Introduction to Bayesian Inference for Statistical Model Fitting (DRAFT)

Luigi Acerbi

Department of Computer Science
University of Helsinki
Finnish Center for Artificial Intelligence FCAI

BAMB! Summer School – Day 2
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Learning objectives

By the end of this lecture/tutorial, we will:

- ...

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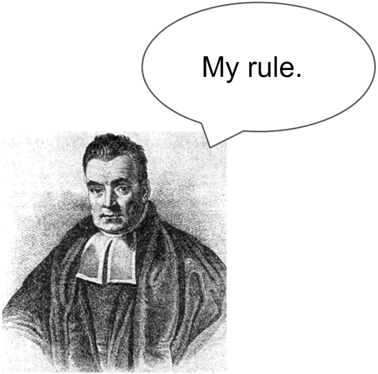

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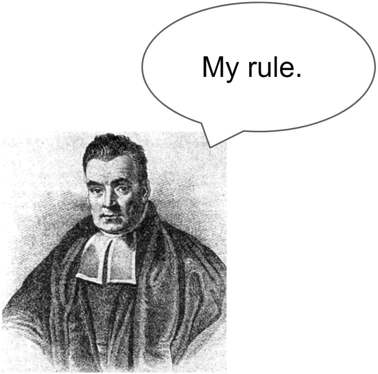

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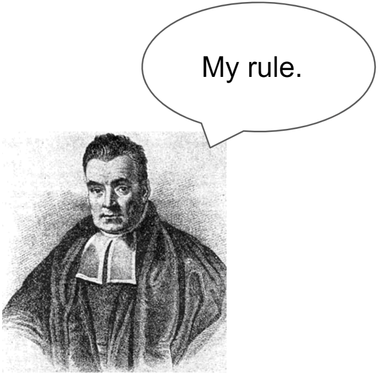

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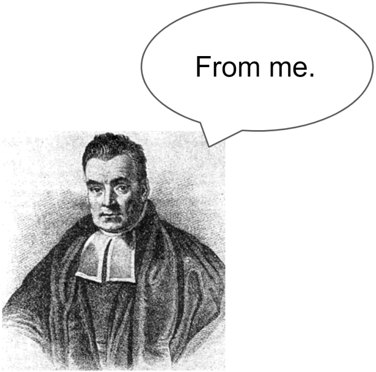

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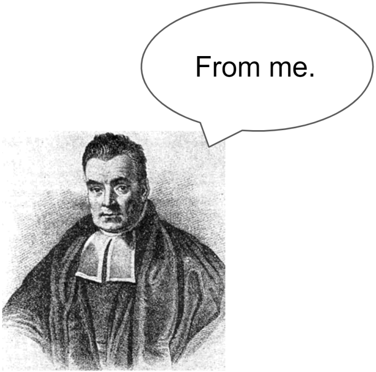

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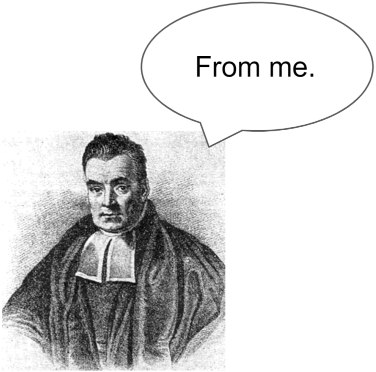

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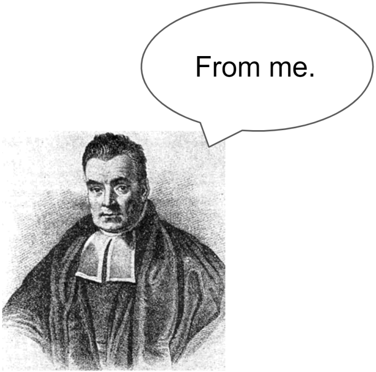

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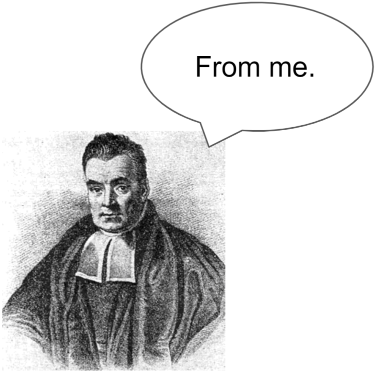

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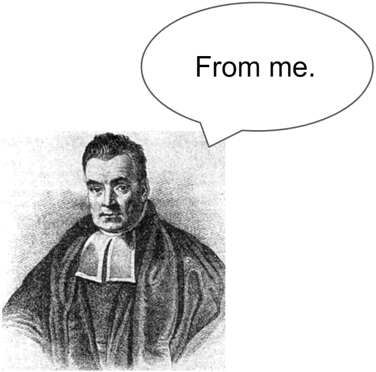

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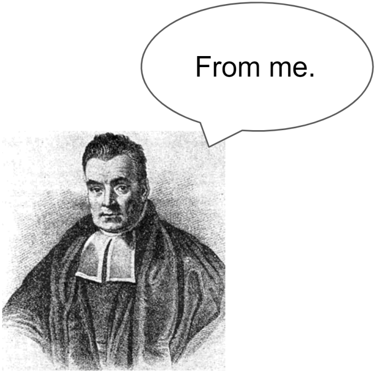

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Bayesian probability

- We are treating both data and $\boldsymbol{\theta}$ as random variables.
- Probability as degree of belief.

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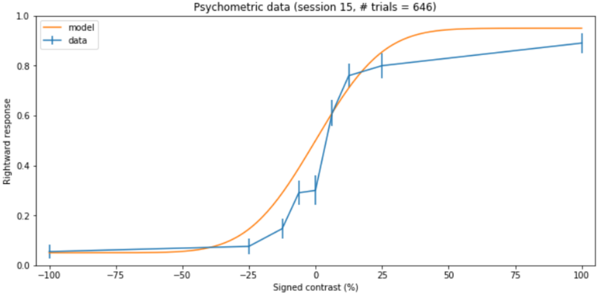
- Better predictions

Data and model

- Same data from before (IBL mouse behavioral data)
- Same model as before (psychometric function model)



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- The normalization is $Z = \int p(\text{data}|\mu_\star, \sigma, \lambda_\star, \gamma_\star)p(\sigma)d\sigma$

Hacking time I

Let's do Bayesian inference by hand!

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- Remember that the prior is a probability distribution $\int p(\theta) d\theta = 1$
- Okay, but how do I pick a prior for each parameter?

Example priors: uniform box

- Bounded parameter
- Uniform in the full range
- **Pros:** Easy to define and to justify (if wide bounds)
- **Cons:** Non-informative

Example priors: tent/trapezoidal

- Bounded parameter
- Uniform in a range, then falls off
- Can use the hard/plausible bounds defined previously
- **Pros:** Still easy to define, "weakly" informative
- **Cons:** Need some thought to define the plausible range

Example priors: smoothed tent/trapezoidal

- Bounded parameter
- Just like tent prior but with smooth edges
- **Pros:** Better numerical properties than tent prior
- **Cons:** More complex to implement (use provided functions)

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Hot take:

- I generally recommend bounded parameters
- Half-bounded / unbounded parameters $\Rightarrow$ numerical issues

Hacking time II

Let's have a look at the priors.

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- Not really – a grid only works in low dimension ($D \sim 1 - 4$)
- Curse of dimensionality: N points per dimension $\Rightarrow N^D$ points
- We need inference algorithms!

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- ...in practice, way more complex algorithms
  - Inference is harder!
  - Need to compute a full distribution instead of a single point

Main families of general-purpose inference algorithms

1. Markov Chain Monte Carlo (MCMC)
2. Variational inference

(there are others)

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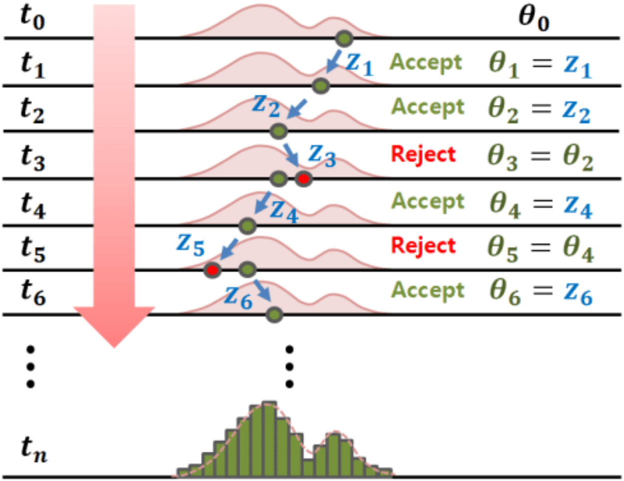
  - In practice, lot of tweaking to ensure convergence of the Markov chain
  - State-of-the-art MCMC methods are (to a degree) self-tuning
  - Still a lot of tweaking involved

Example MCMC algorithm: Metropolis-Hastings



Source: Jin et al. (2019)

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VI casts Bayesian inference into optimization + integration

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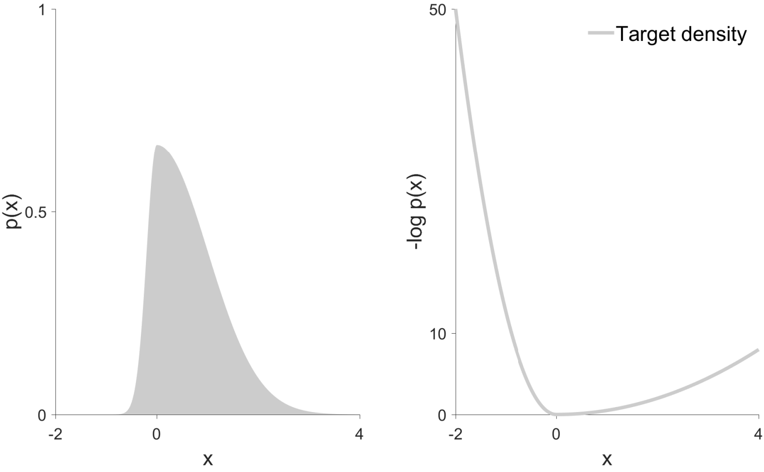

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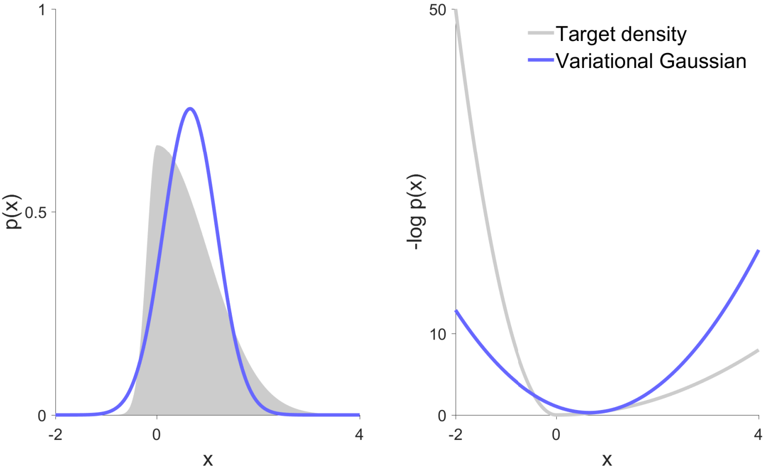

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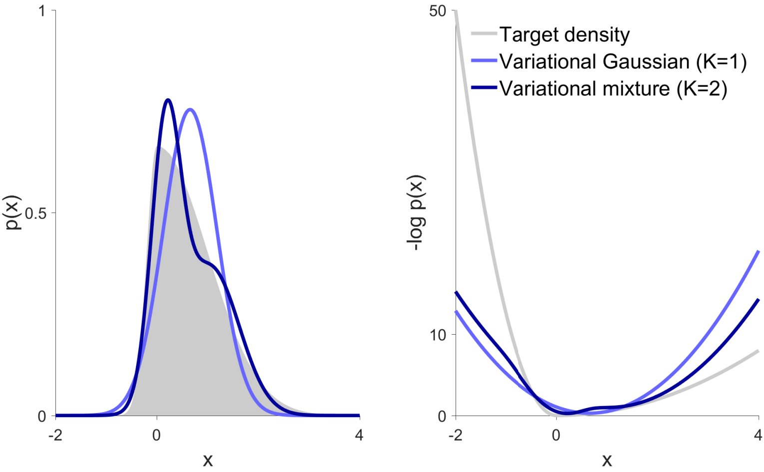

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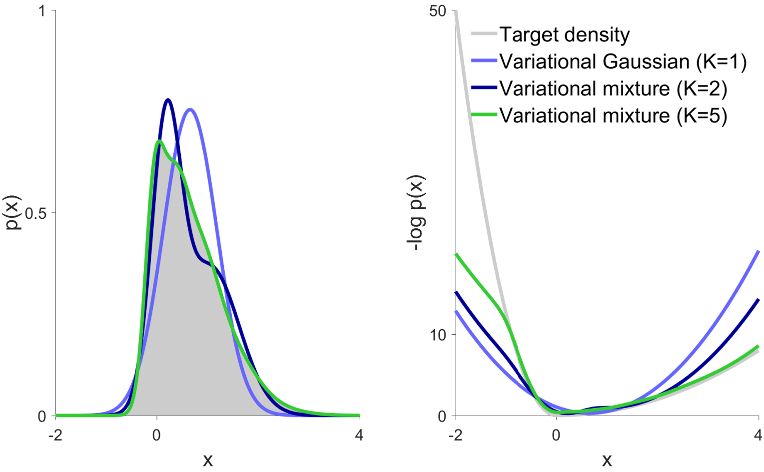

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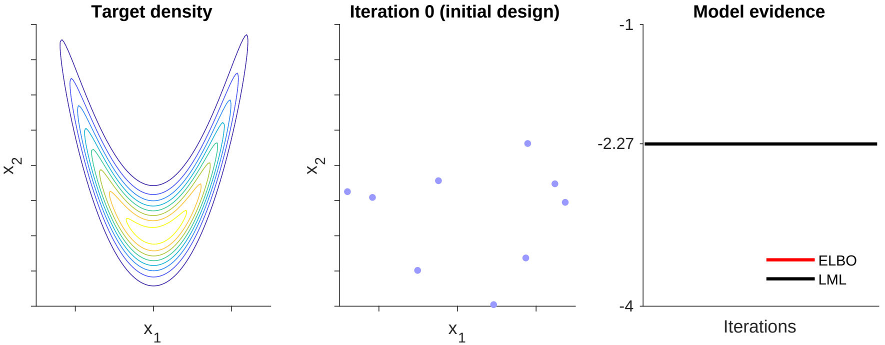

Acerbi, *NeurIPS* (2018; 2020)

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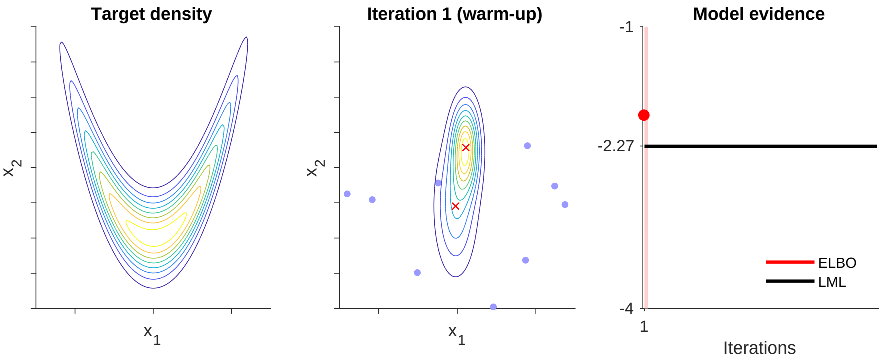

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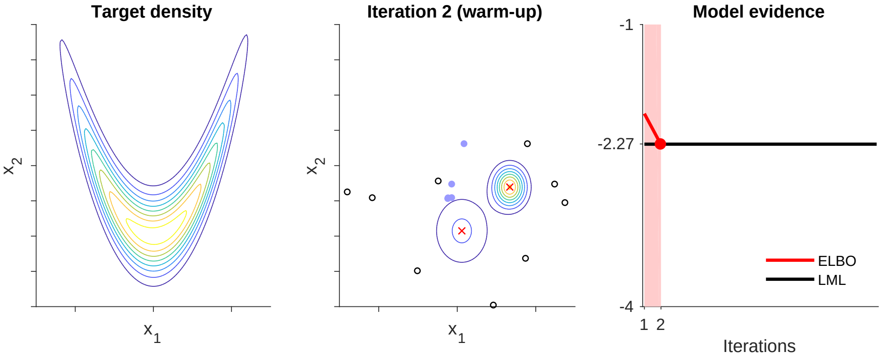

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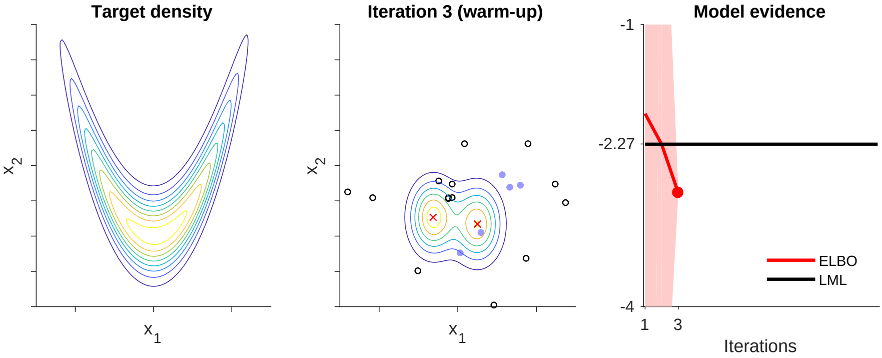

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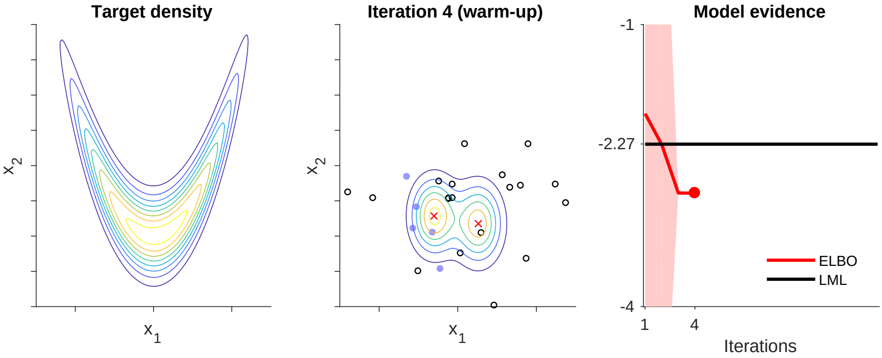

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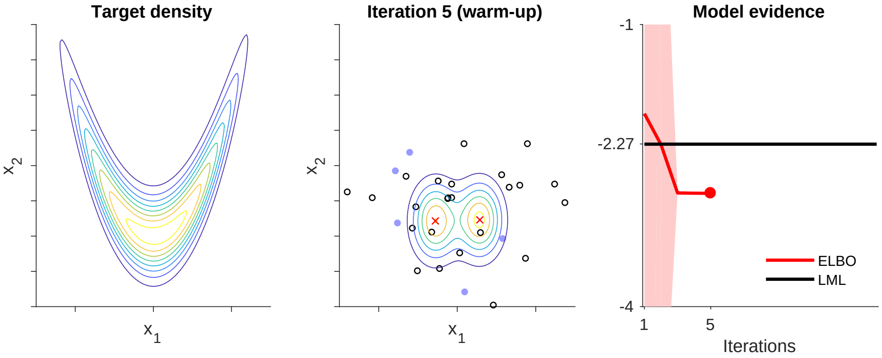

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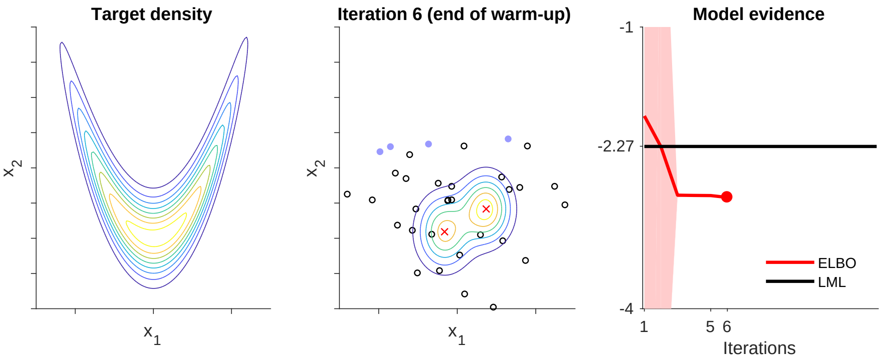

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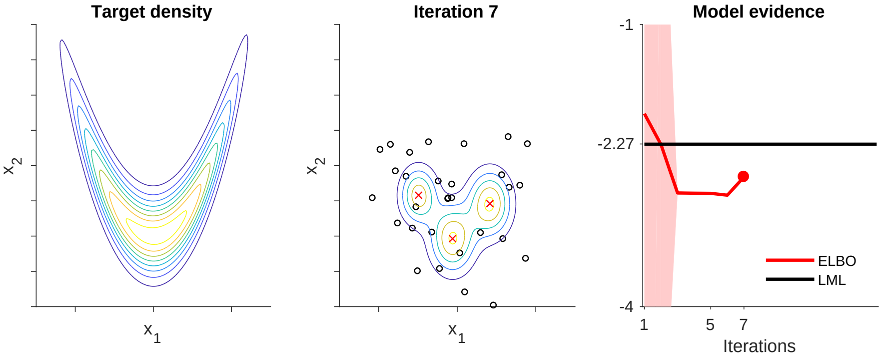

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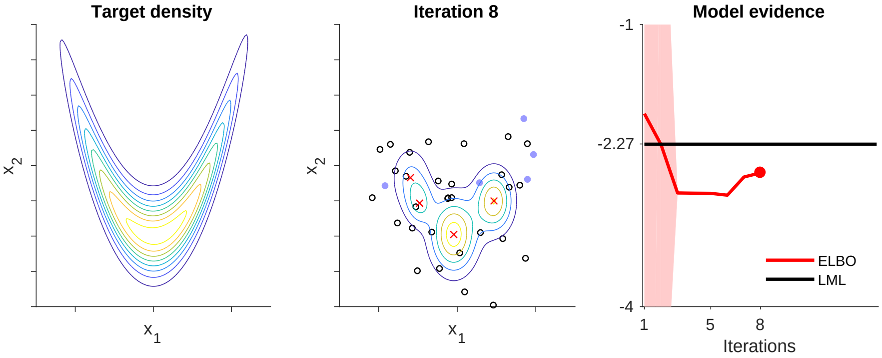

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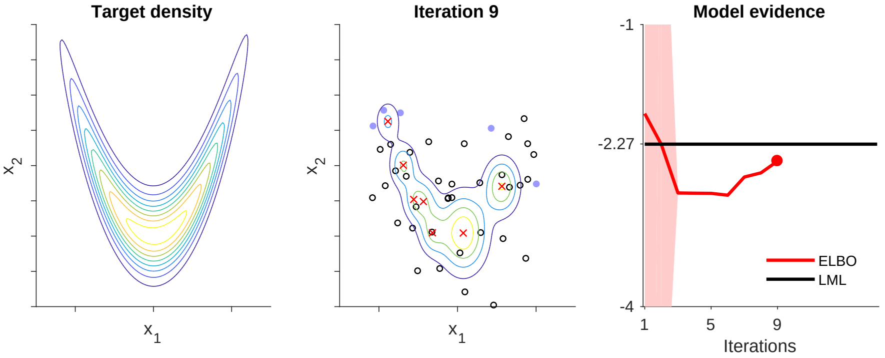

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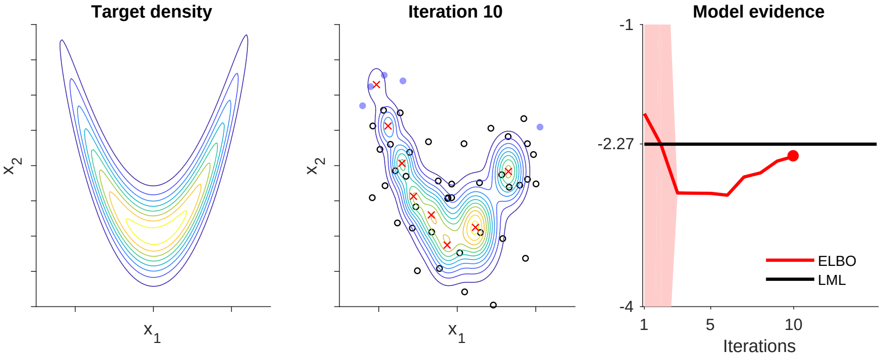
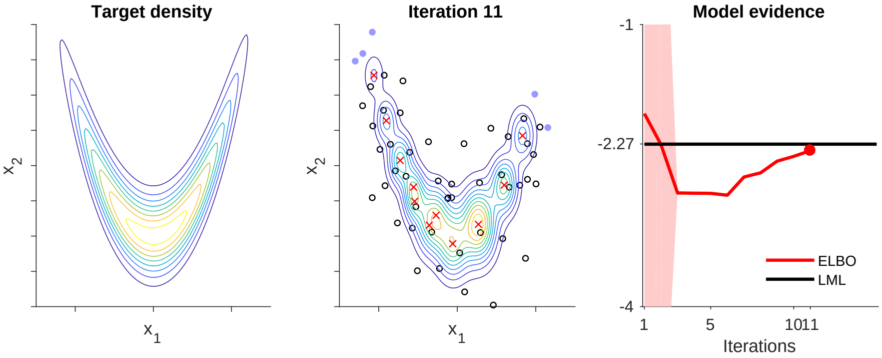


Acerbi, *NeurIPS* (2018; 2020)

Variational Bayesian Monte Carlo (VBMC)



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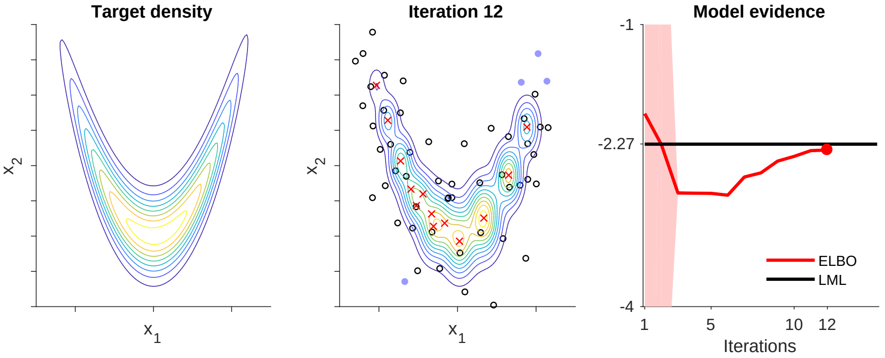

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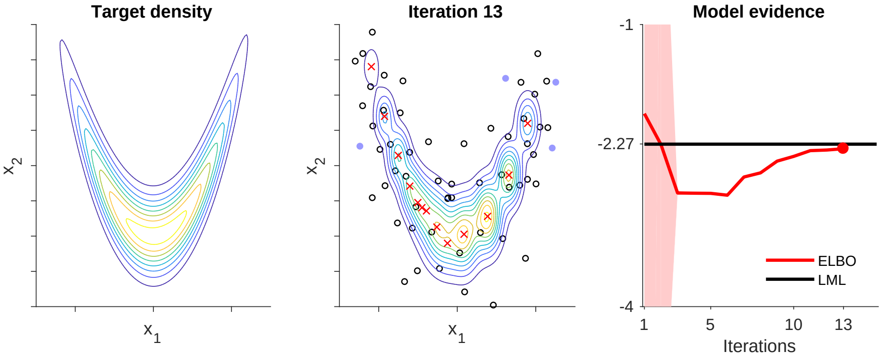

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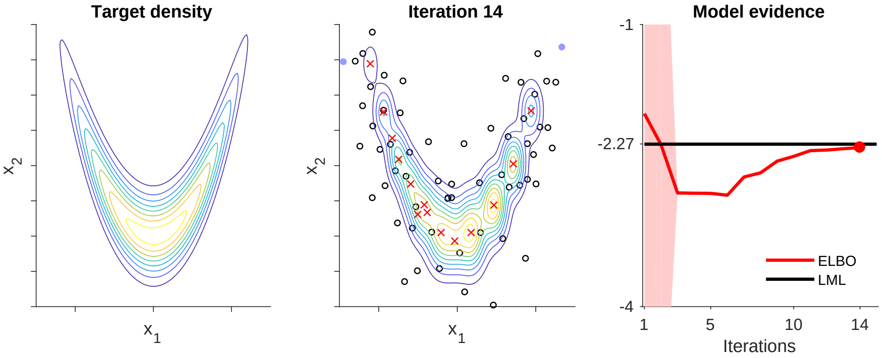

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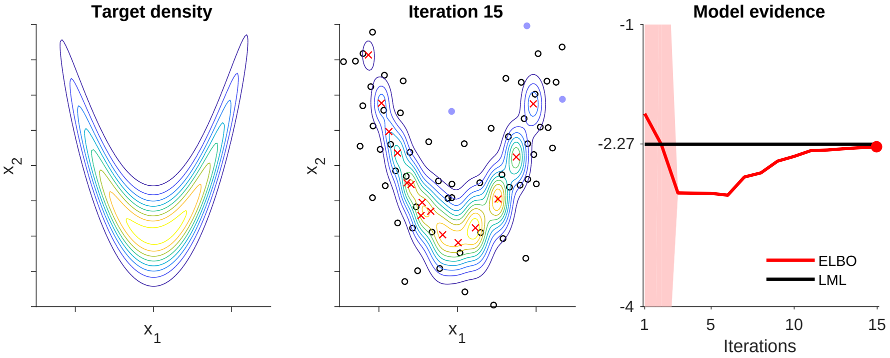

Acerbi, *NeurIPS* (2018; 2020)

Hacking time III

Let's set up and run a Bayesian inference algorithm

OK so we have a posterior what now

. . .

Predictions

...

Final slide

- Contact me at `luigi.acerbi@helsinki.fi`
- Optimization demos: `github.com/lacerbi/optimviz`

MATLAB toolboxes:

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Thanks!

(Time for questions?)