

Extrapolation of the phenomenological nuclear equation of state to high densities

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We propose a modification of the phenomenological Skyrme interaction in order to obtain a nuclear equation of state fitted to the binding energy, saturation density, and incompressibility of nuclear matter and which remains causal at higher densities. The properties of this and other equations of state at transnuclear densities are compared and the implications for stellar collapse and cold neutron structure are briefly discussed.

There has been considerable recent interest in the equation of state (EOS) of nuclear matter,^{1,2} particularly in its extension to transnuclear densities encountered inside neutron stars and in stellar collapse.^{3,4} One form often used⁵⁻⁷ is based on the phenomenological Skyrme interaction⁸ in which the energy per baryon takes the form

$$\frac{E}{N} = \frac{\epsilon}{n} = mc^2 + \delta n^{2/3} - bn + dn^2. \quad (1)$$

Here n and ϵ are the baryon number and energy densities, m is the nucleon mass, and

$$\delta = \frac{3\hbar^2}{10m} \left(\frac{3\pi^2}{2} \right)^{2/3}. \quad (1a)$$

In principle, the attractive term bn^2 originates from the two-body interactions and the repulsive term dn^3 from the three-body interactions. In practice, b and d are simply fitted to the saturation density n_0 and binding energy per particle E/N .

Clearly, Eq. (1) is an approximation based on an expansion in n , and should not be expected to converge at high densities. In fact, if E/N is to have the minimum value 16 MeV at saturation density $n_0 = 0.16 \text{ fm}^{-3}$, then $dn_0^2/bn_0 = 0.4$, suggesting that the expansion is only moderately convergent even at nuclear density.⁹

From the energy density $\epsilon(n)$ we can calculate the chemical potential

$$\mu = \left(\frac{\partial \epsilon}{\partial n} \right)_s, \quad (2)$$

the pressure

$$p = - \left(\frac{\partial (E/N)}{\partial v} \right)_s = n^2 \left(\frac{\partial (\epsilon/n)}{\partial n} \right)_s = n\mu - \epsilon, \quad (3)$$

the incompressibility

$$K = 9 \left(\frac{\partial p}{\partial n} \right)_s, \quad (4)$$

the adiabatic index

$$\Gamma = \left(\frac{\partial \log p}{\partial \log n} \right)_s, \quad (5)$$

and the adiabatic sound speed

$$\left(\frac{c_s}{c} \right)^2 = \left(\frac{\partial p}{\partial \epsilon} \right)_s = \frac{(\partial p / \partial n)_s}{(\partial \epsilon / \partial n)_s} = \left(\frac{\partial \log \mu}{\partial \log n} \right)_s = \frac{p}{p + \epsilon} \Gamma. \quad (6)$$

Since we deal with zero-temperature matter, the entropy S equals 0, and we will hereafter replace partial derivatives with respect to $n = N/V$ by ordinary derivatives. For the original Skyrme interaction of Eq. (1), we plot μ , p , K , and c_s/c in Figs. 1-4 (labeled "Skyrme").

Because the strongly repulsive term dn^2 in Eq. (1) makes nuclear matter very stiff, the nuclear incompressibility $K_0 = 376 \text{ MeV}$ at $n = n_0$ calculated from Eqs. (1) and (4) turns out to be nearly twice the observed value⁷ $K_0 = 220 \text{ MeV}$. The matter becomes superluminal when

$$3dn^2 > mc^2 + \frac{5}{9}\delta n^{2/3}, \quad (7)$$

that is, $n > 3.7n_0$.

The Lattimer-Ravenhall⁶ EOS derived from the Skyrme interaction can be⁷ fitted by the simple form

$$p = A \left[\left(\frac{n}{n_0} \right)^\Gamma - 1 \right], \quad (8)$$

with $A = 2.56 \text{ MeV fm}^{-3}$ and $\Gamma = 3.5$. This EOS becomes superluminal at

$$(n/n_0)^{\Gamma-1} = \frac{mn_0(\Gamma-1)}{A\Gamma(\Gamma-2)}$$

or $n = 3.8n_0$. It predicts a nuclear incompressibility $K_0 = 9A\Gamma/n_0 = 460 \text{ MeV}$. Bethe *et al.*⁷ attributed this excessive value, about twice the observed $K_0 = 220 \text{ MeV}$, to the three-body term in the Skyrme interaction and chose to reduce the pressure by the overall factor $220/460 = 0.48$. We would have pre-

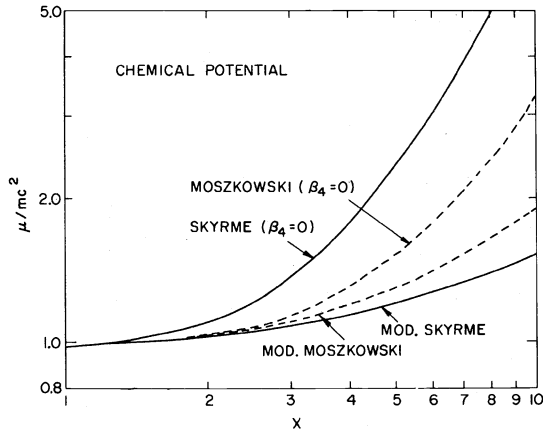


FIG. 1. Chemical potential μ in units of the nucleon mass mc^2 for the Skyrme and Moszkowski models. The curves ($\beta_4=0$) refer to models with no high-density cut-off, hence yielding "stiff" equations of state and superluminal behavior at high density. The "MOD." curves show the effect of imposing subluminal behavior at all densities, with an additional constant β_4 obtained by fitting the nuclear incompressibility.

ferred to obtain the correct incompressibility by retaining the original value $A=2.33 \text{ MeV fm}^{-3}$ and reducing Γ to $(0.48)(3.5)=1.68$, a value for which the EOS (7) becomes causal.

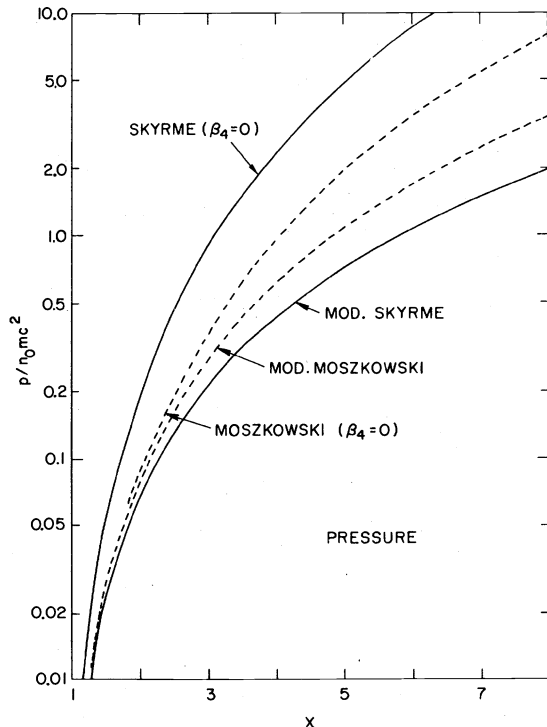


FIG. 2. Pressure p in units of $n_0 mc^2 = 150.4 \text{ MeV fm}^{-3}$, as a function of $x = n/n_0$.

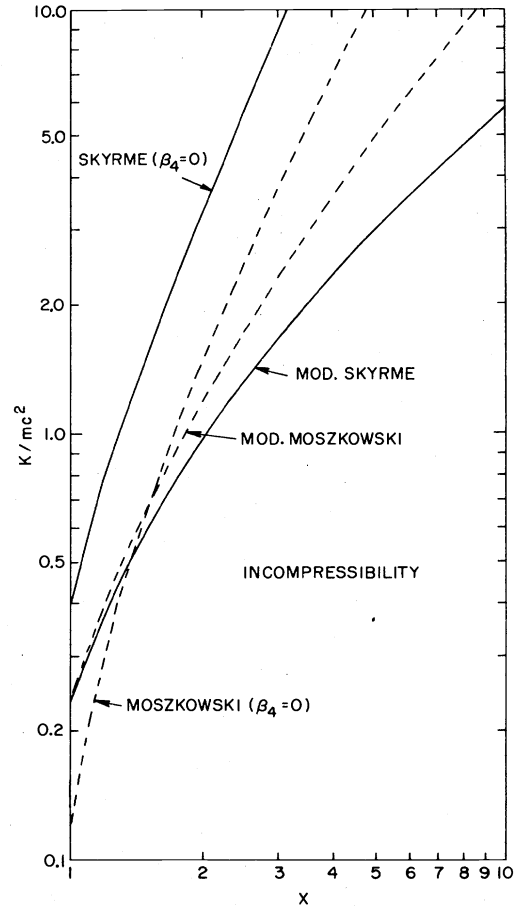


FIG. 3. Incompressibility K , in units of mc^2 , as a function of x .

We have therefore modified the high-density form of the phenomenological Skyrme interaction so as to fit the nuclear incompressibility (as well as saturation density and binding energy) and to

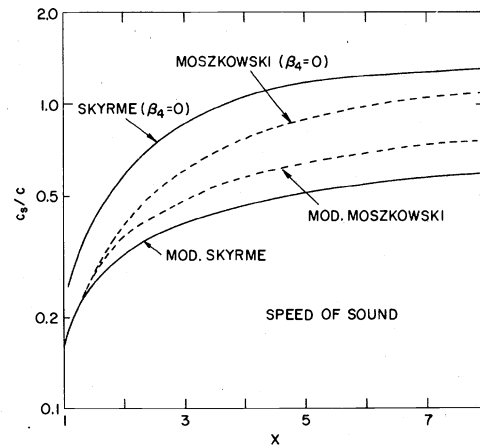


FIG. 4. Adiabatic sound speed (c_s/c) as a function of $x = n/n_0$.

keep the EOS subluminal. If we write

$$\epsilon = nmc^2 + \delta n^{5/3} - bn^2 + \epsilon_c,$$

and $\mu_c = d\epsilon_c/dn$, then

$$\left(\frac{c_s}{c}\right)^2 = \frac{\frac{10}{9}\delta n^{2/3} - 2bn + nd\mu_c/dn}{mc^2 + \frac{5}{3}\delta n^{2/3} - 2bn + \mu_c}, \quad (9)$$

and the causality requirement is

$$\frac{d}{dn}(p_c - \epsilon_c) = n \frac{d\mu_c}{dn} - \mu_c < mc^2 + \frac{5}{9}\delta n^{2/3}.$$

Causality thus requires that μ_c or ϵ_c/n increases slower than n . Note that if $-bn^2$ were the mean-field approximation, then $\epsilon_c < 0$, since the exact energy must be less than that obtained in the mean-field approximation. Stability against the production of an additional baryon-antibaryon pair would then require,¹⁰ in addition, $\mu_c + mc^2 > 0$.

We now consider the particular form

$$\epsilon = n \left(mc^2 + \delta n^{2/3} - bn + \frac{dn^{1+\sigma}}{1 + \alpha n^\sigma} \right), \quad (10)$$

$$\left(\frac{c_s}{c}\right)^2 = \frac{mc^2 x}{\mu} \left[\frac{10}{9}\beta_1 x^{-1/3} - 2\beta_2 + \beta_3 x^\sigma \frac{(1+\sigma)[2+\sigma+\beta_4 x^\sigma(4-\sigma)] + 2\beta_4^2 x^{2\sigma}}{(1+\beta_4 x^\sigma)^3} \right], \quad (14)$$

where $\beta_1 \equiv \delta_0^{2/3}/m = 0.023\,53$, $\beta_2 = bn_0/m$, $\beta_3 = dn_0^2/m$, $\beta_4 = \alpha n_0^\sigma$.

Although Eq. (10) is a purely phenomenological form, it is possible to provide some physical motivation for this choice. For $\sigma=1$, the successive terms $-\beta_2 x^2$, $+\beta_3 x^3$, $-\beta_3 \beta_4 x^4$ in the expansion of ϵ in powers of x , can be roughly interpreted as two-, three-, four-, ... body correlation terms. Since $\beta_4 > 0$ is necessary for stability, the series must be alternating in sign, with an attractive four-body correlation, as expected. In some approximation, it may be possible to sum the various contributions as a geometric series, thereby obtaining Eq. (10). This would be analogous to the Lorenz-Lorentz effect in pion physics,¹³ where the pion self-energy has a correlation contribution proportional to $n^2/(1+\alpha n)$, with $\alpha > 0$. Other mechanisms for a "softening" of the EOS in the region $x > 2$ are also present, such as pion condensation.¹⁴

To fit the three parameters of nuclear matter at nuclear density ($x=1$) requires

$$\frac{d}{dx} \left(\frac{\epsilon}{nmc^2} \right)_{x=1} = 0, \quad (15)$$

$$\left(\frac{\epsilon}{nmc^2} - 1 \right)_{x=1} = -\gamma = -0.017\,02, \quad (16)$$

$$\left(\frac{K}{mc^2} \right)_{x=1} = 0.234, \quad (17)$$

where the denominator will assure causal behavior at high density, and the extra parameter α will be used to fit the incompressibility. Without the convergence factor ($\alpha=0$), these models are similar to those considered by Zamick¹¹; for $\sigma=1$ and $\sigma=\frac{2}{3}$ we recover the original Skyrme form⁸ and the softer form used by Moszkowski,¹² respectively.

In terms of $x \equiv n/n_0$ we can write

$$\frac{\epsilon}{nmc^2} = 1 + \beta_1 x^{2/3} - \beta_2 x + \beta_3 \frac{x^{1+\sigma}}{1 + \beta_4 x^\sigma}, \quad (11)$$

$$\frac{\mu}{mc^2} = 1 + \frac{5}{3}\beta_1 x^{2/3} - 2\beta_2 x + \beta_3 x^{1+\sigma} \frac{(2+\sigma+2\beta_4 x^\sigma)}{(1+\beta_4 x^\sigma)^2}, \quad (12)$$

$$\frac{p}{n_0 mc^2} = x^2 \left[\frac{2}{3}\beta_1 x^{-1/3} - \beta_2 + \beta_3 x^\sigma \frac{(1+\sigma+\beta_4 x^\sigma)}{(1+\beta_4 x^\sigma)^2} \right], \quad (13)$$

or

$$\beta_2 = \beta_1 + \gamma + \beta_3/(1+\beta_4),$$

$$\beta_3 = (1+\beta_4)^3 \xi_1 / \sigma [1 + \sigma + (1-\sigma)\beta_4],$$

$$\beta_4 = \frac{(\sigma+1)\xi_2 - \xi_1}{(\sigma-1)\xi_2 + \xi_1},$$

where $\xi_1 \equiv \frac{1}{9}(K/m + 2\beta_1) = 0.031\,23$, $\xi_2 \equiv \frac{1}{3}\beta_1 + \gamma = 0.024\,86$. For the modified Skyrme and modified Moszkowski forms, respectively,

$$\sigma=1: \beta_2=0.080\,14, \beta_3=0.063\,02, \beta_4=0.592, \quad (18)$$

$$\sigma=\frac{2}{3}: \beta_2=0.0946, \beta_3=0.0782, \beta_4=0.4447. \quad (19)$$

The original Skyrme and Moszkowski forms ($\beta_4=0$) are, respectively,

$$\sigma=1: \beta_2=0.065\,41, \beta_3=0.024\,86, \quad (20)$$

$$\sigma=\frac{2}{3}: \beta_2=0.068\,66, \beta_3=0.028\,11. \quad (21)$$

These forms become noncausal at densities $3.7n_0$ and $6n_0$, respectively, and lead to calculated incompressibilities $K_0=0.40mc^2=376$ MeV and $K_0=0.15mc^2=150$ MeV, respectively.

As shown in Figs. 1-4, the damping factor $(1+\beta_4 x^\sigma)^{-1}$ reduces the chemical potential, pres-

sure, incompressibility, and square of sound speed by about a factor 2 at $\beta_4^{-1}n_0 = 1.7n_0$ in the Skyrme ($\sigma = 1$) case and at $\beta_4^{-3/2}n_0 = 3.4n_0$ in the Moszkowski ($\sigma = \frac{2}{3}$) case. In particular, the sound speed which, in the original Skyrme model, exceeded the speed of light at $n = 3.7n_0$, in the modified Skyrme model reaches only $0.45c$ at the same density.

The damping factor has a larger effect on the stiff Skyrme EOS than on the somewhat softer Moszkowski EOS. For an EOS softer than the Moszkowski form (i.e., for $\sigma < \frac{2}{3}$), damping has even less effect at $n < 5n_0$: the results lie between the original and modified Moszkowski results shown in Figs. 1-4.

For those who prefer the Skyrme phenomenological potential ($\sigma = 1$), we suggest the modified form (10) as a simple softening of the phenomenological EOS in order to fit the incompressibility observed at nuclear density and to remain causal at higher densities. This modified Skyrme EOS can be approximated by the simple form

$$p = A \left[\left(\frac{n}{n_0} \right)^2 - 1 \right], \quad (22)$$

with $A = 1.33 \text{ MeV fm}^{-3}$.

In this paper we have considered the equation of state of nuclear matter for which the proton fraction $Y_e = 0.5$. During neutronization of an Fe-Ni stellar core, this fraction changes moderately from $Y_e = 0.46$ to about $Y_e = 0.30$. How will stellar collapse and the possibility of supernova blowoff be affected by softening the phenomenological EOS? It now appears that if stellar collapse is

arrested and mass ejection takes place, the responsible mechanism must be that of a hydrodynamic reflected shock formed where the infalling outer core strikes the inner core.¹⁵ It had been believed¹⁶ that deep core penetration and a violent bounce shock were necessary for mass ejection. If this were the case, then softening the cold EOS would lead to deeper core penetration and easier ejection of more mass and kinetic energy. We now believe,^{15,17} however, that with realistic equations of state for warm nuclear matter, shock heating leads to sufficient thermal pressure to arrest infall and explosively eject part of the outer core. This shock-heating mechanism^{15,17} operates independently of the inner core EOS, provided it is stiff enough to arrest infall. If this is the efficient reflected shock mechanism, then the principal effect of softening the cold nuclear EOS is to transmit more kinetic energy to whatever mass is ejected.

In cold neutron stars, the softer nuclear EOS would lead to higher central densities and maximum neutron star masses. Because nuclear matter is, in any case, rather incompressible, the effect is not large.

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The term involving t_1 and t_2 arises from the P -wave nucleon-nucleon interaction, which also produces important surface terms in finite nuclei. We could easily extend the approach of the present paper to include a fit to single-particle binding energies and radii of finite nuclei. In addition, one could include additional terms to account for the symmetry energy of $N \neq Z$ system.

⁹When we include the terms in t_1 and t_2 (see Ref. 8), we find $dn_0^2/bn_0 = n_0 t_3/6t_0 \approx 0.36$ for Skyrme force I.

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