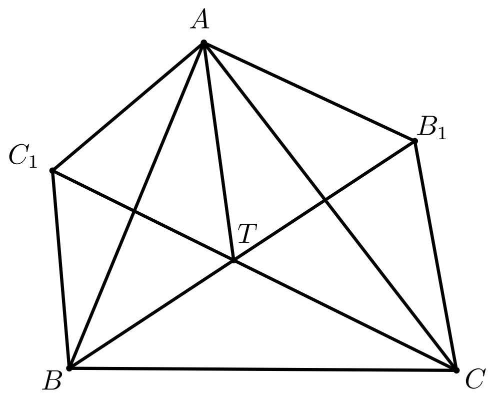
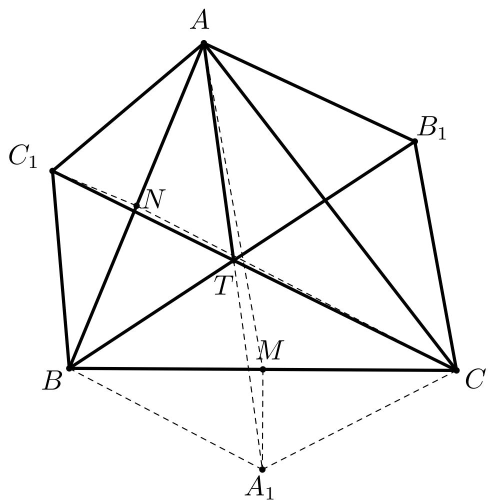
题目(2024 CTST P2):

在锐角$\triangle ABC$中, $\angle A > \angle B > \angle C$. B_1, C_1是平面上的两点, 满足$\triangle AC_1B$和$\triangle CB_1A$分别是以AB, AC为底边, 且顺相似的等腰三角形. 设直线BB_1, CC_1交于点T. 假设上述各点两两不同, 求证: $\angle ATC \neq 90°$.



解: 如图, 作点A_1, 使得$\triangle BA_1C \sim \triangle AC_1B$.



设$\triangle ABC$的内角为A, B, C, 三个等腰三角形的底角为α. 则由正弦定理,

$$\frac{\sin\angle TAB}{\sin\angle TAC} = \frac{A_1B \cdot \sin\angle ABA_1/AA_1}{A_1C \cdot \sin\angle ACA_1/AA_1} = \frac{\sin\angle ABA_1}{\sin\angle ACA_1} = \frac{\sin(B+\alpha)}{\sin(C+\alpha)}.$$

则

$$\prod_{\text{cyc}} \frac{\sin\angle TAB}{\sin\angle TAC} = \prod_{\text{cyc}} \frac{\sin(B+\alpha)}{\sin(C+\alpha)} = 1.$$

由角元塞瓦定理, 知AA_1, BB_1, CC_1共点.

作BC, AB中点M, N, 则$MA_1 \perp BC, NC_1 \perp AB$. 记垂直于纸面的向外的单位向量为$\boldsymbol{e}$, 设$k = MA_1/BC$($A, A_1$在直线$BC$异侧时$k > 0$, 否则$k < 0$). 并且设$\triangle ABC$的三边为$a, b, c$, 面积为$S$, 则由$\angle A > \angle B > \angle C$知$a > b > c > 0$, 且

$$\overrightarrow{AA_1} = \overrightarrow{AM} + ,\overrightarrow{MA_1} = \overrightarrow{AM} + k\overrightarrow{BC} \times \boldsymbol{e}, \overrightarrow{CC_1} = \overrightarrow{CN} + k\overrightarrow{AB} \times \boldsymbol{e}.$$

则

$$\begin{aligned}\overrightarrow{AA_1}\cdot\overrightarrow{CC_1}&=(\overrightarrow{AM}+k\overrightarrow{BC}\times\boldsymbol{e})\cdot(\overrightarrow{CN}+k\overrightarrow{AB}\times\boldsymbol{e})\\&=\frac{1}{4}(\overrightarrow{AB}+\overrightarrow{AC})\cdot(\overrightarrow{CA}+\overrightarrow{CB})+k\overrightarrow{AM}\cdot(\overrightarrow{AB}\times\boldsymbol{e})+k\overrightarrow{CN}\cdot(\underline{\mathrm{B}}\underline{\mathrm{C}}\times\boldsymbol{e})+k^2(\overrightarrow{BC}\times\boldsymbol{e})\cdot(\overrightarrow{AB}\times\boldsymbol{e}).\end{aligned}$$

由点乘、叉乘的性质,

$$\overrightarrow{AM}\cdot(\overrightarrow{AB}\times\boldsymbol{e})=2S_{\triangle AMB},\overrightarrow{CN}\cdot(\overrightarrow{BC}\times\boldsymbol{e})=2S_{\triangle NBC},$$

$$(\overrightarrow{BC}\times\boldsymbol{e})\cdot(\overrightarrow{AB}\times\boldsymbol{e})=\overrightarrow{BC}\cdot\overrightarrow{AB}.$$

故

$$\begin{aligned}&\frac{1}{4}(\overrightarrow{AB}+\overrightarrow{AC})\cdot(\overrightarrow{CA}+\overrightarrow{CB})+k\overrightarrow{AM}\cdot(\overrightarrow{AB}\times\boldsymbol{e})+k\overrightarrow{CN}\cdot(\underline{\mathrm{B}}\underline{\mathrm{C}}\times\boldsymbol{e})+k^2(\overrightarrow{BC}\times\boldsymbol{e})\cdot(\overrightarrow{AB}\times\boldsymbol{e})\\=&-\frac{1}{4}\overrightarrow{AB}\cdot\overrightarrow{AC}+\frac{1}{4}\overrightarrow{BA}\cdot\overrightarrow{BC}-\frac{1}{4}AC^2-\frac{1}{4}\overrightarrow{CA}\cdot\overrightarrow{CB}+2k(S_{\triangle AMB}+S_{\triangle NBC})-k^2\overrightarrow{BC}\cdot\overrightarrow{BA}\\=&-\frac{1}{8}(b^2+c^2-a^2)+\frac{1}{8}(a^2+c^2-b^2)-\frac{1}{8}(a^2+b^2-c^2)-\frac{b^2}{4}+2kS-\frac{k^2}{2}(a^2+c^2-b^2)\\=&-\frac{a^2+c^2-b^2}{2}k^2+2S\cdot k-\frac{5b^2-a^2-c^2}{8}.\end{aligned}$$

这是一个关于k的二次函数, 其判别式为

$$\begin{aligned}&(2S)^2-4\frac{a^2+c^2-b^2}{2}\frac{5b^2-a^2-c^2}{8}\\=&\frac{1}{4}\left(16S^2-(a^2+c^2-b^2)(5b^2-a^2-c^2)\right)\\=&\frac{1}{4}\left(2\sum_{\text{cyc}}a^2b^2-\sum_{\text{cyc}}a^4-(a^2+c^2-b^2)(5b^2-a^2-c^2)\right)\\=&-\left(a^2-b^2\right)\left(b^2-c^2\right)<0.\end{aligned}$$

故对任意的实数k, $\overrightarrow{AA_1}\cdot\overrightarrow{CC_1}$均不为0. 证毕!