题目：对于正实数$x_1,\ x_2,\ \cdots,\ x_n\ (n\geq 2)$，记

$$S_k := 1+\sum_{i=1}^{k}\frac{x_i}{i},\ T_k := \sum_{i=1}^{k}\frac{x_i}{i+1},\ k=1,\ 2,\ \cdots,\ n.$$

求证：

$$\prod_{i=1}^{n}(1+x_i)\leq \sum_{k=1}^{n}\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k.$$

解：注意到恒等式

$$\begin{aligned}
\prod_{i=1}^{n}(1+x_i) &= \prod_{i=1}^{n-1}(1+x_i)+x_n\prod_{i=1}^{n-1}(1+x_i)\\
&= \prod_{i=1}^{n-2}(1+x_i)+x_{n-1}\prod_{i=1}^{n-2}(1+x_i)+x_n\prod_{i=1}^{n-1}(1+x_i)\\
&= \prod_{i=1}^{n-3}(1+x_i)+x_{n-2}\prod_{i=1}^{n-3}(1+x_i)+x_{n-1}\prod_{i=1}^{n-2}(1+x_i)+x_n\prod_{i=1}^{n-1}(1+x_i)\\
&= \cdots\\
&= (1+x_1)+\sum_{k=2}^{n}x_k\prod_{i=1}^{k-1}(1+x_i).
\end{aligned}$$

而对于$2\leq k\leq n$，

$$\begin{aligned}
\sum_{k=2}^{n}x_k\prod_{i=1}^{k-1}(1+x_i) &= (1\cdot 2\cdot\ \cdots\ \cdot k)^2\cdot(k+1)\cdot\left(\prod_{i=1}^{k-1}\frac{1+x_i}{i(i+1)}\right)\cdot\frac{x_k}{k(k+1)}\\
&\leq \frac{((k+1)!)^2}{k+1}\left(\frac{1}{k}\left(\frac{x_k}{k(k+1)}+\sum_{i=1}^{k-1}\frac{1+x_i}{i(i+1)}\right)\right)^k\\
&= \frac{((k+1)!)^2}{(k+1)k^k}\left[x_k\left(\frac{1}{k}-\frac{1}{k+1}\right)+\sum_{i=1}^{k-1}(1+x_i)\left(\frac{1}{i}-\frac{1}{i+1}\right)\right]^k\\
&= \frac{((k+1)!)^2}{(k+1)k^k}\left[\left(1-\frac{1}{k}\right)+\sum_{i=1}^{n}\frac{x_i}{i}-\sum_{i=1}^{n}\frac{x_i}{i+1}\right]^k\\
&= \frac{((k+1)!)^2}{(k+1)k^{2k}}\left[(k-1)+k(S_k-1)-kT_k\right]^k\\
&= \frac{((k+1)!)^2}{(k+1)k^{2k}}\left[k(S_k-T_k)-1\right]^k\leq\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k.
\end{aligned}$$

并且注意到

$$1+x_1=\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k\Bigg|_{k=1}.$$

故

$$\begin{aligned}\prod_{i=1}^{n}(1+x_i)&=(1+x_1)+\sum_{k=2}^{n}x_k\prod_{i=1}^{k-1}(1+x_i)\\&\le\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k\Bigg|_{k=1}+\sum_{k=2}^{n}\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k\\&=\sum_{k=1}^{n}\frac{((k+1)!)^2}{(k+1)k^{2k}}\left(k(S_k-T_k)-\frac{2^k-1}{2^k}\right)^k.\end{aligned}$$

证毕!