一元三次方程的求根公式及其证明

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已知:关于x的三次方程$x^3+bx^2+cx+d=0$,求此方程的精确解.
解:令$x=t-\frac{b}{3}$,则有

$$\begin{aligned}
&x^3+bx^2+cx+d\\
=&(t-\frac{b}{3})^3+b(t-\frac{b}{3})^2+c(t-\frac{b}{3})+d\\
=&t^3-bt^2+\frac{1}{3}tb^2-\frac{b^3}{27}+bt^2-\frac{2}{3}b^2t+\frac{b^3}{9}+ct-\frac{bc}{3}+d\\
=&t^3+(c-\frac{b^2}{3})t+(\frac{2}{27}b^3-\frac{bc}{3}+d)\\
=&0
\end{aligned}$$

令$p=c-\frac{b^2}{3},q=\frac{2}{27}b^3-\frac{bc}{3}+d$,则原方程化为

$$t^3+pt+q=0$$

下面求解这个方程.
已知公式:

$$(t+y+z)(t+\omega y+\omega^2 z)(t+\omega^2 y+\omega z)=t^3+y^3+z^3-3tyz$$

其中$\omega=\frac{-1+\sqrt{3}\mathrm{i}}{2}$. 令

$$\begin{cases}-3yz=p & (1)\\ y^3+z^3=q & (2)\end{cases}$$

则有

$$t^3+pt+q=(t+y+z)(t+\omega y+\omega^2 z)(t+\omega^2 y+\omega z)=0$$

则

$$\begin{cases} t_1 = -y - z \\ t_2 = -\omega y - \omega^2 z \\ t_3 = -\omega^2 y - \omega z \end{cases}$$

由(1)得

$$z = -\frac{p}{3y}$$

代入(2),得

$$y^3 + (-\frac{p}{3y})^3 = q$$

$$y^6 - qy^3 - \frac{p^3}{27} = 0$$

得

$$y = \sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}}$$

代入(1),得

$$z = -\frac{p}{3\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}}} = \sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}}$$

所以,

$$\begin{cases} t_1 = -\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} \\ t_2 = -\omega\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \omega^2\sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} \\ t_3 = -\omega^2\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \omega\sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} \end{cases}$$

即

$$\begin{cases} x_1 = -\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \frac{b}{3} \\ x_2 = -\omega\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \omega^2\sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \frac{b}{3} \\ x_3 = -\omega^2\sqrt[3]{\frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \omega\sqrt[3]{\frac{q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}} - \frac{b}{3} \end{cases}$$

若用b, c, d代替p, q,则会得到以下复杂代数式:

$$\begin{cases} x_1 = -\sqrt[3]{\frac{(y^3+z^3)\pm\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \sqrt[3]{\frac{(y^3+z^3)\mp\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \frac{b}{3} \\ x_2 = -\omega\sqrt[3]{\frac{(y^3+z^3)\pm\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \omega^2\sqrt[3]{\frac{(y^3+z^3)\mp\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \frac{b}{3} \\ x_3 = -\omega^2\sqrt[3]{\frac{(y^3+z^3)\pm\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \omega\sqrt[3]{\frac{(y^3+z^3)\mp\sqrt{(y^3+z^3)^2+\frac{4(-3yz)^3}{27}}}{2}} - \frac{b}{3} \end{cases}$$