# Introduction to Bayesian Inference for Statistical Model Fitting

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BAMB! Summer School – Day 2
September 2022

1. Introduction and motivation
   - Bayes rule
   - Bayesian inference for model fitting

2. Computing the posterior distribution
   - Computing the posterior "by hand"
   - Inference algorithms
   - The prior

3. Making use of a Bayesian posterior
   - Visualizing the posterior
   - Posterior prediction

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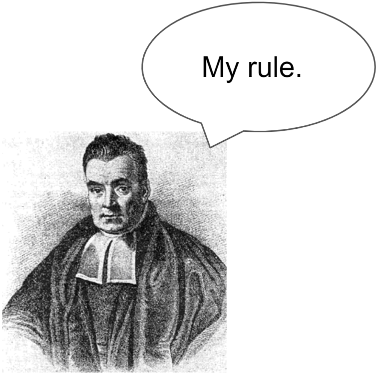


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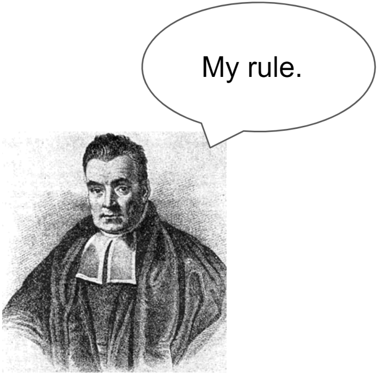


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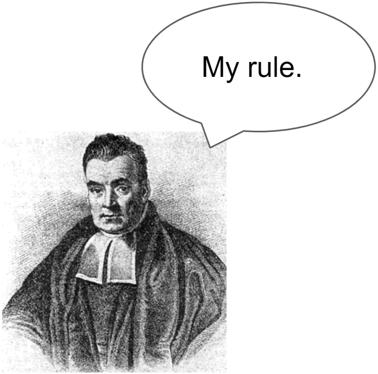


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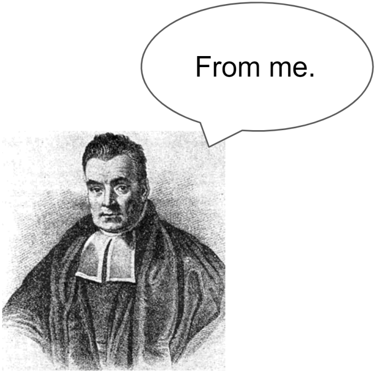

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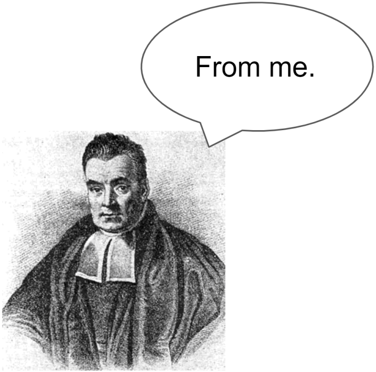


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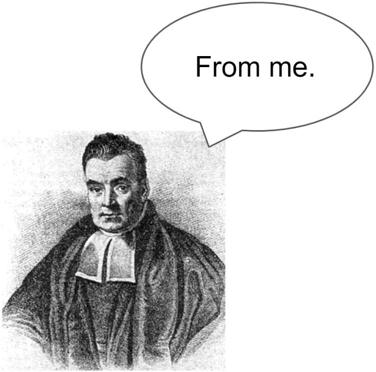


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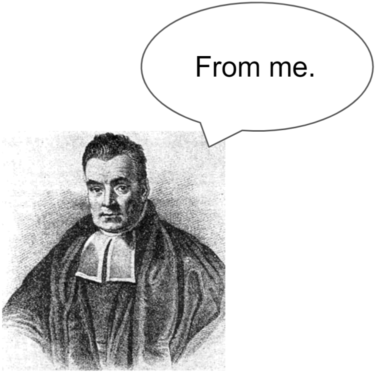


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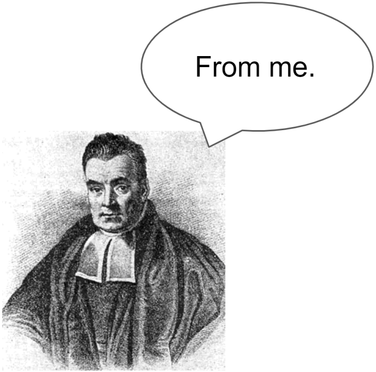


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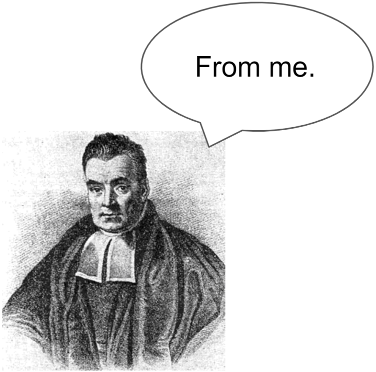


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# What's new in Bayesian inference for model fitting?

The output of Bayesian inference is a probability distribution (posterior) over model parameters:

$$p(\boldsymbol{\theta}|\text{data})$$

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Questions:

1. How do we compute $p(\boldsymbol{\theta}|\text{data})$?
2. What do we do once we have $p(\boldsymbol{\theta}|\text{data})$?
3. Why should we bother?

# Why Bayesian inference?

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- Optimal experiment design
- Robustness
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- Better predictions

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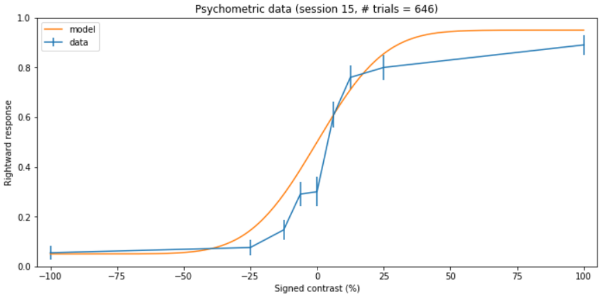
# Data and model

- Same data from before (IBL mouse behavioral data)
- Same model as before (psychometric function model)
  - Parameters $\boldsymbol{\theta} = (\mu, \sigma, \lambda, \gamma)$



# Let's just apply Bayes rule!

- We consider a 1-D posterior (one free parameter, $\sigma$)
- We fix $\mu, \lambda, \gamma$ to some values $\mu_\star, \lambda_\star, \gamma_\star$
- We assume a uniform prior $p(\sigma)$ for $\sigma \in [1, 100]$
- We compute

$$p(\sigma | \mu_\star, \lambda_\star, \gamma_\star, \text{data}) = \frac{p(\text{data} | \mu_\star, \sigma, \lambda_\star, \gamma_\star) p(\sigma)}{Z}$$

- The normalization is $Z = \int p(\text{data} | \mu_\star, \sigma, \lambda_\star, \gamma_\star) p(\sigma) d\sigma$

Let's do this!

# Bayesian inference solved?

- Not really – a grid only works in low dimension ($D \sim 1 - 4$)
- Curse of dimensionality: $N$ points per dimension $\Rightarrow N^D$ points
- We need inference algorithms!

# Inference algorithms

- A general-purpose inference algorithm
  - takes as input an inference problem (likelihood, prior,. . . )
  - returns an approximate posterior
- Abstractly, similar to optimization. . .
  - take as input an optimization problem (target function)
  - return the optimum
- . . . in practice, way more complex algorithms
  - Inference is harder

# Main families of inference algorithms

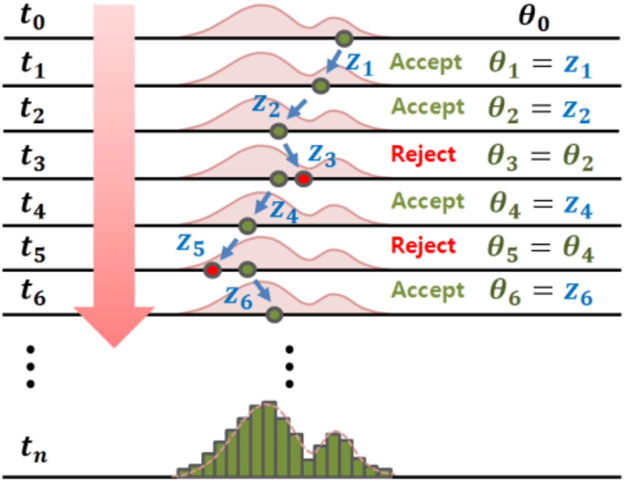
1. Markov Chain Monte Carlo (MCMC)
2. Variational inference

(there are others)

# Markov Chain Monte Carlo (MCMC)

- Generates a random sequence $\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \ldots$ (a Markov chain)
- Various rules for drawing $\boldsymbol{\theta}_{n+1} | \boldsymbol{\theta}_n$ depending on the algorithm
  - These will generally depend on $p(\boldsymbol{\theta}_n, \text{data})$, $p(\boldsymbol{\theta}_{n+1}, \text{data})$
- **Output:** A set of samples $\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_N$
- **If all goes well**, $\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_N \sim p(\boldsymbol{\theta}|\text{data})$
  - In practice, lot of tweaking to ensure convergence of the Markov chain
  - State-of-the-art MCMC methods are (to a degree) self-tuning
  - Still a lot of tweaking involved

# Example MCMC algorithm: Metropolis-Hastings




Source: Jin et al. (2019)

# Variational inference

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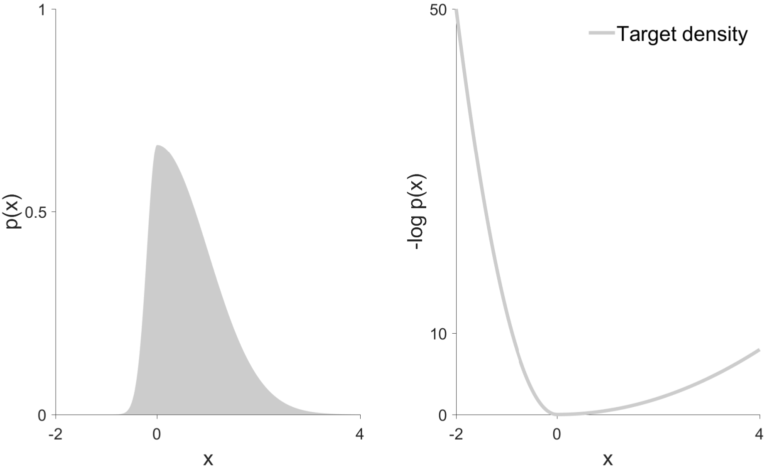
VI casts Bayesian inference into optimization $+$ integration

# Variational inference: example



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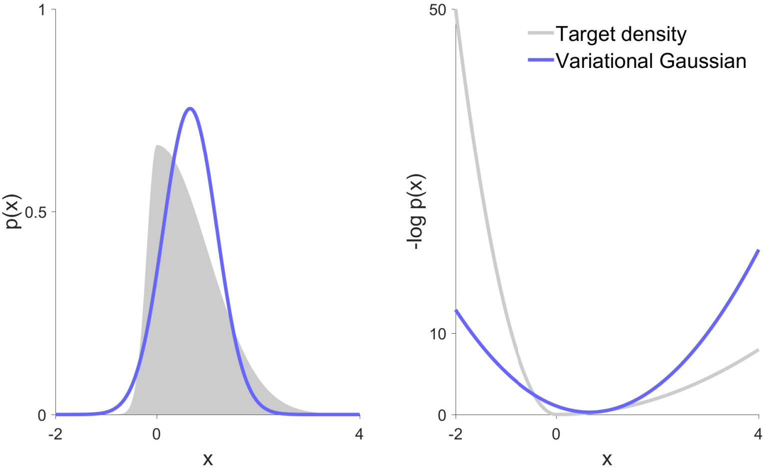


$$q_\phi(x) = \mathcal{N}\left(x, \mu, \sigma^2\right) \qquad \phi = (\mu, \sigma^2)$$

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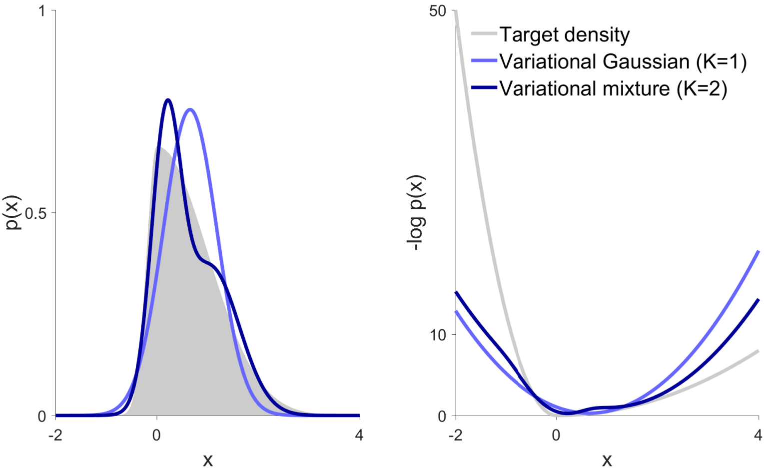


$$q_\phi(x) = \sum_{k=1}^{K} w_k \mathcal{N}\left(x, \mu_k, \sigma_k^2\right) \qquad \phi = \left(w_k, \mu_k, \sigma_k^2\right)_{k=1}^{K}$$

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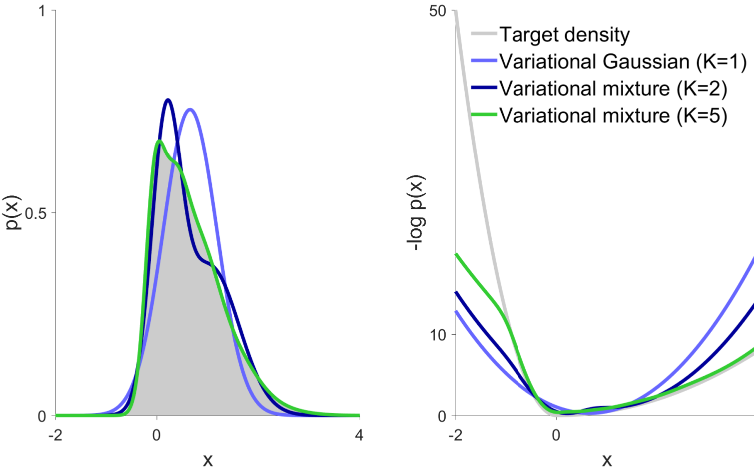


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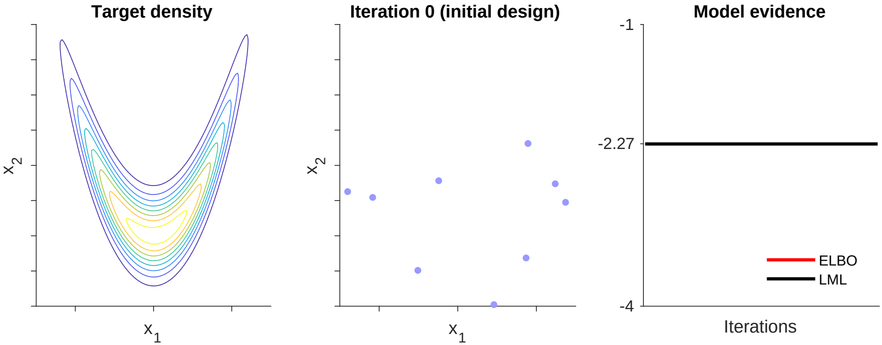


Acerbi, *NeurIPS* (2018; 2020)

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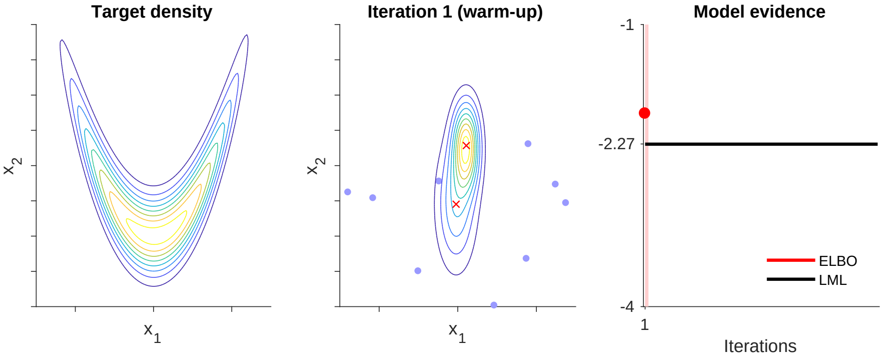


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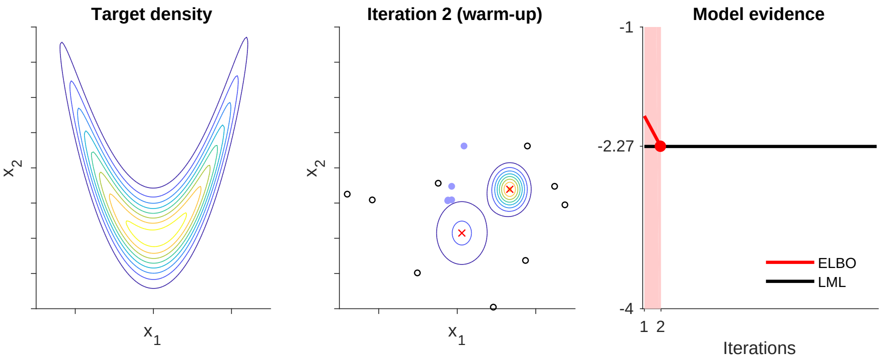


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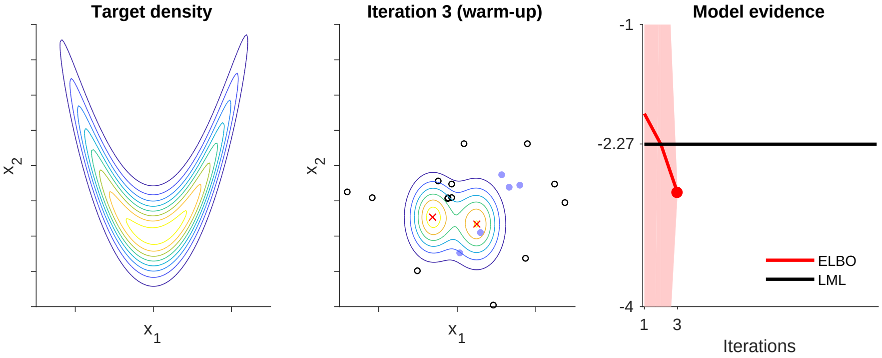


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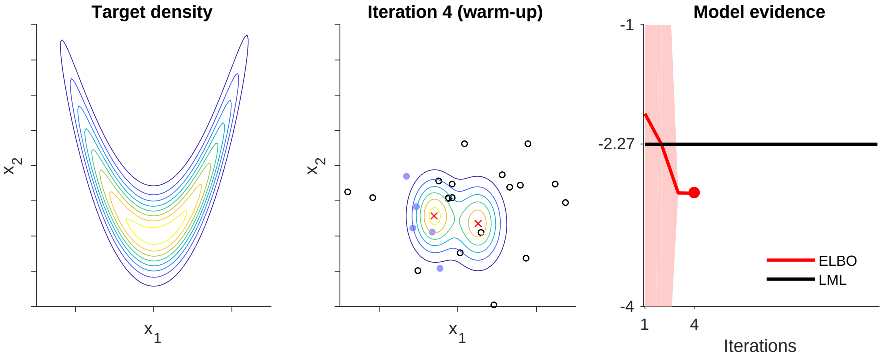


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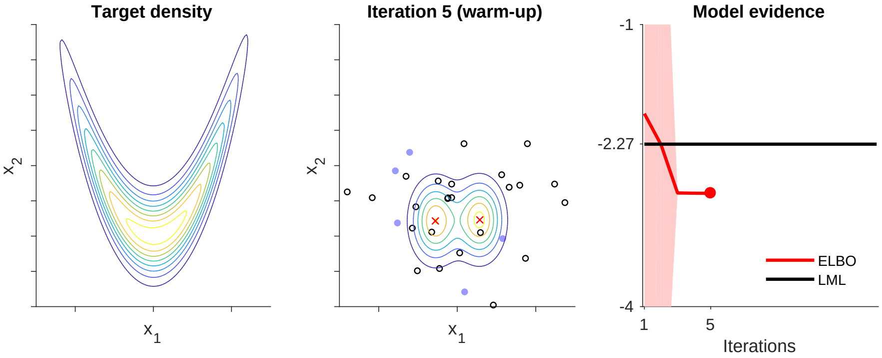


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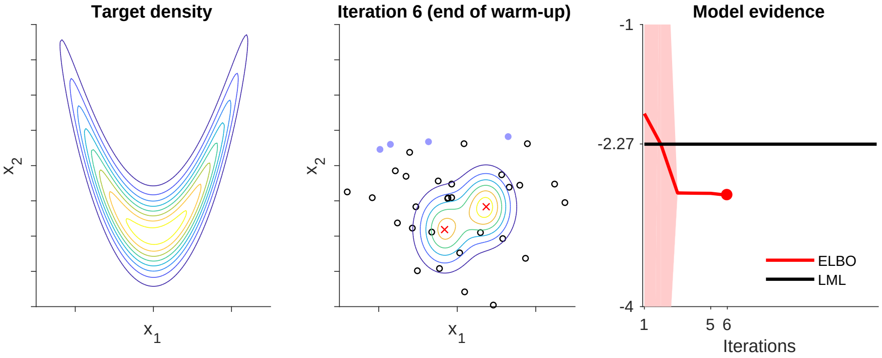


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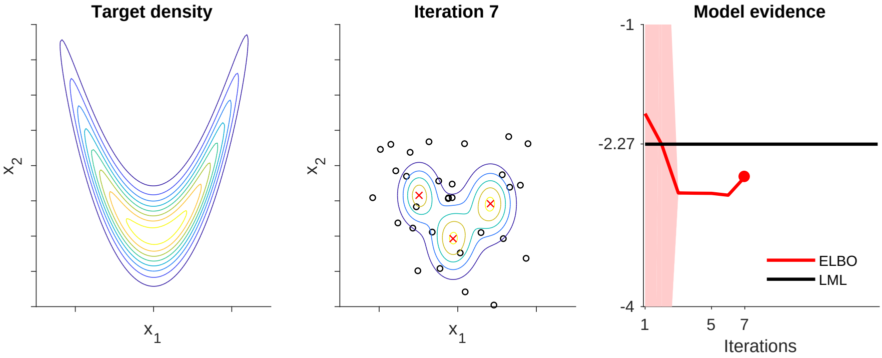


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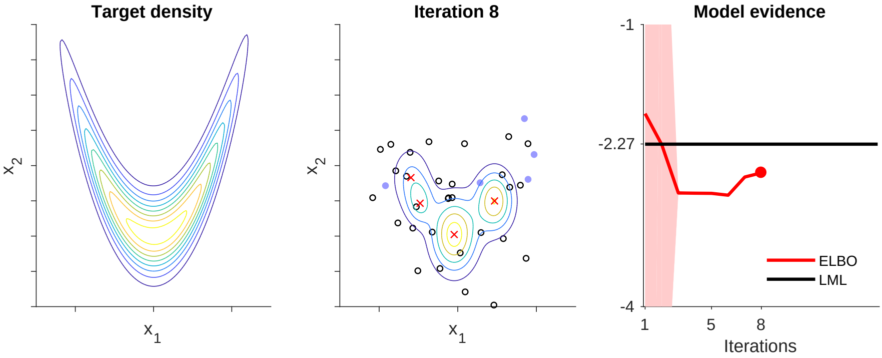


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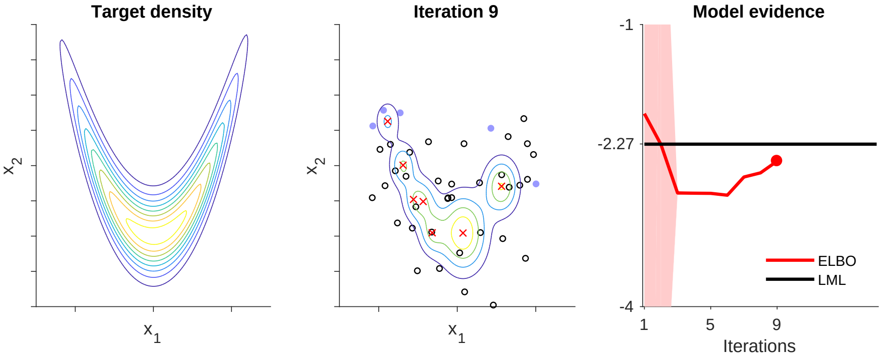


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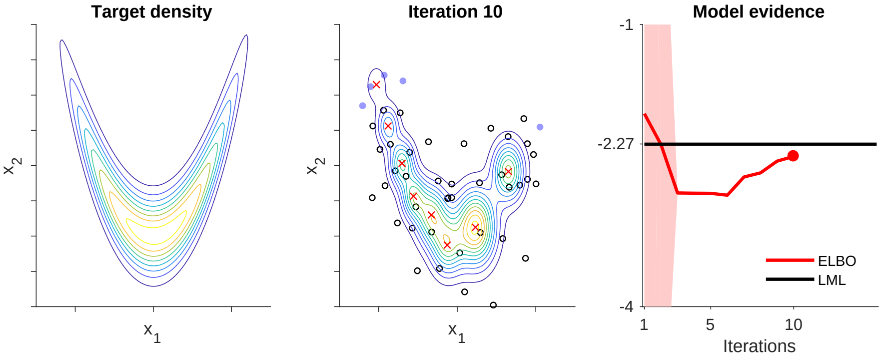


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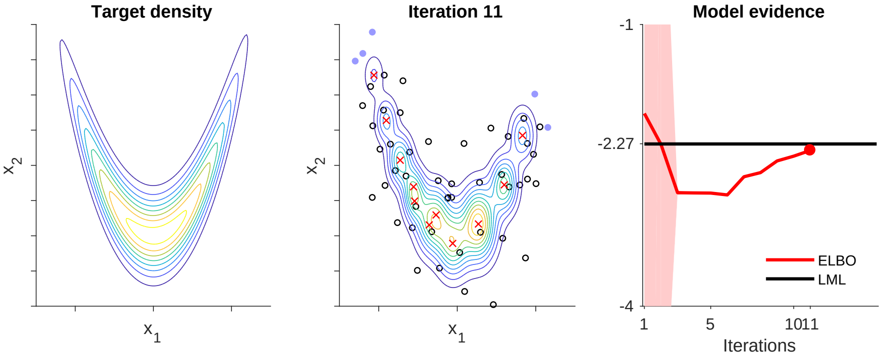


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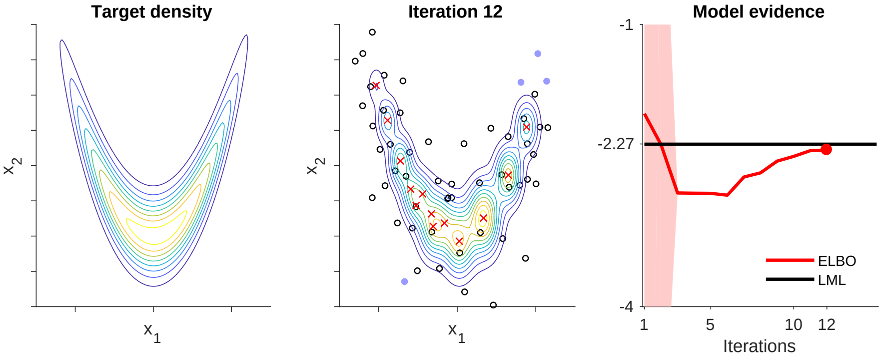


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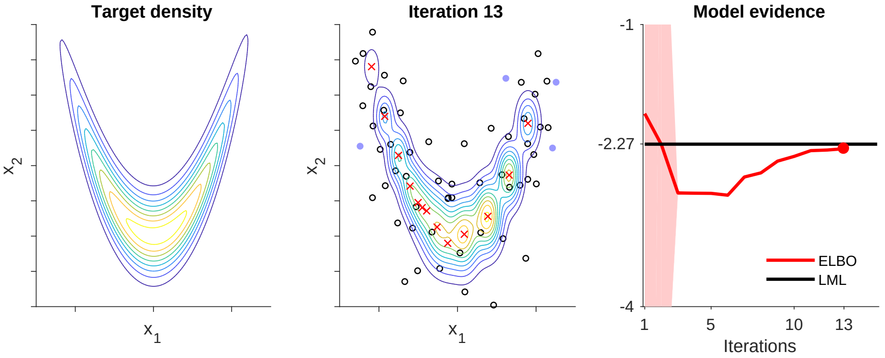


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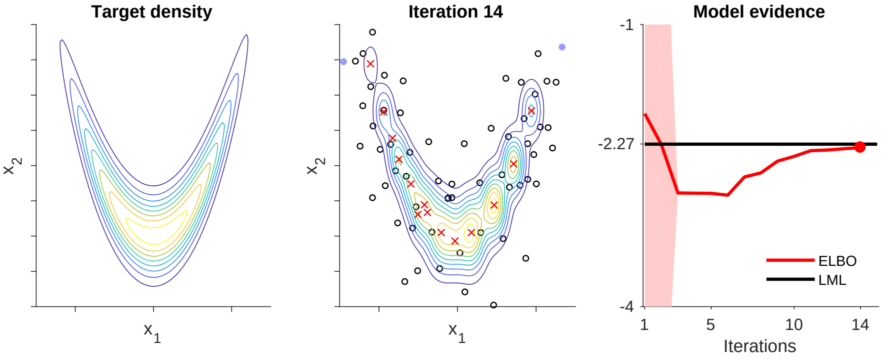


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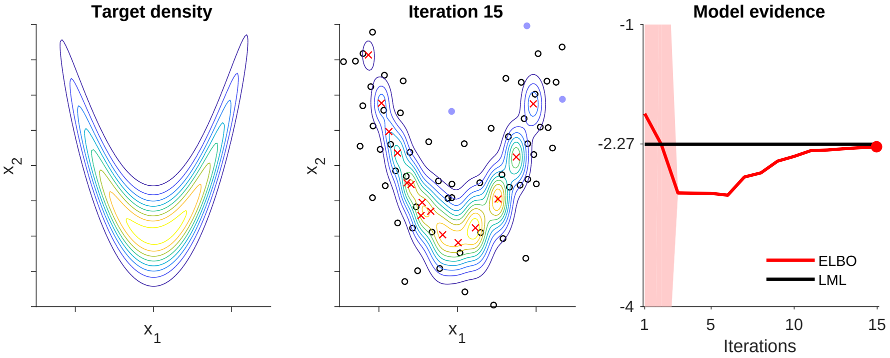


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# Pick your prior

- In Bayesian inference you need a prior over parameters, $p(\boldsymbol{\theta})$
- Common choice: independent priors $p(\boldsymbol{\theta}) = \prod_{d=1}^{D} p(\theta_d)$
  - Choose the prior $p(\theta_d)$ for each parameter
  - Independent prior does not mean that the posterior is independent!
- Remember that the prior is a probability distribution $\int p(\theta) d\theta = 1$
- Okay, but how do I pick a prior for each parameter?

# Example priors: uniform box

- Bounded parameter
- Uniform in the full range
- **Pros:** Easy to define and to justify
- **Cons:** Non-informative

# Example priors: tent

- Bounded parameter
- Uniform in a range, then falls off
- Uses the hard/plausible bounds defined previously
- **Pros:** Still easy to define, “weakly” informative
- **Cons:** Need some thought to define the plausible range

# Example priors: smoothed tent

- Bounded parameter
- Just like tent prior but with smooth edges
- **Pros:** Better numerical properties than tent prior
- **Cons:** More complex to implement (can use provided functions)

# OK so we have a posterior what now

...

# Predictions

. . .

# Final slide

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- Optimization demos: `github.com/lacerbi/optimviz`

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(Time for questions?)