Problem Set II
Econometrics III

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Exercise 1

(a) The likelihood and log-likelihood functions are:

$$L(x_i; \alpha, \beta) = \prod_{i=1}^{n} \alpha \beta x_i^{\beta-1} \exp(-\alpha x_i^\beta)$$

$$\mathcal{L}(x_i; \alpha, \beta) = \sum_{i=1}^{n} \ln\left[\alpha \beta x_i^{\beta-1} \exp(-\alpha x_i^\beta)\right] = n \ln \alpha + n \ln \beta + (\beta - 1) \sum_{i=1}^{n} \ln x_i - \alpha \sum_{i=1}^{n} x_i^\beta$$

(b) The FOCs are

$$\begin{cases} \frac{n}{\hat{\alpha}} - \sum_{i=1}^{n} x_i^{\hat{\beta}} = 0 \\ \frac{n}{\hat{\beta}} + \sum_{i=1}^{n} \ln x_i - \hat{\alpha} \sum_{i=1}^{n} x_i^{\hat{\beta}} \ln x_i = 0 \end{cases} \Rightarrow \begin{cases} \hat{\alpha} = \frac{n}{\sum\limits_{i=1}^{n} x_i^{\hat{\beta}}} \\ \frac{n}{\hat{\beta}} + \sum\limits_{i=1}^{n} \ln x_i - \frac{n \sum\limits_{i=1}^{n} x_i^{\hat{\beta}} \ln x_i}{\sum\limits_{i=1}^{n} x_i^{\hat{\beta}}} = 0 \end{cases}$$

We could use numerical methods to find the solution continue!

(c) Recall that with information equality the asymptotic variance of the ML estimator is the minus inverse of the Hessian. The Hessian is

$$H = \begin{bmatrix} -\frac{n}{\alpha^2} & -\sum\limits_{i=1}^{n} x_i^\beta \ln x_i \\ -\sum\limits_{i=1}^{n} x_i^\beta \ln x_i & -\frac{n}{\beta^2} - \alpha \sum\limits_{i=1}^{n} x_i^\beta (\ln x_i)^2 \end{bmatrix}$$

Hence, I would substitute the ML estimators from the previous part and find the inverse of the matrix. The negative of the resulting matrix would be my estimate of the asymptotic variance matrix of the ML estimators.

Exercise 2

n π_0	50	100	500	1000		n π_0	50	100	500	1000
1	0.0708	0.0662	0.0626	0.0629		1	0.0203	0.0105	0.0021	0.0010
$\frac{1}{6}$	0.2451	0.2443	0.2396	0.2383		$\frac{1}{6}$	1573.5377	13.7510	0.1190	0.0413
	(a) OLS						(b) IV			

Table 1: MSE of estimators across simulations

COMMENT!!!

$$\hat{\beta}^{OLS} - \beta_0 = \sigma_u \frac{\sum\limits_{i=1}^{n} x_i u_i}{\sum\limits_{i=1}^{n} x_i^2} = \sigma_u \frac{\sum\limits_{i=1}^{n} (\pi_0 w_i + \sigma_v v_i) u_i}{\sum\limits_{i=1}^{n} (\pi_0 w_i + \sigma_v v_i)^2} = \sigma_u \frac{\pi_0 \frac{1}{n} \sum\limits_{i=1}^{n} w_i u_i + \sigma_v \frac{1}{n} \sum\limits_{i=1}^{n} v_i u_i}{\pi_0^2 \frac{1}{n} \sum\limits_{i=1}^{n} w_i^2 + 2\pi_0 \sigma_v \frac{1}{n} \sum\limits_{i=1}^{n} w_i v_i + \sigma_v^2 \frac{1}{n} \sum\limits_{i=1}^{n} v_i^2}$$

by WLLN: $\dfrac{1}{n} \sum\limits_{i=1}^{n} w_i u_i \xrightarrow{p} \mathbb{E}(w_i u_i) = 0$

$\dfrac{1}{n} \sum\limits_{i=1}^{n} v_i u_i \xrightarrow{p} \mathbb{E}(v_i u_i) = \rho$

$\dfrac{1}{n} \sum\limits_{i=1}^{n} w_i v_i \xrightarrow{p} \mathbb{E}(w_i v_i) = 0$

$\dfrac{1}{n} \sum\limits_{i=1}^{n} v_i^2 \xrightarrow{p} \mathbb{E}(v_i^2) = 1$

by CMT: $\hat{\beta}^{OLS} - \beta \xrightarrow{p} \dfrac{\sigma_u \sigma_v \rho}{\pi_0^2 + \sigma_v^2} \implies (\hat{\beta}^{OLS} - \beta)^2 \xrightarrow{p} \dfrac{\sigma_u^2 \sigma_v^2 \rho^2}{(\pi_0^2 + \sigma_v^2)^2}$

Similarly, for IV estimator

$$\hat{\beta^{IV}} - \beta_0 = \sigma_u \frac{\frac{1}{n} \sum\limits_{i=1}^{n} w_i u_i}{\frac{1}{n} \sum\limits_{i=1}^{n} w_i x_i} = \sigma_u \frac{\frac{1}{n} \sum\limits_{i=1}^{n} w_i u_i}{\pi_0 \frac{1}{n} \sum\limits_{i=1}^{n} w_i^2 + \sigma_v \frac{1}{n} \sum\limits_{i=1}^{n} w_i v_i} \xrightarrow{p} 0 \implies (\hat{\beta^{IV}} - \beta_0)^2 \xrightarrow{p} 0$$

Hence, as sample size increases MSE of OLS estimates converges in probability to a constant non-zero value, whereas MSE of IV estimates converges in probability to zero. However, if $\rho = 0$, MSE of both estimators converges to zero. The intuition behind is that when $\rho = 0$, x_i becomes an exogenous regressor and OLS estimator becomes consistent. In this case, although IV remains a consistent estimator, it is less efficient than OLS (hence, no justification for use of IV in case of exogenous regressors).