Practical and efficient Bayesian model fitting with Variational Bayesian Monte Carlo (PyVBMC)

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What this is all about

By the end of this lecture/tutorial, we will:

Perform Bayesian inference on a real dataset and model from neuroscience

- Recap the basics of statistical modelling
- Define the psychometric model used in cognitive & neuroscience
- Explain the Bayesian approach to model fitting
- Briefly introduce variational inference algorithms
- Set up and run PyVBMC on a real dataset

1. A recap of statistical modelling
   - Of models and likelihoods
   - The psychometric function

2. Bayesian model fitting
   - Refresher of Bayesian inference
   - Bayesian inference for model fitting

3. Computing the posterior distribution
   - Setting things up
   - Inference algorithms
   - Making use of a Bayesian posterior

4. Hands-on tutorial

What is a model?

The best material model of a cat is another, or preferably the same, cat.

Wiener, *Philosophy of Science* (1945) (with Rosenblueth)

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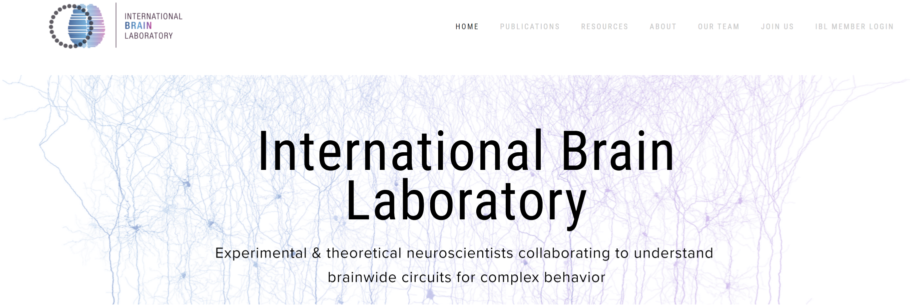
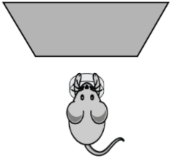
We need some data

Data from International Brain Laboratory (IBL)



`https://www.internationalbrainlab.com`

IBL Task



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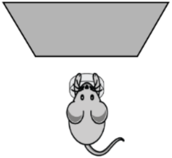

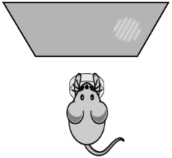

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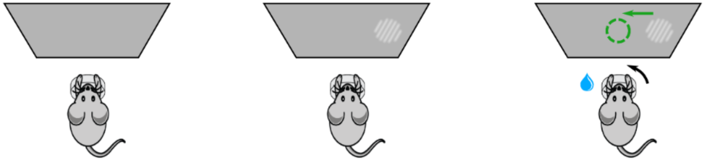

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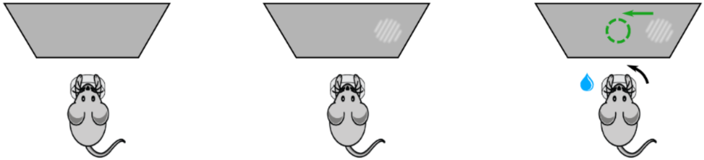

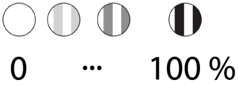

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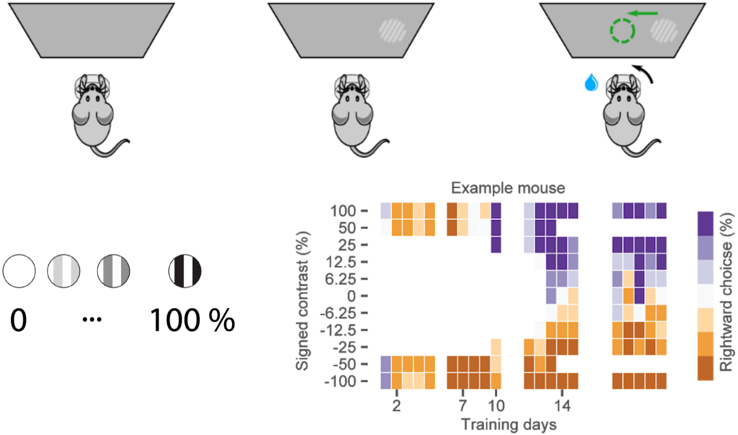

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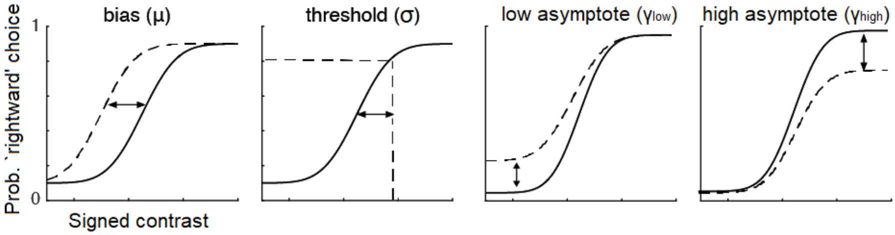
The psychometric function



- Data: (signed contrast, choice) for each trial
- Parameters $\boldsymbol{\theta}$: $(\mu,\ \sigma,\ \gamma_{\text{low}},\ \gamma_{\text{high}})$

$$p(\text{rightward choice}|s, \boldsymbol{\theta}) = \gamma_{\text{low}} + (1 - \gamma_{\text{low}} - \gamma_{\text{high}}) \cdot F(s; \mu, \sigma)$$

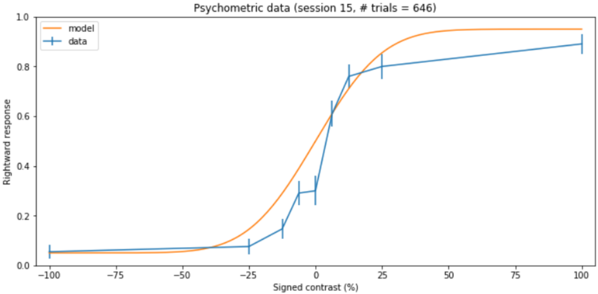
The psychometric function (alt version)

- Default decision process $F(s; \mu, \sigma)$
- Lapses with probability $\lambda \in [0, 1]$ (*lapse rate*)
- If lapse, respond 'rightward' with probability $\gamma \in [0, 1]$ (*lapse bias*)
- Parameters $\boldsymbol{\theta}$: $(\mu,\ \sigma,\ \lambda,\ \gamma)$

$$p(\text{rightward choice}|s, \boldsymbol{\theta}) = \lambda\gamma + (1 - \lambda) \cdot F(s; \mu, \sigma)$$

Metric for model fitting

We need a quantity to measure *goodness of fit*



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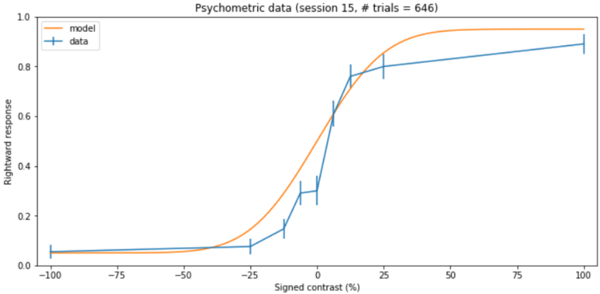

- Mean squared error?

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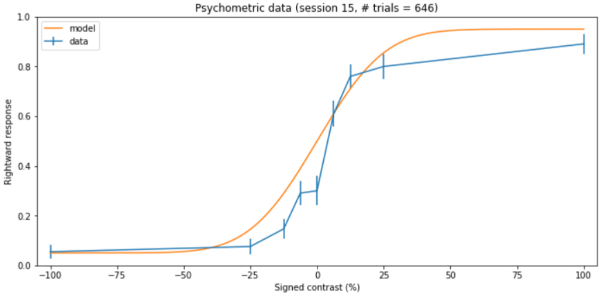

- ~~Mean squared error?~~
- The likelihood $p(\text{data}|\boldsymbol{\theta}) \equiv L(\boldsymbol{\theta}; \text{data})$

The (log) likelihood

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- Simplest case (conditionally independent trials):

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- Model building: Write function with
  - Input: $\boldsymbol{\theta}$ and data
  - Output: $\log p(\text{data}|\boldsymbol{\theta})$

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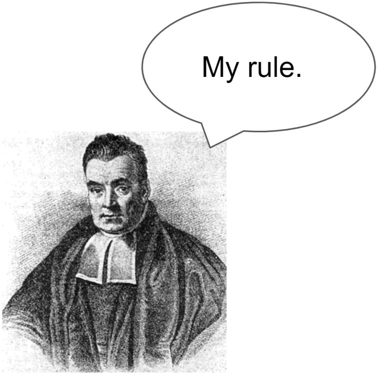

$$p(\boldsymbol{\theta}|\mathrm{data}) = \frac{p(\mathrm{data}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathrm{data})}$$

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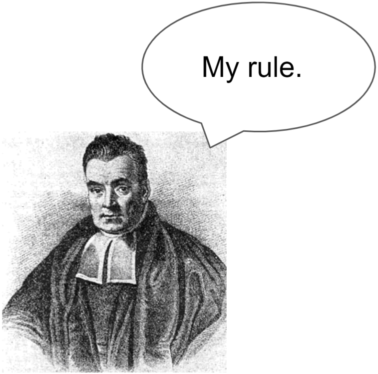

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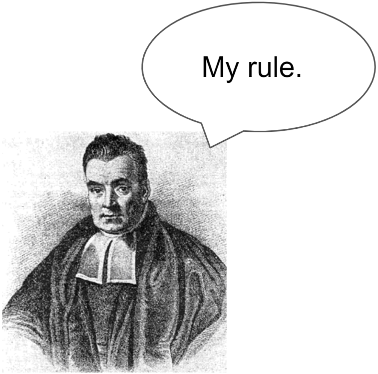

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What's special in Bayesian inference for model fitting?

The output of Bayesian inference is a probability distribution (posterior) over model parameters:

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Instead, other methods (like maximum-likelihood estimation or loss minimization) only return a single best point estimate $\boldsymbol{\theta}_\star$.

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- Optimal experiment design
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- Hyperparameter tuning
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- Better predictions

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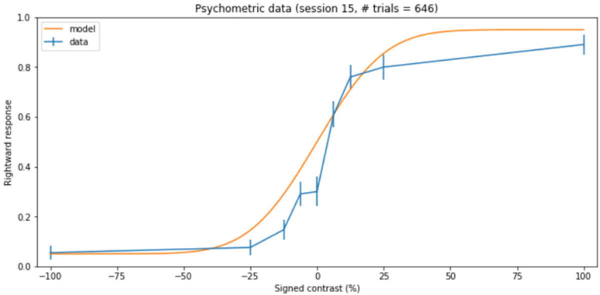
Data and model

- Data: IBL mouse behavioral data
- Model: psychometric function



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  - Independent prior does not mean that the posterior is independent!
- Remember that the prior is a probability distribution $\int p(\theta) d\theta = 1$
- Okay, but how do I pick a prior for each parameter?
  - Bounded parameter: Uniform, ...
  - Unbounded parameter: Gaussian, Student's t...
  - Would deserve a separate lecture

Inference algorithms

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- Example families of algorithms
  1. Markov Chain Monte Carlo (MCMC)
  2. Variational inference
  3. Others

Variational inference

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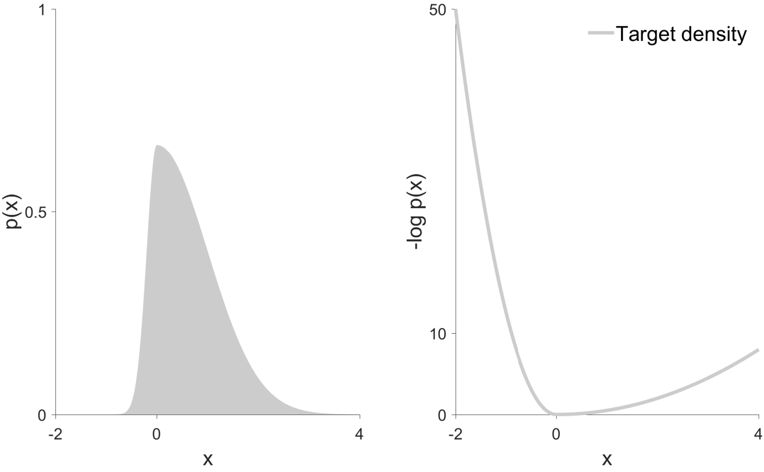
VI casts Bayesian inference into optimization + integration

Variational inference: example



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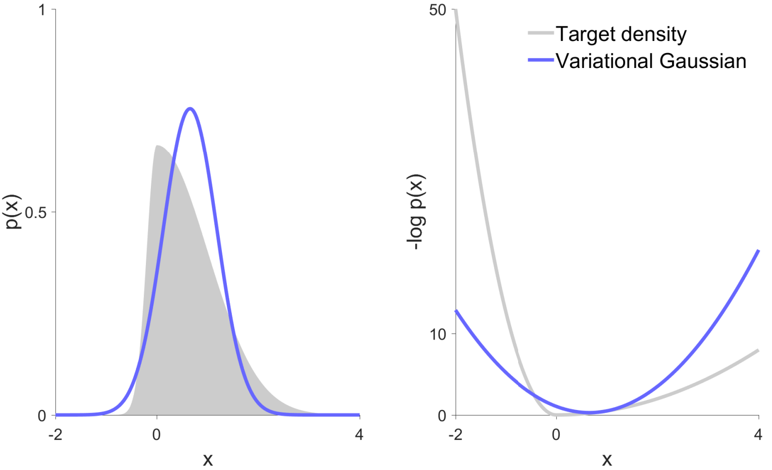

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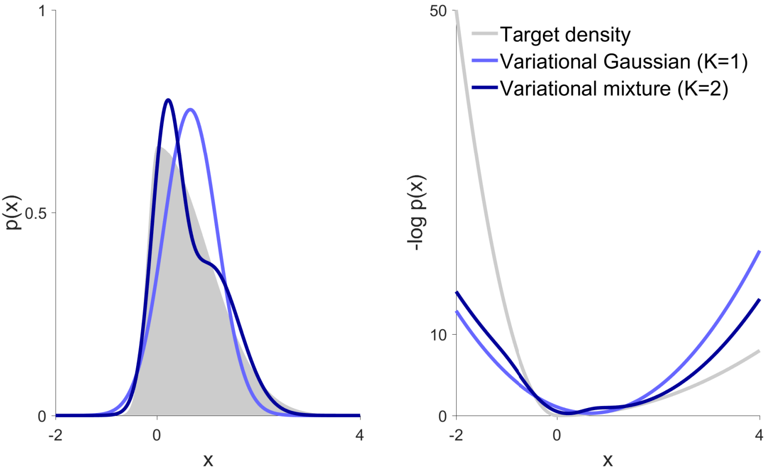

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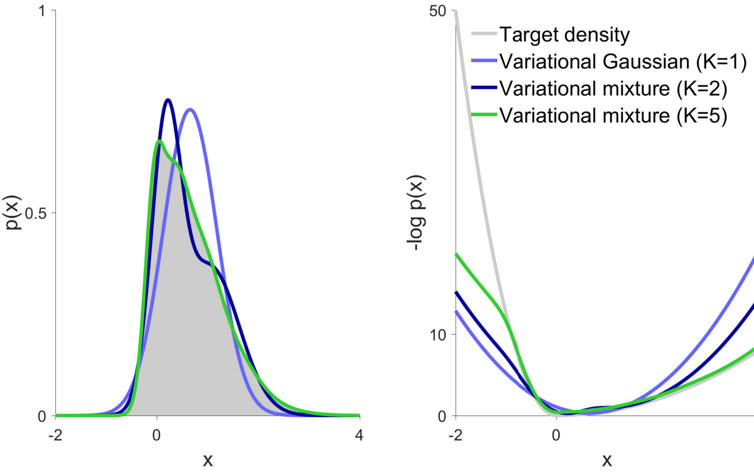

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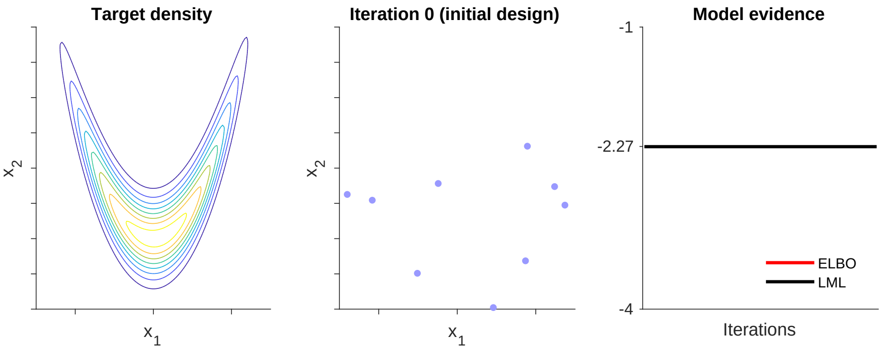
Variational Bayesian Monte Carlo (VBMC)



Acerbi, *NeurIPS* (2018; 2020)

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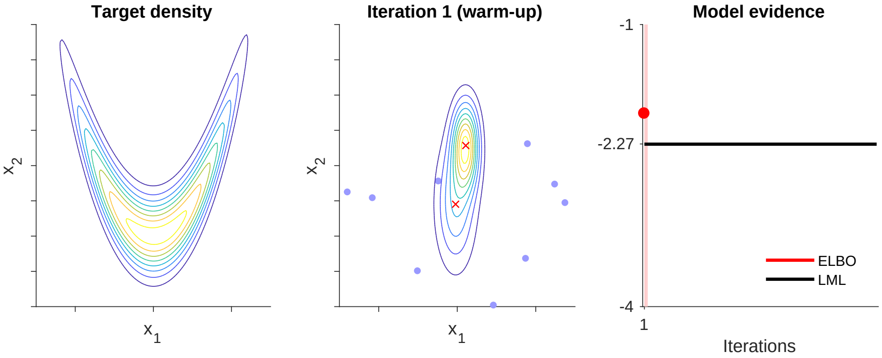

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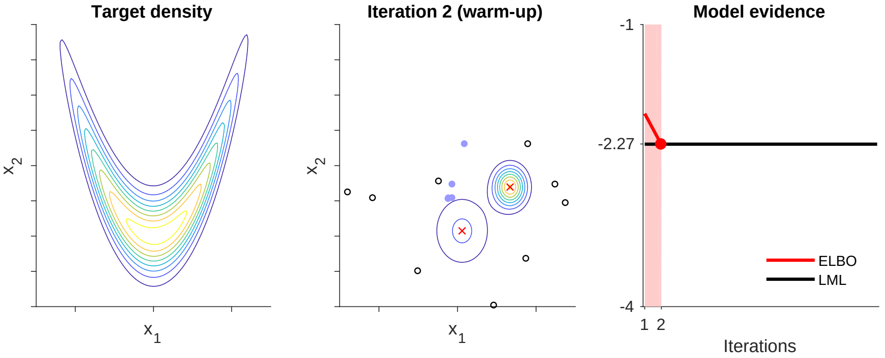

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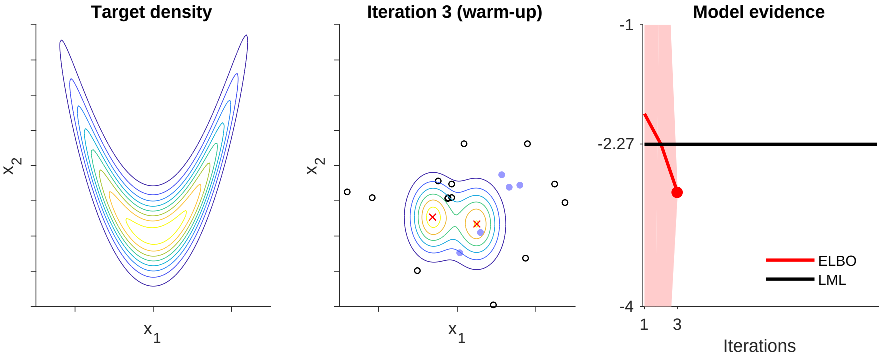

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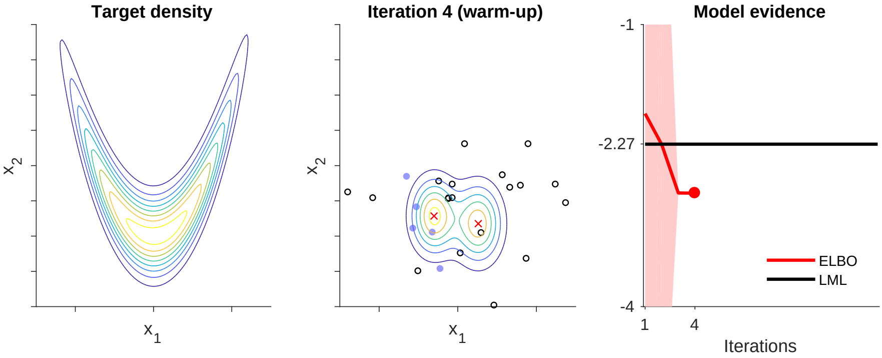

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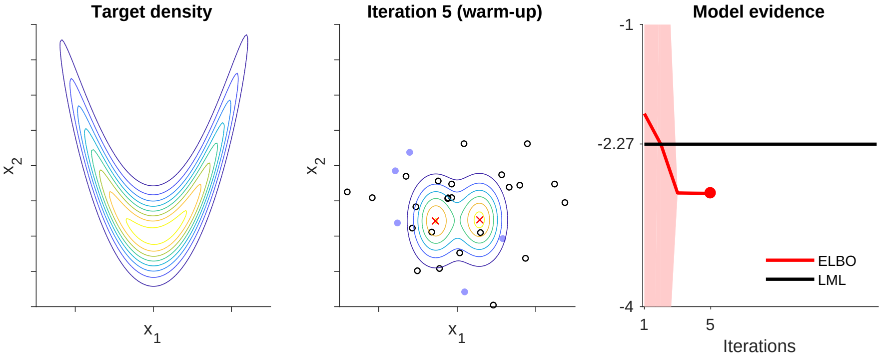

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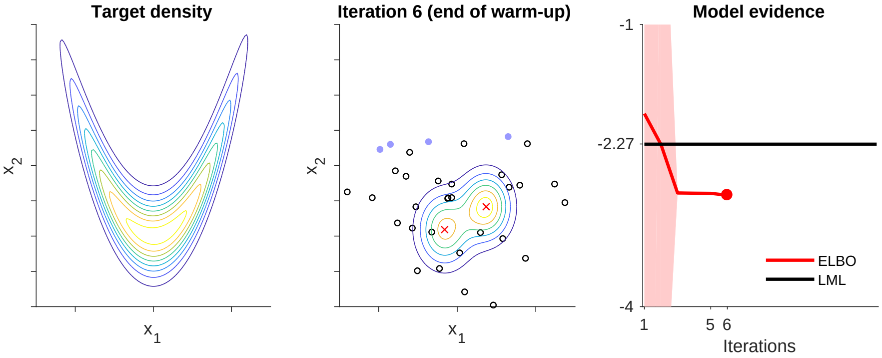

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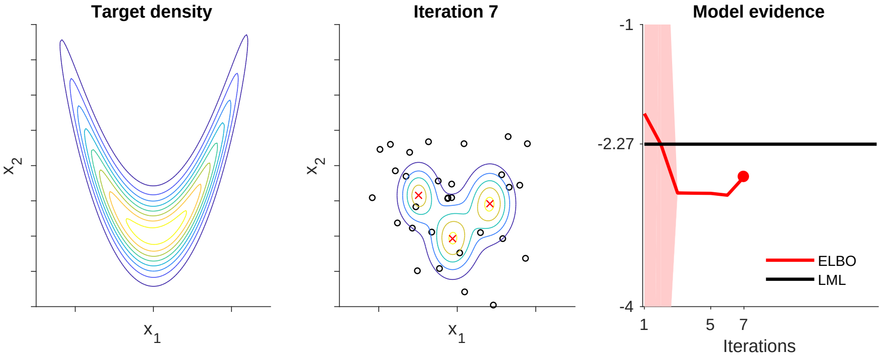

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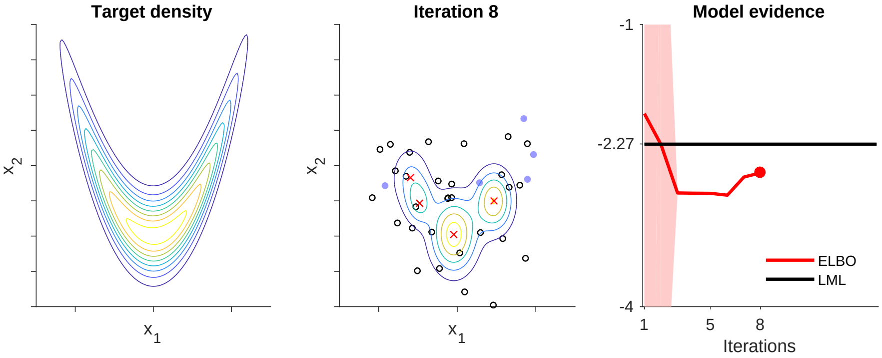

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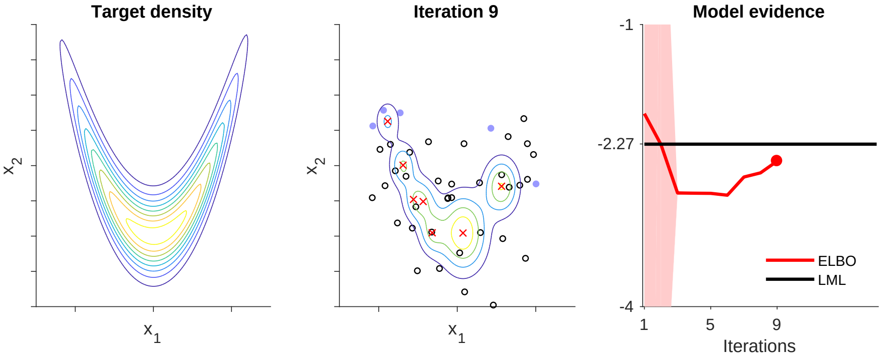

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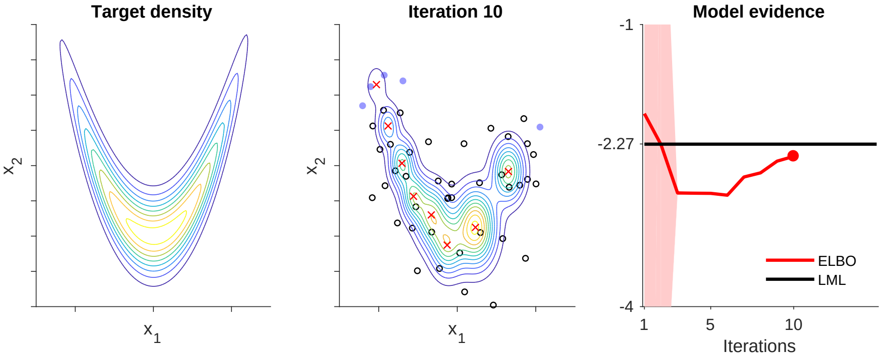

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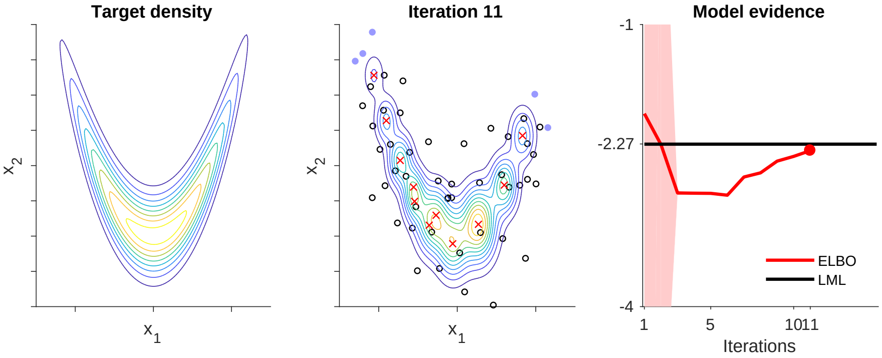

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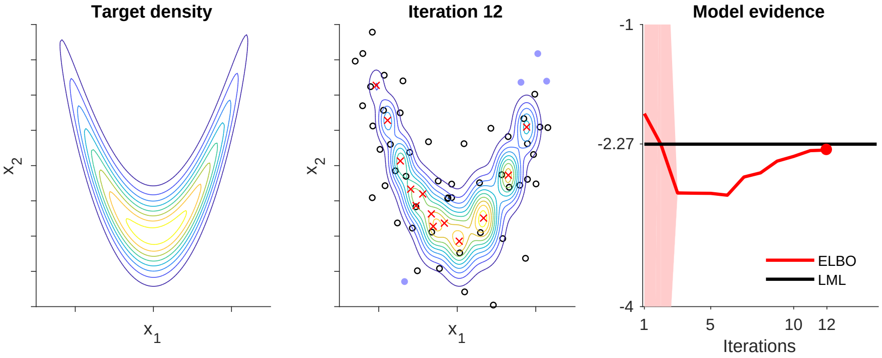

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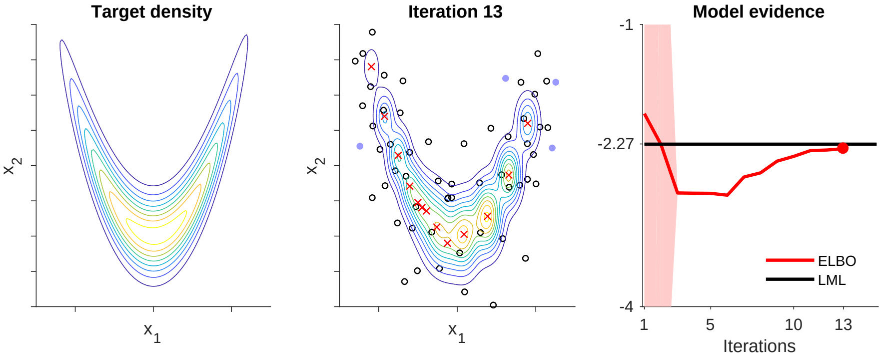

Acerbi, *NeurIPS* (2018; 2020)

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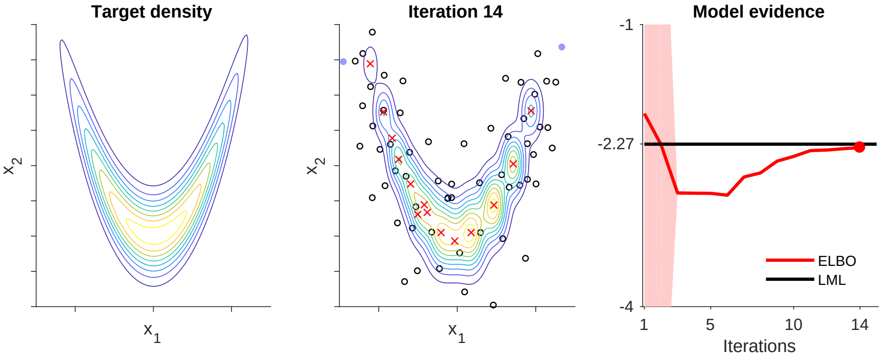

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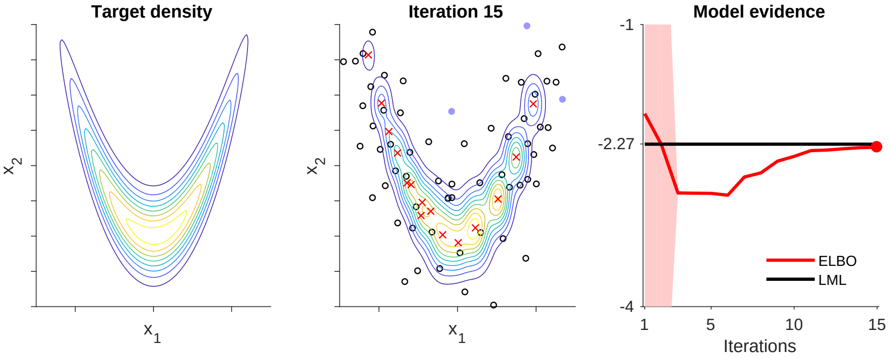

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OK suppose we have a posterior what now

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- Visualize the posterior distribution
- Represent uncertainty (e.g., credible intervals)
- Make posterior predictions ("Bayesian fit") and compare to data

1. A recap of statistical modelling
   - Of models and likelihoods
   - The psychometric function
2. Bayesian model fitting
   - Refresher of Bayesian inference
   - Bayesian inference for model fitting
3. Computing the posterior distribution
   - Setting things up
   - Inference algorithms
   - Making use of a Bayesian posterior
4. Hands-on tutorial

It's Bayes time!

Let's set up and run PyVBMC

`github.com/acerbilab/pyvbmc`

What we learnt

By the end of this lecture/tutorial, we will:

Perform Bayesian inference on a real dataset and model from neuroscience

- Recap the basics of statistical modelling
- Define the psychometric model used in cognitive & neuroscience
- Explain the Bayesian approach to model fitting
- Briefly introduce variational inference algorithms
- Set up and run PyVBMC on a real dataset

This was a lot

This was a lot

You deserve a cat picture

This was a lot

You deserve a cat picture

- Bayesian model fitting could fill the entire day
- This tutorial is just the first steps on the Bayesian way

Final slide

Contacts:

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Acknowledgments:

- The PyVBMC development team
- FCAI

Code:

- PyVBMC: `github.com/acerbilab/pyvbmc`

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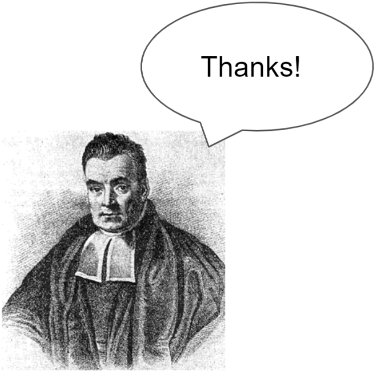

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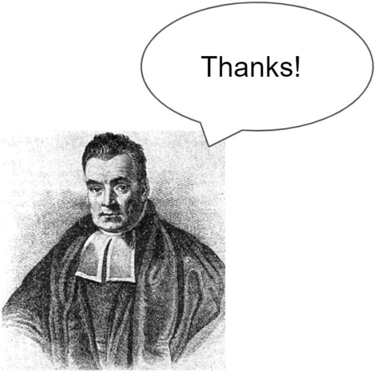

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Questions?