

# Problem Set V

## Microeconomics II

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### Exercise 1

#### 1.1

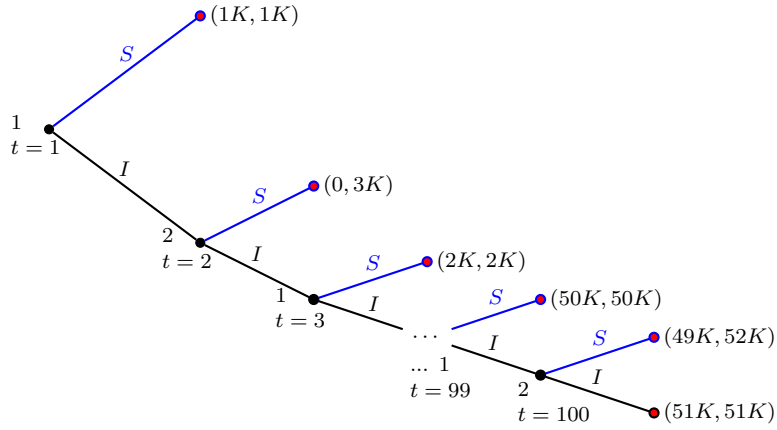


Figure 1: A centipede game in extensive form

There are two players,  $N = \{1, 2\}$ . According to the description of the game, all information sets in this game are singletons ( $\#(H_1) = \#(H_2) = 50$ ). Therefore, strategy of a player has 50 elements (actions at all information sets of the player). Define the strategy spaces and payoff functions:

$$S_1 = S_2 = \{I, S\}^{50}$$

$$\pi_1 : S_1 \times S_2 \rightarrow 1000\{0, 1, \dots, 51\}$$

$$\pi_2 : S_2 \times S_1 \rightarrow 1000\{1, 2, \dots, 52\}$$

Hence, the game in strategic form is given by

$$G(\Gamma) = \{N, S_1 \times S_2, \{\pi_1, \pi_2\}\}$$

and could be represented with a payoff matrix, excerpt of which is presented below

	I, I, ..., I	I, I, ..., S	...	I, S, ..., S	S, S, ..., S
I, I, ..., I	( <u>51K</u> , 51K)	(49K, <u>52K</u> )	...	(1K, 4K)	(0, 3K)
I, I, ..., S	(50K, 50K)	( <u>50K</u> , 50K)	...	(1K, 4K)	(0, 3K)
$\vdots$			$\ddots$		
I, S, ..., S	(2K, 2K)	(2K, 2K)	...	( <u>2K</u> , 2K)	(0, <u>3K</u> )
S, S, ..., S	(1K, <u>1K</u> )	(1K, <u>1K</u> )	...	(1K, <u>1K</u> )	( <u>1K</u> , <u>1K</u> )

Table 1: Part of payoff matrix of the game in strategic form

## 1.2

I'm not sure how to search for Nash equilibria in mixed strategies here, so I present my reasoning for Nash equilibrium in pure strategies. As seen from the above table, if player 1 believes player 2 always invests, then it is optimal for the first player to always invest as well. However, if player 2 believes player 1 always invests it is optimal for him/her to stop at  $t = 100$ . So it cannot be a Nash equilibrium. Similarly, if player 1 believes 2 is going to play  $(I, I, \dots, I, S)$ , his/her best response is to play  $(I, I, \dots, I, S)$ , but given such belief about first player, the best response of the second player would be to stop at  $t = 98$ , just one period before the other player stops. Continuing the same reasoning, it is clear that best responses of the two players intersect only when both of them choose  $(S, S, \dots, S, S)$ . Hence,  $\{(S, S, \dots, S, S), (S, S, \dots, S, S)\}$  is a Nash equilibrium in pure strategies.

By definition, a strategy is a subgame perfect equilibrium if it is NE in every perfect subgame of the game. Consider the smallest subgame at time  $t = 100$ , where player 2 has to decide which action to take. The most optimal action for player 2 in this subgame is to stop and get \$52,000 instead of \$51,000 in case he/she chooses to invest. As mentioned earlier, both players prefer to stop just one period before they believe the other player wants to play stop. Then, in the subgame at  $t = 99$ , it is optimal for player 1 to stop. Iterating backwards, we arrive at time  $t = 1$ , where again player 1 wants to stop because he/she knows that next period player 2 will play stop. Thus,  $\{(S, S, \dots, S, S), (S, S, \dots, S, S)\}$  is also SGPE. This is also illustrated by blue lines in Figure 1.

## Exercise 2

The strategic form representation of the game. There are two players, i.e.,  $N = \{1, 2\}$ . Their strategy spaces:

$$S_1 = \{A, B, C\}$$

$$S_2 = \{a, b\}$$

Profit matrix:

	a	b
A	$(-1, 1)$	$(1, 0)$
B	$(4, 0)$	$(-4, 1)$
C	$(2, 0)$	$(2, 0)$

Notice that strategy  $A$  of player 1 is strictly dominated by a mixed strategy  $\sigma_1 = (0, \frac{1}{7}, \frac{6}{7})$ :

$$\pi_1(\sigma_1, a) = -1 \cdot 0 + 4 \cdot \frac{1}{7} + 2 \cdot \frac{6}{7} > -1 = \pi_1(A, a)$$

$$\pi_1(\sigma_1, b) = 1 \cdot 0 - 4 \cdot \frac{1}{7} + 2 \cdot \frac{6}{7} = \frac{8}{7} > 1 = \pi_1(A, b)$$

## 2.1

*Sorry, I forgot the question was only asking about NE in pure strategies and also found NE in mixed strategies. However, further on I only consider the NE in pure strategies that I've found.*

Using the fact that all Nash equilibria survive IESDS, we can restrict the search of NE to the remaining game. Define the mixed strategy of player 1,  $\sigma_1 = (p, 1 - p)$ , where  $p$  is the probability of player 1 choosing action  $B$ ; and the mixed strategy of player 2,  $\sigma_2 = (q, 1 - q)$ , where  $q$  is the probability of player 2 playing  $a$ .

$$\pi_1(\sigma_1, \sigma_2) = q(4p + 2(1 - p)) + (1 - q)(-4p + 2(1 - p)) = 2 + 2p(4q - 3)$$

$$\pi_2(\sigma_1, \sigma_2) = p(1 - q) = p - pq$$

$$\rho_1(\sigma_2) = \begin{cases} 0 & \text{if } q < \frac{3}{4} \\ [0, 1] & \text{if } q = \frac{3}{4} \\ 1 & \text{if } q > \frac{3}{4} \end{cases}$$

$$\rho_2(\sigma_1) = \begin{cases} 0 & \text{if } p > 0 \\ [0, 1] & \text{if } p = 0 \end{cases}$$

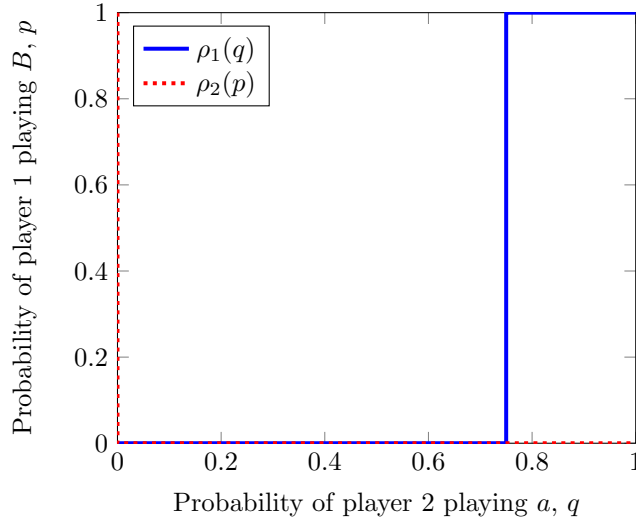


Figure 2: Best response correspondences

The best response correspondences are depicted below in Figure 2.

Hence, there is one Nash equilibrium in pure strategies,  $(C, b)$  and a continuum of Nash equilibria in mixed strategies,  $\sigma_1 \times \sigma_2 = \{(0, 0, 1), (q, 1 - q)\}$ , such that  $q \leq \frac{3}{4}$ .

## 2.2

There is only one proper subgame: the entire game itself. Therefore, the NE  $(C, b)$  is also a SGPE.

## 2.3

Let  $\mu$  denote the probability that player 2 assigns to being at the node following player 1 choosing  $B$ . Consequently, the belief of the second player that he/she is in the node induced by  $A$  is  $1 - \mu$ .

Let's first find conditions for  $\mu$  such that player 2 chooses  $a$  over  $b$  and vice versa:

<p>Player 2 chooses <math>a</math> over <math>b</math> if</p> $\pi_2(a; \mu) = 1 - \mu > \mu = \pi_2(b; \mu)$ $\mu < \frac{1}{2}$	<p>Player 2 chooses <math>b</math> over <math>a</math> if</p> $\pi_2(a; \mu) = 1 - \mu < \mu = \pi_2(b; \mu)$ $\mu > \frac{1}{2}$
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Consider the first case, where  $\mu < \frac{1}{2}$ . Then, by sequential rationality, player 1 chooses  $B$ . This implies that given the strategy profile  $(B, a)$ , the ex-ante probability of reaching the information set of player 2 is 1 and ex-ante probability of reaching the node induced by  $B$  is also equal to one. Then,  $\mu < \frac{1}{2} < \frac{1}{1}$ , which means that such system of beliefs is inconsistent. Hence,  $(B, a)$  cannot be a WPBE.

Consider the second case, where  $\mu > \frac{1}{2}$ , i.e., player 2 chooses  $b$  over  $a$ . Then, sequential rationality implies that the first player chooses  $C$ , in which case, the information set of the second player is never reached. Therefore, any system of beliefs of player 2 is statistically consistent. Hence, the strategy profile  $(C, b)$  is a WPBE in pure strategies.

## Exercise 3

### 3.1

First player has two information sets and his/her strategy space is  $S_1 = \{A, B\} \times \{C, D\}$ . Player 2 has one information set and his/her strategy space is  $S_2 = \{a, b\}$ .

### 3.2

- (i) To find NE in pure strategies, consider the matrix payoff with best responses in pure strategies underlined in Table 2.

Therefore, there are three NE in pure strategies:  $\{(A, C, a), (B, C, b), (B, D, b)\}$ .

	a	b
A, C	(2, <u>-1</u> )	(-1, -2)
A, D	(-10, -2)	(0, <u>-1</u> )
B, C	(1, <u>1</u> )	( <u>1</u> , 1)
B, D	(1, <u>1</u> )	( <u>1</u> , 1)

Table 2: Entire game in strategic form

	a	b
C	(2, <u>-1</u> )	(-1, -2)
D	(-10, -2)	(0, <u>-1</u> )

Table 3: NEs in second proper subgame

- (ii) There are two proper subgames: entire game and subgame that starts at the node where player 2 has to choose an action. Consider the second proper subgame tabulated in Table 3. There are two NE in the second proper subgame,  $\{(C, a), (D, b)\}$ . The first player's best response to  $(C, a)$  is to play  $A$ . Similarly, BR of the first player to  $(D, b)$  is to choose  $B$ . Hence, there are two SGPE,  $\{(A, C, a), (B, D, b)\}$ .
- (iii) Let  $\mu$  denote the probability player 1 assigns to being at the node induced by action  $a$  of player 2. Then,

$$\begin{aligned}\pi_1(C; \mu) &= 2\mu - (1 - \mu) = 3\mu - 1 \\ \pi_1(D; \mu) &= -10\mu\end{aligned}$$

Consider the case when  $\mu > \frac{1}{13}$ , i.e., player 1 chooses  $C$  over  $D$  in his/her second information set. Given this belief, sequential rationality implies that player 2 chooses  $a$  and player 1 plays  $A$  in the first information set. That is, given any belief system such that  $\mu > \frac{1}{13}$ , strategy  $(A, C, a)$  is sequentially rational. This strategy, in turn, implies that the probability of reaching the second information set of player 1 is equal to 1 and probability of reaching the node induced by  $a$  is also equal to 1. Then, the statistically consistent belief system would be  $\mu = \frac{1}{1} = 1 > \frac{1}{13}$ . Hence, a strategy  $(A, C, a)$  and a belief system  $(1, 0)$  constitute a WPBE.

Consider another case, when  $\mu < \frac{1}{13}$ , i.e., when player 1 chooses  $D$  over  $C$ . In this case, player 2 wants to play  $b$  and player 1 prefers  $B$  in the beginning of the game. Given, the strategy  $(B, D, b)$ , the second information set of player 1 is never reached. Hence, any belief system is consistent. Therefore, the strategy  $(B, D, b)$  and any belief system such that  $\mu < \frac{1}{13}$  constitute another WPBE.

Thus, WPBE = SGPE.

### 3.3

The following game in extensive form (with imperfect recall) has the same strategic form representation as in Table 2.

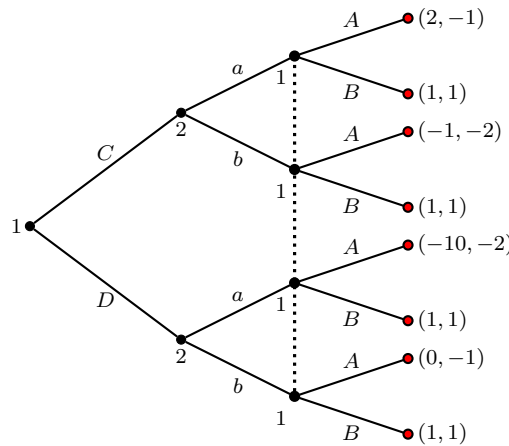


Figure 3: Alternative extensive form game

### 3.4