Stationarity, Ergodicity, etc.

Problem Set 1

1. Show that if a stationary process $\{X_t\}$, is mixing,
$$\lim_{n\to\infty} \Pr(X_1 \le a, X_n \le b) = \Pr(X_1 \le a)\Pr(X_1 \le b),$$
for all $a, b \in \mathbb{R}$,
then it is also ergodic (weakly mixing)
$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \Pr(X_1 \le a, X_{t+1} \le b) = \Pr(X_1 \le a)\Pr(X_1 \le b),$$
for all $a, b \in \mathbb{R}$.

2. Consider the sample average, $\bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$, of a covariance stationary process, and suppose that its autocovariances are such that $\sigma^2 = \lim_{n\to\infty}\sum_{h=-n}^{n}\gamma_h$ is well defined (the limit exists and is finite). Show that
$$\lim_{n\to\infty} \mathrm{var}(\sqrt{n}\bar{X}_n) = \sigma^2.$$
In other words, complete the arguments given during the first lecture where we arrived at $\sum_{h=-n}^{n}\frac{n-|h|}{n}\gamma_h$.

3. Consider the time series, $\{X_t\}_{t=0}^{\infty}$, where X_t has a two-point distribution: $P(X_t = -1) + P(X_t = 1) = 1$, for all t, and the conditional distribution of X_t given the past, is fully described by:
$$\begin{array}{ll} P(X_t = 1|X_{t-1} = 1) = p, & P(X_t = -1|X_{t-1} = 1) = 1 - p, \\ P(X_t = 1|X_{t-1} = -1) = 1 - q, & P(X_t = -1|X_{t-1} = -1) = q, \end{array}$$
where $p, q \in [0, 1]$.
   (a) Is $\{X_t\}$ a martingale difference sequence? For all (p, q)? For some (p, q)? For no (p, q)?

(b) Determine the marginal distribution for X_t that leads to the same marginal distribution for X_{t+1}. Clearly state the possible values for

$$\lambda_{p,q} = P(X_t = 1) = P(X_{t+1} = 1),$$

as a function of p and q.

In the following: Assume that $P(X_0 = 1) = \lambda_{p,q}$.

(c) Suppose $0 < p = q < 1$. Show that $\mathrm{E}(X_t) = 0$.

(d) When $0 < p = q < 1$ it can be shown (you may take it as given) that

$$P(X_{t+k} = 1|X_t = 1) = P(X_{t+k} = -1|X_t = -1) = \tfrac{1}{2}\left[1 + (2p-1)^k\right].$$

Use this to derive

$$\mathrm{E}(X_{t+k}|X_t = 1) \qquad \text{and} \qquad \mathrm{E}(X_{t+k}|X_t = -1).$$

(e) For $0 < p = q < 1$. Derive $\mathrm{E}\left[\mathrm{E}(X_t|X_{t-k})|X_{t-k-1}\right]$ and show that

$$\mathrm{E}\left\{\left[\mathrm{E}(X_t|X_{t-k}) - \mathrm{E}(X_t|X_{t-k-1})\right]^2\right\} = 4p\,(1-p)\,(2p-1)^{2k}.$$

(f) For $0 < p = q < 1$. Compute $\mathrm{cov}(X_t, X_{t-k})$, for $k = 0, 1, 2, \ldots$.

(g) Let $0 < p = q < 1$ and assume (in this question) that $\{X_t\}$ is stationary and ergodic. Show that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t \xrightarrow{d} N(0, \Sigma),$$

and express Σ as a function of p.

In the following: Consider the general situation where p and q can take any values in $[0, 1]$. (Analyze stationarity and ergodicity directly, without relying on results from Markov chains).

(h) Prove or disprove that $\{X_t\}$ is stationary. Does your answer depend on (p, q)?

(i) Prove or disprove that "$\{X_t\}$ is ergodic for all $p, q \in [0, 1]$".

(j) Express X_t as a Markov chain. What is P? What are the eigenvalues and eigenvectors of P?

4. Download the data set PS1SPYdata.xlsx that contains data for an exchange traded index fund (SPDR) that tracks the S&P 500 index. Using the daily returns in the last column, r_t say. Compute the autocorrelations for $h = 0, 1, \ldots, 20$, defined by

$$\rho_h = \frac{\gamma_h}{\gamma_0}$$

for r_t, absolute returns $|r_t|$, and squared returns, r_t^2. Plot them and discuss whether these returns iid or stationary.