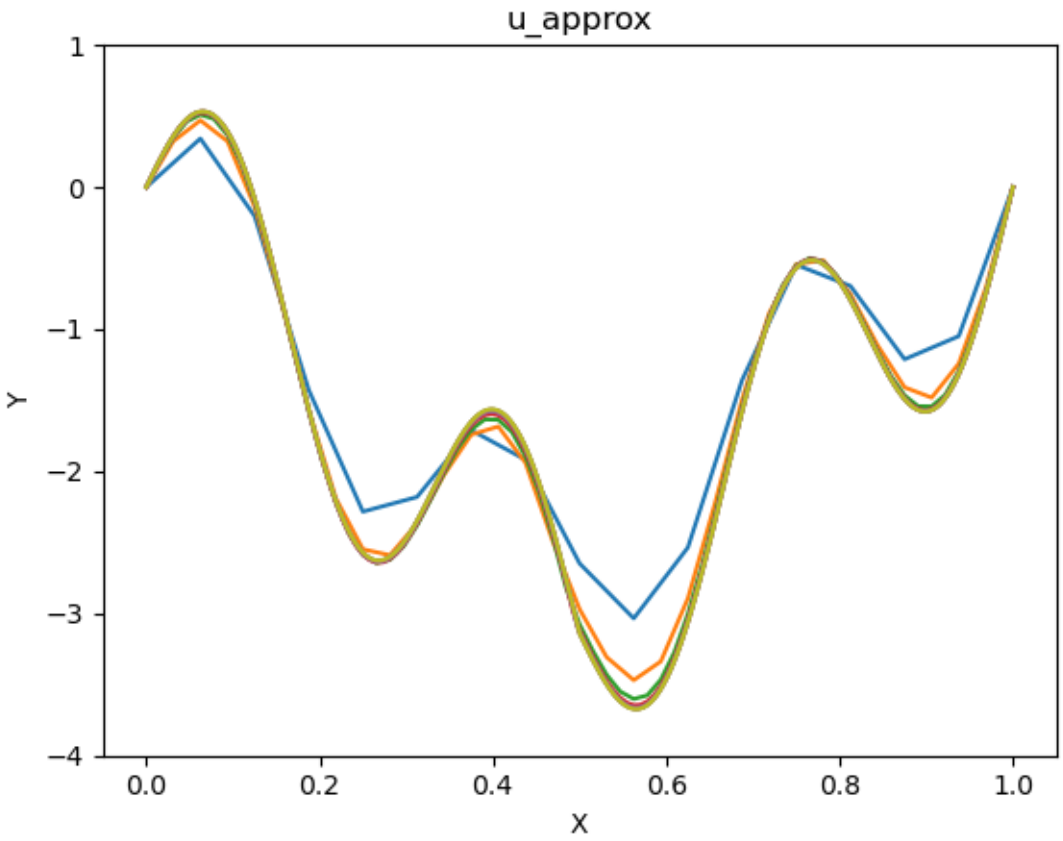
Here we have used $k = 13$, leading to instability for very small values of n, however, convergence is achieved.

In the following test, we let $k = 4$ and $D(x)$ be the step function

$$D(x) = \begin{cases} 1 & x \in [0, 0.5) \\ 2 & x \in [0.5, 1] \end{cases} .$$

[33]:
```
def step_diffusion(x):
    half_nodes = len(x) // 2
    other_half = len(x) - half_nodes
    return np.concatenate((np.ones(half_nodes), 2 * np.ones(other_half)))

TestParams.K_CONST = 6
TestParams.diffusion_function = step_diffusion

n_elems = [2**i for i in range(4, 13)]

for n_elem in n_elems:
    x_coords, u_approx, u_real = galerkin(n_elem, x_start, x_end)
    plt.plot(x_coords, u_approx)
```

```
x_fine = np.linspace(x_start, x_end, 10 ** 4)
plt.ylim((-4, 1))

plt.xlabel("X")
plt.ylabel("Y")
plt.title("u_approx")
plt.show()
```



1.3 Convergence Analysis

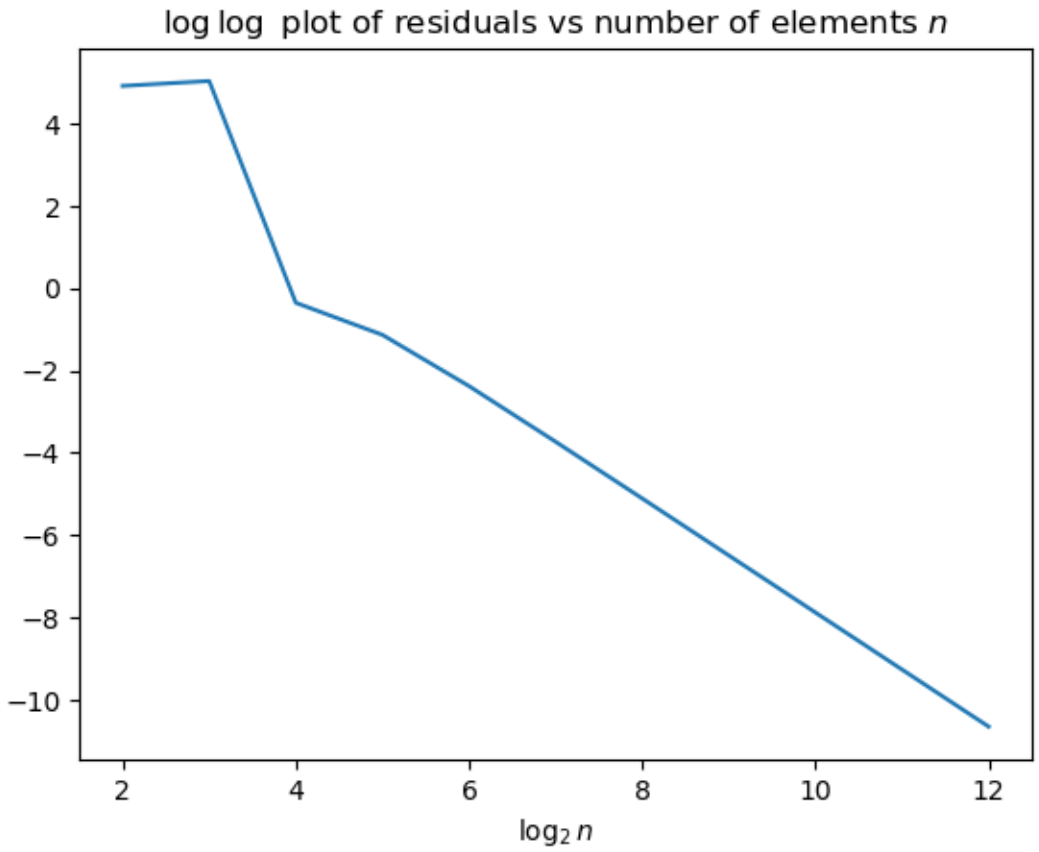
In the first implementation above ($k = 15$ and $D = 1$), we calculated the residuals at each iteration.

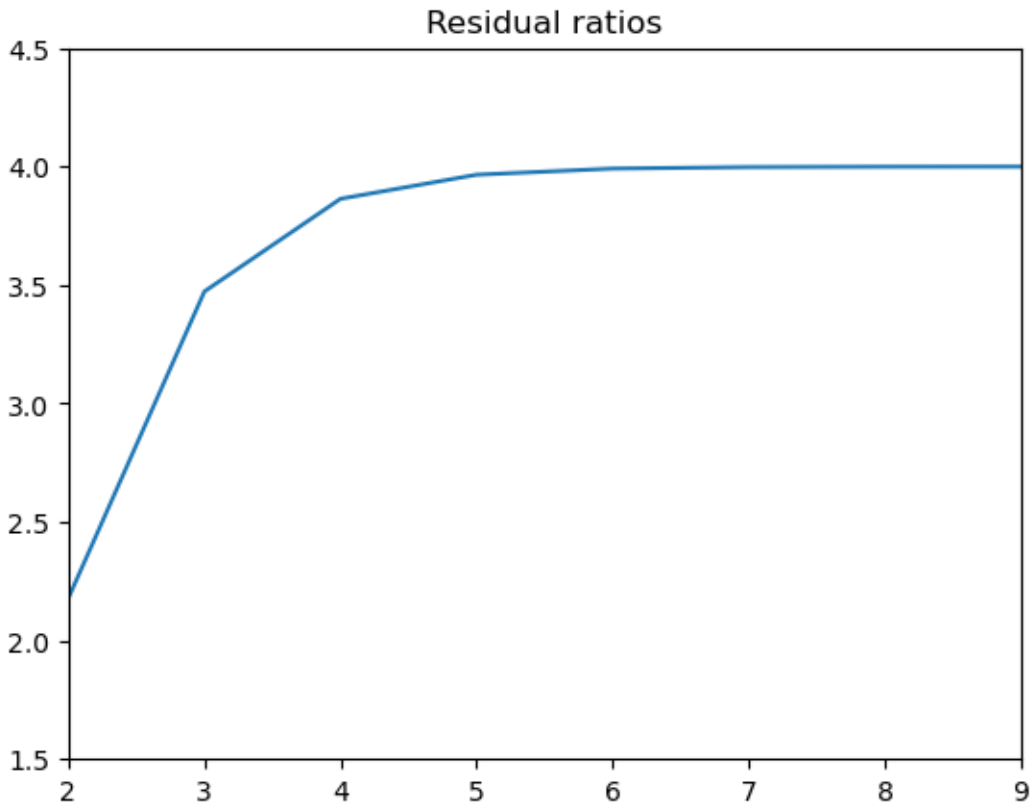
Theory (Nietsche-Aubin) states that the residual error is $O(h^2)$:

$$\|u_{\text{real}} - u_h\|_{L^2} \leq h^2 \|u''\|.$$

Thus if h is halved every time, the ratio of residuals e_i/e_{i+1} should tend to 4. Indeed, as the following plots demonstrate, this is precisely what is obtained:

[59]:
```
residuals = np.array(residuals)
plt.plot(range(2,13), np.log(residuals))
plt.title('$\log \log$ plot of residuals vs number of elements $n$')
plt.xlabel('$\log_2 n$')
plt.show()

ratios = np.divide(residuals[:-1], residuals[1:])
plt.plot(ratios)
plt.title("Residual ratios")
plt.xlim((2,9))
plt.ylim((1.5,4.5))
plt.show()

print(ratios)
```






```
[  0.88758103 218.53112186   2.17898915   3.47306798   3.86407886
   3.96564784   3.99160158   3.9978562    3.99946387   3.99986615]
```