Problem Set V

Microeconomics II

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Exercise 1

1.1

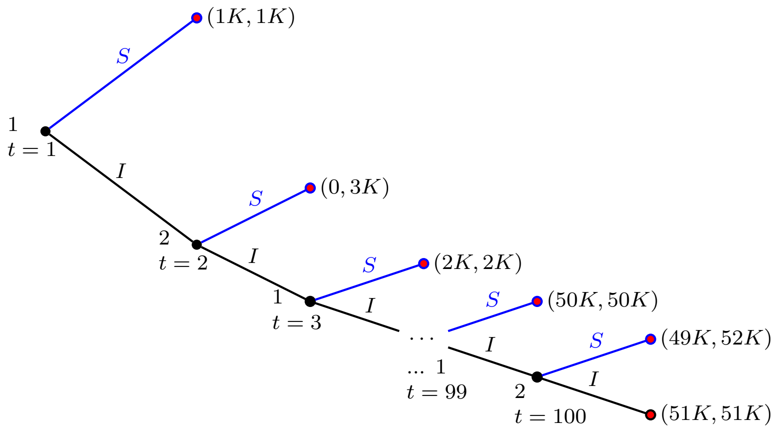


Figure 1: A centipede game in extensive form

There are two players, $N = \{1, 2\}$. According to the description of the game, all information sets in this game are singletons ($\#(H_1) = \#(H_2) = 50$). Therefore, strategy of a player has 50 elements (actions at all information sets of the player). Define the strategy spaces and payoff functions:

$$
\begin{aligned}
S_1 = S_2 &= \{I, S\}^{50} \\
\pi_1 : S_1 \times S_2 &\to 1000\{0, 1, ..., 51\} \\
\pi_2 : S_2 \times S_1 &\to 1000\{1, 2, ..., 52\}
\end{aligned}
$$

Hence, the game in strategic form is given by

$$G(\Gamma) = \{N, S_1 \times S_2, \{\pi_1, \pi_2\}\}$$

and could be represented with a payoff matrix, excerpt of which is presented below

	I, I, ..., I	I, I, ..., S	...	I, S, ..., S	S, S, ..., S
I, I, ..., I	($\underline{51K}$, 51K)	(49K, $\underline{52K}$)	...	(1K, 4K)	(0, 3K)
I, I, ..., S	(50K, 50K)	($\underline{50K}$, 50K)	...	(1K, 4K)	(0, 3K)
⋮			⋱		
I, S, ..., S	(2K, 2K)	(2K, 2K)	...	($\underline{2K}$, 2K)	(0, $\underline{3K}$)
S, S, ..., S	(1K, $\underline{1K}$)	(1K, $\underline{1K}$)	...	(1K, $\underline{1K}$)	($\underline{1K}$, $\underline{1K}$)

Table 1: Part of payoff matrix of the game in strategic form

1.2

I'm not sure how to search for Nash equilibria in mixed strategies here, so I present my reasoning for Nash equilibrium in pure strategies. As seen from the above table, if player 1 believes player 2 always invests, then it is optimal for the first player to always invest as well. However, if player 2 believes player 1 always invests it is optimal for him/her to stop at $t = 100$. So it cannot be a Nash equilibrium. Similarly, if player 1 believes 2 is going to play $(I, I, ..., I, S)$, his/her best response is to play $(I, I, ..., I, S)$, but given such belief about first player, the best response of the second player would be to stop at $t = 98$, just one period before the other player stops. Continuing the same reasoning, it is clear that best responses of the two players intersect only when both of them choose $(S, S, ..., S, S)$. Hence, $\{(S, S, ..., S, S), (S, S, ..., S, S)\}$ is a Nash equilibrium in pure strategies.
By definition, a strategy is a subgame perfect equilibrium if it is NE in every perfect subgame of the game. Consider the smallest subgame at time $t = 100$, where player 2 has to decide which action to take. The most optimal action for player 2 in this subgame is to stop and get \$52,000 instead of \$51,000 in case he/she chooses to invest. As mentioned earlier, both players prefer to stop just one period before they believe the other player wants to play stop. Then, in the subgame at $t = 99$, it is optimal for player 1 to stop. Iterating backwards, we arrive at time $t = 1$, where again player 1 wants to stop because he/she knows that next period player 2 will play stop. Thus, $\{(S, S, ..., S, S), (S, S, ..., S, S)\}$ is also SGPE. This is also illustrated by blue lines in Figure 1.

Exercise 2

The strategic form representation of the game. There are two players, i.e., $N = \{1, 2\}$. Their strategy spaces:

$$S_1 = \{A, B, C\}$$
$$S_2 = \{a, b\}$$

Profit matrix:

	a	b
~~A~~	~~(-1, 1)~~	~~(1, 0)~~
B	(4, 0)	(-4, 1)
C	(2, 0)	(2, 0)

Notice that strategy A of player 1 is strictly dominated by a mixed strategy $\sigma_1 = (0, \frac{1}{7}, \frac{6}{7})$:

$$\pi_1(\sigma_1, a) = -1 \cdot 0 + 4\frac{1}{7} + 2\frac{6}{7} > -1 = \pi_1(A, a)$$
$$\pi_1(\sigma_1, b) = 1 \cdot 0 - 4\frac{1}{7} + 2\frac{6}{7} = \frac{8}{7} > 1 = \pi_1(A, b)$$

2.1

Sorry, I forgot the question was only asking about NE in pure strategies and also found NE in mixed strategies. However, further on I only consider the NE in pure strategies that I've found.
Using the fact that all Nash equilibria survive IESDS, we can restrict the search of NE to the remaining game. Define the mixed strategy of player 1, $\sigma_1 = (p, 1 - p)$, where p is the probability of player 1 choosing action B; and the mixed strategy of player 2, $\sigma_2 = (q, 1 - q)$, where q is the probability of player 2 playing a.

$$\pi_1(\sigma_1, \sigma_2) = q(4p + 2(1 - p)) + (1 - q)(-4p + 2(1 - p)) = 2 + 2p(4q - 3)$$
$$\pi_2(\sigma_1, \sigma_2) = p(1 - q) = p - pq$$
$$\rho_1(\sigma_2) = \begin{cases} 0 & \text{if } q < \frac{3}{4} \\ [0, 1] & \text{if } q = \frac{3}{4} \\ 1 & \text{if } q > \frac{3}{4} \end{cases}$$
$$\rho_2(\sigma_1) = \begin{cases} 0 & \text{if } p > 0 \\ [0, 1] & \text{if } p = 0 \end{cases}$$

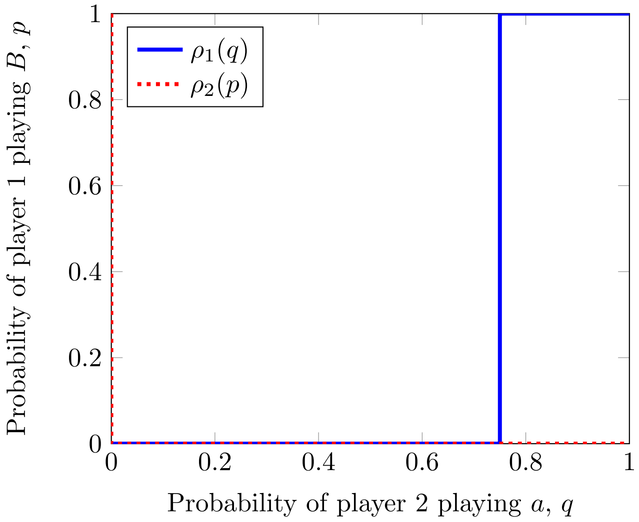


Figure 2: Best response correspondences

The best response correspondences are depicted below in Figure 2.
Hence, there is one Nash equilibrium in pure strategies, (C, b) and a continuum of Nash equilibria in mixed strategies, $\sigma_1 \times \sigma_2 = \{(0, 0, 1), (q, 1 - q)\}$, such that $q \leq \frac{3}{4}$.

2.2

There is only one proper subgame: the entire game itself. Therefore, the NE (C, b) is also a SGPE.

2.3

Let μ denote the probability that player 2 assigns to being at the node following player 1 choosing B. Consequently, the belief of the second player that he/she is in the node induced by A is $1 - \mu$.
Let's first find conditions for μ such that player 2 chooses a over b and vice versa:

Player 2 chooses a over b if	Player 2 chooses b over a if
$\pi_2(a; \mu) = 1 - \mu > \mu = \pi_2(b; \mu)$	$\pi_2(a; \mu) = 1 - \mu < \mu = \pi_2(b; \mu)$
$\mu < \dfrac{1}{2}$	$\mu > \dfrac{1}{2}$

Consider the first case, where $\mu < \frac{1}{2}$. Then, by sequential rationality, player 1 chooses B. This implies that given the strategy profile (B, a), the ex-ante probability of reaching the information set of player 2 is 1 and ex-ante probability of reaching the node induced by B is also equal to one. Then, $\mu < \frac{1}{2} < \frac{1}{1}$, which means that such system of beliefs is inconsistent. Hence, (B, a) cannot be a WPBE.
Consider the second case, where $\mu > \frac{1}{2}$, i.e., player 2 chooses b over a. Then, sequential rationality implies that the first player chooses C, in which case, the information set of the second player is never reached. Therefore, any system of beliefs of player 2 is statistically consistent. Hence, the strategy profile (C, b) is a WPBE in pure strategies.

Exercise 3

3.1

First player has two information sets and his/her strategy space is $S_1 = \{A, B\} \times \{C, D\}$. Player 2 has one information set and his/her strategy space is $S_2 = \{a, b\}$.

3.2

(i) To find NE in pure strategies, consider the matrix payoff with best responses in pure strategies underlined in Table 2.

Therefore, there are three NE in pure strategies: $\{(A, C, a), (B, C, b), (B, D, b)\}$.

	a	b
A, C	($\underline{2}$, $\underline{-1}$)	(-1, -2)
A, D	(-10, -2)	(0, $\underline{-1}$)
B, C	(1, $\underline{1}$)	($\underline{1}$, $\underline{1}$)
B, D	(1, $\underline{1}$)	($\underline{1}$, $\underline{1}$)

Table 2: Entire game in strategic form

	a	b
C	($\underline{2}$, $\underline{-1}$)	(-1, -2)
D	(-10, -2)	($\underline{0}$, $\underline{-1}$)

Table 3: NEs in second proper subgame

(ii) There are two proper subgames: entire game and subgame that starts at the node where player 2 has to choose an action. Consider the second proper subgame tabulated in Table 3. There are two NE in the second proper subgame, $\{(C, a), (D, b)\}$. The first player's best response to (C, a) is to play A. Similarly, BR of the first player to (D, b) is to choose B. Hence, there are two SGPE, $\{(A, C, a), (B, D, b)\}$.

(iii) Let μ denote the probability player 1 assigns to being at the node induced by action a of player 2. Then,

$$\pi_1(C; \mu) = 2\mu - (1 - \mu) = 3\mu - 1$$
$$\pi_1(D; \mu) = -10\mu$$

Consider the case when $\mu > \frac{1}{13}$, i.e., player 1 chooses C over D in his/her second information set. Given this belief, sequential rationality implies that player 2 chooses a and player 1 plays A in the first information set. That is, given any belief system such that $\mu > \frac{1}{13}$, strategy (A, C, a) is sequentially rational. This strategy, in turn, implies that the probability of reaching the second information set of player 1 is equal to 1 and probability of reaching the node induced by a is also equal to 1. Then, the statistically consistent belief system would be $\mu = \frac{1}{1} = 1 > \frac{1}{13}$. Hence, a strategy (A, C, a) and a belief system $(1, 0)$ constitute a WPBE.

Consider another case, when $\mu < \frac{1}{13}$, i.e, when player 1 chooses D over C. In this case, player 2 wants to play b and player 1 prefers B in the beginning of the game. Given, the strategy (B, D, b), the second information set of player 1 is never reached. Hence, any belief system is consistent. Therefore, the strategy (B, D, b) and any belief system such that $\mu < \frac{1}{13}$ constitute another WPBE.

Thus, WPBE = SGPE.

3.3

The following game in extensive form (with imperfect recall) has the same strategic form representation as in Table 2.

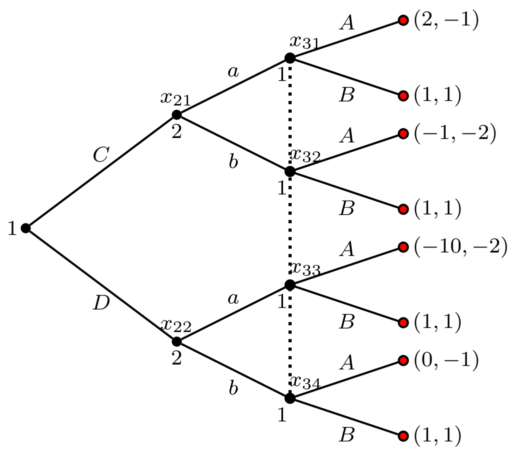


Figure 3: Alternative extensive form game

3.4

(i) Since the game in 3.3 has the same strategic form representation as the original game, the set of Nash equilibria in pure strategies is the same, $\{(A, C, a), (B, C, b), (B, D, b)\}$.

(ii) The game in 3.3 has only one proper subgame: the entire game. Therefore, subgame perfection has no bite and the set of SGPE is the same as the set of NE, $\{(A, C, a), (B, C, b), (B, D, b)\}$. Unlike the original game, where subgame perfection ruled out strategy (B, C, b).

(iii) Let μ denote the probabilities player 1 assigns to being at each of the nodes in his second information set: $\mu = \{\mu(x_{31}), \mu(x_{32}), \mu(x_{33}), 1 - \mu(x_{31}) - \mu(x_{32}) - \mu(x_{33})\}$. Then,

$$\pi_1(A; \mu) = 2\mu(x_{31}) - \mu(x_{32}) - 10\mu(x_{33})$$
$$\pi_1(B; \mu) = \mu(x_{31}) + \mu(x_{32}) + \mu(x_{33}) + 1 - \mu(x_{31}) - \mu(x_{32}) - \mu(x_{33}) = 1$$

For the first player to choose A over B, the following condition must hold: $2\mu(x_{31}) - \mu(x_{32}) - 10\mu(x_{33}) > 1$. Given any belief system such that this condition holds, sequential rationality will imply that player 2 will choose a at x_{21} and b at x_{22}. Then, in the beginning of the game, player 1 will play C. Notice, that in this game player 1 always arrives at his/her second information set. Then, a sequentially rational strategy (A, C, a), implies that $P^\gamma(x_{31}) = 1$ and $P^\gamma(x_{32}) = P^\gamma(x_{33}) = P^\gamma(x_{34}) = 0$. Thus, a belief system should be $\mu = (1, 0, 0, 0)$ to be statistically consistent. This also satisfies the initial condition for A to be preferred to B. Hence, a strategy profile (A, C, a) and a belief system $\mu = (1, 0, 0, 0)$ is a WPBE.

When $2\mu(x_{31}) - \mu(x_{32}) - 10\mu(x_{33}) < 1$, player 2 and 1 are indifferent between choosing a or b, and C or D, respectively. Furthermore, we know that WPBE $\subseteq$ NE[1]. Therefore, any of the following pure strategies $\{(B, C, b), (B, D, b)\}$ satisfies sequential rationality given a belief system such that $2\mu(x_{31}) - \mu(x_{32}) - 10\mu(x_{33}) < 1$. So, need to check consistency of beliefs for each of those pure strategies.

Given a strategy profile (B, C, b), $P^{(B,C,b)}(x_{31}) = P^{(B,C,b)}(x_{33}) = P^{(B,C,b)}(x_{34}) = 0, P^{(B,C,b)}(x_{32}) = 1$. This implies that $\mu(x_{31}) = \mu(x_{33}) = \mu(x_{34}) = 0, \mu(x_{32}) = 1$. Such belief system also satisfies the condition for (B, C, b) to be sequentially rational. Hence, (B, C, b) and the belief system $\mu = (0, 1, 0, 0)$ is a WPBE.

Given a strategy profile (B, D, b), $P^{(B,D,b)}(x_{31}) = P^{(B,D,b)}(x_{32}) = P^{(B,D,b)}(x_{33}) = 0, P^{(B,D,b)}(x_{34}) = 1$. This implies that $\mu(x_{31}) = \mu(x_{32}) = \mu(x_{33}) = 0, \mu(x_{34}) = 1$. Such belief system also satisfies the condition for (B, D, b) to be sequentially rational. Hence, (B, D, b) and the belief system $\mu = (0, 0, 0, 1)$ is a WPBE.

Therefore, the set of WPBE is the same as the set of NE, $\{(A, C, a), (B, C, b), (B, D, b)\}$, unlike in the original game, where the set of WPBE was $\{(A, C, a), (B, D, b)\}$.

Exercise 4

4.1

Notice that player 1 only has to provide one vector, e.g., y, whereas the second vector is automatically determined as $z = (4 - y_a, 4 - y_b)$. The set of all possible choices of the vector y could be written as $y = 4(\alpha, \beta) \implies z = 4(1 - \alpha, 1 - \beta), \quad \forall \alpha \in [0, 1], \forall \beta \in [0, 1]$. Therefore, the strategy space of player 1 is $S_1 = \{(4(\alpha, \beta), 4(1 - \alpha, 1 - \beta)), \forall \alpha \in [0, 1], \forall \beta \in [0, 1]\}$.
Unlike player 1, actions available to player 2 are discrete and the strategy space of player 2 could be written as $S_2 = \{Y, Z\}^{\#(S_1)}$, where Y stands for choosing vector y and Z stands for choosing proposed vector z, for all $(y, z) \in S_1$.

4.2

Notice that player 2 chooses Y if $\min(\alpha, \beta) > \min(1 - \alpha, 1 - \beta)$ and chooses Z if $\min(\alpha, \beta) < \min(1 - \alpha, 1 - \beta)$. Taking this into account, player 1 has to choose α and β to maximize his/her own utility. Consider the following cases

a) $\alpha < \beta$ Hence, $\min(\alpha, \beta) = \alpha$ and $\min(1 - \alpha, 1 - \beta) = 1 - \beta$.

- $\alpha < 1 - \beta$ In this case, player 2 chooses Z and gets $4(1 - \beta)$. Therefore, player 1 gets $16\alpha\beta$.
- $\alpha > 1 - \beta$ Here, player 2 chooses Y and gets 4α. Thus, player 1 gets $16(1 - \alpha)(1 - \beta)$.
- $\alpha = 1 - \beta$ In this case player 2 is indifferent between Y and Z as both of them yield utility of $4\alpha = 4(1 - \beta)$ and player 1 accordingly gets $16(1 - \alpha)(1 - \beta) = 16\alpha\beta$.

[1] At first I thought that any of $\{(B, C, a), (B, C, b), (B, D, a), (B, D, b)\}$ will satisfy sequential rationality and that check for consistency will help to rule out $\{(B, C, a), (B, D, a)\}$. But I could only rule out (B, C, a). And I am lost as to how either sequential rationality or consistency could help to rule out (B, D, a) without imposing the relation between WPBE and NE.

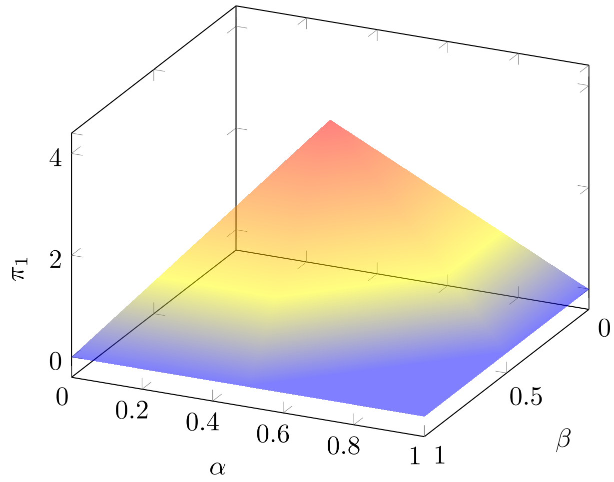
b) $\alpha > \beta$ Hence, $\min(\alpha, \beta) = \beta$ and $\min(1-\alpha, 1-\beta) = 1-\alpha$.

- $\beta < 1-\alpha \Longrightarrow \alpha < 1-\beta$. As seen above, his implies that $\pi_1 = 16\alpha\beta$.
- $\beta > 1-\alpha \Longrightarrow \alpha > 1-\beta \Longrightarrow \pi_1 = 16(1-\alpha)(1-\beta)$.
- $\beta = 1-\alpha \Longrightarrow \alpha = 1-\beta \Longrightarrow \pi_1 = 16(1-\alpha)(1-\beta) = 16\alpha\beta$.

c) $\alpha = \beta$ Hence, $\min(\alpha, \beta) = \alpha = \beta$ and $\min(1-\alpha, 1-\beta) = 1-\alpha = 1-\beta$.

- $\alpha < 1-\alpha$ In this case, player 2 chooses Z and gets $4(1-\alpha)$ and player 1 gets $16\alpha^2$.
- $\alpha > 1-\alpha$ Here, player 2 chooses Y and gets 4α and player 1 gets $16(1-\alpha)^2$.
- $\alpha = 1-\alpha \Longrightarrow \alpha = \frac{1}{2}$. Here, player 2 is indifferent between Y and Z and gets 2. Player 1 gets payoff of 4.

Graphically, these cases could be represented in the chart below:



From the chart, it is straightforward that the best response of player 1 is to set $\alpha = \beta = \frac{1}{2}$. Therefore, the outcome of the game is $y = (2, 2); z = (2, 2)$ with $\pi_1 = 4$ and $\pi_2 = 2$. However, I can't give an explicit expression for SGPE in case where the set of pure strategies is infinite.

4.3