Problem Set V

Macroeconomics I

Nurfatima Jandarova

January 2, 2017

Exercise 1 Corners again

(a)

- time is discrete and infinite
- Define $s_t \in W$ the realization of a stochastic event that defines the wage rate. Hence, history of events is denoted as s^t. Define as well the associated transition matrix, $P(s_{t+1}|s_t)$ and the unconditional probability of each history s^t at time t, $\pi_t(s^t)$.
- Household values $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t[\ln(c_t(s^t)) - (1 - l_t(s^t))] = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)[\ln(c_t(s^t)) - n_t(s^t)], \quad \beta \in (0,1)$. Notice that period utility is a $\mathcal{C}^2$, increasing in both $c_t(s^t)$ and $l_t(s^t)$, and a concave function. Furthermore, period utility function satisfies Inada condition with respect to consumption, but not with respect to leisure: $\lim_{c \to 0} u_c(c, l) = \lim_{c \to 0} \frac{1}{c} = \infty, \forall c$, $\lim_{l \to 0} u_l(c, l) = \lim_{l \to 0} 1 = 1, \forall l$.
- At time t and history s^t, a consumer is paid $w_t(s_t)$. Notice, that wage is said to follow Markov process, i.e., is history-independent. Therefore, the allocations in the economy are as well history-independent.
- Although not directly specified by the problem, I assume sequential trade market structure. Define $a_{t+1}(s^t, s_{t+1})$ the amount of claims on time $t+1$, history s^{t+1} consumption bought at time t, history s^t.
- Consumer has to choose allocations $\{c_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ and asset positions $a_{t+1}(s^t, s_{t+1})$ to

$$\max_{\{c_t(s_t), n_t(s_t), a_{t+1}(s_t, s_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t)[\ln(c_t(s_t)) - n_t(s_t)] \text{ s.t.}$$

$$c_t(s_t) + \sum_{s_{t+1}|s^t} \frac{a_{t+1}(s_t, s_{t+1})}{1+r} = w_t(s_t)n_t(s_t) + a_t(s_t), \qquad \forall t, \forall s^t$$

$$a_{t+1}(s_t, s_{t+1}) \geq B, \qquad \forall t, \forall s_t, \forall s_{t+1}$$

$$a_0(s_0) = 0$$

Hence, the problem could be written recursively as

$$V(a, s) = \max_{c,n,a'} \{\ln(c) - n + \beta \sum_{s' \in W} P(s'|s) V(a'(s'), s')\} \text{ s.t. } c + \sum_{s' \in W} \frac{a'(s')}{1+r} = w(s)n + a$$

$$a'(s') \geq B, \qquad \forall s' \in W$$

or

$$V(a, s) = \max_{n,a'} \left\{ \ln\left(w(s)n + a - \sum_{s' \in W} \frac{a'(s')}{1+r} \right) - n + \beta \sum_{s' \in W} P(s'|s) V(a'(s'), s') \right\} \text{ s.t. } a'(s') \geq B, \quad \forall s' \in W$$

(b) If B is the natural debt limit, then the borrowing constraint will never bind. Suppose it does, which then implies that the household will be bound to consume 0 for the rest of infinite life. However, consumer's period utility satisfies Inada condition with respect to consumption, i.e., even a slightest deviation form zero constitutes a vast utility improvement. Hence, at equilibrium consumer will never want to have zero consumption.

(c) Policy functions for the household are given by $\sigma^c(a,s), \sigma^n(a,s), \sigma^a(a,s,s')$. Then, the FOCs are:

$$
\begin{aligned}
\frac{1}{\sigma^c(a,s)} &= \mu \\
-1 + \mu w(s) &= -1 + \frac{w(s)}{\sigma^c(a,s)} \\
\left.\begin{aligned}\tfrac{\mu}{1+r} &= \beta P(s'|s)\tfrac{\partial V(a'(s'),s')}{\partial a'(s')} \\ \tfrac{\partial V(a,s)}{\partial a} &= \mu\end{aligned}\right. &\implies \frac{1}{1+r} = \beta P(s'|s)\frac{\sigma^c(a,s)}{\sigma^c(\sigma^a(a,s,s'),s')}
\end{aligned}
$$