Part 3a

Sunday, February 28, 2021 12:35 PM

Obtain exact solution for this problem.

$$\frac{\partial^2 u}{\partial x^2} = k^2 \cos\left(\frac{\pi k x}{L}\right) + \alpha(1-k^2)\sin\left(\frac{2\pi k x}{L}\right)$$

B.C.s: ① $X(0) = 0$
② $X(1) = 1$

Integrate Twice to get $u(x)$:

$$k^2 \int \cos\left(\frac{\pi k x}{L}\right) dx + \alpha(1-k^2) \int \sin\left(\frac{2\pi k x}{L}\right) dx$$

$$\frac{k^2 L}{\pi k} \sin\left(\frac{\pi k x}{L}\right) - \frac{\alpha(1-k^2)L}{2\pi k} \cos\left(\frac{2\pi k x}{L}\right) + C_1$$

$$\frac{k^2 L}{\pi k} \int \sin\left(\frac{\pi k x}{L}\right) dx - \frac{\alpha(1-k^2)L}{2\pi k} \int \cos\left(\frac{2\pi k x}{L}\right) dx + C_1 \int dx$$

$$-\frac{k^2 L}{\pi k} \frac{L}{\pi k} \cos\left(\frac{\pi k x}{L}\right) - \frac{\alpha(1-k^2)L}{2\pi k} \frac{L}{2\pi k} \sin\left(\frac{2\pi k x}{L}\right) + C_1 x + C_2$$

Part 3a

Sunday, March 14, 2021 9:41 AM

Apply Boundary Conditions:

① $x(0) = 0$

$$0 = -\frac{L^2}{\pi^2}\cos(0) - \cancel{\frac{\alpha(1-k^2)L^2}{4\pi^2 k^2}\sin(0)}^{\to 0} + \cancel{C_1(0)}^{\to 0} + C_2$$

$$0 = \frac{-L^2}{\pi^2} + C_2$$

$$\boxed{C_2 = \frac{L^2}{\pi^2}}$$

② $x(1) = 1$

$$1 = \frac{-L^2}{\pi^2}\cos\left(\frac{\pi k}{L}\right) - \frac{\alpha(1-k^2)L^2}{4\pi^2 k^2}\sin\left(\frac{2\pi k}{L}\right) + C_1 + \frac{L^2}{\pi^2}$$

$$C_1 = 1 - \frac{L^2}{\pi^2} + \frac{\alpha(1-k^2)L^2}{4\pi^2 k^2}\sin\left(\frac{2\pi k}{L}\right) + \frac{L^2}{\pi^2}\cos\left(\frac{\pi k}{L}\right)$$

Final Equation:

$$u(x) = -\frac{k^2 L}{\pi k}\frac{L}{\pi k}\cos\left(\frac{\pi k x}{L}\right) - \frac{\alpha(1-k^2)L}{2\pi k}\frac{L}{2\pi k}\sin\left(\frac{2\pi k x}{L}\right)$$

$$+\left[1 - \frac{L^2}{\pi^2} + \frac{\alpha(1-k^2)L^2}{4\pi^2 k^2}\sin\left(\frac{2\pi k}{L}\right) + \frac{L^2}{\pi^2}\cos\left(\frac{\pi k}{L}\right)\right]x + \frac{L^2}{\pi^2}$$

Part 3a

Sunday, March 14, 2021 9:49 AM

$$u(x) = \frac{-L^2}{\pi^2}\left[\cos\left(\frac{\pi k x}{L}\right) - \frac{\alpha(1-k^2)}{4k^2}\sin\left(\frac{2\pi k x}{L}\right)\right]$$

$$+\frac{L^2}{\pi^2}x\left[\frac{\pi^2}{L^2} - 1 + \frac{\alpha(1-k^2)}{4k^2}\sin\left(\frac{2\pi k}{L}\right) + \cos\left(\frac{\pi k}{L}\right)\right] + \frac{L^2}{\pi^2}$$

$* \; L = 1, \quad \alpha = 5$

$$\boxed{\begin{aligned} u(x) &= -\frac{1}{\pi^2}\left[\cos(\pi k x) - \frac{5(1-k^2)}{4k^2}\sin(2\pi k x)\right] \\ &+ \frac{1}{\pi^2}x\left[\pi^2 - 1 + \frac{5(1-k^2)}{4k^2}\sin(2\pi k) + \cos(\pi k)\right] + \frac{1}{\pi^2}\end{aligned}}$$