Problem Set II

Econometrics II

Nurfatima Jandarova

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Exercise 2

a) Given an ARMA(1,1) model $X_t = \mu + \phi_1 X_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1}$, where $\epsilon_t \sim iid(0, \sigma_\epsilon^2)$ and $|\phi_1| < 1$. Under these conditions, the process is covariance stationary. Therefore, the mean, variance and autocovariances of the process could be computed as follows:

$$
\begin{aligned}
\mathbb{E}(X_t) &= \mu + \phi_1 \mathbb{E}(X_{t-1}) + \cancelto{0}{\mathbb{E}(\epsilon_t)} - \theta_1 \cancelto{0}{\mathbb{E}(\epsilon_{t-1})} \implies \\
\mathbb{E}(X_t) &= \frac{\mu}{1-\phi_1} \\
\gamma_0 &= Var(X_t) = Var(\mu + \phi_1 X_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1}) \\
&= \phi_1^2 Var(X_{t-1}) + \sigma_\epsilon^2 + \theta_1^2 \sigma_\epsilon^2 + 2\cancelto{0}{Cov(\phi_1 X_{t-1}, \epsilon_t)} - 2Cov(\phi_1 X_{t-1}, \theta_1 \epsilon_{t-1}) - 2\cancelto{0}{Cov(\epsilon_t, \epsilon_{t-1})} \\
&= \phi_1^2 Var(X_{t-1}) + \sigma_\epsilon^2 + \theta_1^2 \sigma_\epsilon^2 - 2\phi_1\theta_1 Cov(\mu + \phi_1 X_{t-2} + \epsilon_{t-1} - \theta_1 \epsilon_{t-2}, \epsilon_{t-1}) \\
&= \phi_1^2 \gamma_0 + \sigma_\epsilon^2 + \theta_1^2 \sigma_\epsilon^2 - 2\phi_1\theta_1 \sigma_\epsilon^2 \implies \\
\gamma_0 &= \frac{\sigma_\epsilon^2 (1 + \theta_1^2 - 2\phi_1\theta_1)}{1-\phi_1^2} \\
\gamma_1 &= Cov(X_t, X_{t-1}) = Cov(\mu + \phi_1 X_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1}, X_{t-1}) \\
&= \phi_1 \gamma_0 - \theta_1 Cov(\epsilon_{t-1}, \mu + \phi_1 X_{t-2} + \epsilon_{t-1} - \theta_1 \epsilon_{t-2}) \\
&= \phi_1 \gamma_0 - \theta_1 \sigma_\epsilon^2 = \frac{\phi_1 \sigma_\epsilon^2 (1 + \theta_1^2 - 2\phi_1\theta_1)}{1-\phi_1^2} - \theta_1 \sigma_\epsilon^2 = \sigma_\epsilon^2 \left(\tfrac{\phi_1(1+\theta_1^2 - 2\phi_1\theta_1)}{1-\phi_1^2} - \theta_1 \right) \\
&= \sigma_\epsilon^2 \frac{\phi_1 + \phi_1\theta_1^2 - \phi_1^2\theta_1 - \theta_1}{1-\phi_1^2} = \sigma_\epsilon^2 \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1-\phi_1^2} \\
\gamma_2 &= Cov(X_t, X_{t-2}) = Cov(\mu + \phi_1 X_{t-1} + \epsilon_t - \theta_1 \epsilon_{t-1}, X_{t-2}) = \phi_1 \gamma_1 = \phi_1 \sigma_\epsilon^2 \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1-\phi_1^2}
\end{aligned}
$$

Similarly, $\forall k \geq 2$

$$
\gamma_k = \phi_1 \gamma_{k-1} = \phi_1^{k-1} \gamma_1 = \phi_1^{k-1} \sigma_\epsilon^2 \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1-\phi_1^2}
$$

Hence,

$$
\begin{aligned}
\sum_{h=-\infty}^{\infty} \gamma_h &= \gamma_0 + 2\gamma_1 + 2\sum_{h=2}^{\infty} \gamma_h = \gamma_0 + 2\gamma_1 + 2\gamma_1 \sum_{h=2}^{\infty} \phi_1^{h-1} = \gamma_0 + 2\gamma_1 + 2\gamma_1 \frac{\phi_1}{1-\phi_1} \\
&= \gamma_0 + \frac{2\gamma_1}{1-\phi_1} = \frac{\sigma_\epsilon^2 (1 + \theta_1^2 - 2\phi_1\theta_1)}{1-\phi_1^2} + \frac{2\sigma_\epsilon^2}{1-\phi_1} \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1-\phi_1^2} \\
&= \frac{\sigma_\epsilon^2}{1-\phi_1^2} \frac{(1+\phi_1)(1-\theta_1)^2}{1-\phi_1} = \sigma_\epsilon^2 \left(\tfrac{1-\theta_1}{1-\phi_1} \right)^2
\end{aligned}
$$

b) From the result above, it is easy to see that when $\theta_1 \neq 1$, the long-run variance is non-zero and when $\phi_1 \neq 1$, the long-run variance is finite.

Exercise 3

In this exercise, I also assume that $\mathbb{E}(\eta_t) = 0$.

a) Substitute the model for Y_t into the expression for X_t:

$$\begin{aligned} X_t &= \varphi Y_{t-1} + \mu + \epsilon_t + \eta_t \\ &= \varphi(X_{t-1} - \eta_{t-1}) + \mu + \epsilon_t + \eta_t \\ &= \varphi X_{t-1} + \mu + \epsilon_t + \eta_t - \varphi\eta_{t-1} \end{aligned}$$

Now, I claim that $Z_t = \epsilon_t + \eta_t - \varphi\eta_{t-1}$ is an MA(1) process. First of all, observe that

$$\mathbb{E}(Z_t) = 0$$

$$\mathbb{E}(Z_t Z_{t-j}) = \begin{cases} \sigma_\epsilon^2 + (1+\varphi^2)\sigma_\eta^2 \text{ for} & j = 0 \\ -\varphi\sigma_\eta^2 \text{ for} & j = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Want to show that it is possible to find θ such that $Z_t = \nu_t - \theta\nu_{t-1}$, where

$$\mathbb{E}(Z_t) = 0$$

$$\mathbb{E}(Z_t Z_{t-j}) = \begin{cases} (1+\theta^2)\sigma_\nu^2 \text{ for} & j = 0 \\ -\theta\sigma_\nu^2 \text{ for} & j = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

For the first-order autocovariances to be equivalent we need the following

$$\begin{aligned} -\theta\sigma_\nu^2 &= -\varphi\sigma_\eta^2 \\ \sigma_\nu^2 &= \frac{\varphi}{\theta}\sigma_\eta^2 \end{aligned}$$

Substitute this into equivalence condition for variances:

$$\begin{aligned} \sigma_\epsilon^2 + (1+\varphi^2)\sigma_\eta^2 &= (1+\theta^2)\sigma_\nu^2 \\ \sigma_\epsilon^2 + (1+\varphi^2)\sigma_\eta^2 &= (1+\theta^2)\frac{\varphi}{\theta}\sigma_\eta^2 \\ \theta^2\varphi + \theta\left(\frac{\sigma_\epsilon^2}{\sigma_\eta^2} + 1 + \varphi^2\right) + \varphi &= 0 \\ \theta &= \frac{-\left(\frac{\sigma_\epsilon^2}{\sigma_\eta^2} + 1 + \varphi^2\right) \pm \sqrt{\left(\frac{\sigma_\epsilon^2}{\sigma_\eta^2} + 1 + \varphi^2\right)^2 - 4\varphi^2}}{2\varphi} \end{aligned}$$

For $\theta > 0$ and $\sigma_\epsilon^2 > 0$ (from Hamilton) there are two real solutions: one invertible ($0 <| \theta |<| \varphi |$), and the other not ($1 <| \frac{1}{\varphi} |<| \theta |$). Hence, indeed we can rewrite $Z_t = \nu_t - \theta\nu_{t-1}$, where $\mathbb{E}(\nu_t) = 0$ and $\mathbb{E}(\nu_t^2) = \sigma_\nu^2 = \frac{\varphi}{\theta}\sigma_\eta^2$. Therefore, the process for X_t could be written as ARMA(1, 1)

$$X_t = \varphi X_{t-1} + \mu + \nu_t - \theta\nu_{t-1}$$

b) First, consider the case when $h = 0$.

$$\hat{\varphi}(0) = \frac{\sum_{t=0}^{n-1}(X_t - \bar{X}_0)X_{t+1}}{\sum_{t=0}^{n-1}(X_t - \bar{X}_0)X_t}$$

This is an OLS estimator from the regression model

$$X_{t+1} = \beta_0 + \varphi(0)X_t + u_t$$

For the cases where $h \geq 1$, the estimator $\hat{\varphi}(h)$ is an IV estimator with X_{t-h} being an instrument for X_t.

$$\begin{aligned} X_{t+1} &= \beta_0 + \varphi(h)X_t + u_t \\ X_t &= \alpha_0 + \alpha_1 X_{t-h} + v_t \end{aligned}$$

c) Notice that we could also rewrite the process for X_t as

$$\begin{aligned}(1-\varphi L)X_t &= \mu + (1-\theta L)\nu_t \\ X_t &= \frac{\mu}{1-\varphi} + \frac{1-\theta L}{1-\varphi L}\nu_t \\ &= \frac{\mu}{1-\varphi} + \sum_{j=0}^{\infty}\varphi^j(\nu_{t-j} - \theta\nu_{t-j-1})\end{aligned}$$

Since ν_t is a function of iid variables, it is also iid. Notice as well that $Var(\nu_t - \theta\nu_{t-1}) = (1+\theta^2)\sigma_\nu^2$. Hence, the process for X_t could be described as a linear process. Therefore, X_t is covariance-stationary, its autocovariances are $\gamma_h = (1+\theta^2)\sigma_\nu^2 \sum\limits_{j=0}^{\infty}\varphi^j\varphi^{j+h} = (1+\theta^2)\sigma_\nu^2 \sum\limits_{j=0}^{\infty}\varphi^{2j+h} < \infty$; and $\sum\limits_{h=-\infty}^{\infty} \mid \gamma_h \mid < \infty$ since $\mid \varphi \mid < 1$. This also implies that X_t is ergodic. Therefore, we can apply Ergodic Theorem to our estimators. When $h=0$ we have $\hat{\varphi}(0) = \frac{\frac{1}{n}\sum_{t=0}^{n-1}X_tX_{t+1} - (\frac{1}{n}\sum_{t=0}^{n-1}X_t)(\frac{1}{n}\sum_{t=0}^{n-1}X_{t+1})}{\frac{1}{n}\sum_{t=0}^{n-1}X_t^2 - (\frac{1}{n}\sum_{t=0}^{n-1}X_t)^2}$. Then, by applying Ergodic Theorem we know that

$$\begin{aligned}\frac{1}{n}\sum_{t=0}^{n-1}X_tX_{t+1} &\overset{a.s.,L_1}{\longrightarrow} \mathbb{E}(X_tX_{t+1}) = \mathbb{E}((Y_t+\eta_t)(Y_{t+1}+\eta_{t+1})) = \mathbb{E}(Y_tY_{t+1}) \\ \frac{1}{n}\sum_{t=0}^{n-1}X_t &\overset{a.s.,L_1}{\longrightarrow} \mathbb{E}(X_t) = \mathbb{Y}_{\approx} \\ \frac{1}{n}\sum_{t=0}^{n-1}X_{t+1} &\overset{a.s.,L_1}{\longrightarrow} \mathbb{E}(X_{t+1}) = \mathbb{E}(Y_{t+1}) \\ \frac{1}{n}\sum_{t=0}^{n-1}X_t^2 &\overset{a.s.,L_1}{\longrightarrow} \mathbb{E}(X_t^2) = \mathbb{E}[(Y_t+\eta_t)^2] = \mathbb{E}(Y_t^2) + \sigma_\eta^2\end{aligned}$$

$$\text{by CMT, } \hat{\varphi}(0) \overset{a.s.,L_1}{\longrightarrow} \frac{\mathbb{E}(Y_tY_{t+1}) - \mathbb{E}(Y_t)\mathbb{E}(Y_{t+1})}{\mathbb{E}(Y_t^2) - (\mathbb{E}(Y_t))^2 + \sigma_\eta^2} \neq \varphi$$

So, $\hat{\varphi}(0)$ is inconsistent estimator for φ. However, notice that following similar reasoning for cases when $h \geq 1$ we obtain the following:

$$\hat{\varphi}(h) \overset{a.s.,L_1}{\longrightarrow} \frac{\mathbb{E}(X_{t+1}X_{t-h}) - \mathbb{E}(X_{t+1})\mathbb{E}(X_{t-h})}{\mathbb{E}(X_tX_{t-h}) - \mathbb{E}(X_t)\mathbb{E}(X_{t-h})} \overset{(1)}{=} \frac{\varphi^{h+1}\gamma_1}{\varphi^h\gamma_1} = \varphi$$

where (1) follows from Exercise 2. So, $\forall h \geq 1, \hat{\varphi}(h)$ is a consistent estimator for φ.

d) I'm sorry, I can't do this one. I find it difficult and too time-consuming to check whether CLT applies to IV estimator or not.

e) As seen in part c) of this question, $\hat{\varphi}(0)$ typically has larger denominator than $\hat{\varphi}(h), \forall h \geq 1$. This observation is confirmed in the table provided.

f) We can use Delta Method here. We have

$$\sqrt{n}(\hat{\varphi}(h) - \varphi) \overset{d}{\Longrightarrow} \mathcal{N}(0, \Sigma)$$

and a function $g(\varphi) = \frac{\ln 0.5}{\ln \varphi} \Longrightarrow g'(\varphi) = -\frac{\ln 0.5}{(\ln\varphi)^2}\frac{1}{\varphi}$.

$$\sqrt{(n)}(\hat{\gamma} - \gamma) \overset{d}{\Longrightarrow} \mathcal{N}(0, \Sigma\left[-\frac{\ln 0.5}{(\ln\varphi)^2}\frac{1}{\varphi}\right]^2)$$

Exercise 4

Table 1: Regression results

	CPI
CPI_{t-1}	1.2216
	(0.1238)
CPI_{t-2}	-0.3934
	(0.1243)
intercept	0.6647
	(0.2760)
Observations	55
σ^2	1.168

In order to compute the roots of the estimated characteristic polynomial has to solve

$$1 - 1.2216z + 0.3934z^2 = 0$$
$$z = \frac{3054 \pm 2\sqrt{127021}i}{1967} \approx 1.5526 \pm 0.3624i$$

Recall that during the lecture we derived that one of the conditions for AR(2) process to be stationary is that $\mid \phi_2 \mid < 1$. From the above we see that the condition is satisfied, hence the process is stationary.