$\boxed{\text{Ex. 5 8. a)}}$

$$\max_{B_1, B_2, d} \; (B_1 B_2)^{1/2}$$

s.t. $p_k^T B p_k - 2 d^T p_k + d^T B^{-1} d \leq 1$, for $k = 1, \dots, K$

$B_1 > 0$

$B_2 > 0$

where $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$

Taking the first set of conditions:

$$p_k^T B p_k - 2 d^T p_k + d^T B^{-1} d \leq 1 \iff (1 - p_k^T B p_k + 2 d^T p_k) - d^T B^{-1} d \geq 0$$

Schur's complement

$$\iff \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0 \, , \quad \mu_k = 1 - p_k^T B p_k + 2 d^T p_k$$

So the problem is equivalent to

$$\max_{B_1, B_2, d} \; (B_1 B_2)^{1/2} \quad \text{s.t.} \quad \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0 \, , \; k = 1, \dots, K \, , \quad B \succ 0$$

epigraph variable $\longleftrightarrow$

$$\max_{B_1, B_2, d, \lambda} \; \lambda \quad \text{s.t.} \quad \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0, \; k = 1, \dots, K \, , \quad B \succ 0 \, , \quad \lambda^2 = B_1 B_2$$

Relaxing the last condition to an inequality:

$$\max_{B_1, B_2, d, \lambda} \quad \lambda$$

$$\text{s.t.} \quad \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0$$

$$B \succ 0$$

$$\lambda^2 \leq B_1 B_2$$

this relaxation does not change the solution, as the maximization of λ is bound by how large $B_1 B_2$ can get, so the solution results in $\lambda^2 = B_1 B_2$.

Developing the last constraint:

$$\lambda^2 \leq B_1 B_2 \Leftrightarrow \|\lambda\|^2 \leq B_1 B_2 \overset{\text{key fact}}{\Leftrightarrow} \left\| \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \right\|_2 \leq B_1 + B_2 \Leftrightarrow$$

because $B_1 > 0$, $B_2 > 0$

$$\Leftrightarrow \left\| \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \right\|_2^2 \leq (B_1 + B_2)^2 \Leftrightarrow \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix}^T \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \leq (B_1 + B_2)^2 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix}^T \mathbb{1} \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \leq (B_1 + B_2)^2 \Leftrightarrow \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix}^T \frac{\mathbb{1}}{(B_1 + B_2)} \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \leq B_1 + B_2 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix}^T \left(\mathbb{1} (B_1 + B_2) \right)^{-1} \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \leq B_1 + B_2 \Leftrightarrow$$

$$\Leftrightarrow (B_1 + B_2) - \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix}^T \left(\mathbb{1} (B_1 + B_2) \right)^{-1} \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \geq 0 \overset{\text{Schur's complement}}{\Leftrightarrow}$$

$$\Leftrightarrow \begin{bmatrix} \mathbb{1}(B_1 + B_2) & \begin{bmatrix} 2\lambda \\ B_1 - B_2 \end{bmatrix} \\ \begin{bmatrix} 2\lambda & B_1 - B_2 \end{bmatrix} & (B_1 + B_2) \end{bmatrix} \succeq 0$$

$$\rightarrow \max_{B_1, B_2, d, \lambda} \quad \lambda$$

$$\text{s.t.} \quad \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0$$

$$B \succeq 0$$

$$\begin{bmatrix} (B_1 + B_2) & 0 & 2\lambda \\ 0 & (B_1 + B_2) & (B_1 - B_2) \\ 2\lambda & (B_1 - B_2) & (B_1 + B_2) \end{bmatrix} \succeq 0$$

Finally, $B \succ 0$ can be relaxed to $B \succeq 0$, as $B_1 = 0 \vee B_2 = 0$ would be an elipse with an infinite axis length, that cannot be the solution for $S \in \mathbb{R}^2$.

This way, the problem is formulated as an SDP as:

$$\max_{B_1, B_2, d, \lambda} \quad \lambda$$

$$\text{s.t} \quad \begin{bmatrix} B & d \\ d^T & \mu_k \end{bmatrix} \succeq 0, \quad k = 1, \dots, K$$

$$B \succeq 0$$

$$\begin{bmatrix} (B_1+B_2) & 0 & 2\lambda \\ 0 & (B_1+B_2) & (B_1-B_2) \\ 2\lambda & (B_1-B_2) & (B_1+B_2) \end{bmatrix} \succeq 0$$

with $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$

and $\mu_k = 1 - p_k^T B p_k + 2 d^T p_k$

$\boxed{\text{Ex. 53}}$

$$\min_{x_{total}} \| H x_{total} - y_{des} \|_\infty \qquad \text{where } ZZ = [ZZ_{Blind} \;\; ZZ_K]$$

$$\text{s.t.} \quad \|x\|_\infty \leq U \qquad \text{and } x_{total} = \begin{bmatrix} x_{Blind} \\ x_K \end{bmatrix}$$

So we re-write

$$\min_{x_{Blind}, x_K} \| ZZ_{Blind} x_{Blind} + ZZ_K x_K - y_{des} \|_\infty$$

$$\text{s.t.} \quad \| x_{Blind} \|_\infty \leq U$$

$$\| x_K \|_\infty \leq U$$

To minimize the average error for a given realization of $H_i, i=1,\dots,K$ we do:

$$\min_{x_{Blind}, x_K} \sum_{i=1}^{K} \| ZZ_i x_{Blind} + ZZ_K x_K - y_{des} \|_\infty \cdot P_i \quad , \quad P_i = P(H = H_i)$$

$$\text{s.t.} \quad \| x_{Blind} \|_\infty \leq U$$

$$\| x_K \|_\infty \leq U$$

where the objective can be re-writen as:

$$\sum_{i=1}^{K} \max_{l \in \{1,\dots,m_T\}} \left(| ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} | \right) \cdot P_i$$

where ZZ_{i_l}, ZZ_{K_l} and y_{des_l} are the l-th line of ZZ_i, ZZ_K and y_{des} respectively.

Introducing epigraph variables λ_i, $i=1,\dots,K$,

$$\lambda_i \geq \max_{l=1,\dots,mT} \left(| ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} | \right)$$

The problem becomes:

$$\min_{x_{Blind},\, x_K,\, \lambda_1,\dots,\lambda_K} \sum_{i=1}^{K} \lambda_i P_i$$

s.t.

$$\| x_{Blind} \|_\infty \leq U$$

$$\| x_K \|_\infty \leq U$$

$$\lambda_i \geq \max_{l=1,\dots,mT} \left(| ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} | \right), \text{ for } i=1,\dots,K$$

$(\Leftrightarrow)$

$$\Leftrightarrow \min_{x_{Blind},\, x_K,\, \lambda_1,\dots,\lambda_K} \sum_{i=1}^{K} \lambda_i P_i$$

$$U \geq \max_{j=1,\dots,mS} \left(| x_{Blind_j} | \right)$$

$$U \geq \max_{j=1,\dots,m(T-S)} \left(| x_{K_j} | \right)$$

$$\lambda_i \geq \max_{l=1,\dots,mT} \left(| ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} | \right), \text{ for } i=1,\dots,K$$

$(\Leftrightarrow)$

$$\Leftrightarrow \min_{x_{Blind},\, x_K,\, \lambda_1,\dots,\lambda_K} \sum_{i=1}^{K} \lambda_i P_i$$

$$U \geq x_{Blind_j}, \text{ for } j=1,\dots,mS$$

$$U \geq -x_{Blind_j}, \text{ for } j=1,\dots,mS$$

$$U \geq x_{K_j}, \text{ for } j=1,\dots,m(T-S)$$

$$U \geq -x_{K_j}, \text{ for } j=1,\dots,m(T-S)$$

$$\lambda_i \geq \left(ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} \right), \text{ for } i=1,\dots,K;\ l=1,\dots,mT$$

$$\lambda_i \geq -\left(ZZ_{i_l} x_{Blind} + ZZ_{K_l} x_K - y_{des_l} \right), \text{ for } i=1,\dots,K;\ l=1,\dots,mT$$

$$\min_{x_{Blend},\, x_K,\, \lambda_1, \dots, \lambda_K} \sum_{i=1}^{K} \lambda_i P_i$$ $\rightarrow$ linear function of $(\lambda_1, \dots, \lambda_K)$

$$U \geq x_{Blend_j}, \text{ for } j = 1, \dots, mS$$
$$U \geq -x_{Blend_j}, \text{ for } j = 1, \dots, mS$$
$$U \geq x_{K_j}, \text{ for } j = 1, \dots, m(T-S)$$
$$U \geq -x_{K_j}, \text{ for } j = 1, \dots, m(T-S)$$
$$\lambda_i \geq (ZZ_{i_l} x_{Blend} + ZZ_{K_l} x_K - Ydbs_l), \text{ for } i = 1, \dots, K;\ l = 1, \dots, mT$$
$$\lambda_i \geq -(ZZ_{i_l} x_{Blend} + ZZ_{K_l} x_K - Ydbs_l), \text{ for } i = 1, \dots, K;\ l = 1, \dots, mT$$

$2(m + Km)T$
Affine inequalities on the optimization variables

This is the formulation of the problem as an LP.