物理笔记

1 Nabla算符

1. $\nabla(fg) = (\nabla f)g + (\nabla g)f$.

$$\nabla(fg) = \partial_i(fg) = f\partial_i g + g\partial_i f = (\nabla f)g + (\nabla g)f.$$

2. $\nabla f\boldsymbol{A} = (\nabla \cdot \boldsymbol{A})f + \boldsymbol{A} \cdot (\nabla f)$.

$$\begin{aligned}(\nabla \cdot \boldsymbol{A})f + \boldsymbol{A} \cdot (\nabla f) &= f\delta_{ij}\partial_j A_i + \delta_{ij}A_i\partial_j f = \delta_{ij}\left(f\partial_j A_i + A_i\partial_j f\right) \\ &= \delta_{ij}\partial_j(fA_i) = \nabla(f\boldsymbol{A}).\end{aligned}$$

3. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \cdot \boldsymbol{B})$.

$$\begin{aligned}\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) &= \varepsilon_{ijk}\partial_i\left(A_j B_k\right) = \varepsilon_{ijk}(A_j\partial_i B_k + B_k\partial_i A_j) \\ &= \varepsilon_{ijk}B_i\partial_j A_k - \varepsilon_{ijk}A_i\partial_j B_k = \boldsymbol{B} \cdot (\nabla \cdot \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \cdot \boldsymbol{B}).\end{aligned}$$

4. $\nabla \times (f\boldsymbol{A}) = (\nabla \times \boldsymbol{A})f - \boldsymbol{A} \times (\nabla f)$.

$$\begin{aligned}\nabla \times (f\boldsymbol{A}) &= \varepsilon_{ijk}\boldsymbol{e}_i\partial_j(fA_k) = \varepsilon_{ijk}\boldsymbol{e}_i(A_k\partial_j f + f\partial_j A_k) \\ &= f\varepsilon_{ijk}\boldsymbol{e}_i\partial_j A_k - \varepsilon_{ijk}\boldsymbol{e}_i A_j\partial_k f = (\nabla \times \boldsymbol{A})f - \boldsymbol{A} \times (\nabla f).\end{aligned}$$

5. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$.

$$\begin{aligned}\nabla \cdot (\nabla \times \boldsymbol{A}) &= \delta_{ij}\partial_i(\nabla \times \boldsymbol{A})_j = \delta_{ij}\partial_i(\varepsilon_{jkl}\partial_k A_l) = \delta_{ij}\varepsilon_{jkl}\partial_{ik}A_l \\ &= \varepsilon_{jkl}(\delta_{ij}\partial_{ik})A_l = \varepsilon_{jkl}\partial_{jk}A_l = \frac{1}{2}(\varepsilon_{jkl}\partial_{jk}A_l + \varepsilon_{kjl}\partial_{kj}A_l) \\ &= \frac{1}{2}(\varepsilon_{jkl} + \varepsilon_{kjl})\partial_{jk}A_l = 0.\end{aligned}$$

6. $\nabla \times (\nabla f) = \boldsymbol{0}$.

$$\nabla \times (\nabla f) = \varepsilon_{ijk}\boldsymbol{e}_i\partial_j(\nabla f)_k = \varepsilon_{ijk}\boldsymbol{e}_i\partial_{jk}f = \frac{1}{2}\boldsymbol{e}_i(\varepsilon_{ijk} + \varepsilon_{ikj})\partial_{jk}f = \boldsymbol{0}.$$

7. $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \cdot \boldsymbol{A}$.

$$\begin{aligned}\nabla \times (\nabla \times \boldsymbol{A}) &= \varepsilon_{ijk}\boldsymbol{e}_i\partial_j(\nabla \times \boldsymbol{A})_k = \varepsilon_{ijk}\varepsilon_{klm}\boldsymbol{e}_i\partial_{jl}A_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\boldsymbol{e}_i\partial_{jl}A_m \\ &= (\delta_{il}\boldsymbol{e}_i)\partial_{jl}(\delta_{jm}A_m) - (\delta_{im}\boldsymbol{e}_i)(\delta_{jl}\partial_j\partial_l)A_m \\ &= \boldsymbol{e}_l\partial_l(\partial_j A_j) - \boldsymbol{e}_m(\delta_{jl}\partial_j\partial_l)A_m = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \cdot \boldsymbol{A}.\end{aligned}$$

2　曲线坐标系的矢量微分

2.1　球坐标系

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\hat{\boldsymbol{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\theta\hat{\boldsymbol{\varphi}}, \quad \mathrm{d}\tau = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi.$$

$$\nabla t = \partial_r t\hat{\boldsymbol{r}} + \frac{1}{r}\partial_\theta t\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\partial_\varphi t\hat{\boldsymbol{\varphi}};$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\partial_\varphi v_\varphi;$$

$$\nabla\times\boldsymbol{v} = \frac{1}{r\sin\theta}\left[\partial_\theta(v_\varphi\sin\theta) - \partial_\varphi v_\theta\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\partial_\varphi v_r - \partial_r(rv_\varphi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\partial_r(rv_\theta) - \partial_\theta v_r\right]\hat{\boldsymbol{\varphi}}.$$

$$\nabla^2 t = \frac{1}{r^2}\partial_r(r^2\partial_r t) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta t) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 t.$$

2.2　柱坐标系

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}s\hat{\boldsymbol{s}} + s\mathrm{d}\varphi\hat{\boldsymbol{\varphi}} + \mathrm{d}z\hat{\boldsymbol{z}}, \quad \mathrm{d}\tau = s\mathrm{d}s\mathrm{d}\varphi\mathrm{d}\theta.$$

$$\nabla t = \partial_s t\hat{\boldsymbol{s}} + \frac{1}{s}\partial_\varphi t\hat{\boldsymbol{\varphi}} + \partial_z t\hat{\boldsymbol{z}};$$

$$\nabla\cdot\boldsymbol{v} = \frac{1}{s}\partial_s(sv_s) + \frac{1}{s}\partial_\varphi v_\varphi + \partial_z v_z.$$

$$\nabla\times\boldsymbol{v} = \left[\frac{1}{s}\partial_\varphi v_z - \partial_z v_\varphi\right]\hat{\boldsymbol{s}} + \left[\partial_z v_s - \partial_s v_z\right]\hat{\boldsymbol{\varphi}} + \frac{1}{s}\left[\partial_s(sv_\varphi) - \partial_\varphi v_s\right]\hat{\boldsymbol{z}}.$$

$$\nabla^2 t = \frac{1}{s}\partial_s(s\partial_s t) + \frac{1}{s^2}\partial_\varphi^2 t + \partial_z^2 t.$$

3　三维形式的Maxwell方程组

3.1　真空中

微分形式:

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\boldsymbol{E} = 4\pi\rho$$

$$\nabla\times\boldsymbol{E} = -\partial_t\boldsymbol{B} \qquad \nabla\times\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{H}$$

$$\nabla\cdot\boldsymbol{B} = 0 \qquad \nabla\cdot\boldsymbol{H} = 0$$

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\partial_t\boldsymbol{E} \qquad \nabla\times\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\partial_t\boldsymbol{E}$$

积分形式:

$$\int_{\partial V}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \frac{Q_{\mathrm{in}}}{\varepsilon_0} \qquad \int_{\partial V}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = 4\pi Q_{\mathrm{in}}$$

$$\oint_{\partial S}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = -\partial_t\int_S\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{a} \qquad \oint_{\partial S}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = -\frac{1}{c}\partial_t\int_S\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{a}$$

$$\oint_S\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{a} = 0 \qquad \oint_S\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{a} = 0$$

$$\oint_{\partial S}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l} = \mu_0\int\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a} + \mu_0\varepsilon_0\partial_t\int\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} \qquad \oint_{\partial S}\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{l} = \frac{4\pi}{c}\int\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{a} + \frac{1}{c}\partial_t\int\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a}$$

3.2 介质中

$$\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P},\quad \boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M}) \qquad\qquad \boldsymbol{D}=\boldsymbol{E}+4\pi\boldsymbol{P},\quad \boldsymbol{B}=\boldsymbol{H}+4\pi\boldsymbol{M}$$

$$\nabla\cdot\boldsymbol{D}=\frac{\rho_{\mathrm{f}}}{\varepsilon_0} \qquad\qquad \nabla\cdot\boldsymbol{D}=4\pi\rho_{\mathrm{f}}$$

$$\nabla\times\boldsymbol{E}=-\partial_t\boldsymbol{B} \qquad\qquad \nabla\times\boldsymbol{E}=-\frac{1}{c}\partial_t\boldsymbol{B}$$

$$\nabla\cdot\boldsymbol{B}=0 \qquad\qquad \nabla\cdot\boldsymbol{B}=0$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_{\mathrm{f}}+\partial_t\boldsymbol{D} \qquad\qquad \nabla\times\boldsymbol{H}=\frac{4\pi}{c}\boldsymbol{J}_{\mathrm{f}}+\frac{1}{c}\partial_t\boldsymbol{D}$$

4 相对性原理

4.1 洛伦兹变换

四维时空中的转动:

$$x=x'\cosh\psi+ct'\sinh\psi,\quad ct=x'\sinh\psi+ct'\cosh\psi,\quad \tanh\psi=\frac{V}{c}.$$

$$x=\frac{x'+Vt}{\sqrt{1-V^2/c^2}},\quad t=\frac{t'+Vx/c^2}{\sqrt{1-V^2/c^2}}$$

洛伦兹收缩: 静止的K系内有一根杆, 在相对于K的运动速度为V的K'系的同一时刻中, 杆两端坐标为

$$x_1=\frac{x_1'+Vt'}{\sqrt{1-V^2/c^2}},\quad x_2=\frac{x_2'+Vt'}{\sqrt{1-V^2/c^2}}.$$

得

$$x_2'-x_1'=\frac{x_2-x_1}{\sqrt{1-V^2/c^2}}.$$

4.2 四维时空

采用**度规**

$$g^{ik}=g_{ik}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

四维时空中的矢量A^i有**逆变分量**(空间分量带有“正确的”正号)和**协变分量**:

$$A_i=g_{ik}A^k.$$

四维矢量A的模长为A_iA^i, 两个矢量A和B的标积为A_iB^i.

洛伦兹变换就是闵可夫斯基空间中的转动, **四维标量**在转动过程中不变. 一个**四维矢量**的模长是一个四维标量. **闵可夫斯基空间**中的转动应当为

$$\begin{pmatrix}A^0\\A^1\\A^2\\A^3\end{pmatrix}=\begin{pmatrix}\cosh\psi&\sinh\psi&0&0\\\sinh\psi&\cosh\psi&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}A'^1\\A'^2\\A'^3\\A'^4\end{pmatrix}.$$

取$\tanh\psi = V/c$, 有

$$A^0 = \frac{A'^0 + VA'^1/c}{\sqrt{1-V^2/c^2}},\quad A^1 = \frac{A'^1 + VA'^0/c}{\sqrt{1-V^2/c^2}},\quad A^2 = A'^2,\quad A^3 = A'^3.$$

5 相对论动力学

能量、动量:

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\boldsymbol{v}^2/c^2}} = c\sqrt{\boldsymbol{p}^2 + m^2c^2},\quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}} = \frac{\mathcal{E}\boldsymbol{v}}{c^2}.$$

此外, 有**四维动量矢量**(在洛伦兹变换下满足四维矢量的变换规律)

$$p^i = (\mathcal{E}/c, \boldsymbol{p}) = mcu_i.$$

自由粒子的**作用量**与**拉格朗日函数**:

$$S = -\int mc\mathrm{d}s = -\int mc^2\sqrt{1-\boldsymbol{v}^2/c^2}\mathrm{d}t,\quad L = -mc^2\sqrt{1-\boldsymbol{v}^2/c^2}.$$

6 运动方程、场方程

6.1 四维电流矢量与四维势

$$J^i = (c\rho, \boldsymbol{J}),\quad A^i = (\varphi, \boldsymbol{A}).$$

连续性方程:

$$\nabla\cdot\boldsymbol{J} + \partial_t\rho = 0 \Leftrightarrow \partial_i J^i = 0.$$

四维势的**洛伦兹规范**:

$$\partial_i A^i = 0.$$

设一个粒子具有电量e, 则在电磁场中的作用量和拉格朗日函数为

$$S = -\int mc\mathrm{d}s - \frac{e}{c}\int A_i\mathrm{d}x^i = \int L\mathrm{d}t,\quad L = -mc\sqrt{1-\boldsymbol{v}^2/c^2} - e\varphi + \frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v}.$$

对于连续的电流情形, 作用量为(其中$\mathrm{d}\Omega$为四维体积微元)

$$S = -\sum\int mc\mathrm{d}s - \frac{1}{c^2}\int A_iJ^i\mathrm{d}\Omega.$$

6.2 运动方程、电磁场张量

可以利用最小作用量原理, 通过变分粒子的轨道, 来推导粒子的运动方程. 设一个粒子的轨道有变分$x^i + \delta x^i$.

$$\delta\mathrm{d}s = \delta\sqrt{\mathrm{d}x_i\mathrm{d}x^i} = \frac{\delta(\mathrm{d}x_i\mathrm{d}x^i)}{2\sqrt{\mathrm{d}x_i\mathrm{d}x^i}} = \frac{\mathrm{d}x_i}{\mathrm{d}s}\delta\mathrm{d}x^i = u_i\delta\mathrm{d}x^i.$$

其中u_i为四维速度. 则

$$\begin{aligned}\delta S &= -\int mc\delta\mathrm{d}s - \frac{e}{c}\int \delta(A_i\mathrm{d}x^i) = -\int mcu_i\delta\mathrm{d}x^i - \frac{e}{c}\int\left(\delta A_i\mathrm{d}x^i + A_i\delta\mathrm{d}x^i\right)\\ &= \int\left\{-mcu_i\mathrm{d}\delta x^i - \frac{e}{c}A_i\mathrm{d}\delta x^i - \frac{e}{c}\left(\partial_k A_i\right)\delta x^k\mathrm{d}x^i\right\}\\ &= -\int\frac{e}{c}\left(\partial_i A_k\right)\delta x^i\mathrm{d}x^k + \int\left\{mc\mathrm{d}u_i\delta x^i + \frac{e}{c}\mathrm{d}A_i\delta x^i\right\}\\ &= -\int\frac{e}{c}\left(\partial_i A_k\right)\delta x^i u_k\mathrm{d}s + \int\left\{mc\frac{\mathrm{d}u_i}{\mathrm{d}s}\mathrm{d}s\delta x^i + \frac{e}{c}\left(\partial_k A_i\right)u^k\mathrm{d}s\delta x^i\right\}\\ &= \int\left\{mc\frac{\mathrm{d}u_i}{\mathrm{d}s} - \frac{e}{c}\left(\partial_i A_k - \partial_k A_i\right)u^k\right\}\delta x^i\mathrm{d}s.\end{aligned}$$

记**电磁场张量**为

$$F_{ik} = \partial_i A_k - \partial_k A_i,$$

则可以得到运动方程

$$\boxed{\frac{e}{c}F_{ik}u^k = mc\frac{\mathrm{d}u_i}{\mathrm{d}s}.}$$

设

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\partial_t\boldsymbol{A},\quad \boldsymbol{H} = \nabla\times\boldsymbol{A},$$

(分别称为**电场**和**磁场**), 则有

$$F_{ik} = \begin{bmatrix} 0 & E_x & E_y & E_z\\ -E_x & 0 & -H_z & H_y\\ -E_y & H_z & 0 & -H_x\\ -E_z & -H_y & H_x & 0\end{bmatrix}, F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z\\ E_x & 0 & -H_z & H_y\\ E_y & H_z & 0 & -H_x\\ E_z & -H_y & H_x & 0\end{bmatrix}.$$

写出上述运动方程的三维形式, 就有

$$\boxed{\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{H}\right).}$$

6.3 电磁场的作用量、场的拉格朗日方程

在**高斯单位制**下, 电磁场及其中粒子的作用量为

$$\boxed{S = -\sum\int mc\mathrm{d}s - \frac{1}{c^2}\int A_i J^i\mathrm{d}\Omega - \frac{1}{16\pi c}\int F^{ik}F_{ik}\mathrm{d}\Omega.}$$

其拉格朗日量的密度为

$$\mathcal{L} = -\frac{1}{c^2}A_i J^i - \frac{1}{16\pi c}F^{ik}F_{ik}.$$

一般地, 对于任意一种形式的场(不仅是电磁场), 设其拉格朗日密度为$\mathcal{L}$, 由一些量$q_{(n)}$(简写为q)所决定(对于电磁场, q就是A^i的分量). 则$\mathcal{L}$可以写为q和$\partial_i q$的函数. 通过变分法可以得到

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\partial_i q)} = 0.$$

6.4 第一对麦克斯韦方程

第一对麦克斯韦方程可以直接从电磁场张量的定义$F_{ik}=\partial_i A_k-\partial_k A_i$得来:

$$\boxed{e^{iklm}\partial_k F_{lm}=0.}\Leftrightarrow\boxed{\partial_i F_{kl}+\partial_k F_{li}+\partial_l F_{ik}=0.}$$

这等价于三维形式的方程:

$$\nabla\times\boldsymbol{E}=\frac{1}{c}\partial_t\boldsymbol{H},\quad \nabla\cdot\boldsymbol{H}=0.$$

6.5 第二对麦克斯韦方程

第二对麦克斯韦方程描述的是电荷如何激发电磁场. 这可以通过在电磁场的作用量中, 变分势A^i而得来:

$$\begin{aligned}\delta S&=-\frac{1}{c^2}\int\delta A_i J^i\mathrm{d}\Omega-\frac{1}{16\pi c}\int\delta\left(F^{ik}F_{ik}\right)\mathrm{d}\Omega\\&=-\frac{1}{c^2}\int J^i\delta A_i\mathrm{d}\Omega-\frac{1}{8\pi c}\int F^{ik}\delta\left(\partial_i A_k-\partial_k A_i\right)\mathrm{d}\Omega\\&=-\frac{1}{c^2}\int J^i\delta A_i\mathrm{d}\Omega+\frac{1}{8\pi c}\int\left\{\partial_i F^{ik}\delta A_k-\partial_k F^{ik}\delta A_i\right\}\mathrm{d}\Omega\\&=-\int\left\{\frac{1}{c^2}J^i+\frac{1}{4\pi c}\partial_k F^{ik}\right\}\delta A_i\mathrm{d}\Omega(\text{对上式交换第二个积分第一项的}i,k\text{并用反对称性})\end{aligned}$$

故有:

$$\boxed{\partial_k F^{ik}=\frac{4\pi}{c}J^i.}$$

这等价于三维形式的方程:

$$\nabla\cdot\boldsymbol{E}=4\pi\rho,\quad\nabla\times\boldsymbol{H}=\frac{4\pi}{c}\boldsymbol{J}+\frac{1}{c}\partial_t\boldsymbol{E}.$$

6.6 能量与动量

由麦克斯韦方程组出发, 通过运算, 可以得到

$$\partial_t\left(\frac{E^2+H^2}{8\pi}\right)+\boldsymbol{J}\cdot\boldsymbol{E}=-\nabla\cdot(\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H}).$$

记**坡印廷矢量**为

$$\boldsymbol{S}=\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H}.$$

对上式在某一体积V内进行积分, 并且利用高斯定理, 注意到$\boldsymbol{J}\cdot\boldsymbol{E}$为动能密度对于时间的导数, 就有

$$\partial_t\left\{\int_V\frac{E^2+H^2}{8\pi}\mathrm{d}V+\mathcal{E}_{\mathrm{kin}}\right\}=-\oint_{\partial V}\boldsymbol{S}\cdot\mathrm{d}\boldsymbol{f}.$$

由此式可知, $\left(E^2+H^2\right)/8\pi$为电磁场的**能量密度**, S为能流密度.

时空的平移对称性表现在场的拉格朗日函数不显含空间坐标, 即

$$\partial_i\mathcal{L}=\frac{\partial\mathcal{L}}{\partial q}\frac{\partial q}{\partial x^i}+\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\frac{\partial(\partial_k q)}{\partial x^i}=\partial_k\left(\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\right)\partial_i q+\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\partial_k(\partial_i q)=\partial_k\left(\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\partial_i q\right).$$

则有

$$\partial_k\left\{\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\partial_i q-\delta_i^k\mathcal{L}\right\}=0.$$

记能量动量张量为
$$T_i^k=\frac{\partial\mathcal{L}}{\partial(\partial_k q)}\partial_i q-\delta_i^k\mathcal{L}.$$
一般来说, 可以加上一个三阶张量的散度, 使得T^{ik}为对称张量:
$$T^{ik}+\partial_l\psi^{ikl},\quad \psi^{ikl}=-\psi^{ilk}.$$

能量动量张量的各分量的物理意义如下:
$$T^{ik}=\begin{bmatrix} W & S_x/c & S_y/c & S_z/c\\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz}\\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz}\\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz}\end{bmatrix}.$$
其中, W为能量密度, $S_\alpha=cT^{0\alpha}$为坡印廷矢量即能流密度, $P_\alpha=T^{0\alpha}/c$为动量密度, $-\sigma_{\alpha\beta}$为动量流密度, $\sigma_{\alpha\beta}$为应力张量.

7 拉普拉斯方程

7.1 球坐标

拉普拉斯算子:
$$\nabla^2=\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2.$$
设欲求的函数$\phi(r,\theta,\phi)=R(r)Y(\theta,\phi)$, 则分离变量得到以下两个方程:
$$\begin{aligned}&\partial_r\left(r^2\partial_r R\right)=l(l+1)R,\\ &\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta Y\right)+\frac{1}{\sin^2\theta}\partial_\phi^2 Y=-l(l+1)Y.\end{aligned}$$
第一个方程可以解出
$$R(r)=A_l r^l+\frac{B_l}{r^{l+1}}.$$
其中A_l,B_l为常数. 对第二个方程继续分离变量:$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$. 则有
$$\begin{aligned}&\partial_\phi^2\Phi=-m^2\Phi,\\ &\partial_\theta^2\Theta+\cot\theta\partial_\theta\Theta+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0.\end{aligned}$$
(因为Φ是周期函数, 故$m\in\mathbb{Z}$.)第一个方程可以解出
$$\Phi(\phi)=\mathrm{e}^{im\phi}.$$
第二个方程的解为$P_l^m(\cos\theta)$, 称为**连带勒让德多项式**(此时要求$|m|\le l$). 若记$x=\cos\theta$, 则有
$$(1-x^2)\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}-2x\frac{\mathrm{d}\Theta}{\mathrm{d}x}+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\Theta=0.$$

故上述拉普拉斯方程的一个特解为
$$\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)Y_{lm}(\theta,\phi).$$
其中Y_{lm}是(归一化的)**球谐函数**.

7.2 球谐函数与勒让德多项式

勒让德多项式可由以下母函数生成:

$$\frac{1}{\sqrt{1-2tx+t^2}}=\sum_{l\geq 0}t^l P_l(x).$$

勒让德多项式是正交完备的函数系:

$$\int_{-1}^{1}P_l(x)P_{l'}(x)\mathrm{d}x=\frac{2}{2l+1}\delta_{ll'},\quad \sum_{l\geq 0}\frac{2l+1}{2}P_l(x)P_l(x')=\delta(x-x').$$

球谐函数为

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta),\quad Y_{l,-m}(\theta,\phi)=(-1)^m Y_{lm}^*(\theta,\phi).\quad (0\leq m\leq l)$$

其也为正交完备的函数系:

$$\int Y_{lm}Y_{l'm'}^*\mathrm{d}o=\delta_{ll'}\delta_{mm'},\qquad \sum_{l\geq 0,\ |m|\leq l}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi')=\delta(\cos\theta-\cos\theta')\delta(\phi-\phi').$$

球谐函数的**加法定理**: 对于单位矢量$\boldsymbol{n}_1,\boldsymbol{n}_2$, 有

$$P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)=\frac{4\pi}{2l+1}\sum_{-l\leq m\leq l}Y_{lm}^*(\boldsymbol{n}_1)Y_{lm}(\boldsymbol{n}_2).$$

因此, 对于两个矢量$\boldsymbol{r},\boldsymbol{r}'$, 满足$r'<r$, 则有

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}\sum_{l\geq 0}\left(\frac{r'}{r}\right)^l P_l(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}}')=\frac{1}{r}\sum_{l\geq 0,\ |m|\leq l}\frac{4\pi}{2l+1}\left(\frac{r'}{r}\right)^l Y_{lm}^*(\hat{\boldsymbol{r}})Y_{lm}(\hat{\boldsymbol{r}}')$$