1 Introduction

We work on a hyper-cubic lattice ($D = 3, 4$ dimensions) with $N = L^3$ sites. We study the Euclidean scalar model defined by the following action

$$S = \sum_{n=1}^{N} \left[\frac{1}{2} B_4 \left(\Box \phi_n \right)^2 + \frac{1}{2} B_2 \sum_{\mu=1}^{D} \left(\partial_\mu \phi_n \right) \left(\partial_\mu \phi_n \right) + \frac{1}{2} r \phi_n^2 + \frac{1}{4} \lambda \phi_n^4 \right] \tag{1}$$

The field index n is an integer valued array with D components, and the discretized derivatives are defined as follows

$$\partial_\mu \phi_n \equiv \phi_{n+\mu} - \phi_n \tag{2}$$

$$\Box \phi_n \equiv \sum_\mu \left(\phi_{n+\mu} + \phi_{n-\mu} - 2\phi_n \right) \tag{3}$$

It is understood that all introduced variables and operators are dimensionless versions of their continuum counterparts, which are obtained by multiplying them by appropriate powers of the lattice spacing a, which does not appear explicitely. The B_2 parameter is dimensionless, while the dimensionful counterpart of the B_4 parameter has the dimensions of a^2, or an inverse square mass.

We will also consider the interaction term with an external current:

$$S_J = \sum_{n=1}^{N} J_n \phi_n \tag{4}$$

where the current is in general not uniform. The Euclidean path integral involves the partition function

$$Z = \int [d\phi] \exp\left(-S[\phi] \right) \tag{5}$$

where the measure is defined by

$$[d\phi] \equiv \prod_{n=1}^{N} d\phi_n \tag{6}$$

The ground state expectation values of operators are then defined as

$$\langle A(\phi) \rangle \equiv \frac{1}{Z} \int [d\phi] A(\phi) \exp\left(-S[\phi] \right) \tag{7}$$

We are interested in studying the case

$$B_4 > 0, B_2 = -1 \tag{8}$$

which is supposed to give rise to a non-uniform field configuration for the ground state. In particular, we are interested in the so-called kinetic condensate

$$K(B_4, B_2, \lambda) \equiv \frac{1}{N} \sum_{n,\mu} \langle (\partial_\mu \phi_n)(\partial_\mu \phi_n) \rangle \tag{9}$$

or, rather, the quantity obtained by subtracting vacuum fluctuations of the field:

$$K_S(B_4, B_2, \lambda) \equiv K(B_4, B_2, \lambda) - K(0, 1, 0) \tag{10}$$

2 Vacuum fluctuations

In order to evaluate the vacuum field fluctuations of the field on the lattice, we consider the following partition function

$$Z(\alpha) \equiv \int [d\phi] \exp(-\alpha S_0[\phi]) \tag{11}$$

S_0 being the standard kinetic term of the action, implying that

$$K(0, 1, 0) = \frac{2}{N} \langle S_0 \rangle \tag{12}$$

where the expectation value is calculated with $S = S_0$ (i.e. $B_4 = 0, B_2 = 1, \lambda = 0$) at $\alpha = 1$. We have that

$$\langle S_0 \rangle = -\frac{\partial}{\partial \alpha} \ln Z(\alpha) \Big|_{\alpha=1} \tag{13}$$

Since S_0 is quadratic in the fields, we can define

$$\chi_n^2 \equiv \alpha \phi_n^2 \tag{14}$$

which gives

$$[d\phi] = \alpha^{-N/2} [d\chi], Z(\alpha) = \alpha^{-N/2} Z(1) \tag{15}$$

from which it follows that

$$K(0, 1, 0) = 1 \tag{16}$$

$$K_S(B_4, B_2, \lambda) = K(B_4, B_2, \lambda) - 1 \tag{17}$$

3 Simulation

We are employing Monte Carlo to evaluate path integral averages as averages over a set of configurations generated with the $\exp(-S)$ distribution. We use the Hybrid Monte Carlo (HMC) approach.

As a check for the algoritghm, we use the Creutz criterion, i.e. we check that for the variations of the HMC Hamiltonian the following holds

$$\langle \exp(-\delta H) \rangle = 1 \tag{18}$$

4 Analytic considerations

Let us consider the following action (in a continuum description):

$$S[\phi] = \int d^d x \left(\frac{1}{2} \phi\, \Omega(-\Box)\, \phi + \frac{u}{4!} \phi^4 \right) \tag{19}$$

with

$$\Omega(-\Box) = \Box + \frac{\Box^2}{2M^2} \tag{20}$$

in particular

$$\Omega(p^2) = -p^2 + \frac{(p^2)^2}{2M^2} \tag{21}$$

Insert the variational ansatz $\phi(x) = A \sin(p_\mu x^\mu + \alpha)$ so that $\Box\phi(x) = -p^2 \phi(x)$ and $\Omega(-\Box)\phi = \Omega(p^2)\phi(x)$, so that

$$S[\phi] = \int d^d x \left(\frac{A^2}{2} \Omega(p^2) \sin^2(p_\mu x^\mu + \alpha) + \frac{u}{4!} A^4 \sin^4(p_\mu x^\mu + \alpha) \right) \tag{22}$$

If $p_\mu = p\delta_{\mu 1}$ we have

$$S[\phi] = \mathcal{V} \left(\frac{A^2}{2} \Omega(p^2) \int_{-\infty}^{+\infty} dx^1 \sin^2(px^1 + \alpha) + \frac{u}{4!} \int_{-\infty}^{+\infty} dx^1 \sin^4(px^1 + \alpha) \right) \tag{23}$$

where $\mathcal{V} = \int dx^2 ... dx^d$. Therefore we can write

$$\frac{S[\phi]}{\mathcal{V}} = \frac{1}{4} A^2 \Omega(p^2) + \frac{3}{8} \frac{u}{4!} A^4 \tag{24}$$

We look for a minimum as a function of p and obtain

$$\frac{d}{dp^2} \left[-p^2 + \frac{(p^2)^2}{2M^2} \right] = -1 + \frac{p^2}{M^2} = 0 \tag{25}$$

so that $p_\mu p^\mu = M^2$ therefore $p_\mu = n_\mu M$ with $n_\mu n^\mu = 1$. Minimizing over A leads instead to $A = \frac{2}{\sqrt{u}} M$ so that the variational solution is

$$\phi(x) = \frac{2}{\sqrt{u}} M \sin(M n_\mu x^\mu + \alpha) \tag{26}$$

with $n \in S^{d-1}$ and $\alpha \in [0, 2\pi)$.