Problem Set V

Macroeconomics I

Nurfatima Jandarova

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Exercise 1 Corners again

(a)

- time is discrete and infinite
- Define $s_t \in W$ the realization of a stochastic event that defines the wage rate. Hence, history of events is denoted as s^t. Define as well the associated transition matrix, $P(s_{t+1}|s_t)$ and the unconditional probability of each history s^t at time t, $\pi_t(s^t)$.
- Household values $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t[\ln(c_t(s^t)) - (1 - l_t(s^t))] = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t)[\ln(c_t(s^t)) - n_t(s^t)], \quad \beta \in (0,1)$. Notice that period utility is a $\mathcal{C}^2$, increasing in both $c_t(s^t)$ and $l_t(s^t)$, and a concave function. Furthermore, period utility function satisfies Inada condition with respect to consumption, but not with respect to leisure: $\lim_{c\to 0} u_c(c,l) = \lim_{c\to 0} \frac{1}{c} = \infty, \forall c$, $\lim_{l\to 0} u_l(c,l) = \lim_{l \to 0} 1 = 1, \forall l$.
- At time t and history s^t, a consumer is paid $w_t(s_t)$. Notice, that wage is said to follow Markov process, i.e., is history-independent. Therefore, the allocations in the economy are as well history-independent.
- Although not directly specified by the problem, I assume sequential trade market structure. Define $a_{t+1}(s^t, s_{t+1})$ the amount of claims on time $t+1$, history s^{t+1} consumption bought at time t, history s^t.
- Consumer has to choose allocations $\{c_t(s^t), n_t(s^t)\}_{t=0}^{\infty}$ and asset positions $a_{t+1}(s^t, s_{t+1})$ to

$$
\begin{aligned}
\max_{\{c_t(s_t), n_t(s_t), a_{t+1}(s_t, s_{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t)[\ln(c_t(s_t)) - n_t(s_t)] \text{ s.t.}& \\
c_t(s_t) + \sum_{s_{t+1}|s^t} \frac{a_{t+1}(s_t, s_{t+1})}{1+r} = w_t(s_t)n_t(s_t) + a_t(s_t), &\qquad \forall t, \forall s^t \\
a_{t+1}(s_t, s_{t+1}) \geq B, \qquad \forall t, \forall s_t, \forall s_{t+1}& \\
a_0(s_0) = 0&
\end{aligned}
$$

Hence, the problem could be written recursively as

$$
V(a,s) = \max_{c,n,a'} \{\ln(c) - n + \beta \sum_{s' \in W} P(s'|s)V(a'(s'), s')\} \text{ s.t. } c + \sum_{s'\in W} \frac{a'(s')}{1+r} = w(s)n + a
$$
$$
a'(s') \geq B, \qquad \forall s' \in W
$$

or

$$
V(a,s) = \max_{n,a'} \left\{ \ln\left(w(s)n + a - \sum_{s' \in W} \frac{a'(s')}{1+r} \right) - n + \beta \sum_{s' \in W} P(s'|s)V(a'(s'), s') \right\} \text{ s.t. } a'(s') \geq B, \quad \forall s' \in W
$$

(b) If B is the natural debt limit, then the borrowing constraint will never bind. Suppose it does, which then implies that the household will be bound to consume 0 for the rest of infinite life. However, consumer's period utility satisfies Inada condition with respect to consumption, i.e., even a slightest deviation form zero constitutes a vast utility improvement. Hence, at equilibrium consumer will never want to have zero consumption.

(c) Policy functions for the household are given by $\sigma^c(a,s), \sigma^n(a,s), \sigma^a(a,s,s')$. Then, the FOCs are:

$$\frac{1}{\sigma^c(a,s)} = \mu$$
$$-1 + \mu w(s) = -1 + \frac{w(s)}{\sigma^c(a,s)}$$
$$\left.\begin{matrix} \frac{\mu}{1+r} = \beta P(s'|s)\frac{\partial V(a'(s'),s')}{\partial a'(s')} \\ \frac{\partial V(a,s)}{\partial a} = \mu \end{matrix}\right. \implies \frac{1}{1+r} = \beta P(s'|s)\frac{\sigma^c(a,s)}{\sigma^c(\sigma^a(a,s,s'),s')}$$

Exercise 2 Irreversible capital accumulation

(a) The problem could be written recursively

$$\begin{aligned} V(a,k) &= \max_{c,k'} \ln(c) + \beta \mathbb{E}_{a'|a} V(a',k') \\ \text{s.t. } & c + k' = ak^\alpha + (1-\delta)k \\ & k' - (1-\delta)k \geq 0 \end{aligned}$$

FOCs:

$$\begin{cases} \frac{1}{c} = \mu \\ \beta \mathbb{E}_{a'|a}\frac{\partial V(a',k')}{\partial k'} - \mu + \lambda = 0 \\ \min\{\lambda, k' - (1-\delta)k\} = 0 \end{cases}$$

Envelope condition

$$\frac{\partial V(a,k)}{\partial k} = \mu(a\alpha k^{\alpha-1} + 1 - \delta) - \lambda(1-\delta)$$

(b) Combining the FOCs and the envelope condition we get

$$\underbrace{\beta \mathbb{E}_{a'|a}\left[\frac{a'\alpha(k')^{\alpha-1}+1-\delta}{c'}\right]}_{\substack{\text{discounted expected marginal} \\ \text{benefit of consumption tomorrow}}} + \underbrace{\lambda - \beta\lambda'(1-\delta)}_{(*)} = \underbrace{\frac{1}{c}}_{\substack{\text{marginal benefit of} \\ \text{consumption today}}} \tag{1}$$

(*) It seems to me that this expression gives an intertemporal value of relaxing irreversibility constraint today versus tomorrow. In a sense, due to depreciation of capital it is "easier" to relax irreversibility constraint in the future than today, but subject to discounting as it takes place in the future.
Suppose that $k' > (1-\delta)k \Rightarrow \lambda = 0$. Then, (1) turns into an Euler equation in standard RBC model:

$$\beta \mathbb{E}_{a'|a}\left[\frac{a'\alpha(k')^{\alpha-1}+1-\delta}{c'}\right] = \frac{1}{c}$$

Now it is easier to see that if $\lambda > 0 \Rightarrow k' = (1-\delta)k$, then an agent ideally would want to start eating up the capital stock, but cannot. Hence, an agent ends up with higher k' and lower c compared to the standard case with reversible investment. This means that

$$\beta \mathbb{E}_{a'|a}\left[\frac{a'\alpha(k')^{\alpha-1}+1-\delta}{c'}\right] < \frac{1}{c}$$

, i.e., again consumer would have been better off consuming more today, but is bound to postpone consumption to the future period.

(c) As long as the depreciation rate of capital $\delta > 0$, irreversibility constraint does not affect the non-stochastic steady-state of capital k^*.
<u>Algebraically:</u>

1. Remove shocks: $a = \mathbb{E}(a) = 1$

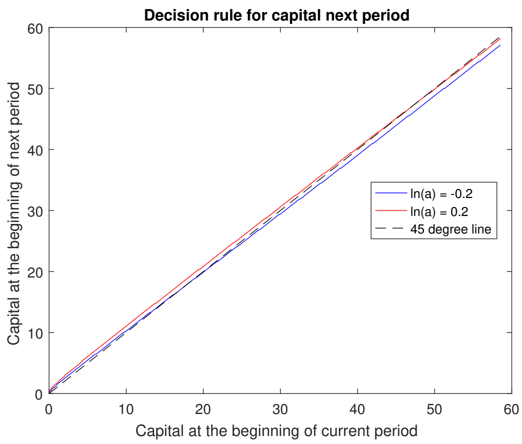
2. k such that $k = k' = k^*$. This means that $k^* - (1-\delta)k^* > 0$ if $\delta > 0$. This also implies that $\Rightarrow c = c' = c^* \Longrightarrow$

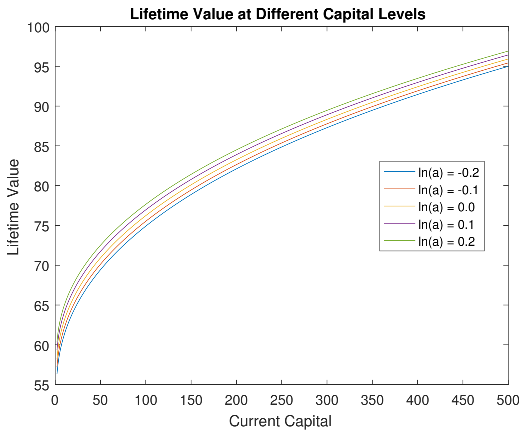
$$\frac{1}{\cancel{c^*}} = \beta\frac{\alpha(k^*)^{\alpha-1} + 1 - \delta}{\cancel{c^*}}$$
$$k^* = \left[\tfrac{1}{\alpha}\left(\tfrac{1}{\beta} - (1-\delta)\right)\right]^{\frac{1}{\alpha-1}}$$

i.e., same as the steady-state capital in the standard model.

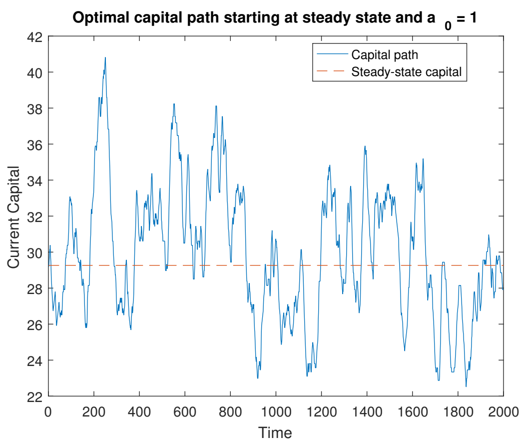
Intuitively: Steady-state capital is defined as the level of capital constant over time. To keep it constant over time in presence of capital depreciation, investment needs to be positive. Thus, irreversibility of investment has no impact on the steady-state level capital.

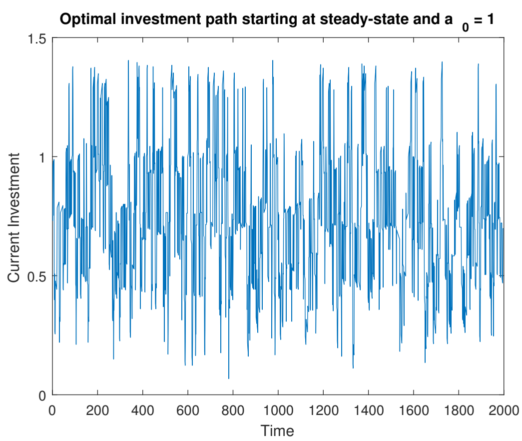
(d) As in standard case, value function is increasing in the productivity level. Higher productivity level allows to produce more for a given level of capital, and hence allows agents to enjoy higher life-time utility. At the same time, higher productivity level also allows agents to both consume and save more. Therefore, the intersection of capital decision rule with 45° line is higher for higher productivity level, i.e., higher productivity level allows agents to accumulate capital longer.





Simulation of the model for 2000 periods starting at the steady-state level shows that capital path is roughly levelled out over time with fluctuations due to productivity shocks. Simulation of the investment path depicts that irreversibility constraint never binds. The reason for this could be that consumer wants to smooth consumption over time. Hence, it is never optimal for a consumer to start eating up capital stock because it increases consumption today at the cost of permanently shifting consumption downwards in the (infinite) future.





Exercise 3 Shocks to depreciation of capital

Let $\delta_t(s_t)$ denote the realization of the stochastic shock to the depreciation rate at time t. Since we are given that the depreciation rate follows a stationary Markov process, the allocations are history-independent. Combining the law of motion and the resource constraint we get $c_t(s_t) + K_{t+1} = f(K_t) + (1-\delta_t(s_t))K_t$. Thus, the planner's problem is to

$$\max_{\{c_t(s_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)\frac{(c_t(s_t))^{1-\sigma}+1}{1-\sigma} \text{ s.t. } c_t(s_t)+K_{t+1}=f(K_t)+(1-\delta_t(s_t))K_t$$
$$c_t(s_t)\geq 0, \forall t$$
$$K_0 \text{ given}$$

The planner's problem could be written recursively as

Exercise 4 Endogenous labour supply

(a) The planner's problem sequentially:

$$\mathcal{L}=\sum_{t=0}^{T}\beta^t\left\{\ln(c_t)-\ln(n_t)+\mu_t\left(n_t^{\alpha}k_t^{1-\alpha}+(1-\delta)k_t-k_{t+1}-c_t\right)\right\}$$

FOCs:

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial c_t} & =\beta^t\left\{\frac{1}{c_t}-\mu_t\right\}=0, \qquad \forall t\leq T-1\\ \frac{\partial\mathcal{L}}{\partial n_t} & =\beta^t\left\{-\frac{1}{n_t}+\mu_t\alpha n_t^{\alpha-1}k_t^{1-\alpha}\right\}=0, \qquad \forall t\leq T-1\\ \frac{\partial\mathcal{L}}{\partial k_{t+1}} & =-\beta^t\mu_t+\beta^{t+1}\mu_{t+1}\left((1-\alpha)n_{t+1}^{\alpha}k_{t+1}^{-\alpha}+1-\delta\right)=0, \qquad \forall t\leq T-1\\ \frac{\partial\mathcal{L}}{\partial \mu_t} & =\beta^t\left(n_t^{\alpha}k_t^{1-\alpha}+(1-\delta)k_t-k_{t+1}-c_t\right)=0, \qquad \forall t\leq T-1\end{cases} \Rightarrow \begin{cases}\frac{1}{c_t} & =\mu_t\\ c_t & =\alpha n_t^{\alpha}k_t^{1-\alpha}\\ \frac{1}{c_t} & =\beta\frac{1}{c_{t+1}}\left((1-\alpha)\left(\frac{k_{t+1}}{n_{t+1}}\right)^{-\alpha}+1-\delta\right)\\ c_t+k_{t+1} & =n_t^{\alpha}k_t^{1-\alpha}+(1-\delta)k_t\end{cases}$$

Using the intratemporal FOC and resource constraint we get the following

$$\begin{cases}c_t=\alpha n_t^{\alpha}k_t^{1-\alpha}\\ (1-\alpha)n_t^{\alpha}k_t^{1-\alpha}=k_{t+1}-(1-\delta)k_t\end{cases} \Rightarrow \begin{cases}c_t=\alpha n_t^{\alpha}k_t^{1-\alpha}\\ n_t=\left[\frac{k_{t+1}-(1-\delta)k_t}{(1-\alpha)k_t^{1-\alpha}}\right]^{\frac{1}{\alpha}}\end{cases} \Rightarrow \begin{cases}c_t=\frac{\alpha}{1-\alpha}\left[k_{t+1}-(1-\delta)k_t\right]\\ n_t=\left[\frac{k_{t+1}-(1-\delta)k_t}{(1-\alpha)k_t^{1-\alpha}}\right]^{\frac{1}{\alpha}}\end{cases}$$

Hence,

$$\begin{cases}\tilde{c}(k_t,k_{t+1}) & =\frac{\alpha}{1-\alpha}\left[k_{t+1}-(1-\delta)k_t\right], \quad \forall t\leq T-1\\ \tilde{n}(k_t,k_{t+1}) & =\left[\frac{k_{t+1}-(1-\delta)k_t}{(1-\alpha)k_t^{1-\alpha}}\right]^{\frac{1}{\alpha}}, \quad \forall t\leq T-1\end{cases}$$

Since the utility function is strictly increasing in c_t nd decreasing in n_t, in the terminal period $t=T$, the following conditions hold:

$$k_{T+1}=0$$
$$n_T=0$$
$$c_T=(1-\delta)k_T$$

The planner's problem recursively:

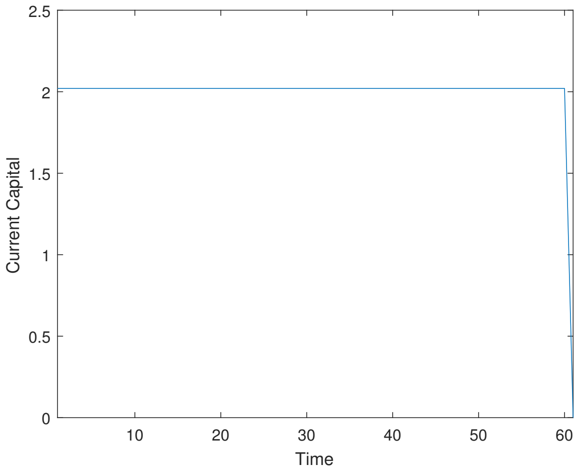
$$V_s(k_{T-s})=\max_{n_{T-s},c_{T-s}}\ln(c_{T-s})-\ln(n_{T-s})+\beta V_{s-1}(n_{T-s}^{\alpha}k_{T-s}^{1-\alpha}+(1-\delta)k_{T-s}-c_{T-s})$$

FOCs:

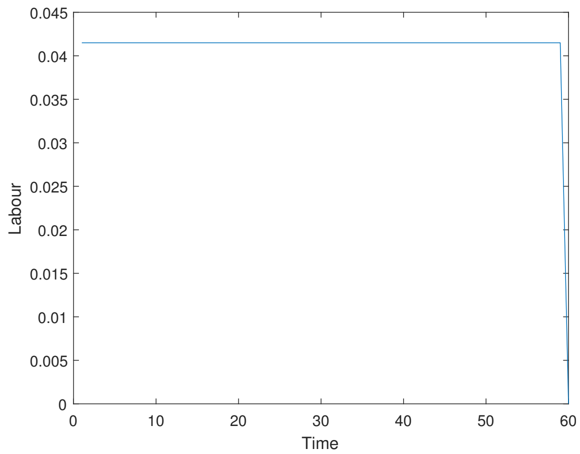
$$\begin{cases}\frac{1}{c_t} & =\beta V_{s-1}'(n_t^{\alpha}k_t^{1-\alpha}+(1-\delta)k_t-c_t), \qquad \forall t\leq T-1\\ \frac{1}{n_t} & =\beta\alpha n_t^{\alpha-1}k_t^{1-\alpha}V_{s-1}'(n_t^{\alpha}k_t^{1-\alpha}+(1-\delta)k_t-c_t), \qquad \forall t\leq T-1\end{cases}$$

To proceed further, one needs to know the functional form of the value functions at each point in time. It seems to me that I'd need to use value function iteration or guess and verify to obtain functional forms of the value functions and optimal decision rules. Envelope condition does not apply because in a finite horizon, value functions do differ across time. So, for the numerical part I continue using first-order conditions derived from sequential problem as both problems should yield same result.

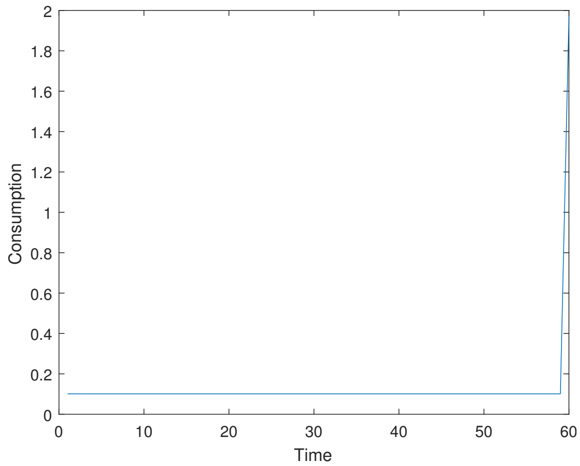
(b) Using the FOC in the previous part, numerical solution yields the following paths for capital, labour and consumption. Notice that optimal choice of capital for the next period is constant over time until it drops to zero when the world ends. Unlike the very simple finite-horizon growth model considered in Problem Set 1, social planner chooses the keep the stock of capital constant because i) a unit of labour supplied results in higher disutility to the agent than an increase in the capital stock, and ii) a lower supply of labour will result in gradual decrease of capital stock and a lower consumption plan for an agent. Using results from the previous part, one can show that $k_{t+1} = k_t \implies n_t = k_t \left(\frac{\delta}{1-\alpha}\right)^{\frac{1}{\alpha}} \approx 0.04$ (see Figure 3b). This allows an agent to enjoy a constant consumption plan for almost an entire life, except the terminal period, when he/she consumes everything that is left before the world ends (see Figure 3c).



(a) Optimal capital path starting at $k_0 = 2$



(b) Optimal path for labour with $k_0 = 2$



(c) Optimal consumption path with $k_0 = 2$

Figure 3: Numerical solution to optimal policy rules