Problem Set III

Econometrics II

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Exercise 1

(a) Recall that OLS estimators are computed as

$$\begin{bmatrix}\hat{\mu}_0 \\ \hat{\mu}_1\end{bmatrix} = \begin{bmatrix}\sum_{t=1}^{n} 1 & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{bmatrix}^{-1} \begin{bmatrix}\sum_{t=1}^{n} Y_t \\ \sum_{t=1}^{n} tY_t\end{bmatrix} = \begin{bmatrix}\mu_0 \\ \mu_1\end{bmatrix} + \begin{bmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{bmatrix}^{-1} \begin{bmatrix}\sum_{t=1}^{n} \varepsilon_t \\ \sum_{t=1}^{n} t\varepsilon_t\end{bmatrix} \tag{1}$$

Pre-multiply (1) by $B = \begin{bmatrix}\sqrt{n} & 0 \\ 0 & n^{\frac{3}{2}}\end{bmatrix}$:

$$\begin{aligned} B\begin{bmatrix}\hat{\mu}_0 - \mu_0 \\ \hat{\mu}_1 - \mu_1\end{bmatrix} &= B\begin{bmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{bmatrix}^{-1} \begin{bmatrix}\sum_{t=1}^{n} \varepsilon_t \\ \sum_{t=1}^{n} t\varepsilon_t\end{bmatrix} = B\begin{bmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{bmatrix}^{-1} BB^{-1}\begin{bmatrix}\sum_{t=1}^{n} \varepsilon_t \\ \sum_{t=1}^{n} t\varepsilon_t\end{bmatrix} \\ &= \left[B^{-1}\begin{pmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{pmatrix}B^{-1}\right]^{-1} B^{-1}\begin{bmatrix}\sum_{t=1}^{n} \varepsilon_t \\ \sum_{t=1}^{n} t\varepsilon_t\end{bmatrix}\end{aligned}$$

Notice that $B^{-1} = \frac{1}{n^2}\begin{bmatrix}n^{\frac{3}{2}} & 0 \\ 0 & \sqrt{n}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{n}} & 0 \\ 0 & \frac{1}{n^{\frac{3}{2}}}\end{bmatrix}$. Hence,

$$B^{-1}\begin{pmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2\end{pmatrix}B^{-1} = \begin{bmatrix} 1 & \frac{1}{n^2}\sum_{t=1}^{n} t \\ \frac{1}{n^2}\sum_{t=1}^{n} t & \frac{1}{n^3}\sum_{t=1}^{n} t^2\end{bmatrix}$$

Claim: $\frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k \longrightarrow \frac{1}{k+1}$. Proof: Rewrite $\frac{1}{n^{k+1}}\sum_{t=1}^{n} t^k = \frac{1}{n}\sum_{t=1}^{n}(\frac{t}{n})^k$. Notice that $\frac{1}{n}(\frac{t}{n})^k$ represents an area under the rectangle with width $\frac{1}{n}$ and height $(\frac{t}{n})^k$. Similar to what we've done in the class, the sum of the areas under this step function converges to the integral

$$\frac{1}{n}\sum_{t=1}^{n}\left(\tfrac{t}{n}\right)^k \longrightarrow \int_0^1 \left(\tfrac{t}{n}\right)^k d\left(\tfrac{t}{n}\right) = \frac{1}{k+1}\left(\tfrac{t}{n}\right)^{k+1}\Big|_{\frac{t}{n}=0}^{1} = \frac{1}{k+1}$$

Therefore,

$$\begin{bmatrix} 1 & \frac{1}{n^2}\sum_{t=1}^{n} t \\ \frac{1}{n^2}\sum_{t=1}^{n} t & \frac{1}{n^3}\sum_{t=1}^{n} t^2\end{bmatrix} \underset{n\to\infty}{\longrightarrow} \begin{bmatrix}1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}$$

Next,

$$B^{-1}\begin{bmatrix}\sum_{t=1}^{n} \varepsilon_t \\ \sum_{t=1}^{n} t\varepsilon_t\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{n}}\sum_{t=1}^{n} \varepsilon_t \\ \frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t\varepsilon_t\end{bmatrix}$$

From FCLT, we know that $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t \overset{L_2}{\Longrightarrow} \sigma_\varepsilon W(1) \sim \mathcal{N}(0,\sigma_\varepsilon^2)$.

Next, observe that $\frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n} t\varepsilon_t$ could be rewritten as $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\frac{t}{n})\varepsilon_t$ and that $(\frac{t}{n})\varepsilon_t$ is a martingale difference process with finite variance:

$$\mathbb{E}[(\frac{t}{n})^2\varepsilon_t^2] = (\frac{t}{n})^2\sigma_\varepsilon^2 \text{ and } \frac{1}{n}\sum_{t=1}^{n}(\frac{t}{n})^2\sigma_\varepsilon^2 \longrightarrow \frac{\sigma_\varepsilon^2}{3}$$

Hence, by CLT for martingale differences,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\frac{t}{n})\varepsilon_t \overset{L_2}{\Longrightarrow} \mathcal{N}(0,\frac{\sigma_\varepsilon^2}{3})$$

For the joint distribution of $\frac{1}{\sqrt{n}}\begin{bmatrix}\sum_{t=1}^{n}\varepsilon_t\\ \sum_{t=1}^{n}(\frac{t}{n})\varepsilon_t\end{bmatrix}$ observe that the limiting distributions of both elements separately is normal. Furthermore, any linear combination of these two elements could be expressed as yet another function of martingale difference process

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\alpha_1+\alpha_2\left(\tfrac{t}{n}\right))\varepsilon_t$$

whose variane satisfies

$$\mathbb{E}(\alpha_1+\alpha_2\left(\tfrac{t}{n}\right))^2\varepsilon_t^2 = \left(\alpha_1^2+2\alpha_1\alpha_2\left(\tfrac{t}{n}\right)+\alpha_2^2\left(\tfrac{t}{n}\right)^2\right)\sigma_\varepsilon^2$$
$$\frac{1}{n}\sum_{t=1}^{n}\left(\alpha_1^2+2\alpha_1\alpha_2\left(\tfrac{t}{n}\right)+\alpha_2^2\left(\tfrac{t}{n}\right)^2\right)\sigma_\varepsilon^2 \longrightarrow \left(\alpha_1^2+2\alpha_1\alpha_2\tfrac{1}{2}+\alpha_2^2\tfrac{1}{3}\right)\sigma_\varepsilon^2 = \begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}\sigma_\varepsilon^2$$

Hence,

$$\frac{1}{\sqrt{n}}\begin{bmatrix}\sum_{t=1}^{n}\varepsilon_t\\ \sum_{t=1}^{n}(\frac{t}{n})\varepsilon_t\end{bmatrix} \overset{L_2}{\Longrightarrow} \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\sigma_\varepsilon^2\right)$$

Combining the results and using Slutsky's theorem we can provide a limiting distribution for the OLS estimators:

$$\begin{bmatrix}\sqrt{n}(\hat{\mu}_0-\mu_0)\\ n^{\frac{3}{2}}(\hat{\mu}_1-\mu_1)\end{bmatrix} \overset{L_2}{\Longrightarrow} \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\sigma_\varepsilon^2\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}^{-1}\right)$$

(b) Rewrite the estimator as

$$\hat{\mu}_k-\mu_k = \frac{\sum_{t=1}^{n}t^k\varepsilon_t}{\sum_{t=1}^{n}t^{2k}} \Longrightarrow n^{k+\frac{1}{2}}(\hat{\mu}_k-\mu_k) = \frac{\frac{1}{n^{k+\frac{1}{2}}}\sum_{t=1}^{n}t^k\varepsilon_t}{\frac{1}{n^{2k+1}}\sum_{t=1}^{n}t^{2k}}$$

Using the results from the previous part, we know that $\frac{1}{n^{2k+1}}\sum_{t=1}^{n}t^{2k} \overset{L_2}{\Longrightarrow} \frac{1}{2k+1}$. For the numerator observe that as in previous part $(\frac{t}{n})^k\varepsilon_t$ is a martingale difference process with variance satisfying

$$\mathbb{E}\left[(\tfrac{t}{n})^{2k}\varepsilon_t^2\right] = (\frac{t}{n})^{2k}\sigma_\varepsilon^2$$
$$\frac{1}{n}\sum_{t=1}^{n}(\frac{t}{n})^{2k}\sigma_\varepsilon^2 \longrightarrow \frac{\sigma_\varepsilon^2}{2k+1}$$

Therefore, applying the CLT for martingale differences we get

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(\frac{t}{n})^k\varepsilon_t \overset{L_2}{\Longrightarrow} \mathcal{N}(0, \frac{\sigma_\varepsilon^2}{2k+1})$$

Finally, applying CMT we get the limiting distribution for the estimator

$$n^{k+\frac{1}{2}}(\hat{\mu}_k - \mu_k) \overset{L_2}{\Longrightarrow} \mathcal{N}(0, \sigma_\varepsilon^2(2k+1))$$

Exercise 3

a) Define a stochastic function $X_n(r)$ as

$$X_n(r) = \begin{cases} 0 & \text{if } 0 \leq r < \frac{1}{n} \\ \frac{\varepsilon_1}{n} = \frac{Y_1}{n} & \text{if } \frac{1}{n} \leq r < \frac{2}{n} \\ \frac{\varepsilon_1+\varepsilon_2}{n} = \frac{Y_2}{n} & \text{if } \frac{2}{n} \leq r < \frac{3}{n} \\ \vdots & \\ \frac{\varepsilon_1+\ldots+\varepsilon_n}{n} = \frac{Y_n}{n} & \text{if } r = 1 \end{cases}$$

Then, the area under this step function could be written as

$$\int_0^1 \sqrt{n}X_n(r)dr = \frac{Y_1}{n^{\frac{3}{2}}} + \ldots + \frac{Y_n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n}Y_t$$

Notice that $\sqrt{n}X_n(r) = \frac{1}{\sqrt{n}}\sum\limits_{t=1}^{\lfloor nr\rfloor}\varepsilon_t = \frac{\sqrt{\lfloor nr\rfloor}}{\sqrt{n}}\frac{1}{\sqrt{\lfloor nr\rfloor}}\sum\limits_{t=1}^{\lfloor nr\rfloor}\varepsilon_t$, which by Functional Central Limit Theorem (FCLT) we know converges in distribution:

$$\sqrt{n}X_n(r) \overset{L_2}{\Longrightarrow} \sigma_\varepsilon W(r)$$

where $W(\cdot)$ is a Brownian motion. Hence, by Continuous Mapping Theorem (CMT) $\int_0^1 \sqrt{n}X_n(r)dr = \frac{1}{n^{\frac{3}{2}}}\sum\limits_{t=1}^{n}Y_t \overset{L_2}{\Longrightarrow} \int_0^1 \sigma_\varepsilon W(r)dr$.

b) Rewrite the expressions as

$$\frac{1}{n}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)\varepsilon_t = \frac{1}{n}\sum_{t=1}^{n}Y_{t-1}\varepsilon_t - \bar{Y}_1\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t = \frac{1}{n}\sum_{t=1}^{n}Y_{t-1}\varepsilon_t - \left(\frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n}Y_{t-1}\right)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varepsilon_t\right)$$
$$\frac{1}{n^2}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)^2 = \frac{1}{n^2}\sum_{t=1}^{n}Y_{t-1}^2 - 2\frac{1}{n}\bar{Y}_1^2 + \frac{1}{n}\bar{Y}_1^2 = \frac{1}{n^2}\sum_{t=1}^{n}Y_{t-1}^2 - \left(\frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n}Y_{t-1}\right)^2$$

As derived in the first part of this exercise, $\frac{1}{n^{\frac{3}{2}}}\sum\limits_{t=1}^{n}Y_{t-1} \overset{L_2}{\Longrightarrow} \int_0^1 \sigma_\varepsilon W(r)dr$. Notice as well that $\frac{1}{n^2}\sum\limits_{t=1}^{n}Y_{t-1}^2 = \int_0^1 n(X_n(r))^2dr \overset{L_2}{\Longrightarrow} \sigma_\varepsilon^2\int_0^1(W(r))^2dr$. According to the derivation in class, $\frac{1}{n}\sum\limits_{t=1}^{n}Y_{t-1}\varepsilon_t \overset{L_2}{\Longrightarrow} \frac{1}{2}\sigma_\varepsilon^2\left[(W(1))^2-1\right]$. Furthermore, $\frac{1}{\sqrt{n}}\sum\limits_{t=1}^{n}\varepsilon_t = \sqrt{n}X_n(1) \overset{L_2}{\Longrightarrow} \sigma_\varepsilon W(1)$. Hence, by CMT:

$$\frac{1}{n}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)\varepsilon_t \overset{L_2}{\Longrightarrow} \frac{1}{2}\sigma_\varepsilon^2\left[(W(1))^2-1\right] - \sigma_\varepsilon^2W(1)\int_0^1 W(r)dr$$
$$\frac{1}{n^2}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)^2 \overset{L_2}{\Longrightarrow} \sigma_\varepsilon^2\int_0^1(W(r))^2dr - \sigma_\varepsilon^2\left[\int_0^1 W(r)dr\right]^2$$

For the asymptotic distribution of the OLS estimator, rewrite the expression

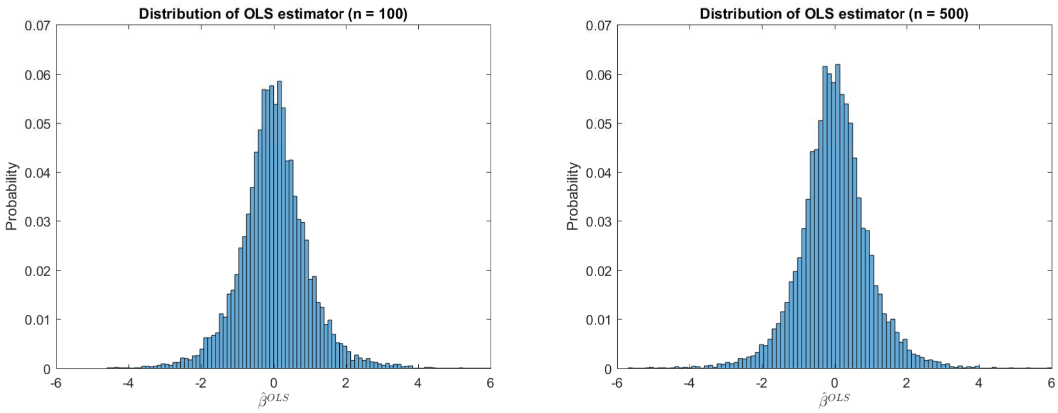
$$\hat{\varphi}_n = \frac{\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)(Y_{t-1}-\varepsilon_t)}{\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)^2} = 1 + \frac{\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)\varepsilon_t}{\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)^2}$$

Notice that we have found limiting distributions of both the scaled numerator and scaled denominator. Hence, by continuous mapping theorem:

$$n(\hat{\varphi}_n - 1) = \frac{\frac{1}{n}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)\varepsilon_t}{\frac{1}{n^2}\sum_{t=1}^{n}(Y_{t-1}-\bar{Y}_1)^2} \xRightarrow{L_2} \frac{\frac{1}{2}\left[(W(1))^2-1\right] - W(1)\int_0^1 W(r)dr}{\int_0^1 (W(r))^2 dr - \left[\int_0^1 W(r)dr\right]^2}$$

Exercise 4

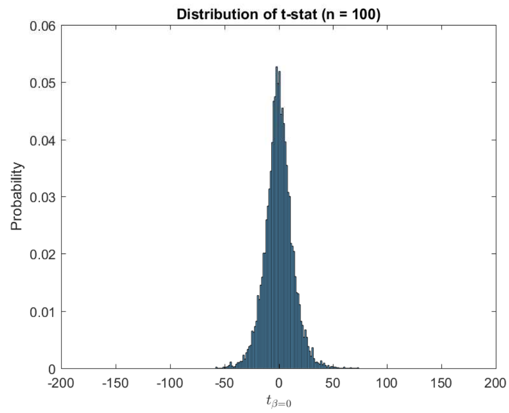
We have generated the data such that one is independent of the other. Hence, by common sense one would anticipate that the regression of these two variables should yield a zero coefficient. However, both of the series were generated according to a random walk process. Therefore, running the regression of one on the other results in a spurious regression. In other words, regression coefficients are statistically significant, as shown in Figure 1 and Figure 2. Although estimators are centred around 0, there is a large likelihood of getting highly statistically significant t-stat (way above 'rule-of-thumb' level of 2). As yet another indication of a spurious regression, the Durbin-Watson statistics is extremely low.
Notice as well the differences across sample sizes. Although it is difficult to see the differences in the distribution of OLS estimator, the distribution of t-statistics became markedly wider as we increased the sample size. That is, the larger the sample size, the more likely that a spurious regression will report coefficients statistically different from zero. Recall from theory that in case of a unit root estimators converge at a faster rate ($n^{\frac{3}{2}}$ instead of $\sqrt{n}$). Therefore, not surprisingly, as sample size grew, the variance of an estimator shrank, evidenced from exploding t-stat. In addition, as the sample size gets larger, the DW statistic approaches zero.



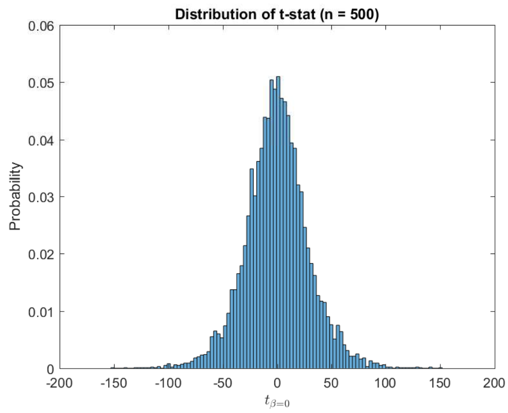
(a) $n = 100$

(b) $n = 500$

Figure 1: Distribution of OLS estimators with different sample sizes

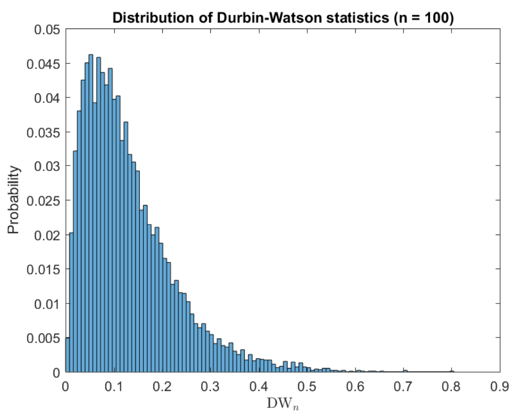


(a) $n = 100$

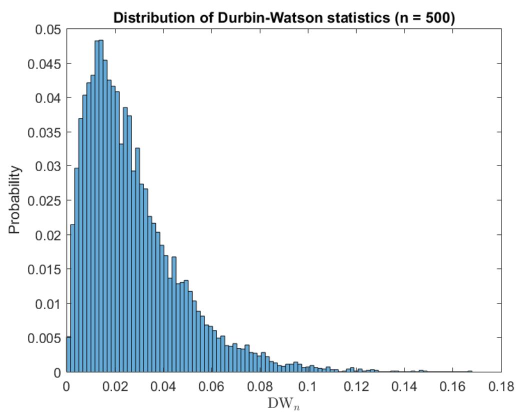


(b) $n = 500$

Figure 2: Distribution of t-statistics with different sample sizes



(a) $n = 100$



(b) $n = 500$

Figure 3: Distribution of Durbin-Watson statistics with different sample sizes