Problem Set I

Macroeconomics I

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Exercise 1

(i) Define X as the set of all possible values for the state variable x_t. Then,

$$\begin{aligned}
\hat{u} : X \times \mathbb{R} &\longrightarrow \mathbb{R}, (x_t, c_t) \longmapsto U(c_t) \\
\hat{\Gamma}(x_t) =& [0, f(x_t) + (1-\delta)x_t] \\
\hat{f} : X \times \mathbb{R} &\longrightarrow X, (x_t, c_t) \longmapsto f(x_t) - c_t + (1-\delta)x_t
\end{aligned}$$

(ii) Given X_t, the set of all possible values for a state variable,

$$\begin{aligned}
&\tilde{F} : X \times X \longrightarrow \mathbb{R}, (x_t, x_{t+1}) \longmapsto U(f(x_t) + (1-\delta)x_t - x_{t+1} \\
&\tilde{\Gamma}(x_t) = [0, (1-\delta)x_t + f(x_t)]
\end{aligned}$$

Exercise 2.1

Since the utility function is monotonically increasing, the budget constraint holds with equality, i.e., $c_t + x_{t+1} = (1-\delta)x_t, \forall t \in [0, T]$. Setup the Lagrangian:

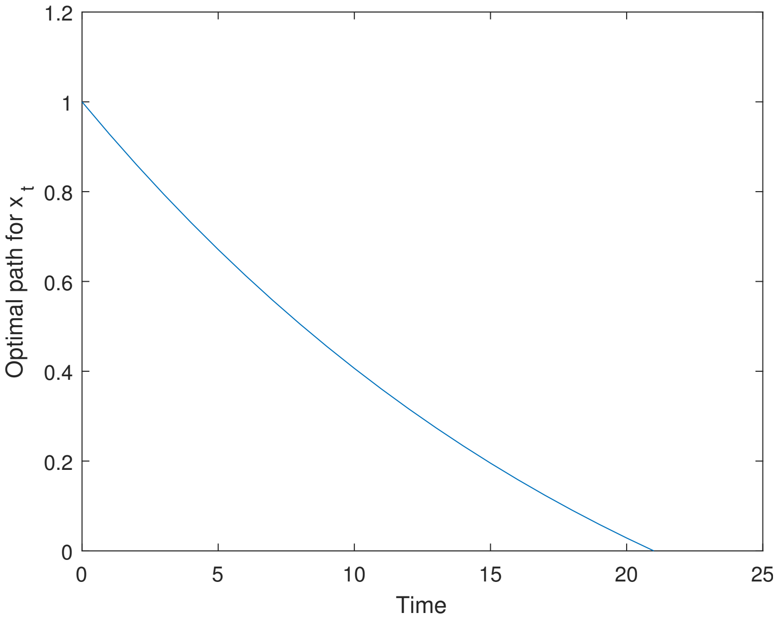
$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \ln(c_t) + \lambda_t((1-\delta)x_t - c_t - x_{t+1})$$

$$\text{FOC: } \begin{cases} \frac{\beta^t}{c_t} = \lambda_t \\ \lambda_{t+1}(1-\delta) = \lambda_t \\ (1-\delta)x_t - c_t - x_{t+1} \end{cases} \Rightarrow$$

$$\begin{aligned}
\text{Euler equation: } \frac{\beta_{t+1}(1-\delta)}{c_{t+1}} &= \frac{\beta_t}{c_t} \\
\frac{c_{t+1}}{c_t} &= \beta(1-\delta)
\end{aligned}$$

Exercise 2.1.2

[See .m file for the code]



Exercise 2.2

First of all, define the following sets: $B(x_t) = [0, (1-\delta)x_t]$ is the set of possible values for c_t and $\Gamma(x_t) = [0, (1-\delta)x_t]$ is the set of possible values for x_{t+1}. Then, using the results from our lecture notes, we could rewrite the optimization problem using Bellman equation:

$$V_s(x_{T-s}) = \max_{c_{T-s}\in B(x_{T-s})} \ln(c_{T-s}) + \beta V_{s-1}((1-\delta)x_{T-s} - c_{T-s}), \forall s \in [0, T]$$

We also have a guess about the functional form of the value function: $V_s(x_{T-s}) = A_s + B_s \ln(x_{T-s})$. Substitute this into the above equation and take the first order condition with respect to c_{T-s}:

$$A_s + B_s \ln(x_{T-s}) = \max_{c_{T-s}\in B(x_{T-s})} \ln(c_{T-s}) + \beta(A_{s-1} + B_{s-1}\ln((1-\delta)x_{T-s} - c_{T-s})), \forall s \in [0, T]$$

$$\text{FOC: } 0 = \frac{1}{c^*_{T-s}} - \frac{\beta B_{s-1}}{(1-\delta)x_{T-s} - c^*_{T-s}}$$

$$\beta B_{s-1} c^*_{T-s} = (1-\delta)x_{T-s} - c^*_{T-s}$$

$$c^*_{T-s} = \frac{1-\delta}{1+\beta B_{s-1}} x_{T-s}$$

Substitute this back to the value function:

$$A_s + B_s \ln(x_{T-s}) = \ln(\frac{1-\delta}{1+\beta B_{s-1}} x_{T-s}) + \beta(A_{s-1} + B_{s-1}\ln((1-\delta)x_{T-s} - \frac{1-\delta}{1+\beta B_{s-1}} x_{T-s})), \forall s \in [0, T]$$

$$A_s + B_s \ln(x_{T-s}) = \ln(\frac{1-\delta}{1+\beta B_{s-1}} x_{T-s}) + \beta(A_{s-1} + B_{s-1}\ln(\frac{\beta B_{s-1}(1-\delta)x_{T-s}}{1+\beta B_{s-1}})), \forall s \in [0, T]$$

$$\text{Hence, } \begin{cases} B_s \ln(x_{T-s}) = \ln(x_{T-s}) + \beta B_{s-1}\ln(x_{T-s}) \\ A_s = (1+\beta B_{s-1})\ln(1-\delta) - \ln(1+\beta B_{s-1}) + \beta A_{s-1} + \beta B_{s-1}\ln \beta B_{s-1} \end{cases}$$

From the first equation we can get a general formula for B_s :

$$B_s = 1 + \beta B_{s-1}$$

$$s = 0 : B_0 = 1$$

$$s = 1 : B_1 = 1 + \beta$$

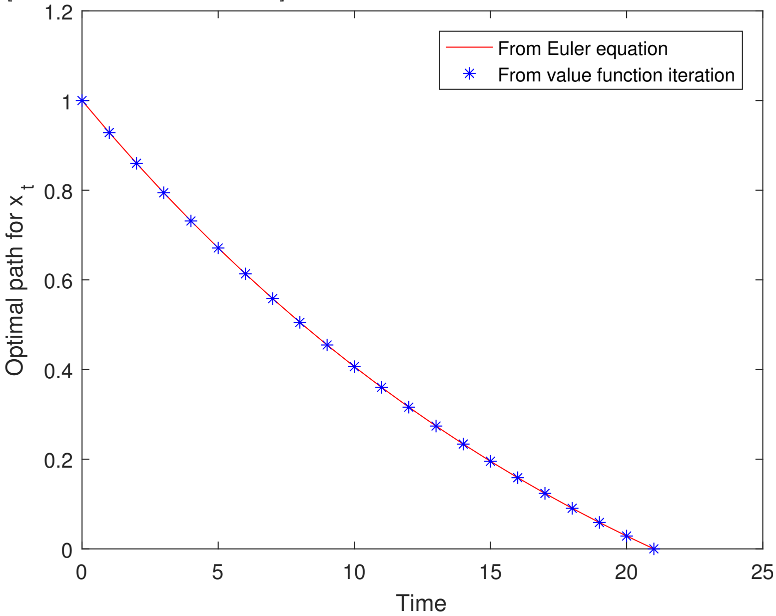
$$\text{Thus, } B_s = \sum_{j=0}^{s} \beta^j \text{ and } A_s = (\sum_{j=0}^{s} \beta^j)\ln(1-\delta) - \ln(\sum_{j=0}^{s}\beta^j) + \beta A_{s-1} + \beta \sum_{j=0}^{s-1}\beta^j \ln(\beta \sum_{j=0}^{s-1})$$

Summarizing, the optimal policy is $\pi^*_T = \{g_t(x_t)\}_{t=0}^T = \left\{\frac{1-\delta}{\sum_{j=0}^{T-t}\beta^j} x_t\right\}_{t=0}^T$ and $V_T(x_0) = A_T + B_T \ln(x_0)$, where $B_T = \sum_{j=0}^{T}\beta^j$ and $A_T = (\sum_{j=0}^{T}\beta^j)\ln(1-\delta) - \ln(\sum_{j=0}^{T}\beta^j) + \beta A_{T-1} + \beta\sum_{j=0}^{T-1}\beta^j \ln(\beta\sum_{j=0}^{T-1})$. Also, the optimal path for $\{x_t\}_{t=1}^T$ could therefore be computed as:

$$x_{t+1}^* = (1-\delta)x_t - c_t^* = (1-\delta)x_t - (1-\delta)x_t\frac{1}{\sum\limits_{j=0}^{T-t}\beta^j} = (1-\delta)x_t\left(\frac{\sum\limits_{j=0}^{T-t}\beta^j - 1}{\sum\limits_{j=0}^{T-t}\beta^j}\right)$$

Exercise 2.2.2

[See .m file for the code]



As witnessed from the chart above, the two methods provide identical optimal paths for x_t.

Exercise 3

Exercise 3.1

Define K as the set of all possible values for capital, k_t and the law of motion, $k_{t+1} = \tilde{f}(k_t, c_t) = f(k_t) + (1-\delta)k_t - c_t = k_t^\alpha + (1-\delta)k_t - c_t$. Define as well the set of all possible values for consumption, $B(k_t) = [0, k_t^\alpha + (1-\delta)k_t]$, and the set of all possible values for future capital, $\Gamma(x_t) = [0, k_t^\alpha + (1-\delta)k_t]$. Then, the dynamic problem could be written as

$$\begin{aligned}
&\max_{k_{t+1}\in\Gamma(k_t)} \sum_{t=0}^T \beta^t u(k_t^\alpha + (1-\delta)k_t - k_{t+1}) \\
&\text{FOC: } -\beta^t u'(k_t^\alpha + (1-\delta)k_t - k_{t+1}) + \beta^{t+1}u'(k_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})(\alpha k_t^{\alpha-1} + 1 - \delta) = 0 \\
&\text{EE: } \beta u'(c_{t+1})(\alpha k_t^{\alpha-1} + 1 - \delta) = u'(c_t)
\end{aligned}$$

It is straightforward to see that our constraint correspondences are non-empty, compact and continuous and that the law of motion, $\tilde{f}(k_t, c_t)$ is continuous. By assumption of the problem, utility function is continuous and bounded. Therefore, by the theory of the maximum a solution to the dynamic problem exists, is continuous and bounded.

Exercise 3.2

Since we are given that $\delta = 1$, the law of motion now is $\tilde{f}(k_t, c_t) = k_t^\alpha - c_t$. Hence, using the results from the lecture notes, we can write the dynamic problem with Bellman equation:

$$\begin{aligned}
V_s(k_{T-s}) &= \max_{c_{T-s}\in B(k_{T-s})} \ln(c_{T-s}) + \beta V_{s-1}(k_{T-s}^\alpha - c_{T-s}), B(k_{T-s}) = [0, k_{T-s}^\alpha] \\
s = 0: V_0(k_T) &= \max_{c_T\in B(k_T)} \ln(c_T), B(k_T) = [0, k_T^\alpha]
\end{aligned}$$

Since the utility function is monotonically increasing, we know the budget constraint is going to bind from above. Also, due to the fact that the agent lives for only T periods, $k^*_{T+1} = g_T(k_T) = 0$. Hence,

$$\begin{aligned} c^*_T &= h_T(k_T) = k^\alpha_T \\ k^*_{T+1} &= g_T(k_T) = 0 \\ V_0(k_T) &= \ln(k^\alpha_T) = \alpha \ln(k_T) \end{aligned}$$

Similarly, for $s = 1$:

$$\begin{aligned} V_1(k_{T-1}) &= \max_{c_{T-1}\in B(k_{T-1})} \ln(c_{T-1}) + \beta V_0(k^\alpha_{T-1} - c_{T-1}) \\ &= \max_{c_{T-1}\in B(k_{T-1})} \ln(c_{T-1}) + \alpha\beta \ln(k^\alpha_{T-1} - c_{T-1}) \\ \text{FOC: } \frac{1}{c^*_{T-1}} &- \frac{\alpha\beta}{k^\alpha_{T-1} - c^*_{T-1}} = 0 \\ c^*_{T-1} = h_{T-1}(k_{T-1}) &= \frac{k^\alpha_{T-1}}{1+\alpha\beta} \\ k^*_T = g_{T-1}(k_{T-1}) &= \frac{\alpha\beta k^\alpha_{T-1}}{1+\alpha\beta} \\ V_1(k_{T-1}) &= \ln(\frac{k^\alpha_{T-1}}{1+\alpha\beta}) + \alpha\beta\ln(\frac{\alpha\beta k^\alpha_{T-1}}{1+\alpha\beta}) \end{aligned}$$

and $s = 2$:

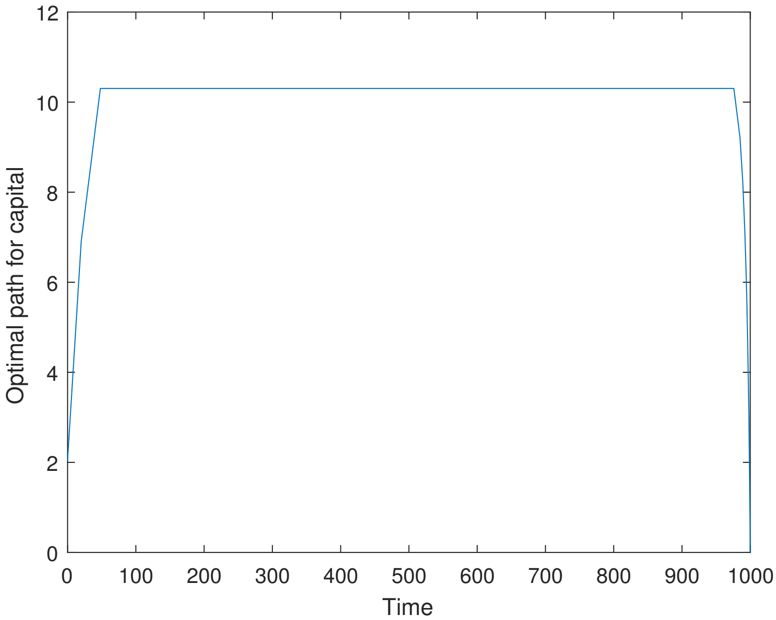
$$\begin{aligned} V_2(k_{T-2}) &= \max_{c_{T-2}\in B(k_{T-2})} \ln(c_{T-2}) + \beta V_1(k^\alpha_{T-2} - c_{T-2}) \\ &= \max_{c_{T-2}\in B(k_{T-2})} \ln(c_{T-2}) + \beta\left(\ln\left(\tfrac{(k^\alpha_{T-2}-c_{T-2})^\alpha}{1+\alpha\beta}\right) + \alpha\beta\ln\left(\tfrac{\alpha\beta(k^\alpha_{T-2}-c_{T-2})^\alpha}{1+\alpha\beta}\right)\right) \\ \text{FOC: } &\frac{1}{c^*_{T-2}} - \frac{\alpha\beta(1+\alpha\beta)}{k^\alpha_{T-2} - c^*_{T-2}} = 0 \\ c^*_{T-2} &= h_{T-2}(k_{T-2}) = \frac{k^\alpha_{T-2}}{1+\alpha\beta+\alpha^2\beta^2} \\ k^*_{T-1} &= g_{T-2}(k_{T-2}) = \frac{(\alpha\beta+\alpha^2\beta^2)k^\alpha_{T-2}}{1+\alpha\beta+\alpha^2\beta^2} \\ V_2(k_{T-2}) &= \ln(\frac{k^\alpha_{T-2}}{1+\alpha\beta+\alpha^2\beta^2}) + \beta\left(\ln\left(\tfrac{(\frac{(\alpha\beta+\alpha^2\beta^2)k^\alpha_{T-2}}{1+\alpha\beta+\alpha^2\beta^2})^\alpha}{1+\alpha\beta}\right) + \alpha\beta\ln\left(\tfrac{\alpha\beta(\frac{(\alpha\beta+\alpha^2\beta^2)k^\alpha_{T-2}}{1+\alpha\beta+\alpha^2\beta^2})^\alpha}{1+\alpha\beta}\right)\right) \\ &= \alpha\ln(k_{T-2}) - \ln(1+\alpha\beta+\alpha^2\beta^2) + (\alpha\beta+\alpha^2\beta^2)\alpha\ln(k_{T-2}) + (\alpha\beta+\alpha^2\beta^2)\ln(\alpha\beta+\alpha^2\beta^2) - \\ &- (\alpha\beta+\alpha^2\beta^2)\ln(1+\alpha\beta+\alpha^2\beta^2) - (1+\alpha\beta)\beta\ln(1+\alpha\beta) + \alpha\beta^2\ln(\alpha\beta) \end{aligned}$$

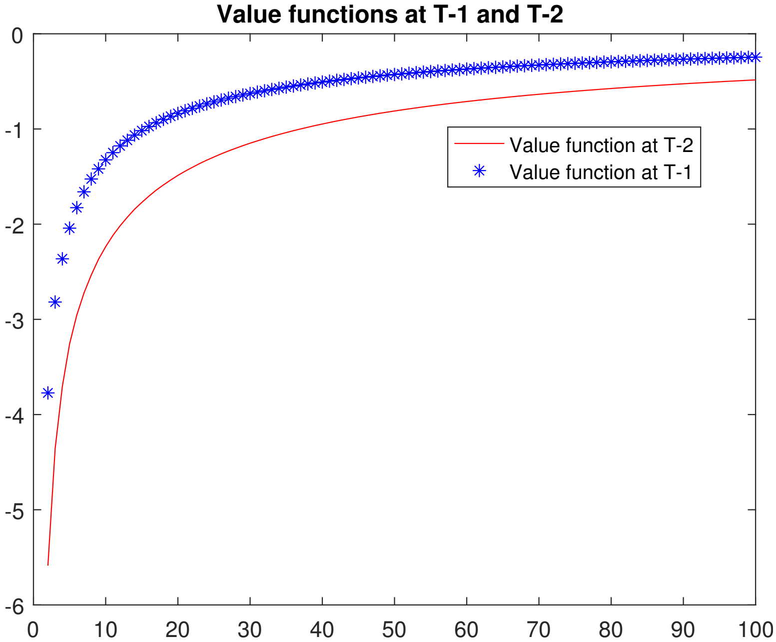
In general, we can give a formula for the optimal decision rules for consumption and savings:

$$\begin{aligned} h_t(k_t) &= \frac{k^\alpha_t}{\sum\limits_{j=0}^{T-t}(\alpha\beta)^j} \\ g_t(k_t) &= k^\alpha_t \frac{\sum\limits_{j=0}^{T-t}(\alpha\beta)^j - 1}{\sum\limits_{j=0}^{T-t}(\alpha\beta)^j} \end{aligned}$$

Exercise 3.3

[See .m file for the code]





In the latter chart, we can see that value functions change through time. In a finite time horizon setup, at each point in time the value functions consists of a continuation value (flow utility) and the remaining optimal life-time utility (V_{s-1}). Hence, as we move closer towards 'the end of the world', the remaining life-time utility decreases. Also, the closer we get to T, the smaller are our incentives to save, hence we eat up the capital stock.

Exercise 4

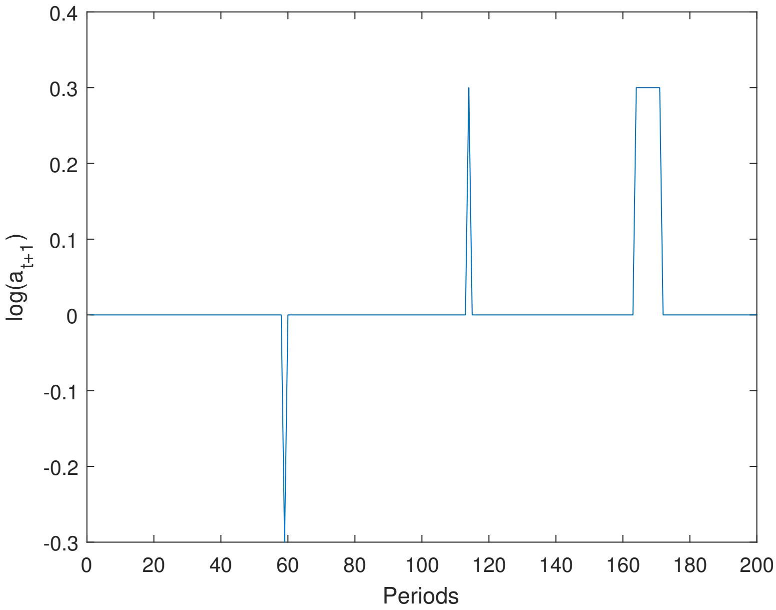
Exercise 4.1

Simulating Markov chain process with 3 nodes for 200 periods.
[See .m file for the code]

```
zstep =

0.3000

Sample mean = 0.012
Sample variance = 0.0043779
```



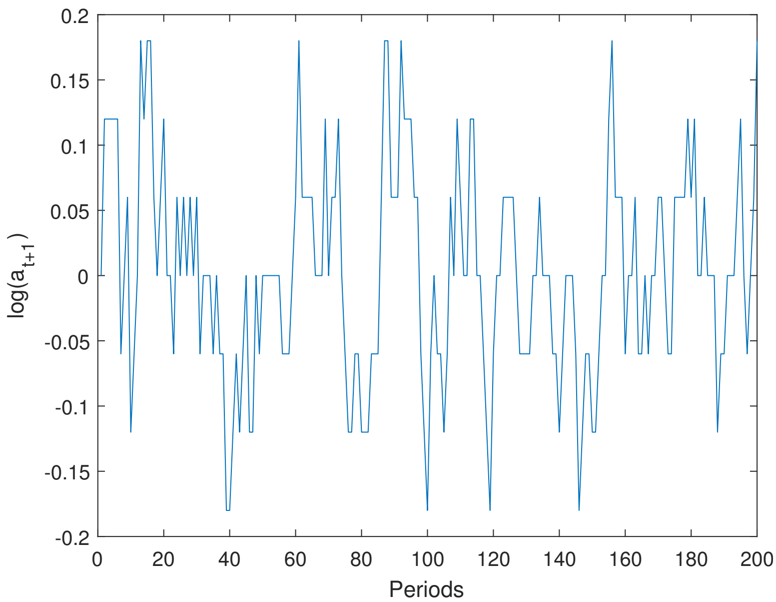
Exercise 4.2

Simulating Markov chain process with 11 nodes for 200 periods.
sufficient

```
zstep =

0.0600

Sample mean = 0.0027
Sample variance = 0.0065233
```



First of all, notice that here we have a zero mean process. Since $|\rho| < 1$, we know the process is stationary. Hence, unconditional mean is zero and variance is $\frac{\sigma^2}{1-\rho^2} = 0.01$. So, in both cases it is easy to see that the sample variances are relatively close to the unconditional variance of a continuous process. And the sample means are within one standard deviation from the population mean.