Problem Set V

Econometrics II

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Exercise 1

(a) The model could be rewritten as

$$\underbrace{\begin{pmatrix}\Delta X_{1t}\\ \Delta X_{2t}\end{pmatrix}}_{\Delta X_t} = \underbrace{\begin{pmatrix}\alpha_1 & -\alpha_1\beta_2\\ 0 & 0\end{pmatrix}}_{\Pi}\underbrace{\begin{pmatrix}X_{1t-1}\\ X_{2t-1}\end{pmatrix}}_{X_{t-1}} + \underbrace{\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}}_{\mu} + \underbrace{\begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix}}_{\varepsilon_t}$$

Hence, also $\Gamma_k = 0, \forall k \geq 1$. The matrix Π could be decomposed as:

$$\underbrace{\begin{pmatrix}\alpha_1\\ 0\end{pmatrix}}_{\alpha}\underbrace{\begin{pmatrix}1 & -\beta_2\end{pmatrix}}_{\beta'} = \begin{pmatrix}\alpha_1 & -\alpha_1\beta_2\\ 0 & 0\end{pmatrix}$$

Notice that

$$\det(\alpha'\alpha) = \alpha_1^2 \neq 0 \text{ if } \alpha_1 \neq 0$$
$$\det(\beta'\beta) = 1 + \beta_2^2 > 0$$

Thus, we can define $\alpha_\perp$ and $\beta_\perp$ such that

$$\begin{cases}\alpha_\perp'\alpha = 0\\ \beta_\perp'\beta = 0\end{cases} \quad \begin{cases}\begin{pmatrix}\alpha_{1\perp} & \alpha_{2\perp}\end{pmatrix}\begin{pmatrix}\alpha_1\\ 0\end{pmatrix} = 0\\ \begin{pmatrix}\beta_{1\perp} & \beta_{2\perp}\end{pmatrix}\begin{pmatrix}1\\ -\beta_2\end{pmatrix} = 0\end{cases} \quad \Rightarrow \alpha_\perp = \begin{pmatrix}0\\ \alpha_{2\perp}\end{pmatrix}, \alpha_{2\perp} \in \mathbb{R}/\{0\}; \qquad \beta_\perp = \begin{pmatrix}\beta_2\\ 1\end{pmatrix}$$

Therefore, using Granger Representation we can define C as

$$C = \beta_\perp(\alpha_\perp'\beta_\perp)^{-1}\alpha_\perp' = \begin{pmatrix}\beta_2\\ 1\end{pmatrix}\left[\begin{pmatrix}0 & \alpha_{2\perp}\end{pmatrix}\begin{pmatrix}\beta_2\\ 1\end{pmatrix}\right]^{-1}\begin{pmatrix}0 & \alpha_{2\perp}\end{pmatrix} = \frac{1}{\alpha_{2\perp}}\begin{pmatrix}0 & \beta_2\alpha_{2\perp}\\ 0 & \alpha_{2\perp}\end{pmatrix} = \begin{pmatrix}0 & \beta_2\\ 0 & 1\end{pmatrix}$$

(b) In section **(a)**, we saw that the process could be written in vector notation, which could further be rearranged as

$$\Delta X_t = \Pi X_{t-1} + \mu + \varepsilon_t$$
$$\underbrace{[(1-L)I - \Pi L]}_{\Phi(L)} X_t = \mu + \varepsilon_t$$

Hence, the root of characteristic polynomial could be found by solving

$$\det\left(\begin{pmatrix}1-z & 0\\ 0 & 1-z\end{pmatrix} - \begin{pmatrix}\alpha_1 z & -\alpha_1\beta_2 z\\ 0 & 0\end{pmatrix}\right) = 0 \Rightarrow (1-(1+\alpha_1)z)(1-z) = 0 \Rightarrow \begin{bmatrix} z & = 1\\ z & = \frac{1}{1+\alpha_1}\end{bmatrix}$$

Hence, if $\mid \frac{1}{1+\alpha_1} \mid < 1 \Rightarrow \alpha_1 \in (-\infty, -2) \cup (0, \infty)$, the process for X_t is explosive; in all other cases when $\alpha_1 \in [-2, 0]$, X_t is I(1) process. Notice that β_2 has no effect on stationarity/nonstationarity of X_t; however, if $\beta_2 = 0$, there is no cointegrating relationship between X_{1t} and X_{2t}.

(c) Recall from section **(b)** that the process for X_t could be written with lag operator as

$$(1-L)IX_t - \Pi L X_t = \mu + \varepsilon_t \Rightarrow (1-L)\beta' X_t - \beta'\alpha\beta' L X_t = \beta'(\mu + \varepsilon_t)$$
$$[(1-L) - \beta'\alpha L]\,\beta' X_t = \beta'(\mu + \varepsilon_t)$$

Check the roots of the characteristic polynomial

$$1 - z - \alpha_1 z = 0 \Rightarrow z = \frac{1}{1+\alpha_1}$$

So, for the process $\beta' X_t$ to be stationary, we need $|z| > 1$:

$$\begin{cases} \frac{1}{1+\alpha_1} > 1 & \text{if } 1+\alpha_1 > 0 \\ \frac{1}{1+\alpha_1} < -1 & \text{if } 1+\alpha_1 < 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 < 0 \\ \alpha_1 > -2 \end{cases}$$

(d) Given the Granger representation given in the problem, there is no linear trend in X_t if $C\mu = 0$:

$$\begin{pmatrix} 0 & \beta_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \beta_2\mu_2 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

That is, the linear trend disappears if the process for X_{2t} is mean-zero: $\mu_2 = 0$.

Exercise 2

(e) Rewrite equations (3) and (4) as

$$\begin{cases} \mathbb{E}_t\left(\text{exch}_{t+1} - \text{exch}_t - i_t^{us} + i_t^{au}\right) = 0 \\ \mathbb{E}_t\left(p_{t+1}^{us} - p_t^{us} - p_{t+1}^{au} + p_t^{au} + i_t^{au} - i_t^{us}\right) = 0 \end{cases}$$

The expression inside the expectations could be written in vector notation:

$$\underbrace{\begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{pmatrix}}_{c_0'} \begin{pmatrix} i_t^{au} \\ i_t^{us} \\ \text{exch}_t \\ p_t^{us} \\ p_t^{au} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}}_{c_1'} \begin{pmatrix} i_{t+1}^{au} \\ i_{t+1}^{us} \\ \text{exch}_{t+1} \\ p_{t+1}^{us} \\ p_{t+1}^{au} \end{pmatrix} = \begin{pmatrix} i_t^{au} - i_t^{us} - \text{exch}_t + \text{exch}_{t+1} \\ i_t^{au} - i_t^{us} - p_t^{us} + p_t^{au} + p_{t+1}^{us} - p_{t+1}^{au} \end{pmatrix}$$

Hence, indeed (3) and (4) are a special case of (2) with

$$c_0 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(f) Rewrite the rational expectations restriction

$$\mathbb{E}_t\left(c_0' X_t + c_1' X_{t+1}\right) = c$$
$$c_0' X_t + c_1' \mathbb{E}_t\left(X_{t+1}\right) = c$$

Substitute (1)

$$c_0' X_t + c_1' \mathbb{E}_t\left((I+\Pi)X_t + \mu + \varepsilon_{t+1}\right) = c$$
$$(c_0' + c_1' + c_1'\Pi)X_t + c_1'\mu + c_1'\overset{0}{\cancel{\mathbb{E}_t(\varepsilon_{t+1})}} = c$$

For the above equality to hold we need X_t to be pre-multiplied by a zero matrix. Hence,

$$c_0' + c_1' + c_1'\Pi = 0$$
$$c_1'\Pi = -(c_0' + c_1') = -(c_0 + c_1)'$$

Then, we are left with condition

$$c_1'\mu = c$$