Problem Set IV

Microeconomics II

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Exercise 1

There are n firms in a market for a homogeneous good. The demand function $F : \mathbb{R}_+ \to \mathbb{R}_+, p \mapsto F(p)$, which satisfies the Law of Demand, i.e., it could be inverted $P(Q) = p \Leftrightarrow F(p) = Q$.Every firm i displays a cost function $C_i : \mathbb{R}_+ \to \mathbb{R}_+$, which is assumed increasing and strictly convex.

a) Competitive market
In a competitive environment, firms are price takers and hence each firm's objective is to maximize $\pi_i(q_i) = \bar{p}q_i - C_i(q_i)$. Firms choose their output q_i^* such that $\bar{p} = C_i'(q_i^*)$.

b) Cournot oligopoly
In an oligopolistic market, firms internalize their effect on the market price. Therefore, now their objective is to maximize $\pi_i(q_i, q_{-i}) = P(\sum\limits_{i=1}^{n} q_i)q_i - C_i(q_i)$ with respect to q_i. Assuming interior solution, the FOC is

$$C_i'(\hat{q}_i) - P(\hat{Q}) = P'(\hat{Q})\hat{q}_i$$

Recall that the demand function satisfies the Law of Demand, i.e., $F'(p) < 0$. Then, we also know that $P'(Q) < 0$. Hence, the right-hand side is negative and at the optimum $C_i'(\hat{q}_i) - P(\hat{Q}) < 0$.

Notice that the (type of) mark-up function $M(q_i, q_{-i}) = C_i'(q_i) - P(\sum\limits_{i=1}^{n} q_i)$ is increasing in q_i:

$$\frac{\partial M(q_i, q_{-i})}{\partial q_i} = \underbrace{\frac{\partial^2 C_i(q_i)}{\partial q_i^2}}_{\substack{>0 \text{ since } C_i \\ \text{is strictly convex}}} - \underbrace{P'(Q)}_{<0} > 0$$

Recall that in a competitive market $M(q_i, q_{-i})$ is equal to zero at equilibrium, while in Cournot oligopoly it is negative. Hence, it must be that $\hat{q}_i < q_i^* \implies \hat{Q} < Q^*$.

Exercise 2

Recall that we have

- a demand function $F : \mathbb{R}_+ \to \mathbb{R}_+, p \mapsto F(p)$. By assumption, the demand function is affine, i.e., $F(p) = \tilde{a} - \tilde{b}p$. Since the demand function should also satisfy the Law of Demand, then we can invert the function and write $P(Q) = a - bQ \Leftrightarrow F(p) = Q$, where $a = \frac{\tilde{a}}{\tilde{b}}$, $b = \frac{1}{\tilde{b}}$, and $Q = \sum\limits_{i=1}^{n} q_i$.
- a linear cost function (same for each firm, by assumption of the problem) $C : \mathbb{R}_+ \to \mathbb{R}_+, q_i \mapsto C(q_i) = cq_i$.

A Cournot-Nash equilibrium is a vector $q^* = (q_t^*, ..., q_n^*)$ such that each firm i maximizes its profits:

$$\max_{q_i} \quad \pi_i(q_i) = P(Q)q_i - C_i(q_i) = (a - b\sum_{i=1}^{n} q_i)q_i - cq_i$$

$$\text{FOC:} \; -bq_i^* + (a - bQ^*) - c = 0$$

$$\text{Sum over all firms:} \; n(a - c) = bQ^* + nbQ^*$$

$$Q^* = \frac{n}{1+n}\frac{a-c}{b}$$

Since each firm has the same cost function and faces the same market demand, the optimal output supply is the same for all firms, i.e., $q_i = q_j, \forall i,j \in \{1,...,n\}$ and $q_i^* = \frac{1}{n}Q^* = \frac{1}{1+n}\frac{a-c}{b}, \forall i \in \{1,...,n\}$.
Then, $\pi_i(q_i) = \frac{1}{1+n}\frac{a-c}{b}(a - \cancel{b}\frac{n}{1+n}\frac{a-c}{\cancel{b}} - c) = \frac{1}{1+n}\frac{a-c}{b}\frac{a-c}{1+n} = \frac{1}{b}\left(\frac{a-c}{1+n}\right)^2$.
Next, observe that $\lim\limits_{n\to\infty}\frac{n}{1+n} = 1$. Hence, as $n\to\infty$, NE allocation approaches competitive equilibrium allocation, derivation of which is shown below:

$$\max_{q_i} \quad \pi_i(q_i) = \bar{p}q_i - cq_i$$
$$\text{FOC: } \bar{p} = c$$
$$\text{Market clearing condition: } P(\sum_{i=1}^n \hat{q}_i(\bar{p})) = a - b\sum_{i=1}^n \hat{q}_i(\bar{p}) = \bar{p}$$
$$\hat{Q}(\bar{p}) = \frac{a-\bar{p}}{b} = \frac{a-c}{b}$$
$$\text{and } \hat{q}_i(\bar{p}) = \frac{1}{n}\hat{Q}(\bar{p}) = \frac{1}{n}\frac{a-c}{b}$$

Exercise 3

$[\Rightarrow]$ Suppose we have a Bertrand-Nash equilibrium defined by p^*. Want to show that this implies $(\theta(p*) = c \wedge \#\{j \in N : p_j^* = \theta(p^*)\} \geq 2)$. This is equivalent to showing that $\theta(p*) \neq c \vee \#\{j \in N : p_j^* = \theta(p^*)\} < 2 \Longrightarrow p^*$ is not NE.

$\theta(p^*) < c$: then $\exists i \in N : p_i^* = \theta(p^*) < c$. Consequently, this firm's profit is given by $\pi_i(p^*) = \underbrace{(\theta(p^*) - c)}_{<0}\underbrace{\frac{F(\theta(p^*))}{\eta(p^*)}}_{>0} < 0$. Consider a deviation to $\hat{p}_i > \theta(p^*)$. Then, $\pi_i(\hat{p}_i, p_{-i}^*) = 0$, which implies that $\hat{p}_i$ constitutes a profitable deviation. But it contradicts the fact that p^* is NE.

$\theta(p^*) > c$: i.e., $\exists j \in N : p_j^* = \theta(p^*) > c$. Then, there exists firm j and $\varepsilon > 0$ small enough such that $\hat{p}_j = \theta(p^*) - \varepsilon \in (c, \theta(p^*))$ and

$$\pi_j(\hat{p}_j, p_{-j}^*) = (\theta(p^*) - \varepsilon - c)F(\theta(p^*) - \varepsilon) > (\theta(p^*) - c)\frac{F(\theta(p^*))}{\eta(p^*)} = \pi_j(p^*)$$

This again contradicts the assumption in the very beginning that p^* is a NE.

$\#\{j \in N : p_j^* = \theta(p^*)\} < 2$: i.e., there is one firm j, which charges $p_j^* = \theta(p^*)$. Define p_i^* as a second lowest price charged by some firm i. Then, $\exists\varepsilon > 0$ such that $\hat{p}_j = \theta(p^*)+\varepsilon \in (\theta(p^*), p_i^*)$ and

$$\pi_j(\hat{p}_j, p_{-j}^*) = (\theta(p^*) + \varepsilon - c)F(\theta(p^*) + \varepsilon) > (\theta(p^*) - c)F(\theta(p^*))$$

In other words, by increasing the price infinitesimally the firm j could still absorb the whole market demand at a slightly higher mark-up, i.e., a profitable deviation. $\lightning$

Thus, (p^* is Bertrand-Nash equilibrium) $\Longrightarrow (\theta(p*) = c \wedge \#\{j \in N : p_j^* = \theta(p^*)\} \geq 2)$.

$[\Leftarrow]$ Take firm j such that $p_j^* = \theta(p^*)$ and consider a unilateral deviation to $\tilde{p}_j > \theta(p^*)$. Then, $\#\{j \in N : p_j^* = \theta(p^*)\} \geq 1$ and by deviating to $\tilde{p}_j$, firm j reduces its profit to 0. Consider another deviation of this firm to $\tilde{p}_j < \theta(p^*) = c$. Then, $\pi_j(\tilde{p}_j, p_{-j}^*) = (\tilde{p}_j - c)F(\tilde{p}_j) < 0$, not a profitable deviation. Similar reasoning also helps to eliminate deviation of any other firm to a price lower than marginal cost. Hence, any price system deviation away from the one characterized by $(\theta(p*) = c \wedge \#\{j \in N : p_j^* = \theta(p^*)\} \geq 2)$ cannot constitute a profitable deviation. Therefore, any price system p^* that satisfies $(\theta(p*) = c \wedge \#\{j \in N : p_j^* = \theta(p^*)\} \geq 2)$ is a Bertrand-Nash equilibrium.

Exercise 4

Define the strategy profile space $S = \{(C,C),(C,F),(F,C),(F,F)\}$ and probability distribution over the strategy profiles $q = \{p_1,p_2,p_3,p_4\}$. Then, the expected payoffs of the players are

$$\pi_1(q) = p_1 2 + p_2 0 + p_3 0 + p_4 5 = 2p_1 + 5p_4$$
$$\pi_2(q) = p_1 5 + p_2 0 + p_3 0 + p_4 2 = 5p_1 + 2p_4$$

Player 1 gets recommendation to play C with probability p_1+p_2 and to play F with probability p_3+p_4. Then, given that first player gets recommendation to play C, probability of the second player to be advised to play C is $\frac{p_1}{p_1+p_2}$. Similarly, probability of the second player to be recommended with action F given the first player was recommended to play F is $\frac{p_4}{p_3+p_4}$.

i) According to the definition, q is a correlated equilibrium if $\forall i \in \{1,2\}$ and $\forall \eta_i : \Sigma_i \to \Sigma_i$

$$\sum_{\sigma\in\Sigma} q(\sigma)\pi_i(\sigma) \geq \sum_{\sigma\in\Sigma} q(\sigma)\pi_i(\eta_i(\sigma_i), \sigma_{-i})$$

Applied to this example, the above is equivalent to the following set of inequalities

$$\begin{cases} 2\frac{p_1}{p_1+p_2} + 0\frac{p_2}{p_1+p_2} \geq 0\frac{p_1}{p_1+p_2} + 5\frac{p_2}{p_1+p_2} \\ 0\frac{p_3}{p_3+p_4} + 5\frac{p_4}{p_3+p_4} \geq 2\frac{p_3}{p_3+p_4} + 0\frac{p_4}{p_3+p_4} \\ 5\frac{p_1}{p_1+p_3} + 0\frac{p_3}{p_1+p_3} \geq 0\frac{p_1}{p_1+p_3} + 2\frac{p_3}{p_1+p_3} \\ 0\frac{p_2}{p_2+p_4} + 2\frac{p_4}{p_2+p_4} \geq 5\frac{p_2}{p_2+p_4} + 0\frac{p_4}{p_2+p_4} \end{cases} \implies \begin{cases} p_1 \geq \frac{5}{2}p_2 \\ p_4 \geq \frac{2}{5}p_3 \\ p_1 \geq \frac{2}{5}p_3 \\ p_4 \geq \frac{5}{2}p_2 \end{cases} \tag{1}$$

We are also asked to find a correlated equilibrium such that it maximizes the expected payoff of player 1

$$\max_q 2p_1 + 5p_4$$

It is clear from above that the maximum possible payoff for player 1 is attained when $p_4 = 1$. Moreover, $q = \{0,0,0,1\}$ satisfies the set of inequalities in (1). Hence, $q = \{0,0,0,1\}$ is libertarian equilibrium for player 1.

ii) To find the libertarian equilibrium for player 2, have to find q such that satisfies (1) and

$$\max_q 5p_1 + 2p_4$$

Again, if $p_1 = 1$ results in a highest possible payoff for the second player and

$$\begin{cases} 1 \geq 0 \\ 0 \geq 0 \\ 1 \geq 0 \\ 0 \geq 0 \end{cases}$$

Therefore, $q = \{1,0,0,0\}$ is a libertarian equilibrium for player 2.

iii) Now q should satisfy (1) and

$$\max_q 2p_1 + 5p_4 + 5p_1 + 2p_4 = \max_q 7(p_1+p_4)$$

It is again possible to see that the maximum is attained when $p_1 + p_4 = 1 \implies p_4 = 1 - p_1$. Let's check if (1) is satisfied:

$$\begin{cases} p_1 \geq 0 \\ 1 - p_1 \geq 0 \\ p_1 \geq 0 \\ 1 - p_1 \geq 0 \end{cases}$$

Therefore, all distributions over strategy profiles $q = \{p_1, 0, 0, 1-p_1\}$, where $p_1 \in [0,1]$, define a set of utilitarian equilibria.

iv) I'm not sure if I understood the concept of egalitarian equilibrium, but after some googling and staring at the definition in the problem set, I think it is the probability distribution that satisfies (1) and

$$\max_q \min\{2p_1 + 5p_4, 5p_1 + 2p_4\}$$

Graphically, the maximization problem looks like in Figure 1

Hence, the solution is found at the point where the two expected payoffs are equal:

$$2p_1 + 5p_4 = 5p_1 + 2p_4 \implies p_1 = p_4$$

For these to be probability measures we also need them to add up to 1. So, $p_4 = 1 - p_1 \implies 2p_1 = 1 \implies p_1 = p_4 = \frac{1}{2}$. Notice that $q = (\frac{1}{2}, 0, 0, \frac{1}{2})$ does indeed satisfy conditions for correlated equilibria given in (1), and hence is an egalitarian equilibrium.

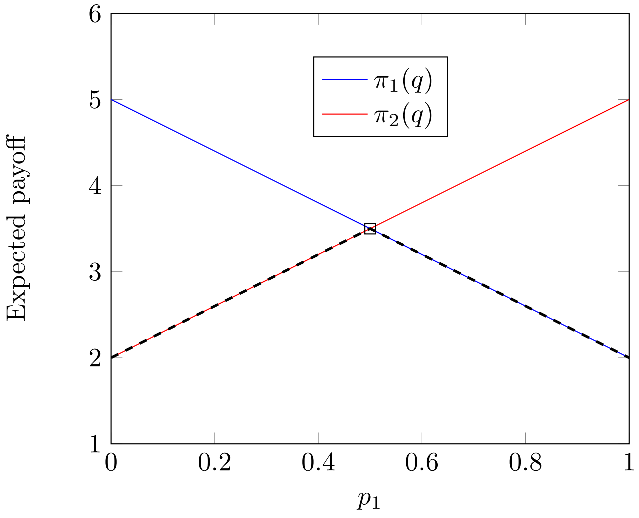

Figure 1: Egalitarian equilibrium

v) Figure 2 depicts the set of two Nash equilibria in pure strategies (which are also the respective libertarian equilibria for player 1 LE_1 and for player 2 LE_2), Nash equilibrium in mixed strategies (ME), set of utilitarian equilibria (UE) and egalitarian equilibrium (EE).

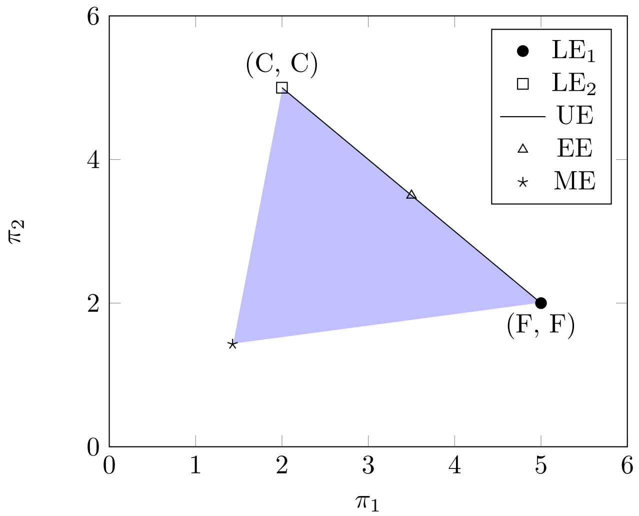

Figure 2: The set of attainable payoffs

Recall that any point within the shaded area could be obtained as a correlated equilibrium with public information about the outcome of a stochastic device. Then, notice that the line connecting two NE in pure strategies also defines the set of Pareto efficient outcomes under public signal: it is not possible to achieve a higher payoff for one player without making the other worse off. Thus, any utilitarian equilibrium, including the two libertarian and egalitarian equilibria, is a Pareto efficient outcome.