Problem Set I

Econometrics II

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Exercise 3

a) Since we are given that the process only depends on the most recent realization of itself, conditioning on natural filtration is equivalent to conditioning on the previous period's value. With this in mind, let's compute conditional expectation of the process:

$$\mathbb{E}(X_t|X_{t-1}=1) = P(X_t=1|X_{t-1}=1) - P(X_t=-1|X_{t-1}=1) = p-(1-p) = 2p-1$$
$$\mathbb{E}(X_t|X_{t-1}=-1) = P(X_t=1|X_{t-1}=-1) - 1P(X_t=-1|X_{t-1}=-1) = 1-q-q = 1-2q$$

Hence, $\{X_t\}_{t=0}^{\infty}$ is a martingale difference sequence if $p=q=\frac{1}{2}$.

b) Solve the following for $\lambda_{p,q}$:

$$\begin{aligned} P(X_t=1) &= P(X_t=1|X_{t-1}=1)P(X_{t-1}=1) + P(X_t=1|X_{t-1}=-1)P(X_{t-1}=-1) \\ \lambda_{p,q} &= p\lambda_{p,q} + (1-q)(1-\lambda_{p,q}) \\ \lambda_{p,q} &= \frac{1-q}{1-p+1-q} \end{aligned}$$

c) First of all, notice that when $P(X_0=1)=\lambda_{p,q}$, $P(X_t=1)=\lambda_{p,q}, \forall t\geq 0$:

$$\begin{aligned} \begin{bmatrix} P(X_1=1) & P(X_1=-1) \end{bmatrix} &= \begin{bmatrix} \frac{1-q}{1-q+1-p} & \frac{1-p}{1-q+1-p} \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \\ &= \begin{bmatrix} \frac{p(1-q)}{1-q+1-p} + \frac{(1-p)(1-q)}{1-p+1-q} & \frac{(1-p)(1-q)}{1-p+1-q} + \frac{(1-p)q}{1-p+1-q} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1-q}{1-p+1-q} & \frac{1-p}{1-q+1-p} \end{bmatrix} \end{aligned}$$

Notice that for $t=2$ the computation of unconditional probabilities is the same. Thus, $\forall t>0$

$$\begin{bmatrix} P(X_t=1) & P(X_t=-1) \end{bmatrix} = \begin{bmatrix} \frac{1-q}{1-p+1-q} & \frac{1-p}{1-q+1-p} \end{bmatrix}$$

Given $p=q$, we also have that $\lambda_{p,q}=\frac{1}{2}$. Hence, $\mathbb{E}(X_t)=P(X_t=1)-P(X_t=-1)=\frac{1}{2}-\frac{1}{2}=0, \forall t\geq 0$.

d) Compute $\mathbb{E}(X_{t+k}|X_t=1)$ and $\mathbb{E}(X_{t+k}|X_t=-1)$

$$\begin{aligned} \mathbb{E}(X_{t+k}|X_t=1) &= P(X_{t+k}=1|X_t=1) - P(X_{t+k}=-1|X_t=1) \\ &= \frac{1}{2}\left[1+(2p-1)^k\right] - 1 + \frac{1}{2}\left[1+(2p-1)^k\right] \\ &= 1+(2p-1)^k - 1 = (2p-1)^k \\ \mathbb{E}(X_{t+k}|X_t=-1) &= P(X_{t+k}=1|X_t=-1) - P(X_{t+k}=-1|X_t=-1) \\ &= 1 - \frac{1}{2}\left[1+(2p-1)^k\right] - \frac{1}{2}\left[1+(2p-1)^k\right] \\ &= 1-1-(2p-1)^k = -(2p-1)^k \end{aligned}$$

e) Since conditioning on X_{t-k-1} results in a smaller set than conditioning on X_{t-k}, by Law of Iterated Expectations we know that $\mathbb{E}\left[\mathbb{E}(X_t|X_{t-k})|X_{t-k-1}\right] = \mathbb{E}(X_t|X_{t-k-1})$. From 3.d) we know that $\mathbb{E}(X_t|X_{t-k-1}) = \begin{cases} (2p-1)^{k+1} \text{ if } X_{t-k-1}=1 \\ -(2p-1)^{k+1} \text{ if } X_{t-k-1}=-1 \end{cases}$

Now, rewrite $\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k}) - \mathbb{E}(X_t|X_{t-k-1})\right]^2\right\}$ as

$$\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k})\right]^2\right\} - 2\mathbb{E}\left\{\mathbb{E}(X_t|X_{t-k})\mathbb{E}(X_t|X_{t-k-1})\right\} + \mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k-1})\right]^2\right\}$$

Notice that each component could be computed as:

$$\begin{aligned}
\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k})\right]^2\right\} &= \left[\mathbb{E}(X_t|X_{t-k}=1)\right]^2 P(X_{t-k}=1) + \left[\mathbb{E}(X_t|X_{t-k}=-1)\right]^2 P(X_{t-k}=-1) \\
&= ((2p-1)^k)^2\frac{1}{2} + (-(2p-1)^k)^2\frac{1}{2} = (2p-1)^{2k} \\
\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k-1})\right]^2\right\} &= \left[\mathbb{E}(X_t|X_{t-k-1}=1)\right]^2 P(X_{t-k-1}=1) + \left[\mathbb{E}(X_t|X_{t-k-1}=-1)\right]^2 P(X_{t-k-1}=-1) \\
&= ((2p-1)^{k+1})^2\frac{1}{2} + (-(2p-1)^{k+1})^2\frac{1}{2} = (2p-1)^{2(k+1)} \\
\mathbb{E}\left\{\mathbb{E}(X_t|X_{t-k})\mathbb{E}(X_t|X_{t-k-1})\right\} &\overset{L.I.E.}{=} \mathbb{E}\left\{\mathbb{E}\left[\mathbb{E}(X_t|X_{t-k})\mathbb{E}(X_t|X_{t-k-1})|X_{t-k-1}\right]\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[\mathbb{E}(X_t|X_{t-k})|X_{t-k-1}\right]\mathbb{E}(X_t|X_{t-k-1})\right\} \\
&= \mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k-1})\right]^2\right\} = (2p-1)^{2(k+1)}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k}) - \mathbb{E}(X_t|X_{t-k-1})\right]^2\right\} &= (2p-1)^{2k} - 2(2p-1)^{2(k+1)} + (2p-1)^{2(k+1)} \\
&= (2p-1)^{2k}(1-(2p-1)^2) = (2p-1)^{2k}(1-4p^2+4p-1) \\
&= 4p(1-p)(2p-1)^{2k}
\end{aligned}$$

f) Compute covariances

$$\begin{aligned}
Cov(X_t, X_{t-k}) &= \mathbb{E}(X_tX_{t-k}) - \cancelto{0}{\mathbb{E}(X_t)}\cancelto{0}{\mathbb{E}(X_{t-k})} \\
&= \mathbb{E}(X_t|X_{t-k}=1)P(X_{t-k}=1) - \mathbb{E}(X_t|X_{t-k}=-1)P(X_{t-k}=-1) \\
&= (2p-1)^k\frac{1}{2} + (2p-1)^k\frac{1}{2} = (2p-1)^k
\end{aligned}$$

g) Given the assumption that $\{X_t\}$ is stationary and ergodic we can try to apply a general CLT to the time process. Notice that $\mathbb{E}(X_t|X_{t-1}=1) = (2p-1)^k \xrightarrow{p} 0 = \mathbb{E}(X_t)$ as $k\to\infty$ because $\mid 2p-1 \mid \in (0,1)$. Next, using the result above we can show that $\sum_{k=0}^{\infty}\sqrt{\mathbb{E}(r_{t,t-k}^2)} = \sum_{k=0}^{\infty}\sqrt{\mathbb{E}\left\{\left[\mathbb{E}(X_t|X_{t-k}) - \mathbb{E}(X_t|X_{t-k-1})\right]^2\right\}} = \sum_{k=0}^{\infty}\sqrt{4p(1-p)(2p-1)^{2k}}$. Since $\mid 2p-1 \mid \in (0,1)$, then as $k\to\infty$ the infinite sum converges, hence is finite. Therefore, we can apply CLT to conclude that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(X_t - 0) \overset{d}{\Longrightarrow} \mathcal{N}\left(0, \sum_{k=-\infty}^{\infty} Cov(X_t, X_{t-k})\right) \sim \mathcal{N}\left(0, \sum_{k=-\infty}^{\infty}(2p-1)^k\right)$$

h) It is easy to show that as long as the condition $P(X_t=1) = P(X_{t+k}=1), \forall k\geq 1$ is met, the process is covariance stationary. Notice also that since X_t is a process such that conditional probabilities are time invariant, conditional expectations are also time ivariant.

$$\mathbb{E}(X_t) = P(X_t=1) - P(X_t=-1) = 2P(X_t=1) - 1 = 2P(X_{t+k}=1) - 1 = \mathbb{E}(X_{t+k}), \forall k\geq 1$$

$$\begin{aligned}
Var(X_t) &= \cancelto{1}{P(X_t=1) + P(X_t=-1)} - (2P(X_t=1)-1)^2 \\
&= 1 - (2P(X_{t+k}=1)-1)^2 = Var(X_{t+k}), \forall k\geq 1 \\
Cov(X_t, X_{t+k}) &= \mathbb{E}(X_tX_{t+k}) - \mathbb{E}(X_t)\mathbb{E}(X_{t+k}) \\
&= \mathbb{E}(X_{t+k}|X_t=1)P(X_t=1) - \mathbb{E}(X_{t+k}|X_t=-1)P(X_t=-1) - (2P(X_t=1)-1)^2
\end{aligned}$$

only depends on lag length.

i) The statement is false. Take, for example, $p=q=1$ and $P(X_t=1)=\frac{1}{2}$. In this case, $P(X_{t+k}=1|X_t=1)=1$, $P(X_{t+k}=-1|X_t=1)=0$, $P(X_{t+k}=1|X_t=-1)=0$, and

$P(X_{t+k} = -1|X_t = -1) = 1$, $\forall k \geq 1$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \Pr(X_1 \leq -1, X_{t+q+1} \leq -1) &= \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \Pr(X_{t+q+1} \leq -1|X_1 \leq -1) \Pr(X_1 \leq -1) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \Pr(X_1 \leq -1) \\
&= \Pr(X_1 \leq -1) = \frac{1}{2} \neq \frac{1}{4} = \Pr(X_1 \leq -1) \Pr(X_{t+q+1} \leq -1)
\end{aligned}$$

Thus, the process is not ergodic for all (p, q).

j) Expressing $\{X_t\}$ as a Markov chain is equivalent to finding the associated transition matrix, P:

$$P = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

To calculate the eigenvalues, solve $det(P - \lambda I) = 0$ for λ:

$$\begin{aligned}
det(\begin{bmatrix} p-\lambda & 1-p \\ 1-q & q-\lambda \end{bmatrix}) &= (p-\lambda)(q-\lambda) - (1-p)(1-q) = 0 \\
\cancel{pq} - \lambda(p+q) + \lambda^2 - 1 + p + q - \cancel{pq} &= \lambda^2 - \lambda(p+q) - (1-p-q) = 0 \\
\lambda &= \frac{p+q \pm \sqrt{(p+q)^2 + 4 - 4(p+q)}}{2} \\
&= \frac{p+q \pm \sqrt{(p+q-2)^2}}{2} \\
&= \frac{p+q \pm (p+q-2)^2}{2} \\
&= \big(p+q-1, 1\big)
\end{aligned}$$

To find the associated eigenvectors, solve $(P - \lambda_i I)v_i = 0$ for v_i. First, for $\lambda_1 = 1$:

$$\begin{bmatrix} p-1 & 1-p \\ 1-q & q-1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = 0$$

$$\begin{cases} (p-1)v_1^1 + (1-p)v_2^1 = 0 \\ (1-q)v_1^1 + (q-1)v_2^1 = 0 \end{cases} \Rightarrow \begin{cases} (1-p)(v_2^1 - v_1^1) = 0 \\ (1-q)(v_1^1 - v_2^1) = 0 \end{cases} \Rightarrow \begin{cases} v_2^1 = v_1^1 \\ v_1^1 = v_2^1 \end{cases} \Rightarrow v^1 = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \in \mathbb{R}$$

Similarly, for $\lambda_1 = p + q - 1$:

$$\begin{bmatrix} p+1-p-q & 1-p \\ 1-q & q+1-p-q \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1-q & 1-p \\ 1-q & 1-p \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = 0$$

$$\begin{cases} (1-q)v_1^2 + (1-p)v_2^2 = 0 \\ (1-q)v_1^2 + (1-p)v_2^2 = 0 \end{cases} \Rightarrow \begin{cases} (1-p)(v_2^2 - v_1^2) = 0 \\ (1-q)(v_1^2 - v_2^2) = 0 \end{cases} \Rightarrow \begin{cases} v_1^2 = -\frac{(1-p)v_2^2}{1-q} \\ v_1^2 = -\frac{(1-p)v_2^2}{1-q} \end{cases} \Rightarrow v^2 = b \begin{bmatrix} \frac{1-p}{1-q} \\ 1 \end{bmatrix}, b \in \mathbb{R}$$

Thus, the associated eigenvectors are $a(1,1), a \in \mathbb{R}$ and $b(\frac{1-p}{1-q}, 1), b \in \mathbb{R}$.

Table of Contents

Econometrics II, Problem Set I, Question 4

Nurfatima Jandarova

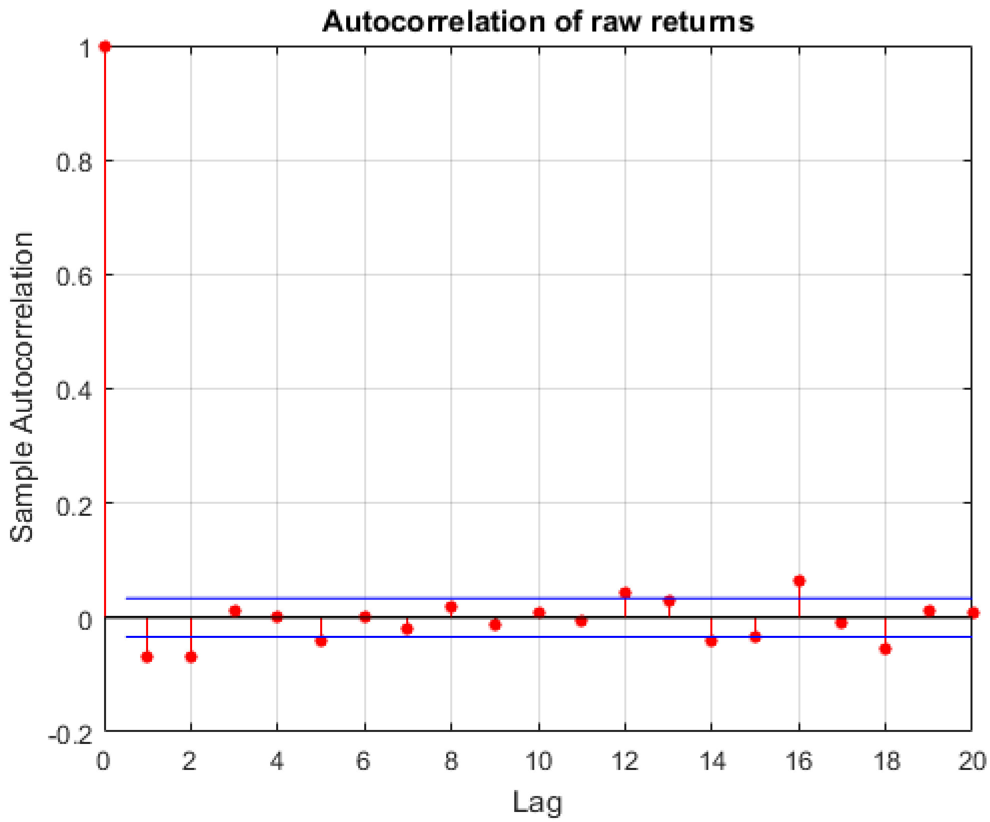
```
clear all

% Loading data
data=xlsread('PS1SPYdata.xlsx');
h = 20; % lag length
```

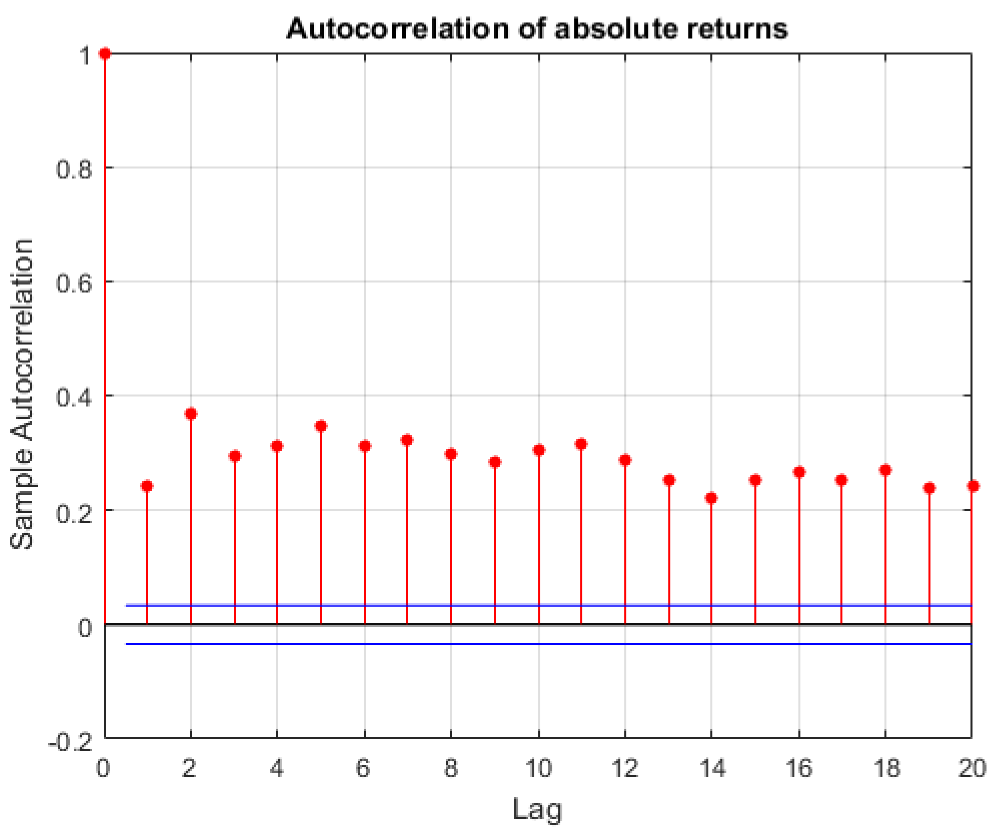
Computing autocorrelations

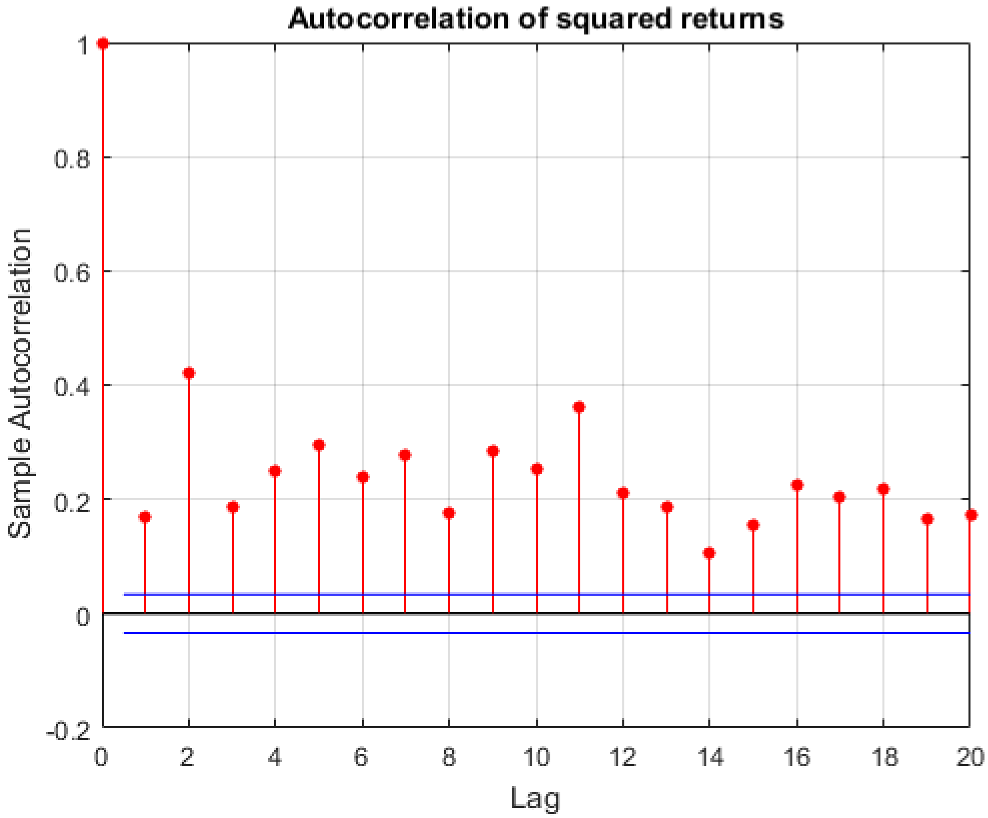
```
corr = autocorr(data(:,7), h); % Autocorrelation of raw returns
corabs = autocorr(abs(data(:,7)),h); % Autocorrelation of absolute
 returns
corsq = autocorr((data(:,7)).^2,h); % Autocorrelation of squared
 returns
```

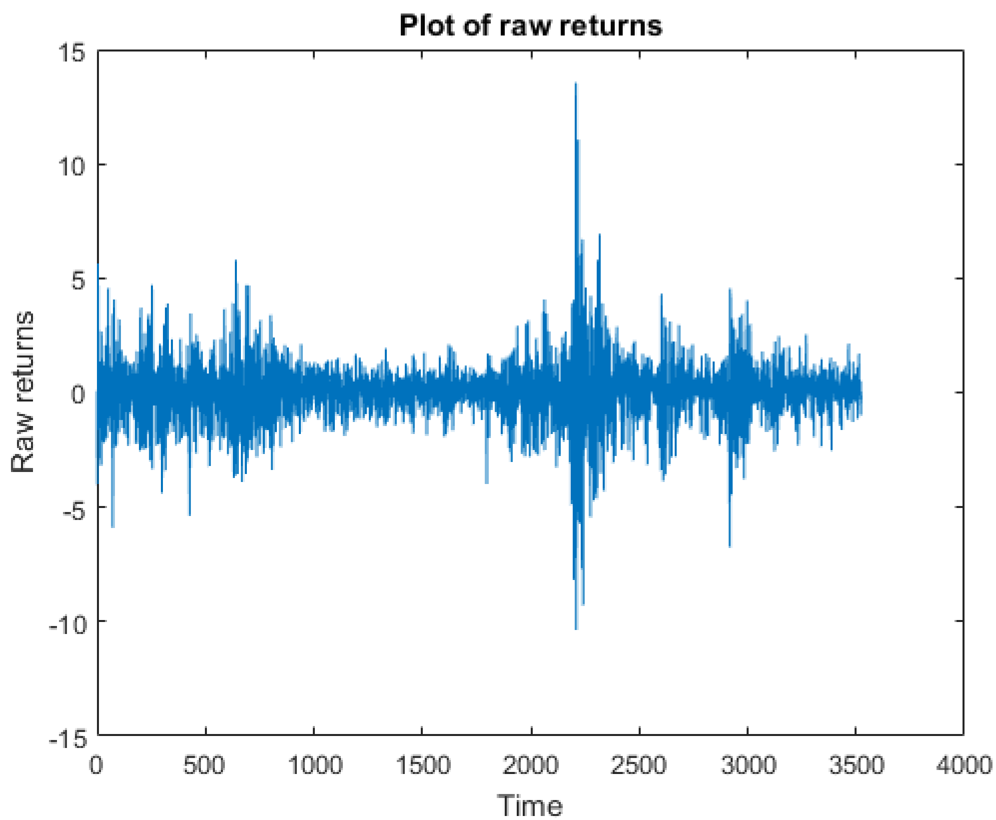
Plotting figures

```
figure(1)
autocorr(data(:,7), h)
title('Autocorrelation of raw returns')

figure(2)
autocorr(abs(data(:,7)),h)
title('Autocorrelation of absolute returns')

figure(3)
autocorr((data(:,7)).^2,h)
title('Autocorrelation of squared returns')

figure(4)
plot(1:3524,data(:,7))
xlabel('Time')
ylabel('Raw returns')
title('Plot of raw returns')
```


Autocorrelation of raw returns
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
0 2 4 6 8 10 12 14 16 18 20


Autocorrelation of absolute returns
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
0 2 4 6 8 10 12 14 16 18 20


Autocorrelation of squared returns
Sample Autocorrelation
Lag
1
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20


Plot of raw returns
Raw returns
Time
15
10
5
0
-5
-10
-15
0
500
1000
1500
2000
2500
3000
3500
4000

So, from the last chart we see that raw returns are unlikely to be stationary as variance of the series changes with time (and similarly, absolute returns and squared returns). From the ACF plots, we can see that autocorrelations are significantly different from zero and don't decay as lag length increases. Hence, the series is not iid and not ergodic.

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