问题征解（1）答案

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2022 年 2 月 27 日

1.证明:素数有无穷多个.

解:假设素数的个数有限,设这些素数为$p_1, p_2, \ldots p_n$.

考虑$k = p_1 p_2 p_3 \ldots p_n + 1$,则$p_1 \nmid k, p_2 \nmid k, \ldots p_n \nmid k$.

若k为素数,则与假设矛盾.

若k为合数,$\because p_1 \nmid k, p_2 \nmid k, \ldots p_n \nmid k, \therefore$k有比$p_1, p_2, \ldots p_n$更大的素因子,与假设矛盾.

$\therefore$假设不成立,即素数的个数无限.

2.证明:数列$10001, 100010001, 1000100010001, \ldots$中,每一个数都是合数.

解:(1) $10001 = 73 \times 137, \therefore 10001$为合数.

(2)$n > 1$时,$1\underbrace{00010001\ldots0001}_{共n个1} = 10^{4n} + 10^{4n-4} + \cdots + 1 = \dfrac{10^{4n+4}-1}{10^4-1} = \dfrac{(10^{2n+2}-1)(10^{2n+2}+1)}{9999}$

$\because (10^{2n+2}+1) > (10^{2n+2}-1) > 9999, \therefore 1\underbrace{00010001\ldots0001}_{共n个1}$为合数.

综上,数列$10001, 100010001, 1000100010001, \ldots$中,每一个数都是合数.

3.求所有的正整数x, y, z,满足$3^x + 4^y = 5^z$.

解:对原方程两边同时模3,得到

$$2^z \equiv (-1)^z \equiv 1 \pmod 3$$

所以$2 \mid z$,设$z = 2z_1$.

对原方程两边同时模4,得到

$$3^x \equiv (-1)^x \equiv 1 \pmod 4$$

所以$2 \mid x$,设$x = 2x_1$.

则原方程$\Rightarrow 3^{2x_1} + 2^{2y} = 5^{2z_1}$,即

$$\begin{cases} 3^{2x_1} = (5^{z_1})^2 - (2^y)^2 \\ 2^{2y} = (5^{z_1})^2 - (3^{x_1})^2 \end{cases}$$

得到

$$\begin{cases} 3^{2x_1} = (5^{z_1} - 2^y)(5^{z_1} + 2^y) & (1) \\ 2^{2y} = (5^{z_1} - 3^{x_1})(5^{z_1} + 3^{x_1}) & (2) \end{cases}$$

对于(1):

$\because (5^{z_1} - 2^y, 5^{z_1} + 2^y) = (5^{z_1} - 2^y, 2^{y+1}), 3 \nmid 2^{y+1}$,

$\therefore 3 \nmid (5^{z_1} - 2^y, 5^{z_1} + 2^y)$,即$5^{z_1} - 2^y$与$5^{z_1} + 2^y$不同时含有素因子3.$\therefore 5^{z_1} - 2^y$与$5^{z_1} + 2^y$一个为$3^{2x_1}$,一个为1.

又$5^{z_1} - 2^y < 5^{z_1} + 2^y$,$\therefore$有

$$\begin{cases} 5^{z_1} - 2^y = 1 \\ 5^{z_1} + 2^y = 3^{2x_1} \end{cases}$$

得

$$5^{z_1} = 2^y + 1 \tag{3}$$

对于(2):

$\because (5^{z_1} - 3^{x_1}, 5^{z_1} + 3^{x_1}) = (5^{z_1} - 3^{x_1}, 2 \times 3^{x_1}), 2 \mid 5^{z_1} - 3^{x_1}, 2 \mid 2 \times 3^{x_1}$而$4 \nmid 2 \times 3^{x_1}$

$\therefore (5^{z_1} - 3^{x_1}, 5^{z_1} + 3^{x_1}) = 2$,又$5^{z_1} - 3^{x_1} < 5^{z_1} + 3^{x_1})$,$\therefore$有

$$\begin{cases} 5^{z_1} - 3^{x_1} = 2 & (4) \\ 5^{z_1} + 3^{x_1} = 2^{2y-1} & (5) \end{cases}$$

$(4) + (5)$,得

$$2 \times 5^{z_1} = 2 + 2^{2y-1}$$

得

$$5^{z_1}=2^{2y-2}+1 \tag{6}$$

由(3), (6)得

$$2^y+1=2^{2y-2}+1$$

即

$$y=2y-2$$

得$y=2$.

将$y=2$代入(3),得

$$5^{z_1}=5$$

得$z_1=5, z=2$.

将$y=2, z=2$代入原方程,立得$x=2$.

综上:原方程的正整数解为$x=y=z=2$.

4.已知

$$a+b+c=5, a^2+b^2+c^2=15, a^3+b^3+c^3=47$$

求

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

解:展开$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$,得

$$3a^2b^2c^2+\sum_{sym}a^4b^2+2\sum_{sym}a^3b^2c+\sum_{cyc}a^4bc+\sum_{cyc}a^3b^3$$

其中$\sum\limits_{sym}$与$\sum\limits_{cyc}$详见脚注.[1]

$\because 2\sum\limits_{cyc}ab=(a+b+c)^2-(a^2+b^2+c^2)=5^2-15=10$

[1] $\sum\limits_{sym}$ 指将此符号后的式子中的字母随意交换,将得到的所有式子求和,例如$\sum\limits_{sym}a^2b=a^2b+b^2c+c^2a+ab^2+bc^2+ca^2$; $\sum\limits_{cyc}$指将此符号后的式子中的字母依次轮换(例如$a\to b, b\to c, c\to a$),将得到的所有式子求和,例如$\sum\limits_{cyc}ab=ab+bc+ca$.

$\therefore \sum_{cyc} ab = 5.$

$\because$ 由欧拉公式,

$$
\begin{aligned}
3abc &= (\sum_{cyc} a^3) - (\sum_{cyc} a)(\sum_{cyc} a^2 - \sum_{cyc} ab) \\
&= 47 - 5 \times (15 - 5) \\
&= -3
\end{aligned}
$$

$\therefore abc = -1.$

$$
\begin{aligned}
\sum_{cyc} a^4bc &= \sum_{cyc} (abc \times a^3) \\
&= abc(\sum_{cyc} a^3) \\
&= (-1) \times 47 \\
&= -47
\end{aligned}
$$

$\therefore \sum_{cyc} a^4bc = -47.$

$$
\begin{aligned}
\sum_{cyc} a^2b^2 &= (\sum_{cyc} ab)^2 - 2\sum_{cyc} (ab \times bc) \\
&= (\sum_{cyc} ab)^2 - 2\sum_{cyc} (abc \times a) \\
&= (\sum_{cyc} ab)^2 - 2abc\sum_{cyc} a \\
&= 5^2 - 2 \times (-1) \times 5 \\
&= 35
\end{aligned}
$$

$\therefore \sum_{cyc} a^2b^2 = 35.$

$$
\begin{aligned}
\sum_{sym} a^4b^2 &= (\sum_{cyc} a^2)(\sum_{cyc} a^2b^2) - 3a^2b^2c^2 \\
&= 15 \times 35 - 3 \times (-1)^2 \\
&= 522
\end{aligned}
$$

$\therefore \sum_{sym} a^4b^2 = 522$

$$\begin{aligned}\sum_{cyc} a^2b &= (\sum_{cyc} ab)(\sum_{cyc} a) - 3abc \\ &= 5 \times 5 - 3 \times (-1) \\ &= 28\end{aligned}$$

$\therefore \sum_{sym} a^2b = 28.$

$$\begin{aligned}\sum_{sym} a^3b^2c &= \sum_{sym} (abc \times a^2b) \\ &= abc \times \sum_{sym} a^2b \\ &= (-1) \times 28 \\ &= -28\end{aligned}$$

$\therefore \sum_{sym} a^3b^2c = -28.$

$$\begin{aligned}\sum_{cyc} a^3b^3 &= (\sum_{cyc} ab)(\sum_{cyc} a^2b^2) - \sum_{sym} a^3b^2c \\ &= 5 \times 35 - (-28) \\ &= 203\end{aligned}$$

$\therefore \sum_{cyc} a^3b^3 = 203$

$\therefore$原式

$$\begin{aligned}&= 3a^2b^2c^2 + \sum_{sym} a^4b^2 + 2\sum_{sym} a^3b^2c + \sum_{cyc} a^4bc + \sum_{cyc} a^3b^3 \\ &= 3 \times (-1)^2 + 522 + 2 \times (-28) + (-47) + 203 \\ &= 625\end{aligned}$$

综上:原式$= 625$.