Problem Set VI

Microeconomics II

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Exercise 1

The set of NE in pure strategies is $\{(M,l),(L,r)\}$.

1. $\gamma_1 = (0,1,0), \gamma_2 = (1,0)$ Define $\gamma_{1k} = (\rho\varepsilon_{1k}, 1-(1+\rho)\varepsilon_{1k}, \varepsilon_{1k}), \gamma_{2k} = (1-\varepsilon_{2k}, \varepsilon_{2k})$ such that $\lim\limits_{k\to\infty} \varepsilon_{1k} = \lim\limits_{k\to\infty} \varepsilon_{2k} = 0$, for some $\rho > 0$.

   Then, induced belief system is $\mu_{2k} = (\frac{1-(1+\rho)\varepsilon_{1k}}{1-\rho\varepsilon_{1k}}, \frac{\varepsilon_{1k}}{1-\rho\varepsilon_{1k}}) \to (1,0)$ as $k \to \infty$.

   Given the belief system $\mu_2 = (1,0)$, player 2 prefers to choose l and player 1's best response then is to choose M. Hence, $(\gamma_1,\gamma_2) = ((0,1,0),(1,0))$ and $\mu_2 = (1,0)$ constitute a sequential equilibrium.

2. $\gamma_1 = (1,0,0)$ and $\gamma_2 = (0,1)$. Define $\gamma_{1k} = (1-(1+\rho)\varepsilon_{1k}, \rho\varepsilon_{1k}, \varepsilon_{1k}), \gamma_{2k} = (\varepsilon_{2k}, 1-\varepsilon_{2k})$, where $\lim\limits_{k\to\infty} \varepsilon_{1k} = \lim\limits_{k\to\infty} \varepsilon_{2k} = 0$, for some $\rho > 0$.

   Induced belief system is $\mu_{2k} = (\frac{\rho\varepsilon_{1k}}{(1+\rho)\varepsilon_{1k}}, \frac{\varepsilon_{1k}}{(1+\rho)\varepsilon_{1k}}) = (\frac{\rho}{1+\rho}, \frac{1}{1+\rho}) \to (\frac{\rho}{1+\rho}, \frac{1}{1+\rho})$ as $k \to \infty$.

   $$\pi_2(l|\mu_2, H_2) = \frac{\rho}{1+\rho} \leq \pi_2(r|\mu_2, H_2) = \frac{1}{1+\rho} \iff \rho \leq 1$$

   Then, player 1's best response is to play L. Therefore, $(\gamma_1,\gamma_2) = ((1,0,0),(0,1))$ and any belief system $\mu_2 = (\frac{\rho}{1+\rho}, \frac{1}{1+\rho})$ such that $\rho \in (0,1]$ is a sequential equilibrium given.

3. $\gamma_1 = (1,0,0)$ and $\gamma_2 = (q, 1-q), \forall q \in [0,1]$. Define $\gamma_{1k} = (1-2\varepsilon_{1k}, \varepsilon_{1k}, \varepsilon_{1k}), \gamma_{2k} = (q, 1-q)$, where $\lim\limits_{k\to\infty} \varepsilon_{1k} = 0$.

   Induced belief system is $\mu_{2k} = (\frac{1}{2}, \frac{1}{2}) \to (\frac{1}{2}, \frac{1}{2})$ as $k \to \infty$.

   $$\pi_2(l|\mu_2, H_2) = \frac{1}{2} = \pi_2(r|\mu_2, H_2)$$

   Hence, $\gamma_2 = (q, 1-q)$ is optimal for player 2 in his information set given the belief system $\mu_2 = (\frac{1}{2}, \frac{1}{2})$. Let's compare payoff of player 1 following strategy γ_1 and deviating to strategy $\hat{\gamma_1} = (1-\nu, \nu, 0)$ for some $\nu > 0$:

   $$\pi_1(\gamma_1, \gamma_2|\mu_2) = 2 \geq 2 - \nu(1-2q) = \pi_1(\hat{\gamma_1}, \gamma_2|\mu_2) \iff q \leq \frac{1}{2}$$

   Hence, all strategy profiles $(\gamma_1 = (1,0,0), \gamma_2 = (q, 1-q)), \forall q \leq \frac{1}{2}$ given the belief system $\mu_2 = (\frac{1}{2}, \frac{1}{2})$ constitute sequential equilibria.

Exercise 2

Let's first find all NE in pure strategies. From Table 1, we can see that the set of NE in pure strategies is $\{(M,r,a),(R,l,a)\}$.

(M,r,a) Define completely mixed strategies of players $\gamma_{1k} = (\rho\varepsilon_{1k}, 1-(1+\rho)\varepsilon_{1k}, \varepsilon_{1k}), \gamma_{2k} = (\varepsilon_{2k}, 1-\varepsilon_{2k}), \gamma_{3k} = (1-\varepsilon_{3k}, \varepsilon_{3k})$ for some $\rho > 0$ such that $\varepsilon_{sk} \underset{k\to\infty}{\longrightarrow} 0, \forall s \in \{1,2,3\}$. Thus, $\lim\limits_{k\to\infty} \gamma_{1k} = (0,1,0), \lim\limits_{k\to\infty} \gamma_{2k} = (0,1), \lim\limits_{k\to\infty} \gamma_{3k} = (1,0)$.

These strategies induce the following beliefs:

$$\mu_{2k} = (\frac{\rho\varepsilon_{1k}}{1-\varepsilon_{1k}}, \frac{1-(1+\rho)\varepsilon_{1k}}{1-\varepsilon_{1k}}) \underset{k\to\infty}{\longrightarrow} (0,1) = \mu_2$$
$$\mu_{3k} = (\varepsilon_{2k}, 1-\varepsilon_{2k}) \underset{k\to\infty}{\longrightarrow} (0,1) = \mu_3$$

Player 3 plays a

1 \ 2	l	r
L	$(0, \underline{1}, \underline{0})$	$(0, 0, \underline{0})$
M	$(0, 1, \underline{1})$	$(\underline{2}, \underline{2}, \underline{1})$
R	$(\underline{1}, \underline{0}, \underline{0})$	$(1, \underline{0}, \underline{0})$

Player 3 plays b

1 \ 2	l	r
L	$(0, \underline{1}, \underline{0})$	$(0, 0, \underline{0})$
M	$(0, \underline{1}, 0)$	$(0, 0, 0)$
R	$(\underline{1}, 0, \underline{0})$	$(\underline{1}, 0, \underline{0})$

Table 1: Strategic form of the game

Given this belief system

$$\pi_3(a|\mu_2, \mu_3, H_3) = 1 > 0 = \pi_3(b|\mu_2, \mu_3, H_3) \Rightarrow \gamma_3^* = (1, 0)$$
$$\pi_2((l, a)|\mu_2, \mu_3, H_2) = 1 < 2 = \pi_2((r, a)|\mu_2, \mu_3, H_2) \Rightarrow \gamma_2^* = (0, 1)$$
$$\begin{cases} \pi_1((L, r, a)|\mu_2, \mu_3, H_1) = 0 < 2 = \pi_1((M, r, a)|\mu_2, \mu_3, H_1) \\ \pi_1((M, r, a)|\mu_2, \mu_3, H_1) = 2 > 1 = \pi_1((R, r, a)|\mu_2, \mu_3, H_1) \end{cases} \Rightarrow \gamma_1^* = (0, 1, 0)$$

Hence, a strategy profile $\gamma^* = (\gamma_1^*, \gamma_2^*, \gamma_3^*)$ and a belief system $\mu^* = (\mu_2, \mu_3)$ is a sequential equilibrium.

(R, l, a) Define completely mixed strategies of players $\gamma_{1k} = (\rho\varepsilon_{1k}, \varepsilon_{1k}, 1 - (1 + \rho)\varepsilon_{1k}), \gamma_{2k} = (1 - \varepsilon_{2k}, \varepsilon_{2k}), \gamma_{3k} = (1 - \varepsilon_{3k}, \varepsilon_{3k})$ for some $\rho > 0$ such that $\varepsilon_{sk} \underset{k\to\infty}{\longrightarrow} 0, \forall s \in \{1, 2, 3\}$. Thus, $\lim\limits_{k\to\infty} \gamma_{1k} = (0, 0, 1), \lim\limits_{k\to\infty} \gamma_{2k} = (1, 0), \lim\limits_{k\to\infty} \gamma_{3k} = (1, 0)$.

These strategies induce following beliefs:

$$\mu_{2k} = (\frac{\rho}{1+\rho}, \frac{1}{1+\rho}) \underset{k\to\infty}{\longrightarrow} (\frac{\rho}{1+\rho}, \frac{1}{1+\rho}) = \mu_2$$
$$\mu_{3k} = (1 - \varepsilon_{2k}, \varepsilon_{2k}) \underset{k\to\infty}{\longrightarrow} (1, 0) = \mu_3$$

Given this belief system

$$\pi_3(a|\mu_2, \mu_3, H_3) = 1 > 0 = \pi_3(b|\mu_2, \mu_3, H_3) \Rightarrow \gamma_3^* = (1, 0)$$
$$\pi_2((l, a)|\mu_2, \mu_3, H_2) = 1 > \frac{2}{1+\rho} = \pi_2((r, a)|\mu_2, \mu_3, H_2) \Rightarrow \gamma_2^* = (1, 0) \iff \rho > 1$$
$$\pi_1((L, l, a)|\mu_2, \mu_3, H_1) = \pi_1((M, l, a)|\mu_2, \mu_3, H_1) = 0 < 1 = \pi_1((R, l, a)|\mu_2, \mu_3, H_1) \Rightarrow \gamma_1^* = (0, 0, 1) \iff \rho > 1$$

Hence, a strategy profile $\gamma^* = (\gamma_1^*, \gamma_2^*, \gamma_3^*)$ and a belief system $\mu^* = (\mu_2, \mu_3)$ such that $\rho > 1$ is another sequential equilibrium.

Exercise 3

The game could be formalized as a Bayesian game $BG = \{N, T, A, P, \{\pi_i\}_{i\in N}\}$, where

$N = \{1, 2\}$;

$T = T_1 \times T_2 = \{b_1, b_2\} \times \{t_2\}$, where b_1 refers to the first box and b_2 refers to the second box (notice that there's only one type of player 2);

$A = \{A, B\} \times \{C, D\}$;

$P : T \to [0, 1], \begin{matrix} (b_1, t_2) \mapsto \frac{1}{2} \\ (b_2, t_2) \mapsto \frac{1}{2} \end{matrix}$;

$\pi_i : T \times A \to \mathbb{R}$ specified by the table in the problem.

Suppose player 2 thinks that player 1's strategy is $\gamma_1(b_1) = (p_1, 1 - p_1)$ and $\gamma_1(b_2) = (p_2, 1 - p_2)$. Then,

$$\mathbb{E}\pi_2(C) = \frac{1}{2}[\pi_2((b_1, t_2), \gamma_1(b_1), C) + \pi_2((b_2, t_2), \gamma_1(b_2), C)] = \frac{1}{2}(p_1 + 2p_2 + 1 - p_2) = \frac{1}{2}(1 + p_1 + p_2)$$
$$\mathbb{E}\pi_2(D) = \frac{1}{2}[\pi_2((b_1, t_2), \gamma_1(b_1), D) + \pi_2((b_2, t_2), \gamma_1(b_2), D)] = \frac{1}{2}(4p_2 + 3(1 - p_2)) = \frac{1}{2}(3 + p_2)$$
$$\mathbb{E}\pi_2(D) - \mathbb{E}\pi_2(C) = \frac{1}{2}(3 + p_2 - 1 - p_1 - p_2) = \frac{1}{2}(2 - p_1) > 0, \forall p_1 \in [0, 1], \forall p_2 \in [0, 1]$$

Hence, player 2 maximizes expected payoff by choosing D and $\gamma_2^* = (0,1)$.
Then, player 1 makes decision based on

$$\frac{1}{2}[\pi_1((b_1,t_2),A,\gamma_2^*) + \pi_1((b_2,t_2),A,\gamma_2^*)] = 0 < \frac{1}{2} = \frac{1}{2}[\pi_1((b_1,t_2),B,\gamma_2^*) + \pi_1((b_2,t_2),B,\gamma_2^*)] \Rightarrow \gamma_1^* = (0,1)$$

Therefore, a strategy profile $\gamma^* = \{\gamma_1^*, \gamma_2^*\}$ is a BNE in pure strategies.