LINE BUNDLES, BUNDLE GERBES AND HIGHER GEOMETRY

Lachlan Simpson

Supervisor: Dr Raymond Vozzo

October 2021

Thesis submitted for the degree of Honours Bachelor of Mathematical and Computer Sciences in Pure Mathematics.

SCHOOL OF MATHEMATICAL SCIENCES

THE UNIVERSITY OF ADELAIDE AUSTRALIA

Declaration

Except where stated this thesis is, to the best of my knowledge, my own work and my supervisor has approved its submission.

Signed by student:

Date:

Signed by supervisor:

Date:

Acknowledgements

I would like to firstly thank my supervisor Dr Raymond Vozzo. Thank you for dedicating so much of your time to meet with me every week and patiently guide me through this material. Without your supervision this thesis would not have been possible and this has been an invaluable experience. Thank you to my family and friends for your support and encouragement in my pursuits.

Abstract

In this thesis we review the theory of complex line bundles, principal bundles and bundle gerbes. It is shown that line bundles and principal $\mathbb{C}^\times$-bundles provide a geometric realisation of degree-two integer cohomology and bundle gerbes provide a realisation of degree-three. We show that the categories of complex line bundles and $\mathbb{C}^\times$-bundles are equivalent. We also provide a detailed study of an important example of a bundle gerbe known as the basic bundle gerbe.

Contents

Chapter 1

Introduction

In mathematical problems understanding the structure of the space you are working in is of central importance. One way this can be done is to study cohomology groups, in particular Cech and de Rham cohomology. However, these groups can be quite hard to study from simply an algebraic perspective.

One way to study these is to associate geometric objects to them. For example, principal G-bundles over M provide what is called a *geometric realisation* of the cohomology group $H^1(M, G)$. By geometric realisation we mean geometric objects whose equivalence classes are isomorphic to $H^1(M, G)$ [10]. Characteristic classes give a way of associating to objects such as principal bundles and vector bundles cohomology classes in $H^p(M, \mathbb{Z})$. However, they are not always complete invariants. When G is abelian and $p = 2$ there is a well known correspondence between elements of $H^2(M, \mathbb{Z})$ and a line bundle over M via the *first chern class*. Which gives a bijection between elements of $H^2(M, \mathbb{Z})$ to isomorphism classes of line bundles.

In [1] Brylinski details Giraud's theory of *gerbes* which provides a geometric realisation of $H^3(M, \mathbb{Z})$. However, as $H^2(M, \mathbb{Z})$ is related to line bundles, one might think that $H^3(M, \mathbb{Z})$ should have some 'higher dimensional' analogue of a line bundle. This is the type of object Michael Murray introduced in his paper *Bundle Gerbes* [16]. Bundle gerbes provide a geometric realisation of $H^3(M, \mathbb{Z})$ via the *Dixmier-Douady class*. Bundle gerbes have interesting applications in physics and mathematics and we refer the interested reader to, for example, ([4], [5], [6]).

The outline of the thesis is as follows. In chapter two we provide a basic overview of some of the background needed. We start with a brief overview of cochain complexes and then set up de Rham and Cech cohomology. We don't spend much time on cohomology as we only really need the basic result that functions that satisfy the cocycle condition represent a class in Cech cohomology and by the de Rham isomorphism, closed differential forms represent a class in real cohomology.

This result forms the crucial idea that bundles and related objects provide a geometric realisation of cohomology, as the transition functions of a bundle satisfy the cocycle condition.

We then provide enough basic category theory to provide the definition of what it means for two categories to be equivalent. This is useful in the thesis as in chapter four we will prove that the category of complex line bundles and principal $\mathbb{C}^{\times}$-bundles are in fact equivalent. The remainder of the background chapter covers some of the basics of Lie groups and Lie algebras as we will need that when discussing principal bundles.

In chapter three we go over the theory of complex line bundles. Line bundles are interesting in their own right, but for us they serve as the building block for the objects we are really interested in which are bundle gerbes. We start by providing the definition of a line bundle and then work our way through the various operations and constructions that are possible. Specifically we look at sections of a line bundle which provides us with an abstract notion of a vector valued function, this is proceeded by discussing what it means for line bundles to be isomorphic. We then focus on connection and curvature which provides us with some notion of 'directional derivative' on a line bundle. The remainder of the chapter illustrates how we can use line bundles to provide a geometric understanding of degree-two integer cohomology $H^2(M, \mathbb{Z})$ via the first chern class.

Chapter four concerns itself with principal bundles where instead of the fibres being complex vector spaces, we have that the fibres are Lie groups. The majority of this chapter focuses on the same aspects of chapter three but for principal bundles. However, at the end of chapter four we demonstrate the amazing result that the categories of complex line bundles and principal $\mathbb{C}^{\times}$-bundles are equivalent. This is quite an important result as it means we can move between the languages of these two objects without losing any data.

In chapter five we are finally ready to provide an overview of the theory of bundle gerbes. All the ideas in chapters three and four are again present here in their own form. We go through the definition of a bundle gerbe, isomorphisms, the connective structure and three-curvature which is the higher analogue of connection and curvature. We finish by looking at the characteristic class of a bundle gerbe called the Dixmier-Douady class which shows how bundle gerbes correspond to elements in $H^3(M, \mathbb{Z})$.

In the final chapter we bring together all the theory from previous chapters and focus on a particular bundle gerbe called the basic bundle gerbe. We go through the calculations of the connective structure, three-curvature and its characteristic class. We hope that this thesis will be a useful resource for future students who may encounter bundle gerbes.

Chapter 2

Background

In this chapter we give some background theory that will be needed later and set up our notation and conventions. This is not all the background needed just some key definitions and results which will be referred to throughout. We assume the reader is comfortable with differential geometry, Lie groups, de Rham and Cech cohomology and some category theory. The reader familiar with these topics can safely skip this chapter.

2.1 De Rham and Čech Cohomology

Here we give an overview of de Rham and Čech cohomology as we will need some of the theory in the coming chapters.

2.1.1 De Rham Cohomology

Definition 2.1.1. [3] A cochain complex is a collection of modules and maps $(A^\bullet, d^\bullet)$ satisfying $d_n \circ d_{n+1} = 0$.

$$\ldots \to A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \to \ldots \tag{2.1.1}$$

The maps d are called the differentials.

It follows from the fact that $d^2 = 0$ that $\mathrm{im}(d_n) \subseteq \ker(d_{n+1})$.

Definition 2.1.2. [3] We call a cochain complex *exact* if for all $n \in \mathbb{N}$ we have $\mathrm{im}(d_n) = \ker(d_{n+1})$.

Suppose we have a cochain complex where $A_n = 0$ for all $n < k$:

$$\ldots \to 0 \xrightarrow{d_n} A_k \xrightarrow{d_k} A_{k+1} \to \ldots$$

Then, $d_k \colon A_k \to A_{k+1}$ is injective.

On the other hand if we have a cochain complex where $A_n = 0$ for all $n > k$:

$$\ldots \to A_{k-1} \xrightarrow{d_{k-1}} A_k \xrightarrow{d_k} 0 \to \ldots$$

Then, $d_{k-1}\colon A_{k-1} \to A_k$ is surjective.

Combing these two types of cases we get the following.

Definition 2.1.3. [3] A *short exact sequence* is a cochain complex of the form

$$0 \to A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} A_3 \to 0 \tag{2.1.2}$$

Where $d_1\colon A_1 \to A_2$ is injective and $d_2\colon A_2 \to A_3$ is surjective.

Definition 2.1.4. [3] Let M be a smooth n-dimensional manifold. Let $\Omega^p(M)$ be the space of differential p-forms on M and $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ be the exterior derivative. $\Omega^p(M)$ is a real vector space under addition and forms a module over the ring of real-valued functions. For $p > n$ we set $\Omega^p(M) = 0$ and for any p-form $d^2\omega = 0$.

Definition 2.1.5. [3] The *de Rham complex* is the collection $(\Omega^\bullet(M), d^\bullet)$

$$\ldots \to \Omega^{p-1}(M) \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \to \ldots \tag{2.1.3}$$

Where the differential d is exterior derivative.

Definition 2.1.6. [3] A p-form $\omega \in \Omega^p(M)$ is said to be *closed* if $d\omega = 0$ and *exact* if there exists some $(p-1)$-form ω' such that $d\omega' = \omega$.

We have the following

1. $\ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid d\omega = 0\}$
2. $\operatorname{im}(d\colon \Omega^{p-1}(M) \to \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega = d\omega' \text{ for some } \omega' \in \Omega^{p-1}(M)\}$

Definition 2.1.7. [3] The p-th *de Rham cohomology group* is denoted $H^p_{dR}(M)$ and defined as

$$H^p_{dR}(M) = \ker(d\colon \Omega^p(M) \to \Omega^{p+1}(M))/\operatorname{im}(d\colon \Omega^{p-1}(M) \to \Omega^p(M)). \tag{2.1.4}$$

A closed p-form is also called a *p-cocycle* for the de Rham cohomology and an exact p-form is called a *p-coboundary* for the de Rham cohomology.

2.1.2 Čech Cohomology

Let M be a differentiable manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M.

Definition 2.1.8. [3] For $p \geq 0$ a *p-cochain* relative to $\mathcal{U}$ is a collection $f = \{f_{ij\ldots k}\}$ of functions such that

1. $f_{ij\ldots k}$ has $p+1$ indices and is defined on $U_i \cap \ldots \cap U_k$.
2. The collection f contains one function $f_{ij\ldots k}$ for each ordered set of $p+1$ indices such that $U_i \cap \ldots \cap U_k$ is nonempty.
3. $f_{ij\ldots k}$ is skew-symmetric under permutation of its indices.

Note that cochains can take values in various spaces, for example abelian groups or sheaves. The cases we will be most interested in are codomains valued in $\mathbb{Z}$, $\mathbb{C}^\times$ and the sheaf of smooth functions with values in $\mathbb{C}^\times$ which we denote by $\underline{\mathbb{C}^\times}$.

Definition 2.1.9. [3] The set of p-cochains is denoted $\mathcal{C}^p(\mathcal{U})$ which is a real vector space and we define $\mathcal{C}^p(\mathcal{U}) = 0$ for $p < 0$.

Definition 2.1.10. [3] Define the differential $\delta \colon \mathcal{C}^p(\mathcal{U}) \to \mathcal{C}^{p+1}(\mathcal{U})$ by,

$$(\delta c)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j c_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_{p+1}} \tag{2.1.5}$$

for $c = c_{\alpha_0\ldots\alpha_p} \in \mathcal{C}^p(\mathcal{U})$ and $\hat{\alpha}_j$ means α_j is omitted.

Lemma 2.1.11. [3] $\delta \circ \delta = 0$

So we get another cochain complex $(\mathcal{C}^\bullet(\mathcal{U}), \delta^\bullet)$, the Čech complex with respect to $\mathcal{U}$

$$\ldots \to \mathcal{C}^{p-1}(\mathcal{U}) \xrightarrow{\delta} \mathcal{C}^p(\mathcal{U}) \xrightarrow{\delta} \mathcal{C}^{p+1}(\mathcal{U}) \to \ldots \tag{2.1.6}$$

So we can now define the p-th *Čech cohomology group* with respect to $\mathcal{U}$.

Definition 2.1.12. [3] The p-th *Čech cohomology group* with respect to $\mathcal{U}$.

$$\check{H}^p(\mathcal{U}) = \ker(\delta \colon \mathcal{C}^p(\mathcal{U})) \to \mathcal{C}^{p+1}(\mathcal{U}))/\operatorname{im}(\delta \colon \mathcal{C}^{p-1}(\mathcal{U}) \to \mathcal{C}^p(\mathcal{U})) \tag{2.1.7}$$

The problem with this definition is that it is dependent on the choice of the open cover of our manifold M. We would like this to not be the case.

Definition 2.1.13. [3] A *direct system of groups* is a collection of groups $\{G_i\}_{i\in I}$ such that for any pair $a < b$ there is a group homomorphism $f^a_b \colon G_a \to G_b$ satisfying

1. $f^a_a = id$

2. $f_c^a = f_c^b \circ f_b^a$ if $a < b < c$.

Take the disjoint union $\bigsqcup_{i\in I} G_i$ and let $g_a \in G_a$ and $g_b \in G_b$. We introduce the following equivalence relation. We say $g_a \sim g_b$ if there exists some $c \in I$ such that $a, b < c$ and

$$f_c^a(g_a) = f_c^b(g_b) \in G_c.$$

Definition 2.1.14. [3] Let $\{G_i\}_{i\in I}$ be a direct system of groups. *The direct limit* of the direct system is the quotient of the disjoint union of G_i by the above equivalence relation. We denote this as follows

$$\varinjlim_{i\in I} G_i = \bigsqcup_{i\in I} G_i / \sim$$

We have from [3] that $\{\check{H}^p(\mathcal{U})\}$ is a direct system of groups and so we have the following definition.

Definition 2.1.15. [3] The p-th *Čech cohomology group* $\check{H}^p(M)$ is the direct limit over all open overs of $\mathcal{U}$ of M.

$$\check{H}^p(M) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U}) \tag{2.1.8}$$

Theorem 2.1.16. [3] *The de Rham isomorphism*

$$H^p_{dR}(M) \simeq \check{H}^p(M) \tag{2.1.9}$$

Note that throughout this thesis we will usually denote H^p_{dR} and $\check{H}^p$ by H^p if there is no chance of confusion.

2.2 Some Category Theory

Definition 2.2.1. [13] A category $\mathcal{C}$ consists of a collection of objects denoted $ob(\mathcal{C})$ and for each pair a objects X, Y a collection of morphisms denoted $\mathcal{C}(X, Y)$ such that:

1. For every $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ a specific morhpism $g \circ f \in \mathcal{C}(X, Z)$.

2. For each object an identity map $id_X \in \mathcal{C}(X, X)$.

The morphisms must satisfy the following:

1. For each triple $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ and $h \in \mathcal{C}(W, X)$ composition is associative i.e. $g \circ (f \circ h) = (g \circ f) \circ h$.

2. For each object X and $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Z, X)$ we have, $f \circ id_X = f$ and $id_X \circ g = g$.

A morphism $f \in \mathcal{C}(X,Y)$ is an *isomorphism* is there exists another morphism $f^{-1} \in \mathcal{C}(Y,X)$ such that, $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$

Definition 2.2.2. [13] Given two categories $\mathcal{C}$ and $\mathcal{D}$ a functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following:

1. For every object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$.

2. For each morphism $f \in \mathcal{C}(X,Y)$ a morphism $F(f) \in \mathcal{D}(F(X), F(Y))$

Such that:

1. For every $X \in ob(\mathcal{C})$, $F(id_X) = id_{F(X)}$

2. $F(g \circ f) = F(g) \circ F(f)$ for all $g \in \mathcal{C}(Y,Z)$ and $f \in \mathcal{C}(X,Y)$

Definition 2.2.3. [13] A functor $F \to \mathcal{C} \to \mathcal{D}$ is faithful (respectively, full) if for each $C, C' \in \mathcal{C}$ the function

$$\begin{gathered} \mathcal{C}(C,C') \to \mathcal{D}(F(C), F(C')) \\ f \mapsto F(f) \end{gathered}$$

is injective (respectively, surjective).

Definition 2.2.4. [13] A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *essentially surjective on objects* if for all $D \in \mathcal{D}$, there exists a $C \in \mathcal{C}$ such that $F(C) \simeq D$.

Definition 2.2.5. [13] A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if the following two conditions are satisfied.

1. For objects $X, Y \in \mathcal{C}$ the map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is bijective.

2. Every object $X \in \mathcal{D}$ is isomorphic to $F(Y)$ for some $Y \in \mathcal{C}$.

The following theorem is helpful in proving that a functor is an equivalence.

Theorem 2.2.6. [13] *$F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is full, faithful and essentially surjective on objects.*

The notion of equivalent categories will be useful when talking about the categories of complex line bundles and principal $\mathbb{C}^{\times}$-bundles which will be shown to be equivalent in section (4.7.3).

2.3 Some Lie Group Theory

Here we give some basic Lie group theory needed for discussing connections on principal bundles.

Definition 2.3.1. [8] A Lie algebra is a vector space L over a field F together with a map $L \times L \to L$ denoted $(x, y) \mapsto [x, y]$ which satisfies the following

1. The bracket operation is bilinear.

2. $[x, x] = 0 \ \forall x \in L$.

3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \ \forall x, y, z \in L$.

The last condition is called the Jacobi identity.

Definition 2.3.2. [8] A Lie algebra homomorphism is a linear map between Lie algebras $f \colon L \to H$ such that it is compatible with the Lie bracket.

$$f([x, y]) = [f(x), f(y)]$$

We say that a Lie algebra homomorphism is an isomorphism if it is a bijection.

Definition 2.3.3. [11] A Lie group is a set G such that:

1. G is a group

2. G is a smooth manifold such that taking the product of two elements and taking the inverse are smooth.

So the maps $\mu \colon G \times G \to G$ where $\mu(g_1, g_2) = g_1 g_2$ and $v \colon G \to G$ where $v(g) = g^{-1}$ are smooth. From fixing either the left or right component of μ we get smooth left and right multiplication maps.

$L_g \colon G \to G$ where $L_g(x) = (gx)$.

$R_g \colon G \to G$ where $R_g(x) = (xg)$.

Let P be a space with a free, right G-action. Let $\mathfrak{X}(P)$ be the Lie algebra of vector fields on P.

Definition 2.3.4. [11] A vector field $X \in \mathfrak{X}(P)$ is left-invariant if $(L_g)_*(X) = X \ \forall g \in G$.

The set of left-invariant vector fields on G is denoted $\mathfrak{g}$ which is a real vector space.

Proposition 2.3.5. [11] $\mathfrak{g}$ *is a Lie algebra.*

We call $\mathfrak{g}$ the Lie algebra of G.

Define the map $\iota\colon \mathfrak{g} \to \mathfrak{X}(P)$ by

$$\iota_p(X) = \frac{d}{dt}(p \cdot \exp(tX))\Big|_{t=0} \text{ for } p \in P. \tag{2.3.1}$$

So the one-parameter subgroup $\exp(tX)$ on $\mathfrak{g}$ induces a vector field on P.

Proposition 2.3.6. [11] $\iota\colon \mathfrak{g} \to \mathfrak{X}(P)$ *is a Lie algebra isomorphism.*

Definition 2.3.7. [11] For $X \in \mathfrak{g}$ the associated vector field $\iota(X) \in \mathfrak{X}(P)$ is called the *fundamental vector field corresponding to X.*

Definition 2.3.8. [11] $\mathrm{ad}_g(h) \coloneqq ghg^{-1}$ is called the *Adjoint map.*

Lemma 2.3.9. [11] $(R_g)_*\iota(X) = \iota(ad_{g^{-1}}(X))$

Definition 2.3.10. [11] The *Maurer-Cartan* form on G is the left invariant $\mathfrak{g}$-valued 1-form defined by

$$\Theta_g(v) = (L_{g^{-1}})_*(v) \text{ for } g \in T_gG \tag{2.3.2}$$

The Maurer-Cartan form is sometimes referred to as the canonical one-form on G.

Chapter 3

Line bundles

We begin our journey into bundle gerbes by looking at one of the geometric objects a bundle gerbe is built out of, namely complex line bundles. This object will highlight a core idea that we can associate a geometric object to an algebraic one, specifically elements in degree two cohomology. By having a geometric interpretation for these elements we can perform calculations by utilizing the tools from differential geometry to give us insight into our space. We begin by providing a definition of a complex line bundle which is proceeded with the definition of a section which is a way to abstract the idea of a vector valued function. We then give the definition of connection and curvature which provides us with an idea of 'directional derivative' on a line bundle. This chapter is finished with the relationship between line bundles and degree two cohomology. Examples are provided throughout to help illustrate the theory.

3.1 Definition of a line bundle

Definition 3.1.1. [18] A complex line bundle over manifold M is a manifold L with a surjection $\pi\colon L \to M$ called the *projection* such that:

1. The *fibres* $L_m \coloneqq \pi^{-1}(m)$ are one-dimensional complex vector spaces.

2. For every $m \in M$ there is an open neighbourhood $U \subseteq M$ with a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\varphi(L_m) \subseteq \{m\} \times \mathbb{C}$ for all $m \in M$.

3. $\varphi\big|_{L_m} \colon L_m \to \{m\} \times \mathbb{C}$ is a linear isomorphism.

M is called the base space and L is called the total space. The map φ in condition two is called a *local trivialisation* of L over U and condition three is sometimes referred to as being *linear over the fibres*. We denote a line bundle as (L, π, M) or $L \to M$.

3.2 Examples of line bundles

Here we give some examples of line bundles and some ways to construct new line bundles from old that will be of importance in the coming chapters.

Example 3.2.1. [18] **The trivial bundle** $M \times \mathbb{C}$.

Let $\pi\colon M \times \mathbb{C} \to M$ be projection in the first factor. Then $\pi^{-1}(m) = \{m\} \times \mathbb{C}$, which is a copy of $\mathbb{C}$ so as a complex vector space it is one dimensional. If we choose our open neighbourhood to be M then, $\pi^{-1}(M) = M \times \mathbb{C}$. So the diffeomorphism is just the identity map $id_{M\times\mathbb{C}}$.

Example 3.2.2. [18] **The Hopf bundle**.

We start by defining the complex projective space $\mathbb{C}P^1 := (\mathbb{C}^2 \backslash \{0\})/\sim$. Where $\sim$ is the equivalence relation

$$x = (x^1, x^2) \sim y = (y^1, y^2) \iff x = \lambda y$$

for some $\lambda \in \mathbb{C}^\times$. $\mathbb{C}P^1$ has the quotient topology and $[x] := [x^1, x^2]$.

Now we construct a line bundle $H \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$ over $\mathbb{C}P^1$.

$H := \{([x], v) \mid v \in span\{x\}\}$ with projection $\pi\colon H \to \mathbb{C}P^1$ where $\pi(([x], v)) = [x]$.

So we now want to show that this actually is a line bundle. Clearly π is surjective as it is projection of the first factor. The equivalence classes $[x]$ can be thought of as lines going through the origin and the point x as all points in the equivalence class are scalar multiples of x. So elements of the fibre

$$H_{[x]} = \{([x], \lambda x) \mid \lambda \in \mathbb{C}\}$$

can be thought of as points on the line $[x]$. Meaning $H_{[x]}$ can be identified with the line $[x]$.

$H_{[x]}$ is given the following vector space structure. Let $\alpha, \beta \in \mathbb{C}$ and $([x], \lambda x), ([x], \lambda' x) \in H_{[x]}$. Then

$$\alpha([x], \lambda x) + \beta([x], \lambda' x) = ([x], (\alpha\lambda + \beta\lambda')x) \in H.$$

So we have

$$H_{[x]} = span\{([x], x)\}.$$

Which makes $H_{[x]}$ a one dimensional complex vector space.

Define open sets $U_i = \{[x] \mid x^i \neq 0\}$, so $\mathbb{C}P^1 = U_1 \cup U_2$. We have a local trivialisation

$$h_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$

given by

$$h_i(([x], v)) = ([x], v_i)$$

We need to check this is a local trivialisation. Firstly, we have

$$\begin{aligned} h_i(H_{[x]}) &= \{h_i([x], \lambda x) \mid \lambda \in \mathbb{C}\} \\ &= \{([x], \lambda x^i) \mid \lambda \in \mathbb{C}\} \\ &\subseteq [x] \times \mathbb{C} \end{aligned}$$

Now to show $h_i\big|_{H_{[x]}} : H_{[x]} \to [x] \times \mathbb{C}$ is a linear isomorphism. Firstly h_i is linear as

$$\begin{aligned} h_i(([x], \lambda x) + ([x], \lambda' x)) &= ([x], (\lambda + \lambda')x^i) \\ &= h_i(([x], \lambda x)) + h_i(([x], \lambda' x)). \end{aligned}$$

For scalars we have

$$\begin{aligned} \alpha h_i([x], \lambda x) &= ([x], \alpha \lambda x^i) \\ &= h_i(\lambda([x], \lambda x)) \end{aligned}$$

Let $([x], v) \in [x] \times \mathbb{C}$ and $[x] \in U_i$ for either $i = 1$ or $i = 2$. Let $\lambda = v/x^i$ since $x^i \neq 0$ this is defined. Therefore, $h_i([x], \lambda x) = ([x], v)$, so $h_i\big|_{H_{[x]}}$ is surjective.

Now suppose $h_i([x], \lambda x) = h_i([x], \lambda' x)$ for some $\lambda, \lambda' \in \mathbb{C}^\times$. So $([x], \lambda x^i) = ([x], \lambda' x^i)$, it follows that $\lambda = \lambda'$. Hence, $([x], \lambda x) = ([x], \lambda' x)$. So $h_i\big|_{H_{[x]}}$ is injective and hence a linear isomorphism.

So we have a line bundle $(H, \pi, \mathbb{C}P^1)$. Furthermore, $\mathbb{C}P^1$ is diffeomorphic to the sphere S^2, so H is also a line bundle over S^2.

Example 3.2.3. [18] **The tangent bundle to the sphere**.

The sphere $S^2 \subseteq \mathbb{R}^3$ is a surface in $\mathbb{R}^3$ and thus a manifold. The tangent space for a point $x \in S^2$ is the set of vectors orthogonal to x i.e. the tangent plane to the sphere at x.

So the tangent space is

$$T_x S^2 := \{v \in \mathbb{R}^3 \mid x \cdot v = 0\}.$$

We define the *tangent bundle* of S^2 to be

$$TS^2 := \bigsqcup_{x \in S^2} T_x S^2 = \{(x, y) \mid x \in S^2, y \in T_x S^2\}.$$

The projection maps $\pi\colon TS^2 \to S^2$ are defined by $\pi(x, y) = x$. Let's start by looking at the tangent spaces.

T_xS^2 is already a real vector space as it is a subspace of $\mathbb{R}^3$. We can make T_xS^2 into a complex vector space by defining complex scalar multiplication as follows.

$$(\alpha + i\beta)y := (\alpha y + \beta x) \times x.$$

For $\alpha, \beta \in \mathbb{R}$, $x \in S^2$ and $y \in T_xS^2$.

Since, T_xS^2 is already a real vector space, the only thing that needs to be checked is that T_xS^2 respects complex scalar multiplication. In fact it is enough to show $i(iy) = (i^2)y = -y$.

$$i(iy) = i(0y + 1y \times x) = i(x \times y) = 0(x \times y) + x \times (x \times y) = (x \cdot y)x - (x \cdot x)y = 0 - 1y = -y$$

Since T_xS^2 is a plane to a surface it is a two-dimensional real vector space. Hence as a complex vector space it is one dimensional. The fibres are of the form

$$\pi^{-1}(x) = \{(x, y) \in TS^2 \mid \pi(x, y) = x\} = \{x\} \times T_xS^2$$

So the fibres are one dimensional complex vector spaces. The projection π is surjective as it is just projection in the first factor. The next section will show local trivialisation as we will have an easier way to do so in terms of sections.

There are also a number of ways of constructing new line bundles from old.

Example 3.2.4. [1] **The dual**.

Let $L \to M$ be a line bundle, then the *dual* of L denoted $L^* \to M$, is defined on the fibres to be $L^*_x := (L_x)^*$, where $(L_x)^*$ is the dual of the vector space. Since $\dim(L_x) = \dim((L_x)^*)$ in finite dimensions, we have that the fibres of L^* are 1-dimensional $\mathbb{C}$ vector spaces.

Example 3.2.5. [1] **The pullback**.

Let $L \to M$ be a line bundle and $f\colon N \to M$ be a smooth map. Define the pullback of L by f to be

$$f^{-1}(L) = \{(n, l) \in N \times L \mid \pi(l) = f(n)\}.$$

Let the projection map be

$$\mathrm{pr}_1\colon f^{-1}(L) \to N$$

Let $n \in N$ looking at the fibre

$$\begin{aligned} f^{-1}(L)_n &= \{(n, l) \in N \times L \mid \mathrm{pr}_1(n, l) = m\} \\ &= \{l \in L \mid f(n) = \pi(l)\} \\ &= L_{f(n)} \simeq \mathbb{C} \end{aligned}$$

So the fibres are 1-dimensional complex vector spaces. The projection is surjective as it is projection onto the first factor. Let (U, φ) be a local trivialisation of L we want to show that $(f^{-1}(U), \psi)$ where $\psi(n, l) = (n, \mathrm{pr}_2(\varphi(l)))$ is a local trivialisation of $f^{-1}(L)$. Firstly, $\psi\colon \mathrm{pr}_1^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{C}$ is a diffeomorphism because φ is. Let $n \in N$ and choose $U \subseteq M$ so that $n \in f^{-1}(U)$. Then, $\psi(f^{-1}(L)_n) \subseteq \{n\} \times \mathbb{C}$ by definition of ψ. $\psi\big|_{f^{-1}(L)_n}\colon f^{-1}(L)_n \to \{n\} \times \mathbb{C}$ is a linear isomorphism due to the fact that φ is a linear isomorphism.

Example 3.2.6. [1] **The tensor product**.

Let $L_1 \to M$ and $L_2 \to M$ be line bundles. The *tensor product* of L_1 and L_2 is denoted $L_1 \otimes L_2$ and is defined on the fibres as $(L_1 \otimes L_2)_x = (L_1)_x \otimes (L_2)_x$.

3.3 Sections of line bundles

Definition 3.3.1. [18] Given a line bundle (L, π, M) a *section* is a smooth map $\sigma\colon M \to L$ such that: $\pi \circ \sigma = id_M$ or equivalently $\sigma(m) \in L_m$ for all $m \in M$.

Definition 3.3.2. [18] A *local section* of a line bundle is a section $\sigma\colon U \to L$ where $U \subseteq M$ is an open subset.

The set of sections of L over M is denoted $\Gamma(M, L)$ which is a complex vector space under pointwise addition and scalar multiplication.

Proposition 3.3.3. *([12], p.g. 112) Local non-vanishing sections gives rise to a local trivialization.*

Proof. Given $x \in M$ and an open neighbourhood $U \subseteq M$ of x. Suppose we have a non-vanishing local section $\sigma\colon U \to L$. Define the map $\psi\colon U \times \mathbb{C} \to \pi^{-1}(U)$ by $\psi(x, y) = y\sigma(x)$.

Since ψ is a bijection it suffices to show ψ is a local diffeomorphism. Since we have a line bundle we have a local trivialisation. Let $V \subseteq M$ be an open neighbourhood of x. Making V smaller if needed, we can assume $V \subseteq U$.

For $x \in V \subseteq U \subseteq M$ we have a diffeomorphism $\phi\colon \pi^{-1}(V) \to V \times \mathbb{C}$. So $\phi \circ \sigma\big|_V\colon V \to V \times \mathbb{C}$ is smooth. So there must be a smooth function $\widetilde{\sigma}\colon V \to \mathbb{C}$ such that $(\phi \circ \sigma)(x) = (x, \widetilde{\sigma}(x))$. So $(\phi \circ \psi)\big|_{V \times \mathbb{C}}(x, y) = (x, y\widetilde{\sigma}(x))$ and $(\phi \circ \psi)^{-1}\big|_{V \times \mathbb{C}}(x, y) = (x, y\widetilde{\sigma}^{-1}(x))$ are smooth since $\widetilde{\sigma}$ is smooth.

So we have diffeomorphisms $(\phi \circ \psi)\big|_{V \times \mathbb{C}}\colon V \times \mathbb{C} \to V \times \mathbb{C}$ and $\phi\colon \pi^{-1}(V) \to V \times \mathbb{C}$. Since $\psi = (\psi \circ \phi) \circ \phi^{-1}$ that implies $\psi\colon V \times \mathbb{C} \to \pi^{-1}(V)$ is a diffeomorphism. Thus, $\psi^{-1}\colon \pi^{-1}(V) \to V \times \mathbb{C}$ is a diffeomorphism. So now we want to show ψ^{-1} is a local trivialization.

Let $y \in L_x$. Since ψ is surjective, there exists $(v, c) \in V \times \mathbb{C}$ such that $\psi(v, c) = c\sigma(v) = y$.

So, $\sigma(v) = c^{-1}y \in L_x$. Then $v = \pi(\sigma(v)) = \pi(c^{-1}y) = m$. So, $(m, c) = \psi^{-1}(\psi(m, c)) = \psi^{-1}(y)$. If $c = 0$ then $\psi(v, 0) = 0\sigma(v) = 0 = y$. So $\psi(m, 0) = 0$. Hence $\psi^{-1}(\psi(m, 0)) = (m, 0) = \psi^{-1}(0)$. Hence, $\psi^{-1}(L_m) \subseteq \{m\} \times \mathbb{C}$.

Suppose $\psi\big|_{\{x\}\times\mathbb{C}}(x, y) = \psi\big|_{\{x\}\times\mathbb{C}}(x, z)$. So $y\sigma(x) = z\sigma(x)$ and since σ is non-vanishing $y = z$. So ψ is injective and clearly linear. So $\psi\big|_{\{x\}\times\mathbb{C}}$ is a linear injective map over vector spaces of the same dimension. Thus, $\psi\big|_{\{x\}\times\mathbb{C}}$ is a linear isomorphism. So, $\psi^{-1}\big|_{L_x} : L_x \to \{x\} \times \mathbb{C}$ is a linear isomorphism. Therefore, ψ^{-1} is a local trivialization. □

3.4 Examples of sections

Here we look at sections of the line bundles given in section 3.2.

Example 3.4.1. [18] **The trivial bundle.**

The fibres of $M \times \mathbb{C}$ are $\{x\} \times \mathbb{C}$ so all sections are of the form, $\sigma(x) = (x, f(x))$ for some smooth function $f \colon M \to \mathbb{C}$.

Example 3.4.2. [18] **The tangent bundle to the sphere** TS^2.

A section of TS^2 would be a map $\sigma \colon S^2 \to \mathbb{R}^3$ such that, $\forall x \in S^2$, $\sigma(x) \in TS^2_x$ so $\sigma(x) \cdot x = 0$.

We can find local non-vanishing sections in the following way. Let $x = (x_1, x_2, x_3) \in S^2$ and suppose $x_2 > 0$. Take the open neighbourhood of x to be the set

$$U = \{y \in S^2 \mid y_2 > 0\}.$$

Define a local section by

$$\sigma(x_1, x_2, x_3) = (1, -x_1/x_2, 0).$$

So σ is locally non-vanishing and $\sigma(x) \cdot x = 0$ for all $x \in U$. If $x_2 < 0$ take the western hemisphere instead. If $x_2 = 0$ take the northern and southern hemisphere with

$$\sigma(x_1, x_2, x_3) = (0, 1, -x_2/x_3).$$

So we have local non-vanishing sections, which gives a local trivialisation of TS^2.

Example 3.4.3. [18] **The Hopf bundle**.

The fibres of H are of the form $H_{[x]} = \{([x], \lambda x) \mid \lambda \in \mathbb{C}\}$. So the sections have the form $\sigma([x]) = ([x], f([x]))$ where $f([x]) = \lambda x$ for some $\lambda \in \mathbb{C}$.

Example 3.4.4. [24] **The pullback**.

Let $L \to M$ be a line bundle, $f\colon N \to M$ be a smooth map and $\sigma\colon M \to L$ be a section of L. Then define a section $s\colon N \to f^{-1}(L)$ of $f^{-1}(L)$ by

$$s(n) = (n, \sigma(f(n))).$$

This map is a section as $\pi(\sigma(f(n))) = f(n)$ so $s(n) \in f^{-1}(L)$ and

$$\mathrm{pr}_1(s(n)) = \mathrm{pr}_1(n, \sigma(f(n))) = n.$$

This section is also denoted as $f^{-1}\sigma$.

Example 3.4.5. [24] **The dual**. Let $L \to M$ be a line bundle and $\sigma\colon M \to L$ be a section. Since the fibres of L^* are linear maps $L^*_m = \{L_m \to \mathbb{C}\}$. Given any $\mathbb{C}$-linear map $\lambda\colon L \to \mathbb{C}$ and the fact that $\sigma(m) \in L_m$ we have a section of L^* by $\lambda \circ \sigma\colon M \to L^*$.

Example 3.4.6. [24] **The tensor product**. Let $L_1, L_2 \to M$ be line bundles and take their tensor product $L_1 \otimes L_2$. Take a section σ_1 of L_1 and σ_2 of L_2 and take their product $\sigma_1 \otimes \sigma_2$. Then for $m \in M$ we have $\sigma_1(m) \otimes \sigma_2(m) \in (L_1)_m \otimes (L_2)_m = (L_1 \otimes L_2)_m$. So $\sigma_1 \otimes \sigma_2$ is a section of $L_1 \otimes L_2$.

3.5 Isomorphism of line bundles

Definition 3.5.1. [18] Two line bundles (L, π_1, M) and (J, π_2, M) are isomorphic if there is a diffeomorphism $\varphi\colon L \to J$ such that:

1. $\varphi(L_m) \subseteq J_m$ for every $m \in M$
2. $\varphi\big|_{L_m}\colon L_m \to J_m$ is a linear isomorphism.

A line bundle is said to be *trivial* if it is isomorphic to the trivial bundle $(M \times \mathbb{C}, \mathrm{pr}_1, M)$. Where pr_1 is projection in the first factor. Such an isomorphism is called a trivialisation of L.

There is a useful way to see if a line bundle is trivial by looking at its sections.

Proposition 3.5.2. *[18] A line bundle L is trivial if and only if it has a nowhere vanishing section.*

Proof. Suppose L is trivial, let $\varphi\colon M \times \mathbb{C} \to L$ be a trivialisation of L. $\sigma(m) = (m, 1)$ is a section of $M \times \mathbb{C}$ and no-where zero. So, $(\varphi \circ \sigma)(m) \in L_m$ and since φ and σ are smooth $\varphi \circ \sigma\colon M \to L$ is smooth and thus a section of L. Notice that $(\varphi \circ \sigma)(m) = \varphi(m, 1) \neq 0$ since φ is injective.

Conversely, suppose we have a non-vanishing section σ for L. Define the map $\varphi : M \times \mathbb{C} \to L$ by $\varphi(m, \lambda) = \lambda\sigma(m)$. Let $(m, \lambda) \in M \times \mathbb{C}_m$, so $\varphi(m, \lambda) = \lambda\sigma(m) \in L_m$ as L_m

is a complex vector space. Thus $\varphi(M \times \mathbb{C}_m) \subseteq L_m$. Now suppose, $\varphi\big|_{\{m\}\times\mathbb{C}}(m,\lambda) = \varphi\big|_{\{m\}\times\mathbb{C}}(m,z)$. Hence, $(\lambda - w)\sigma(m) = 0$, but since $\sigma \neq 0$ this implies $\lambda = w$ so $\varphi\big|_{\{m\}\times\mathbb{C}}$ is injective and clearly linear. Thus $\varphi\big|_{\{m\}\times\mathbb{C}}$ is a linear isomorphism as it is a map over complex vector spaces of the same dimension. □

3.6 Connections on a line bundle

We would like to perform calculus on line bundles, however we need some notion of a derivative. This is the role of a connection, it provides a way to differentiate a section.

Definition 3.6.1. [18] A connection ∇ on a line bundle $L \to M$ is a linear map $\nabla\colon \Gamma(M,L) \to \Gamma(M, T^*M \otimes L)$ that satisfies the Leibniz rule. Meaning for all $\sigma \in \Gamma(M,L)$ and $f \in C^\infty(M,L)$ we have

$$\nabla(fs) = f\nabla s + df \otimes s.$$

Example 3.6.1. [18] **The trivial bundle**.

A section on the trivial bundle is of the form $\sigma(x) = (x, f(x))$ for some smooth function $f\colon M \to \mathbb{C}$. We can think of $\Gamma(M, T^*M \otimes L)$ as one-forms with values in the bundle. So if we set $\nabla\sigma = df$ we get a one-form with values in the trivial bundle. By definition of the exterior derivative, ∇ satisfies the Leibniz rule and is linear and therefore a connection.

Example 3.6.2. [18] **The tangent bundle to the sphere**.

Since a section $s\colon S^2 \to \mathbb{R}^3$ is real-valued we can differentiate it using ordinary calculus on $\mathbb{R}^3$. However $ds(x) \notin TS^2$ in general. This is fixed by defining $\nabla\sigma = \pi(ds)$, where π is orthogonal projection onto the tangent plane of x. Let $x \in S^2$, then

$$\begin{aligned}\pi(ds(x)) \cdot x &= (ds(x) - (x \cdot ds(x))x) \cdot x \\ &= ds(x) \cdot x - (x \cdot ds(x)) \\ &= 0\end{aligned}$$

So $\pi(ds(x)) \in T_xS^2$ and hence a section. Let $s, s'\colon S^2 \to TS^2$ be two sections and $\alpha, \beta \in \mathbb{C}$. Then we have

$$\begin{aligned}\nabla(\alpha s(x) + \beta s'(x)) &= \pi(\alpha ds(x) + \beta ds'(x)) \\ &= (\alpha ds(x) - (x \cdot \alpha ds(x))x) + (\beta ds(x) - (x \cdot \beta ds(x))x) \\ &= \alpha\nabla(s) + \beta\nabla(s')\end{aligned}$$

Then since ds is regular differentiation ∇ satisfies the Leibniz rule.

Example 3.6.3. [18] **The Hopf bundle**.

Recall that sections of H have the form $\sigma([x]) = ([x], f([x]))$ where $f([x]) = \lambda x$ for some $\lambda \in \mathbb{C}^\times$. So really, sections can be identified with $\sigma([x]) = \lambda x$. So we can follow the same process and differentiate σ as a $\mathbb{C}^\times$-valued function and use Hermitian projection. Meaning that for $\pi(v) = x - \langle x, v\rangle x$, we use the Hermitian inner product.

Example 3.6.4. [1] **The dual**.

Let $L \to M$ be a line bundle with connection ∇. We get an induced connection ∇^* on L^* called the *dual connection* defined in the following way. Let $s \in \Gamma(M, L)$ and $\sigma \in \Gamma(M, L^*)$. Then we define $(\nabla^*\sigma)(s) = d(\sigma(s)) - \sigma(\nabla s)$.

Example 3.6.5. [1] **The pullback**.

Let $L \to M$ be a line bundle with connection ∇ and $f\colon N \to M$ be a smooth map. Then ∇ induces a connection on $f^{-1}L$ denoted $f^{-1}\nabla$ and is defined by $(f^{-1}\nabla)(f^{-1}s) = f^*\nabla(s)$ for any section $s\colon M \to L$.

The linearity of the pullback connection follows from the connection ∇ and the pullback f^* being linear. So we need to check the Leibniz rule holds. Let $h \in C^\infty(M)$ and $s \in \Gamma(M, L)$. So

$$\begin{aligned}
(f^{-1}\nabla)(f^{-1}h \cdot f^{-1}s) &= f^*(\nabla(hs)) \\
&= f^*(dh \otimes s + h\nabla s)) \\
&= f^*(dh \otimes s) + f^*(h\nabla s) \\
&= f^*(dh) \otimes f^*(s) + hf^*(\nabla s) \\
&= d(f^{-1}h) \otimes f^{-1}s + h(f^{-1}\nabla)(f^{-1}s)
\end{aligned}$$

So the Leibniz rule holds.

Example 3.6.6. [1] **The tensor product**.

Let $L_1 \to M$ and $L_2 \to M$ be line bundles with connections ∇_1 and ∇_2. Let $s \in \Gamma(L_1, M)$ and $\sigma \in \Gamma(L_2, M)$. Then we can form a connection on $L_1 \otimes L_2$ denoted $\nabla_1 + \nabla_2$ by

$$(\nabla_1 + \nabla_2)(s \otimes \sigma) = s \otimes \nabla_2(\sigma) + \nabla_1(s) \otimes \sigma.$$

The linearity of $\nabla_1 + \nabla_2$ follows from the linearity of both ∇_1 and ∇_2.

Let f be a smooth $\mathbb{C}$-valued function. Then

$$\begin{aligned}
(\nabla_1 + \nabla_2)(f \cdot (s \otimes \sigma)) &= s \otimes \nabla_2(f\sigma) + \nabla_1(s) \otimes f\sigma \\
&= s \otimes (f\nabla_2(\sigma) + df \otimes \sigma) + \nabla_1(s) \otimes f\sigma \\
&= s \otimes f\nabla_2(\sigma) + \nabla_1(s) \otimes f\sigma + df \otimes s \otimes \sigma \\
&= f(s \otimes \nabla_2(\sigma) + \nabla_1(s)\sigma) + df \otimes s \otimes \sigma \\
&= f(\nabla_1 + \nabla_2)(s \otimes \sigma) + df \otimes (s \otimes \sigma)
\end{aligned}$$

So $\nabla_1 + \nabla_2$ is a connection.

3.7 Transition functions

It is useful to understand how two different coordinate charts on a manifold behave in their overlap, and how to move between two charts. The same is true for local trivialisations of a line bundle. Or equivalently, local nowhere vanishing sections.

Let (L, π, M) be a line bundle. Given a collection $\{U_\alpha, \sigma_\alpha\}_{\alpha \in I}$ where U_α is an open cover of M and $\sigma_\alpha \colon U_\alpha \to L$ are local non-vanishing sections of L. We define smooth maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ by $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$. The maps $g_{\alpha\beta}$ are called the transition functions of L [18].

From the definition of $g_{\alpha\beta}$ we get three conditions.

$$
\begin{aligned}
&1)\ g_{\alpha\alpha} = 1 \\
&2)\ g_{\alpha\beta} = g_{\beta\alpha}^{-1} \\
&3)\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma
\end{aligned}
$$

Proof.
1) $\sigma_\alpha = g_{\alpha\alpha}\sigma_\alpha$ and since σ_α is non-vanishing on U_α must have $g_{\alpha\alpha} = 1$.
2) $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta = g_{\alpha\beta}g_{\beta\alpha}\sigma_\alpha$. So $g_{\alpha\beta}g_{\beta\alpha} = 1$ which implies $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.
3) $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta = g_{\alpha\beta}g_{\beta\gamma}\sigma_\gamma = g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\sigma_\alpha$, thus $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$. □

These three conditions are equivalent to saying that the transition functions satisfy the Čech cocycle condition.

$$g_{\beta\gamma}g_{\alpha\gamma}^{-1}g_{\alpha\beta} = 1 \text{ for } \ U_\alpha \cap U_\beta \cap U_\gamma \tag{3.7.1}$$

Therefore, transition functions are 1-cocycles and hence give a class in $H^1(M, \underline{\mathbb{C}^\times})$.

3.8 Curvature of a connection

Let $L \to M$ be a line bundle with connection ∇ and $\{U_\alpha\}_{\alpha \in I}$ be an open cover of M. Let $\sigma_\alpha \colon U_\alpha \to L$ be local nowhere vanishing sections and define local one-forms A_α on U_α by $\nabla\sigma_\alpha = A_\alpha \otimes \sigma_\alpha$.

Let $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ be the transition functions for the given open cover and local sections. So we have $\nabla\sigma_\alpha = \nabla g_{\alpha\beta}\sigma_\beta$ and hence by Leibniz $\nabla\sigma_\alpha = dg_{\alpha\beta} \otimes \sigma_\beta + g_{\alpha\beta}\nabla\sigma_\beta$. So we get $A_\alpha \otimes \sigma_\alpha = dg_{\alpha\beta} \otimes (g_{\alpha\beta}^{-1}\sigma_\alpha) + g_{\alpha\beta}(A_\beta \otimes g_{\alpha\beta}^{-1}\sigma_\alpha)$. Therefore,

$$A_\alpha = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + A_\beta \tag{3.8.1}$$

The one forms A_α are called the connection one-forms of ∇.

Now consider the two-forms dA_α. By linearity of the exterior derivative and (3.8.1) we get,

$$\begin{aligned} dA_\alpha &= d(g_{\alpha\beta}^{-1} dg_{\alpha\beta}) + dA_\beta \\ &= dg_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} d(dg_{\alpha\beta}) + dA_\beta \\ &= (-g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1}) \wedge dg_{\alpha\beta} + dA_\beta \end{aligned}$$

Then by bilinearity of the wedge product and the fact that $dg \wedge dg = 0$, we get that $dA_\alpha = dA_\beta$.

Since the 2-forms dA_α are equal on the intersection of sets in the open cover we get a 2-form dA defined on all of M. We call this 2-form the *curvature* of ∇ and is denoted F_∇ [18].

Let us now look at the corresponding curvature of the connections calculated in section (3.6).

Example 3.8.1. [18] **The trivial bundle**.

Since all sections for the trivial bundle are of the form $\sigma = (id_M, f)$ for some smooth $f\colon M \to \mathbb{C}$ and our connection is defined as $\nabla\sigma = df$ taking our one-form to be $A = df$ then $F_\nabla = dA = d^2 f = 0$.

Example 3.8.2. [18] **Tangent bundle to the sphere**.

We will do this calculation using polar coordinates. The tangent vectors are

$$\begin{aligned} \frac{\partial}{\partial\theta} &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\ \frac{\partial}{\partial\phi} &= (\cos(\theta)\sin(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \end{aligned}$$

To get a unit normal vector to the sphere, we take the cross product of the tangent vectors and normalise. Which gives the following

$$\hat{n} = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) = \sin(\phi)\frac{\partial}{\partial\phi} \times \frac{\partial}{\partial\theta}$$

Then taking our non-vanishing section $s\colon S^2 \to \mathbb{R}^3$ to be $s = (-\sin(\theta), \cos(\theta), 0)$ and differentiating $ds = (-\cos(\theta), -\sin(\theta), 0)d\theta$. Then putting this into our connection

$$\begin{aligned}
\nabla(s) &= \pi(ds) \\
&= ds - (ds \cdot \hat{n})d\theta \\
&= (-\cos(\theta), -\sin(\theta), 0)d\theta + \sin(\phi)\hat{n}d\theta \\
&= -\cos(\phi)\frac{\partial}{\partial\phi} \\
&= \cos(\phi)\hat{n} \times s \\
&= i\cos(\phi)s
\end{aligned}$$

So we take $A = i\cos(\phi)d\theta$ and hence the curvature is

$$dA = F = i\sin(\phi)d\theta \wedge d\phi.$$

Example 3.8.3. [1] **The dual**.

If F_∇ is the curvature of some line bundle $L \to M$ then the curvature on $L^* \to M$ is $-F_\nabla$.

Example 3.8.4. [1] **The pullback**.

Let $L \to M$ be a line bundle with curvature F_∇ then the curvature of the pullback bundle $f^{-1}(L) \to N$ is $f^*(F_\nabla)$, where $f^*(F_\nabla)$ means the pullback of the 2-form F_∇ by f.

Let s_α be a local section of ∇ and A_α be the corresponding connection one-forms. Then we have

$$\begin{aligned}
(f^{-1}\nabla)(f^{-1}s_\alpha) &= f^*(\nabla s_\alpha) \\
&= f^*(A_\alpha \otimes s_\alpha) \\
&= f^*(A_\alpha) \otimes f^{-1}s_\alpha
\end{aligned}$$

So, $F_{f^{-1}\nabla} = f^*(F_\nabla)$.

Example 3.8.5. [1] **The tensor product**.

Let $L_1 \to M$ and $L_2 \to M$ be line bundles with connections ∇_1 and ∇_2. Then the curvature of $\nabla_1 + \nabla_2$ on $L_1 \otimes L_2$ is $F_{\nabla_1} + F_{\nabla_2}$.

Let s_α and σ_α be local sections of L_1 and L_2 respectively. We have that $\nabla_1(s_\alpha) = A_\alpha \otimes s_\alpha$ and $\nabla_2(\sigma_\alpha) = B_\alpha \otimes \sigma_\alpha$. Where A_α and B_α are the connection one-forms on ∇_1 and ∇_2.

Now we have that

$$\begin{aligned}(\nabla_1 + \nabla_2)(s_\alpha \otimes \sigma_\alpha) &= \nabla_1(s_\alpha) \otimes \sigma_\alpha + s_\alpha \otimes \nabla_2(\sigma_2)\\ &(A_\alpha \otimes s_\alpha) \otimes \sigma_\alpha + s_\alpha \otimes (B_\alpha \otimes \sigma_\alpha)\\ &= A_\alpha \otimes (s_\alpha \otimes \sigma_\alpha) - (B_\alpha \otimes \sigma_\alpha) \otimes s_\alpha\\ &= A_\alpha \otimes (s_\alpha \otimes \sigma_\alpha) + B_\alpha \otimes (s_\alpha \otimes \sigma_\alpha)\\ &= (A_\alpha + B_\alpha) \otimes (s_\alpha \otimes \sigma_\alpha).\end{aligned}$$

Therefore, $F_{\nabla_1+\nabla_2} = F_{\nabla_1} + F_{\nabla_2}$.

3.9 The first Chern class

Here we introduce the first Chern class associated to a complex line bundle. Which in a sense provides us with an algebraic way to view and classify complex line bundles.

Let (L, π, M) be a line bundle with $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover of M and $\{g_{\alpha\beta}\}_{\alpha,\beta\in I}$ transition functions.

We have the following result from Brylinski which relates sheaf cohomology and Cech cohomology in degrees one and two respectively.

Theorem 3.9.1. [1] $H^1(M, \underline{\mathbb{C}^\times}) \simeq H^2(M, \mathbb{Z})$

Let $g_{\alpha\beta} \colon U_{\alpha\beta} \to \mathbb{C}^\times$ be the transition functions with respect to some open over $\{U_\alpha\}$ of M. Then we have a Cech 1-cocycle and hence a class in $H^1(M, \underline{\mathbb{C}^\times})$. Choose a branch of the logarithm of $g_{\alpha\beta}$ over $U_{\alpha\beta}$ and define

$$\begin{aligned}\mu_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} &\to \mathbb{Z}(1)\\ g_{\alpha\beta} &\mapsto -\mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta})\end{aligned}$$

Here $\mathbb{Z}(1) = (2\pi i)\mathbb{Z}$ i.e. all integer multiples of $2\pi i$.

Lemma 3.9.2. *$\mu_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathbb{Z}(1)$ as defined above is a Cech 2-cocycle.*

Proof. Firstly

$$\exp(\mu_{\alpha\beta\gamma}) = \exp(-\mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta})) = \exp(\mathrm{Log}(\frac{g_{\alpha\gamma}}{g_{\beta\gamma}g_{\alpha\beta}})).$$

Since $g_{\alpha\beta}$ is a cocycle we have that

$$g_{\alpha\gamma} = g_{\beta\gamma}g_{\alpha\beta}.$$

So $\exp(\mu_{\alpha\beta\gamma}) = \exp(\mathrm{Log}(1))$ which implies that $\mu_{\alpha\beta\gamma} = 2\pi k_{\alpha\beta\gamma} i$ for some $k_{\alpha\beta\gamma} \in \mathbb{Z}^\times$. Then

$$\mu_{\beta\gamma\delta}\mu^{-1}_{\alpha\gamma\delta}\mu_{\alpha\beta\delta}\mu^{-1}_{\alpha\beta\gamma} = k_{\beta\gamma\delta}k^{-1}_{\alpha\gamma\delta}k_{\alpha\beta\delta}k^{-1}_{\alpha\beta\gamma}.$$

Now by the cocycle condition we have

$$\begin{aligned}\mu_{\alpha\gamma\delta} &= -\mathrm{Log}(g_{\gamma\delta}) + \mathrm{Log}(g_{\alpha\beta}g_{\beta\delta}) - \mathrm{Log}(g_{\alpha\beta}g_{\beta\gamma})\\ &= -\mathrm{Log}(g_{\gamma\delta}) + \mathrm{Log}(g_{\alpha\beta}) + \mathrm{Log}(g_{\beta\delta}) - \mathrm{Log}(g_{\alpha\beta}) - \mathrm{Log}(g_{\beta\gamma})\\ &= -\mathrm{Log}(g_{\gamma\delta}) + \mathrm{Log}(g_{\beta\delta}) - \mathrm{Log}(g_{\beta\gamma})\\ &= \mu_{\beta\gamma\delta}.\end{aligned}$$

Therefore $k_{\alpha\gamma\delta} = k_{\beta\gamma\delta}$ and a similar calculation shows that

$$k_{\alpha\beta\delta} = k_{\alpha\beta\gamma}.$$

So

$$\mu_{\beta\gamma\delta}\mu_{\alpha\gamma\delta}^{-1}\mu_{\alpha\beta\delta}\mu_{\alpha\beta\gamma}^{-1} = k_{\beta\gamma\delta}k_{\alpha\gamma\delta}^{-1}k_{\alpha\beta\delta}k_{\alpha\beta\gamma}^{-1} = 1$$

Hence $\mu_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to \mathbb{Z}(1)$ is a Cech 2-cocycle and hence a class in $H^2(M, \mathbb{Z}(1))$. □

So we have a mapping $H^1(M, \underline{\mathbb{C}^\times}) \to H^2(M, \mathbb{Z}(1))$

Definition 3.9.3. [1] Given a line bundle (L, π, M) the *first Chern class* denoted $c_1(L)$ is the class under the above mapping $H^1(M, \underline{\mathbb{C}^\times}) \to H^2(M, \mathbb{Z}(1))$.

Definition 3.9.4. [1] The *Picard group* denoted $Pic^\infty(M)$ is the group of isomorphism classes of line bundles on M.

We now have a remarkable fact that the Picard group is completely described by the first Chern class. Thus from calculating the the transition functions of a line bundle we have a representative of the first Chern class and hence the isomorphism classes of the line bundle.

Theorem 3.9.5. [1] $Pic^\infty(M) \simeq H^1(M, \underline{\mathbb{C}^\times})$

So the first Chern class classifies line bundles up to isomorphism. Below we will see how we can reconstruct a line bundle from transition functions and an open cover of your base manifold.

3.10 The clutching construction

The clutching construction is a way to get a line bundle from only having an open cover of your manifold and transition functions. The construction is outlined in the proof of the following proposition.

Proposition 3.10.1. [18] *Given an open cover $\{U_\alpha\}_{\alpha\in I}$ of M and functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ that satisfy the Čech cocycle condition, we can construct a line bundle $L \to M$ with transition functions $g_{\alpha\beta}$.*

Proof. Let $\tilde{M} = \{(m, \alpha) \mid \alpha \in I \text{ and } m \in U_\alpha\} \subseteq M \times I$. Now take $\mathbb{C} \times \tilde{M}$ and define an equivalence relation on the set by $(\lambda, m, \alpha) \sim (\mu, n, \beta)$ if and only if $m = n$ and $g_{\alpha\beta}(m)\lambda = \mu$. The reason that $\sim$ is an equivalence relation comes from the cocycle condition of the transition functions. Let $(\lambda, \alpha, m) \in \mathbb{C} \times \tilde{M}$ then $g_{\alpha\alpha}(m)\lambda = 1 \cdot \lambda = \lambda$ so $\sim$ is reflexive. Now suppose $(\lambda, \alpha, m) \sim (\mu, n, \beta)$. So $m = n$ and $g_{\alpha\beta}(m)\lambda = g^{-1}_{\alpha\beta}(m)\lambda = \mu$ hence, $g_{\beta\alpha}(m)\mu = \lambda$. So $\sim$ is symmetric. Finally, suppose $(\lambda, \alpha, m) \sim (\mu, n, \beta)$ and $(\mu, n, \beta) \sim (\tilde{\lambda}, \gamma, p)$. Then $g_{\alpha\gamma}(m)\lambda = g^{-1}_{\gamma\alpha}(m)g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m)\lambda = g_{\alpha\beta}(m)g_{\beta\gamma}(m)\lambda = g_{\beta\gamma}(m)\mu = \tilde{\lambda}$ and $m = n = p$. So the relation is transitive and therefore an equivalence relation.

Let $L = (\mathbb{C} \times \tilde{M})/\sim$ and define addition and scalar multiplication by $[(\lambda, m, \alpha)] + [(\mu, m, \alpha)] = [(\lambda + \mu, m, \alpha)]$ and $z[(\lambda, m, \alpha)] = [(z\lambda, m, \alpha)]$ respectively. Define the projection by $\pi([(\lambda, m, \alpha)]) = m$.

Define local sections by $\sigma_\alpha(m) = [(1, m, \alpha)] = [(g_{\alpha\beta}(m), m, \beta)] = g_{\alpha\beta}(m)\sigma_\beta(m)$. So the transition functions satisfy the requirements. With this we can make L into a line bundle. Firstly define a local trivialisation by $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to \mathbb{C} \times U_\alpha$ by $\psi_\alpha([\lambda, m, \alpha]) = (\lambda, m)$. We need to show this map is well defined, so let $[(\mu, n, \beta)]$ be another representative of $[(\lambda, n, \alpha)]$. Firstly $[(\mu, n, \beta)] \in \pi^{-1}(U_\alpha)$ so $\beta = \alpha$ and $n \in U_\alpha$. Now since $(\lambda, m, \alpha) \sim (\mu, n, \beta)$ we have that $m = n$ and $g_{\alpha\beta}(m)\lambda = \mu$. But, $\alpha = \beta$ so, $\mu = g_{\alpha\beta}(m)\lambda = g_{\alpha\alpha}(m)\lambda = \lambda$. So $\psi_\alpha([(\lambda, m, \alpha)]) = \psi_\alpha([\mu, n, \beta])$. So ϕ_α is well defined. Finally to show L is a differentiable manifold take coordinate charts (V_α, ϕ_α) on $U_\alpha \times \mathbb{C}$ and define a chart on L by $(\phi^{-1}_\alpha(V_\alpha), \phi_\alpha \circ \psi_\alpha)$. This chart gives an atlas and therefore L is a differentiable manifold. Thus making $L \to M$ a line bundle. $\square$

So if we want to get an isomorphism class of line bundles and a representative of this class, the clutching construction and first Chern class give us a way to do so. Therefore, the transition functions capture all the topological information of a line bundle. Another important fact is that we can now give a geometric representation of elements in $H^2(M, \mathbb{Z}(1))$.

Chapter 4

Principal Bundles

In this chapter we provide an overview of principal bundles. These can be considered as a more general vector bundle where instead of the fibres being vector spaces, the fibres are a Lie group. As such all the same ideas in the previous chapter are present here and will progress much in the same way. We finish this chapter with the result that the category of principal $\mathbb{C}^{\times}$-bundles is equivalent to the category of complex line bundles.

4.1 Definition of a principal bundle

Definition 4.1.1. [11] Let M be a manifold and G be a Lie group. A *principal G-bundle* is a manifold P with a $G-$action such that the following hold.

1. G acts freely on the right $\rho\colon P \times G \to P$ where we write $\rho(u, g) = ug$.

2. M is isomorphic the orbit space P/G and $\pi\colon P \to M$ is a smooth surjection.

3. For all $x \in M$ there is an open neighbourhood $U \subseteq M$ with a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$ with $\varphi\colon \pi^{-1}(U) \to G$ satisfying $\varphi(ug) = \varphi(u)g$ for all $u \in \pi^{-1}(U)$ and $g \in G$.

M is called the base space, P the total space, π the projection and G the structure group. If $x \in M$, $P_x \coloneqq \pi^{-1}(x)$ is called the fibre over x. Principal bundles are denoted $P(M, G, \pi)$ or for brevity $P \to M$, where the structure group and projection are understood.

4.2 Examples of principal bundles

Here we give some examples of principal bundles and some constructions we can do with principal bundles to generate more principal bundles.

Example 4.2.1. [9] **The trivial bundle.**

Let $P = M \times G$ then $P(M, G, \mathrm{pr}_1)$ is a G-bundle with the trivial action $(x, g) \cdot g' = (x, gg')$.

This action is clearly free as if $(m, g) \cdot g' = (m, g)$ for all $(m, g) \in M \times G$. Then we would have $gg' = g$ for all $g \in G$ and then by uniqueness of the group identity we have $g' = e_G$. The orbit space $(M \times G)/G$ is clearly isomorphic to M as the orbits are equivalence classes

$$[(x, g)] = \{(x, h) \in \{x\} \times G \mid \exists g' \text{ s.t. } gg' = h\}.$$

So the isomorphism between $(M \times G)/G \to M$ is clear. Lastly let $m \in M$ and let $U \subseteq M$ be an open neighbourhood of m. Take the diffeomorphism $\psi \colon \pi^{-1}(U) \to U \times G$ to be $\psi(x, g) = (\mathrm{pr}_1(x, g), \mathrm{pr}_2(x, g)) = id_{U \times G}$. We have that $\mathrm{pr}_2((x, g)g') = gg' = \mathrm{pr}_2(x, g)g'$. So the trivial bundle is a G-bundle.

Example 4.2.2. [9] **Pullback.**

Let $P \to M$ be a G-bundle and $f \colon N \to M$ be a smooth map. The *pullback* of P by f is a $G-$bundle denoted $f^{-1}(P)$ where

$$f^{-1}(P) \coloneqq \{(n, p) \in N \times P \mid \pi(p) = f(n)\}.$$

With projection map $\mathrm{pr}_1 \colon f^{-1}(P) \to N$.

Since $P \to M$ is a G-bundle it comes with a free right action. So let G act on $f^{-1}(P)$ by $(n, p) * g = (n, p \cdot g)$. Where $\cdot$ is the right action on P. The action $*$ is free because $\cdot$ is. The projection pr_1 is clearly smooth and surjective. Now we want $f^{-1}(P)/G \simeq N$.

Let $(n, p) \in f^{-1}(P)$ and take the orbit $O_{(n,p)} \in f^{-1}(P)/G$. Then we have a mapping $f^{-1}(P)/G \to N$ by taking a representative of this orbit (n, p) and apply the projection $\mathrm{pr}_1(n, p) = n$.

Firstly this mapping is well-defined as if you take another representative $(m, q) \in O_{(n,p)}$ then $m = n$. So both orbits are mapped to the same point in N. Now suppose for $O_{(n,p)}, O_{(m,q)} \in f^{-1}(P)/G$ we have $\mathrm{pr}_1((n, p)) = \mathrm{pr}_1((m, q))$. So $n = m$ which implies $\pi(p) = f(n) = f(m) = \pi(q)$ and therefore, $p, q \in P_p$. So there must be some $g \in G$ such that $p = qg$ therefore $(m, q) \in O_{(n,p)}$. Hence, $O_{(n,p)} = O_{(m,q)}$ and so the map is injective.

The map is surjective as if $n \in N$, $f(n) \in M$ and since π is surjective there must be some $p \in P$ such that $f(n) = \pi(p)$. So $(n, p) \in f^{-1}(P)$ and so we can take the orbit $O_{(n,p)}$ which is mapped to n. Therefore, $f^{-1}(P)/G \simeq N$.

Since $P \to M$ is a G-bundle, for all $m \in M$ there is an open neighbourhood $U \subseteq M$ of m and a diffeomorphism $\phi \colon \pi^{-1}(U) \to U \times G$. Now let $n \in N$ and choose $U \subseteq M$ such that $n \in f^{-1}(U)$. Then define a diffeomorphism $\psi \colon \mathrm{pr}_1^{-1}(f^{-1}(U)) \to f^{-1}(U) \times G$ by

$\psi(n,p) = (n, \mathrm{pr}_2(\phi(p)))$. This is a diffeomorphism because ϕ is. Now $\mathrm{pr}_2(\phi(p)) = \varphi(p)$ for some $\varphi\colon \pi^{-1}(U) \to G$ such that $\varphi(pg) = \varphi(p)g$ for all $g \in G$. Therefore, $\mathrm{pr}_2(\phi(pg)) = \mathrm{pr}_2(\phi(p))g$ for all $g \in G$. Therefore, $f^{-1}(P) \to N$ is a G-bundle.

Example 4.2.3. **pre-G-bundle.**

Definition 4.2.1. [5] A pre-G-bundle is a pair $(Y, \hat{g})$, where $\pi_Y\colon Y \to M$ is a surjective submersion and $\hat{g}\colon Y^{[2]} \to G$ is a map such that $\hat{g}(y_1, y_3) = \hat{g}(y_1, y_2)\hat{g}(y_2, y_3)$ for all y_1, y_2, y_3 in the same fibre of $\pi_Y\colon Y \to M$.

If $P \to M$ is a principal G-bundle we can form a pre-G-bundle $(P, \hat{g})$ by taking the canonical difference map $\hat{g}\colon P^{[2]} \to G$ defined by $p_1\hat{g}(p_1, p_2) = p_2$. Conversely given a pre-G-bundle $(Y, \hat{g})$ we can form a principal G-bundle over M in the following way. Take $Y \times G$ and define an equivalence relation $(y_1, h_1) \sim (y_2, h_2)$ if and only if $\pi_Y(y_1) = \pi_Y(y_2)$ and $h_1\hat{g}(y_1, y_2) = h_2$. Then $P = (Y \times G)/\sim$ is a G-bundle over M with the right action given by $[(y,h)] \cdot g = [(y, hg)]$.

Firstly we need this group action to be free. Suppose we had some $g \in G$ such that $[(y,h)] \cdot g = [(y, hg)]$ for all $[(y,h)] \in P$. Then that would imply that $h\hat{g}(y,y) = gh$. Note that by definition of $\hat{g}$ we have, $\hat{g}(y,y) = \hat{g}(y,y)\hat{g}(y,y)$ and so $\hat{g}(y,y) = e$. So $h\hat{g}(y,y) = h = gh$. Therefore $g = e$. So the action is free. For the projection we already have a surjective submersion $\pi_Y\colon Y \to M$ so define $\pi\colon P \to M$ by $\pi([(y,h)]) = \pi_Y(y)$. This projection is well defined as if you take some other representative $(x,g) \in [(y,h)]$ by definition we have $\pi([(y,h)]) = \pi_Y(y) = \pi_Y(x) = \pi([(x,g)])$. The projection is smooth and surjective because π_Y is.

Now we require $P/G \simeq M$. To show this we need the fact that for $u, v \in P$, $v = u \cdot c$ for some $c \in G$ if and only if $\pi(u) = \pi(v)$. Suppose firstly that, $\pi(u) = \pi(v)$. Let (x,g) and (y,h) be representatives for u and v respectively. Then $\pi_Y(x) = \pi_Y(y)$. Let $c = g^{-1}h\hat{g}(y,x)$, then $gc = h\hat{g}(y,x)$ and therefore, $(y,h) \sim (x, gc)$. Hence $[(y,h)] = [(x,g)] \cdot c$. So $v = u \cdot c$. Now suppose that $v = u \cdot c$. Then $(y,g) \sim (x, gc)$ so, $\pi_Y(x) = \pi_Y(x)$ and therefore $\pi(v) = \pi(u)$. We then have a mapping $P/G \to M$ by taking an orbit $O_v \in P/G$ for $v \in P$ and applying π. This mapping is injective as if we have two orbits $O_u, O_v \in P/G$ and supposing that $\pi(u) = \pi(v)$ then we have $u = v \cdot c$ for some $c \in G$. So $O_u = O_v$. This mapping is surjective because π is. So we have $P/G \simeq M$.

Finally, let $m \in M$ and $m \in U \subseteq M$ be an open neighbourhood. Let $p \in \pi^{-1}(U)$ and let $[(y,h)]$ be a representative for p. Define $\varphi\colon \pi^{-1}(U) \to G$ by $\varphi([(y,h)]) = h$. Then $\varphi([(y,h)] \cdot g) = hg = \varphi([y,h]) \cdot g$. So define a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times G$ by $\psi(p) = (\pi(p), \varphi(p))$. So we have a local trivialisation and hence a G-bundle.

So any pre-G bundle gives rise to a G-bundle. An important thing to note is that many of the ideas in the above calculation will be used for the clutching construction of a principal bundle.

We defer the rest of our examples to the coming section on $\mathbb{C}^\times$-bundles.

4.3 Isomorphism of principal bundles

Definition 4.3.1. [9] Let $P(M, G, \pi)$ and $P'(M',G',\pi')$ by two principal bundles. A *homomorphism* of principal bundles consists of a pair of maps (u, f) where $u\colon P \to P'$ and $f\colon G \to G'$ is a homomorphism such that $u(pg) = u(p)f(g)$ for all $p \in P$ and $g \in G$.

If we have two principal bundles with the same structure group G then we only require a map $u\colon P \to P'$ such that $u(pg) = u(p)g$. We say the G-bundles are *isomorphic* if the map $u\colon P \to P'$ is an isomorphism. Note that in this case we have $M = M'$.

Definition 4.3.2. [9] A principal G-bundle is *trivialisable* if it is isomorphic to the trivial G-bundle.

Theorem 4.3.3. [9] *Let $u\colon P \to P'$ be a smooth homomorphism of principal G-bundles $P(M, G, \pi)$ and $P'(M, G, \pi')$. Then $u\colon P \to P'$ is an isomorphism.*

Proof. Firstly suppose $u(p_1) = u(p_2)$ for some $p_1, p_2 \in P$. Then $\pi(p_1) = \pi'(u(p_1)) = \pi'(u(p_2)) = \pi(p_2)$. So p_1, p_2 lie in the same fibre and hence $p_1 = p_2 \cdot g$ for some $g \in G$. Then $u(p_1) = u(p_2 \cdot g) = u(p_2) \cdot g = u(p_1) \cdot g$, but the group action is free so we must have $g = e_G$ and therefore, $p_1 = p_2$. So u is injective.

Let $p' \in P'$ and since π and π' are surjective choose $p \in P$ such that $\pi'(p') = \pi(p)$. Then, $\pi'(u(p)) = \pi(p) = \pi'(p')$. So $u(p), p'$ are in the same fibre and hence $p' = u(p) \cdot g$ for some $g \in G$. So we have $p' = u(p \cdot g)$ and hence u is surjective.

So $u\colon P \to P'$ is bijective and therefore an inverse exists. What remains to be shown is that $u^{-1}\colon P' \to P$ is smooth. This is shown in ([9], p.g. 131), □

Corollary 4.3.4. [9] *A principal G-bundle $P \to M$ is trivial if there is a homomorphism to the trivial bundle $M \times G$.*

Proof. This follows immediately from above as if we have a homomorphism $P \to M \times G$ then by theorem (4.3.3) we have an isomorphism and hence $P \to M$ is trivial. □

4.4 Sections of principal bundles

Definition 4.4.1. [9] Given a principal bundle $P(M, G, \pi)$ a *section* is a smooth map $\sigma\colon M \to P$ such that $\pi \circ \sigma = id_M$.

Unlike line bundles, principal bundles very rarely have a section. In fact, if a principal bundle has a section it is trivialisable.

Proposition 4.4.2. [9] *A principal bundle has a section if and only if it is trivialisable.*

Proof. Let $P \to M$ be a G-bundle. Firstly suppose the G-bundle is trivialisable so we have a diffeomorphism $u\colon M \times G \to P$ then define a map $\sigma\colon M \to P$ by $\sigma(x) = u(x, e_G)$ as u is a diffeomorphism σ is smooth. Let $(x, e_G) \in M \times G$ so then

$$\pi(\sigma(x)) = \pi(u(x, e_G)) = \mathrm{pr}_1(x, e_G) = x.$$

So $\sigma\colon M \to P$ is a smooth section.

Now suppose we have a smooth section $\sigma\colon M \to P$. Define a smooth function $f\colon P \to G$ by $p = \sigma(\pi(p)) \cdot f(p)$ for $p \in P$. Now define $u\colon P \to M \times G$ by $u(p) = (\pi(p), f(p))$, since π and f is smooth u is smooth. Since $\pi(p \cdot g) = \pi(p)$ we have

$$p \cdot g = \sigma(\pi(p)) \cdot f(p) \cdot g = \sigma(\pi(p \cdot g)) \cdot f(p \cdot g)$$

Therefore, $f(p \cdot g) = f(p) \cdot g$. Then

$$\begin{aligned} u(p \cdot g) &= (\pi(p), f(p \cdot g)) \\ &= (\pi(g), f(p) \cdot g) \\ &= (\pi(g), f(p)) \cdot g \end{aligned}$$

Thus, $u\colon P \to M \times G$ is an G-bundle homomorphism and thus an isomorphism. So $P \to M$ is trivialisable. $\square$

4.5 Transition functions

Just like with line bundles we can use the clutching construction to construct a principal G-bundle given an open cover and transition functions. Firstly, we need to make precise what the transition functions of a G-bundle are.

Let $P \to M$ be a G-bundle and $\{U_\alpha\}$ be an open over of M. So on each U_α we have diffeomorphisms $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $\psi_\alpha(u) = (\pi(u), \varphi_\alpha(u))$ with $\varphi_\alpha(ug) = \varphi_\alpha(u)g$. On the overlaps we define maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ to compute the difference of φ_α and φ_β in the following way $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$. Like with line bundles the maps $g_{\alpha\beta}$ are called *transition functions* and satisfy the 1-cocycle condition.

The next theorem shows that, like with line bundles, we can construct a G-bundle from transition functions.

Theorem 4.5.1. [11] *Let M be a differentiable manifold with open cover $\{U_\alpha\}$ and let G be a lie group. Given functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ which satisfy the Čech 1-cocycle condition we can construct a principal G-bundle with transition functions $g_{\alpha\beta}$.*

Proof. Let $X_\alpha = U_\alpha \times G$ and define $X = \bigcup_{\alpha\in I} X_\alpha$. Elements of X are triples (α, x, a) for $\alpha \in I$, $x \in U_\alpha$ and $a \in G$. Define an equivalence relation on X in the following way, $(\alpha, x, a) \sim (\beta, y, b)$ if and only if $x = y \in U_\alpha \cap U_\beta$ and $b = g_{\beta\alpha}(x)a$. Since each X_α is a differentiable manifold X is. Now let $P = X/\sim$ and define the right action by G as $[(\alpha, x, a)]\cdot c = [(\alpha, x, a\cdot c)]$. This action is well-defined as if we take another representative (β, y, b) we have that $(\beta, y, b) \sim (\alpha, x, a)$ and so $y = x$ and $b = g_{\beta\alpha}(x)a$. Therefore $b \cdot c = g_{\beta\alpha}(y) \cdot c$ so, $(\beta, y, b \cdot c) \sim (\alpha, x, a \cdot c)$. Now we need to show that $M \simeq P/G$ and G acts freely. Firstly the action by G is free as if we had some $c \in G$ such that $[(\alpha, x, a)] \cdot c = [(\alpha, x, a)]$ for all $[(\alpha, x, a)] \in P$. That would imply that $\forall g \in G\ g \cdot c = g$ and then by uniqueness of the group identity $c = e$. Define the projection $\pi\colon P \to M$ by $\pi([(\alpha, x, a)]) = x$. This projection is well defined as if we take another representative $(\beta, y, b) \in [(\alpha, x, a)]$ we have that $x = y$ so $\pi([(\beta, y, b)]) = \pi([(\alpha, x, a)])$.

To show $P/G \simeq M$ we will first show that for $u, v \in P$, $\pi(u) = \pi(v)$ if and only if $v \sim u\cdot c$ for some $c \in G$. Let (α, x, a) and (β, y, b) be representatives for u and v respectively. Clearly if $v \sim u$ then $x = y \in U_\alpha \cap U_\beta$ and so $\pi(u) = x = y = \pi(v)$. Conversely, suppose that $\pi(u) = \pi(v)$ and let $c = a^{-1}g_{\beta\alpha}(x)^{-1}b$. Then, $(\alpha, x, a) \cdot c = (\alpha, x, g_{\beta\alpha}(x)^{-1}b)$ and since $x = y$ we have $x, y \in U_\alpha \cap U_\beta$ and $g_{\beta\alpha}(x)g_{\beta\alpha}(x)^{-1}b = b$. So $u \cdot c \sim v$ where this equivalence relation is on orbits. Let O_v be the orbit of some $v \in P$. Then we get a map into M by taking a representative for v, say $[(\alpha, x, a)]$ and using π to map into M. This mapping is injective as if we take two orbits O_v and O_u for some $u, v \in P$ and suppose that $\pi(u) = \pi(v)$. Then, from above we have that $v \sim u \cdot c$ for some $c \in G$ and so $O_u = O_v$. The mapping $P/G \to M$ is surjective because π is. As for any $m \in M$ there is a corresponding $u \in P$ such that $\pi(u) = m$ and therefore a corresponding orbit O_u. Therefore, $P/G \simeq M$.

Finally, for each $m \in M$ we require an open neighbourhood $m \in U \subseteq M$ and a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times G$ where $\psi = (\pi, \varphi)$ and $\varphi(ug) = \varphi(u)g$. For each $m \in M$ we already have an open neighbourhood $m \in U_\alpha$ so let $\varphi_\alpha([(\alpha, x, a)]) = a$, which clearly satisfies $\varphi_\alpha(u \cdot g) = \varphi_\alpha(u) \cdot g$. Define the diffeomorphism $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ by $\psi_\alpha([(\alpha, x, a)]) = (x, a)$. Therefore, we have a G-bundle $P \to M$. For the given open cover we need transition functions so that $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$, but this is exactly requiring that $b = g_{\beta\alpha}(x)a$. So the transition functions are exactly the $g_{\alpha\beta}$ given. $\square$

4.6 Connections on a principal bundle

We would like a way to compare points in neighbouring fibres that is not dependent on a particular trivialization of the bundle. So we need a way of moving between these fibres.

Definition 4.6.1. [9] For $p \in P$, the vertical subspace $V_pP \subseteq T_pP$ is the set of vectors tangent to the fibre at p. So, $V_p(P) \coloneqq \{v \in T_pP \mid \pi_*(v) = 0\}$. A vector field $v \in \mathfrak{X}(P)$ is said to be *vertical* if $v(p) \in V_pP$ for all $p \in P$.

This subspace only allows us to move vertically along a fibre not horizontally. This is what a connection resolves and like with line bundles gives, a notion of directional derivative in the direction of a vector field.

Definition 4.6.2. [9] A connection on a principal bundle $P(M, G, \pi)$ is a smooth assignment for each $p \in P$ a horizontal subspace H_pP of T_pP complement to V_pP such that:

1. $T_pP = V_pP \oplus H_pP$, for all $p \in P$.

2. $(R_g)_*(H_pP) = H_{pg}P$, for all $g \in G$ and $p \in P$.

Condition one shows that any vector in the tangent space can be decomposed into its vertical and horizontal parts. Given a tangent vector $v \in T_pP$ we denote the horizontal and vertical part by $hor(v)$ and $vert(v)$ respectively.

The connection gives rise to a $\mathfrak{g}$-valued one-form ω on P, called the connection one form.

Definition 4.6.3. [9] The connection one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ is a $\mathfrak{g}$-valued one form on P with $v \in T_pP$ defined as

$$\omega(v) \coloneqq \iota^{-1}(vert(v)). \tag{4.6.1}$$

Where ι is as defined in (2.3.1).

Proposition 4.6.4. [9] *The connection one form ω satisfies the following:*

1. $\omega_p(\iota(X)) = X$ *for all* $p \in P$ *and* $X \in \mathfrak{g}$.

2. $(R_g)^*(\omega) = Ad_{g^{-1}}(\omega)$ *for all* $g \in G$.

3. $v \in H_pP$ *if and only if* $\omega_p(v) = 0$ *for* $p \in P$.

Proof.
1) Let $X \in \mathfrak{g}$ since ι is an isomorphism from $\mathfrak{g}$ into V_pP, $\iota(X) \in V_pP$. So $vert(\iota(X)) = \iota(X)$, thus $\omega_p(\iota(X)) = \iota^{-1}(\iota(X)) = X$.

2) As $\omega_p(v)$ is linear we can then suppose either v is vertical or horizontal. Suppose v is horizontal and $p \in P$, then $vert(v) = 0$ and so $\omega_p(v) = 0$. Thus $(R_g)^*(\omega(v)) = Ad_{g^{-1}}(\omega(v)) = 0$. Now suppose $v \in V_pP$. Then we have, $(R_g)^*_p(\omega(v)) = \omega_p((R_g)_*(v))$ and since ι is an isomorphism from $\mathfrak{g}$ into V_pP then we may assume $v = \iota(X)$ for some $X \in \mathfrak{g}$. So, $\omega_p((R_g)_*(v)) = \omega_p(\iota(Ad_{g^{-1}}(X))) = Ad_{g^{-1}}(X)$ and $X = \iota^{-1}(v) = \iota^{-1}(vert(v)) = \omega_p(v)$. Therefore, $(R_g)^*(\omega) = Ad_{g^{-1}}(\omega)$.

3) Suppose $\omega_p(v) = 0$, then $\iota^{-1}(vert(v)) = 0$ and since ι^{-1} is injective, $vert(v) = 0$. Conversely, suppose $v \in H_pP$. So $vert(v) = 0$ which implies $w_p(v) = 0$. □

Note 4.6.1. Not only does this one-form arise from a connection on a principal bundle, we may define a connection in terms of a $\mathfrak{g}$-valued one form on P satisfying the above proposition (4.6.4).

So we have the following definition.

Definition 4.6.5. [9] A connection one-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ on a principal bundle P is a one-form which satisfies proposition (4.6.4).

Example 4.6.1. **Connection on the trivial bundle**. Let $P = M \times G$ be the trivial G-bundle. Then $\omega = \mathrm{pr}_2^*\Theta$ is a connection one-form where Θ is the Maurer-Cartan form.

We can also describe a connection one-form locally in the following way.

Let $\{U_\alpha\}$ be an open cover of M with a collection of isomorphisms $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and transition functions $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$. Let $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$ be local sections on U_α where $e \in G$ is the identity. Let Θ be the Maurer-Cartan $\mathfrak{g}$-valued 1-form on G. On the overlaps $U_\alpha \cap U_\beta$ define $\Theta_{\alpha\beta} = g_{\alpha\beta}^*\Theta$. Finally, for each α define a $\mathfrak{g}$-valued 1-form on U_α by $\omega_\alpha = \sigma_\alpha^*\omega$.

Proposition 4.6.6. [11] *With the notation above,*

$$\omega_\beta = ad_{\psi_{\alpha\beta}^{-1}}(\omega_\alpha) + \Theta_{\alpha\beta} \tag{4.6.2}$$

Proof. Let $x \in U_\alpha \cap U_\beta$ then $(\omega_\beta)_x = (\sigma_\beta^*\omega)_x = \omega_{\sigma_\beta(x)}(\sigma_\beta)_*$. From the cocycle condition we get $\sigma_\beta(x) = \sigma_\alpha(x)g_{\alpha\beta}(x)$. So we can view s_β as the image of $\rho(\sigma_\alpha, g_{\alpha\beta})$ where $\rho \colon P \times G \to G$ is the right action of the G-bundle. So you can think of $(\sigma_\beta(x), g_{\alpha\beta}(x))$ as a map $(\sigma_\alpha, g_{\alpha\beta}) \colon U_\alpha \cap U_\beta \to P \times M$. So

$$\begin{aligned}
(\sigma_\beta)_*(x) &= (\rho \circ (\sigma_\alpha, g_{\alpha\beta}))_*(x) \\
&= \rho_*(\sigma_\beta(x), g_{\alpha\beta}(x)) \circ (\sigma_\beta, g_{\alpha\beta})_*(x) \\
&= \rho_*(\sigma_\beta(x), g_{\alpha\beta}(x)) \circ ((\sigma_\beta)_*(x), (g_{\alpha\beta})_*(x)).
\end{aligned}$$

So we now need an expression for $\rho_*(p,g)$ for $(p,g) \in P \times G$. From ([9], p.g. 93) we have the following isomorphism $T_{(p,g)}(P \times G) \simeq T_p(P) \oplus T_g(G)$. Let $u \in T_p(P)$ and γ be an integral curve of u. Then

$$\begin{aligned}
\rho(p,g)_*(u,0) &= \frac{d}{dt}\Big|_{t=0} \rho(\gamma(t), g) \\
&= \frac{d}{dt}\Big|_{t=0} \gamma(t) \cdot g \\
&= \frac{d}{dt}\Big|_{t=0} R_g(\gamma(t)) \\
&= R_g(p)_*(u).
\end{aligned}$$

Now if we take $Y \in T_g G$ we have that $X = \Theta_g(Y) \in \mathfrak{g}$ is the unique Lie algebra element such that

$$\frac{d}{dt}\Big|_{t=0} g \exp(tX) = T_e L_g(X) = Y.$$

So $t \mapsto g\exp(t\Theta_g(Y))$ is an integral curve of Y. So we get

$$\begin{aligned}
\rho(p,g)_*(0,Y) &= \frac{d}{dt}\Big|_{t=0} \rho(p, g\exp(t\Theta_g(Y))) \\
&= \frac{d}{dt}\Big|_{t=0} p \cdot g \exp(t\Theta_g(Y)) \\
&= \frac{d}{dt}\Big|_{t=0} (p \cdot g) \cdot \exp(t\Theta_g(Y)) \\
&= \iota_{p \cdot g}(\Theta_g(Y)).
\end{aligned}$$

So we have

$$\rho(p,g)_*(u,Y) = (R_g(p))_*(u) + \iota_{p \cdot g}(\Theta_g(Y)).$$

Now letting $p = \sigma_\alpha(x)$, $g = g_{\alpha\beta}(x)$ and $v \in T_x(U_\alpha \cap U_\beta)$ we get

$$\begin{aligned}
(\sigma_\beta)_*(v) &= \rho_*(\sigma_*(v), (g_{\alpha\beta})_*(v)) \\
&= (R_{g_{\alpha\beta}(x)})_*((\sigma_\alpha)_*(v)) + \iota_{\sigma_\alpha(x) g_{\alpha\beta}(x)}(\Theta_{g_{\alpha\beta}(x)}((g_{\alpha\beta})_*(v)) \\
&= (R_{g_{\alpha\beta}(x)})_*((\sigma_\alpha)_*(v)) + \iota_{\sigma_{\beta(x)}}((g_{\alpha\beta}^* \Theta)_x(v))
\end{aligned}$$

Now evaluating both sides by $\omega_{\sigma_\beta(x)}$. Firstly we have

$$\omega_{\sigma_\beta}(\iota_{\sigma_\beta}(g_{\alpha\beta}^* \Theta(v)) = g_{\alpha\beta}^* \Theta(v) = \Theta_{\alpha\beta}(v).$$

For the other term we have

$$\begin{aligned}
\omega_{\sigma_\beta}((R_{g_{\alpha\beta}})_*((\sigma_\alpha)_*(v))) &= \omega_{\sigma_\alpha g_{\alpha\beta}}((R_{g_{\alpha\beta}})_*((\sigma_\alpha)_*(v))) \\
&= (R_{g_{\alpha\beta}}^*(\omega))_{\sigma_\alpha}((\sigma_\alpha)_*(u)) \\
&= \mathrm{ad}_{g_{\alpha\beta}^{-1}}(\omega_{\sigma_\alpha}(\sigma_\alpha(u))) \\
&= \mathrm{ad}_{g_{\alpha\beta}^{-1}}((\sigma_\alpha^* \omega)(u)) \\
&= \mathrm{ad}_{g_{\alpha\beta}^{-1}}(\omega_\alpha(u)).
\end{aligned}$$

Therefore,

$$\omega_\beta = \mathrm{ad}_{g_{\alpha\beta}^{-1}}(\omega_\alpha) + \Theta_{\alpha\beta}$$

□

So we have three different ways of describing a connection on a principal bundle. We could talk about a connection as it was defined or as a one form which we can now describe locally and globally.

Definition 4.6.7. [9] The curvature of a connection $D\omega \in \Omega^2(P) \otimes \mathfrak{g}$ is $\mathfrak{g}$-valued 2-form on P defined as

$$D\omega := d\omega \circ hor. \tag{4.6.3}$$

Lemma 4.6.8. [9] *Let $p \in P$ and suppose $X \in T_pP$ is horizontal and $Y \in \mathfrak{g}$. Then $[X, \iota_p(Y)]$ is horizontal.*

This is calculation can be found in ([9], p.g. 176).

Theorem 4.6.9. [9]

$$D\omega_p(X,Y) = d\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)]. \tag{4.6.4}$$

Proof.
Since exterior derivative is linear we can consider three cases of X and Y. Both X and Y are horizontal, vertical or one is vertical and the other is horizontal.

Firstly, suppose X and Y are horizontal. Then by proposition (4.6.4), $\omega_p(X) = \omega_p(Y) = 0$ so $[\omega_p(X), \omega_p(Y)] = 0$. Since X, Y are horizontal, $d\omega_p(X,Y) = d\omega_p(hor(X), hor(Y)) = D\omega_p(X,Y)$. So the above holds.

Now suppose both X and Y are vertical. Since X and Y are vertical we can suppose $X = \iota(A)$ and $Y = \iota(B)$ for some $A, B \in \mathfrak{g}$.

So,

$$\begin{aligned} d\omega_p(X,Y) + [\omega_p(X), \omega_p(Y)] &= d\omega_p(\iota(A), \iota(B)) + [\omega_p(\iota(A)), \omega_p(\iota(B))] \\ &= \iota(A)(\omega_p(\iota(B))) - \iota(B)(\omega_p(\iota(A))) - \omega_p([\iota(A), \iota(B)]) + [A,B] \\ &= \iota(A)(B) - \iota(B)(A) - \omega_p(\iota([A,B])) + [A,B] \\ &= -\omega_p(\iota([A,B])) + [A,B] \\ &= -[A,B] + [A,B] = 0. \end{aligned}$$

So, $D\omega_p(X,Y) = 0$ as X and Y are vertical. So the above holds.

Finally, suppose X is horizontal and Y is vertical. As Y is vertical, $hor(Y) = 0$ so $D\omega_p(X,Y) = 0$. As X is horizontal, $vert(X) = 0$ so $\omega_p(X) = 0$, which implies that $[\omega_p(X), \omega_p(Y)] = 0$. So we want $d\omega_p(X,Y) = 0$. Letting $Y = \iota(B)$ for some $B \in \mathfrak{g}$ and the fact that $\omega(X) = 0$, the above calculation for both X and Y vertical shows that, $d\omega_p(X,Y) = -\omega_p([X, \iota(B)])$. Since $[X, \iota(B)]$ is horizontal by lemma (4.6.8), $\omega([X, \iota(B)]) = 0$. Therefore, $d\omega_p(X,Y) = 0$. □

From rearranging the above result in terms of $d\omega$ we have a way to compute $d\omega$ if we know the curvature form

$$d\omega(X,Y) = D\omega(X,Y) - [\omega(X), \omega(Y)]. \tag{4.6.5}$$

This the called the *structural equation.*

4.7 Principal $\mathbb{C}^\times$-bundles

We now look at the specific case where the structure group is $\mathbb{C}^\times$. We look at the theory outlined above with some specific examples and finishing with the connection between $\mathbb{C}^\times$-bundles and complex line bundles.

As all the main definitions remain the same we will not restate them. The great fact about $\mathbb{C}^\times$ is that it is abelian so a lot of the hard terms to calculate in connections and curvature vanish or are made easier.

Proposition 4.7.1.

$$D\omega_p(X,Y) = d\omega_p(X,Y) \tag{4.7.1}$$

Proof.
This follows from theorem 4.6.9 and as the one-form ω is valued in $\mathbb{C}^\times$ whose Lie algebra is $\mathbb{C}$ which is abelian and therefore, $[\omega(X), \omega(Y)] = 0$. □

This means that if we know the connection one form we immediately get its curvature without much work.

Proposition 4.7.2. *The connection one-form ω can be expressed locally as*

$$\omega_\beta = \omega_\alpha + \Theta_{\alpha\beta} \tag{4.7.2}$$

Proof. This follows from theorem 4.6.6 as $\mathbb{C}^\times$ is abelian the adjoint term $Ad_{\psi^{-1}_{\alpha\beta}}$ cancels. □

4.7.1 Examples of $\mathbb{C}^\times$-bundles

Here we give a few examples of $\mathbb{C}^\times$-bundles and look at their connection and curvature.

Example 4.7.1. [1] **Contracted product.**

Let $P \to M$ and $Q \to M$ be $\mathbb{C}^\times$-bundles. The *contracted product* of P and Q is a $\mathbb{C}^\times$-bundle over M and is denoted $P \otimes Q$. The construction is as follows.

Take the fibre product $P \times_M Q$ and quotient by the $\mathbb{C}^\times$-action $\lambda \cdot (p, q) = (\lambda^{-1} \cdot p, \lambda \cdot q)$. Define the projection $\pi \colon P \otimes Q \to M$ by $\pi([p, q]) = \pi_P(p)$. This is well defined as if we took another representative $[x, y] = [p, q]$, then $(x, y) = (\lambda^{-1}p, \lambda q)$ for some $\lambda \in \mathbb{C}^\times$. Then

$$\begin{aligned}
\pi([x, y]) &= \pi_P(x) \\
&= \pi_P(\lambda^{-1}p) \\
&= \pi_P(p) \\
&= \pi([p, q])
\end{aligned}$$

Define the right action on $P \otimes Q$ by $\lambda \cdot [(p, q)] = [(\lambda \cdot p, q)]$. This action is free from both the actions on P and Q being free. Since P and Q are $\mathbb{C}^\times$-bundles we have $P/\mathbb{C}^\times \simeq M$. If we take the orbit of $[(p, q)] \in P \otimes Q$ we have

$$O_{[(p,q)]} = \{[x, y] \in P \otimes Q \mid [\lambda x, y] = [p, q] \text{ for some } \lambda \in \mathbb{C}^\times\}.$$

Notice that since $[\lambda x, y] = [p, q]$ there exists some $\alpha \in \mathbb{C}^\times$ such that $\alpha^{-1}\lambda x = p$. So $x \in O_p$. Hence we have a natural map

$$O_{[(p,q)]} \to O_p$$

It then follows from the fact that $P/\mathbb{C}^\times \simeq M$ that $(P \otimes Q)/\mathbb{C}^\times \simeq M$.

Let $[(p, q)] \in P \otimes Q$ and $m \in M$. Since P and Q are principal bundles we have open neighbourhood $U \subseteq M$ for m. We also have diffeomorphisms $\psi_P \colon \pi_P^{-1}(U) \to U \times \mathbb{C}^\times$ and $\psi_Q \colon \pi_Q^{-1}(V) \to V \times \mathbb{C}^\times$.

So we can define a diffeomorphism

$$\psi \colon \pi^{-1}(U) \to U \times \mathbb{C}^\times.$$

By

$$\psi([p, q]) = (\pi_P(p), \varphi_P(p)\varphi_Q(q)).$$

So $P \otimes Q$ is $\mathbb{C}^\times$-bundle.

Let $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ and $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ be the transition functions of P and Q respectively. Then we have

$$(\varphi_P)_\alpha = g_{\alpha\beta}(\varphi_P)_\beta$$
$$(\varphi_Q)_\alpha = h_{\alpha\beta}(\varphi_Q)_\beta$$

Thus the transition functions of $P \otimes Q$ must be $g_{\alpha\beta}h_{\alpha\beta}$.

If ω_P and ω_Q are connection one-forms on P and Q respectively, then $P \otimes Q$ has an induced connection $\omega_P \otimes \omega_Q$. The curvature of this connection $d\omega_P + d\omega_Q$.

Example 4.7.2. [23] **Dual.**

Let $P \to M$ be a $\mathbb{C}^\times$-bundle then the *dual* is denoted $P^* \to M$ and is the same as P but the right action is given by $(p, z) = pz^{-1}$. Note that this is only a right action because $\mathbb{C}^\times$ is abelian.

From $P \to M$ we get a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times \mathbb{C}^\times$ such that $\psi(u) = (\pi(u), \varphi(u))$. Where $\varphi\colon \pi^{-1}(U) \to \mathbb{C}^\times$ such that $\varphi(u \cdot z) = \varphi(u)z$ for $z \in \mathbb{C}^\times$. However on the dual bundle the group action is the inverse. So we need $\varphi\colon \pi^{-1}(U) \to \mathbb{C}^\times$ such that $\varphi(uz^{-1}) = \varphi(u)z^{-1}$. So give the local trivialisation of P^* by $\psi(u) = (\pi(u), \frac{1}{\varphi(u)})$.

Therefore, if $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$ are the transition functions of P, the transition functions of P^* are $g_{\alpha\beta}^{-1}$

If $P \to M$ is a $\mathbb{C}^\times$-bundle and $\omega \in \Omega^1(P)$ is a connection one-form then there is an induced connection one-form on $P^* \to M$ is $-\omega$ with curvature $-d\omega$.

Example 4.7.3. [23]

Let $P \to M$ be a $\mathbb{C}^\times$-bundle. $P \otimes P^*$ is canonically trivial. Recall that a principal bundle is trivial if and only if it has a global section. We can see $P \otimes P^*$ is trivial by taking the section $s\colon M \to P \otimes P^*$ to be $s(m) = [(p, p^*)]$ where $p \in P_m$ and $p^* = p$ but taken as a point in P^*. This is well defined as if you took another representative $[(q, q^*)]$ say, then for some $\lambda \in \mathbb{C}^\times$ we have $[(q, q^*)] = [(\lambda p, \lambda^{-1} p^*)] = [(p, p^*)]$.

Example 4.7.4. [2] **Central extensions.** Let G and A be groups. A *central extension* of G by A is another group $\hat{G}$ and a homomorphism $\pi\colon \hat{G} \to G$ whose kernel is isomorphic to A and in the centre of $\hat{G}$.

Let G be a Lie group. Consider the following central extension

$$\mathbb{C}^\times \xrightarrow{\iota} \hat{G} \xrightarrow{p} G$$

We can view this central extension as a $\mathbb{C}^\times$-bundle in the following way. Firstly define the free right group action $\hat{G} \times \mathbb{C}^\times \to \hat{G}$ by $(\hat{g}, z) = \hat{g} \cdot \iota(z)$. The group action is free because ι

is injective. Let the projection be $p\colon \hat{G} \to G$ which is smooth and surjective by definition of the central extension. We want $\hat{G}/\mathbb{C}^\times \simeq G$. Define the following map $\phi\colon G \to \hat{G}/\mathbb{C}^\times$ by $\phi(g) = [\hat{g}]$ where $p(\hat{g}) = g$. Let $g, h \in G$ and suppose that $\phi(g) = \phi(h)$. Then $[\hat{g}] = [\hat{h}]$, so there must be some $z \in \mathbb{C}^\times$ such that $\hat{g} = \hat{h} \cdot \iota(z)$. Then $p(\hat{g}) = p(\iota(z))p(\hat{h})$ and since $\mathrm{im}(\iota) = \ker(p)$ we have that $g = h$ so ϕ is injective. Now let $[\hat{g}] \in \hat{G}/\mathbb{C}^\times$ then $p(\hat{g}) \in G$ so $\phi(p(\hat{g})) = [\hat{g}]$. So ϕ is surjective and clearly a well defined homomorphism. Therefore, $\hat{G}/\mathbb{C}^\times \simeq G$. Lastly we need the local triviality condition. Let $g \in G$ and $U \subseteq G$ be an open neighbourhood of g. Then define a diffeomorphism $\varphi\colon p^{-1}(U) \to U \times \mathbb{C}^\times$ by $\varphi(u) = (p(u), \iota^{-1}(u))$ where $\iota^{-1}\colon p^{-1}(U) \to \mathbb{C}^\times$ is the left inverse of ι. Then $\iota^{-1}(u \cdot z) = \iota^{-1}(u \cdot \iota(z)) = \iota^{-1}(u)z$. Therefore $\hat{G} \to G$ is a principal $\mathbb{C}^\times$-bundle.

4.7.2 The first Chern class

We saw above that given transition functions which satisfy the 1-cocycle condition and an open cover we could construct a principal bundle whose transition functions are the ones given. So just like with complex line bundles we have that the transition functions of a $\mathbb{C}^\times$-bundle define a class $[g_{\alpha\beta}] \in H^1(M, \underline{\mathbb{C}^\times}) \simeq H^2(M, \mathbb{Z})$ called the first Chern class and from the results above classify $\mathbb{C}^\times$-bundles up to isomorphism. So we have another way to represent elements in $H^2(M, \mathbb{Z})$.

Theorem 4.7.3. [1] *Let $P \to M$ and $Q \to M$ be $\mathbb{C}^\times$-bundles. The first Chern class has the following properties.*

1. $c_1(P \otimes Q) = c_1(P) + c_1(Q)$.

2. $c_1(P^*) = -c_1(P)$.

3. *If $f\colon N \to M$ then* $c_1(f^{-1}(P)) = f^{-1}(c_1(P))$.

Proof. 1) If $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$, $h_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$ are the transition functions of P and Q respectively, then $P \otimes Q$ has transition functions $g_{\alpha\beta}h_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$. Then we have

$$\begin{aligned}
\mu_{\alpha\beta\gamma} &= -\mathrm{Log}(g_{\beta\gamma}h_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}h_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta}h_{\alpha\beta}) \\
&= -\mathrm{Log}(g_{\beta\gamma}) - \mathrm{Log}(h_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) + \mathrm{Log}(h_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta}) - \mathrm{Log}(h_{\alpha\beta}) \\
&= (-\mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta})) + (-\mathrm{Log}(h_{\beta\gamma}) + \mathrm{Log}(h_{\alpha\gamma}) - \mathrm{Log}(h_{\alpha\beta}))
\end{aligned}$$

So $c_1(P \otimes Q) = c_1(P) + c_1(Q)$.

2) If $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$ are the transition functions for $P \to M$ then the corresponding transition functions for $P^* \to M$ are $g_{\alpha\beta}^{-1}$. Then under the mapping

$$\begin{aligned}
\hat{\mu}_{\alpha\beta\gamma} &= -\mathrm{Log}(g_{\beta\gamma}^{-1}) + \mathrm{Log}(g_{\alpha\gamma}^{-1}) - \mathrm{Log}(g_{\alpha\beta}^{-1}) \\
&= \mathrm{Log}(g_{\beta\gamma}) - \mathrm{Log}(g_{\alpha\gamma}) + \mathrm{Log}(g_{\alpha\beta}) \\
&= -\mu_{\alpha\beta\gamma}.
\end{aligned}$$

So $c_1(P^*) = -c_1(P)$.

3) If $g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$ are the transition functions for $P \to M$ then the corresponding transition functions for $f^{-1}(P) \to N$ are $f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f$. We have

$$\begin{aligned} f^*(\mu_{\alpha\beta\gamma}) &= f^*(-\mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta})) \\ &= -f^*(\mathrm{Log}(g_{\beta\gamma})) + f^*(\mathrm{Log}(g_{\alpha\gamma})) - f^*(\mathrm{Log}(g_{\alpha\beta})) \\ &= -\mathrm{Log}(g_{\beta\gamma} \circ f) + \mathrm{Log}(g_{\alpha\gamma} \circ f) - \mathrm{Log}(g_{\alpha\beta} \circ f) \end{aligned}$$

So $c_1(f^{-1}(P)) = f^{-1}(c_1(P))$. □

Proposition 4.7.4. *Let $P \to M$ be a $\mathbb{C}^\times$-bundle. Then P is trivialisable if and only if it has zero chern class.*

Proof. Suppose $P \to M$ is trivialisable with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$. Then it is isomorphic to the trivial bundle $M \times \mathbb{C}^\times$. We have the following result from Woodhouse ([24], p.g. 267) that if two bundles P and Q with transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$ respectively are isomorphic then, there exists functions $f_\alpha\colon U_\alpha \to \mathbb{C}^\times$ such that

$$g_{\alpha\beta} = f_\alpha f_\beta^{-1} h_{\alpha\beta}.$$

However, in the case of the trivial bundle $M \times \mathbb{C}^\times$ the transition functions are clearly identically one. So $h_{\alpha\beta} = 1$ and so we have

$$g_{\alpha\beta} = f_\alpha f_\beta^{-1}.$$

So we have

$$\begin{aligned} &- \mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta}) \\ &= -\mathrm{Log}(f_\beta f_\gamma^{-1}) + \mathrm{Log}(f_\alpha f_\gamma^{-1}) - \mathrm{Log}(f_\alpha f_\beta^{-1}) \\ &= -\mathrm{Log}(f_\beta) - \mathrm{Log}(f_\gamma^{-1}) + \mathrm{Log}(f_\alpha) + \mathrm{Log}(f_\gamma^{-1}) - \mathrm{Log}(f_\alpha) - \mathrm{Log}(f_\beta^{-1}) \\ &= (-\mathrm{Log}(f_\beta) - \mathrm{Log}(f_\beta^{-1})) + (-\mathrm{Log}(f_\gamma^{-1}) + \mathrm{Log}(f_\gamma^{-1})) + (\mathrm{Log}(f_\alpha) - \mathrm{Log}(f_\alpha)) \\ &= 0 \end{aligned}$$

Therefore, $c_1(P) = 0$.

Now suppose that $c_1(P) = 0$. If we take the trivial bundle $M \times \mathbb{C}^\times$ then the transition functions are identically one. Hence

$$\begin{aligned} &- \mathrm{Log}(g_{\beta\gamma}) + \mathrm{Log}(g_{\alpha\gamma}) - \mathrm{Log}(g_{\alpha\beta}) \\ &= -\mathrm{Log}(1) + \mathrm{Log}(1) - \mathrm{Log}(1) \\ &= 0 \end{aligned}$$

Therefore, P and $M \times \mathbb{C}^\times$ belong to the same chern class. From the section on the first chern class of line bundles we saw that the first chern class classifies line bundles up to isomorphism and the same is true for principal bundles. Therefore $P \simeq M \times \mathbb{C}^\times$ and so P is trivialisable. □

4.7.3 Complex line bundles and $\mathbb{C}^\times$-bundles

Here we show that for each complex line bundle L we can associate a principal $\mathbb{C}^\times$-bundle and that the categories of principal $\mathbb{C}^\times$-bundles and complex line bundles are equivalent. This result means that we can move between the languages of complex line bundles and $\mathbb{C}^\times$-bundles without losing any data. This helps as in a given situation using the language of line bundles may be easier than principal bundles or vice versa. We can also move between various operations on on these bundles for example the tensor product of line bundles becomes the contracted product. The following section is based on section 2.1 of Brylinksi [1].

Let $L \to M$ be a complex line bundle. Then as each fibre L_m is a complex vector space, it has a zero vector 0_m. Let $0\colon M \to L$ be the zero-section which sends each point in M to its corresponding zero-vector. Define L^+ to be the complement of L by the image of the zero-section, i.e.

$$L^+ = L \backslash \{0(M)\}.$$

We make L^+ into a $\mathbb{C}^\times$-space by letting $\mathbb{C}^\times$ act by dilations on the fibres. So for $l \in L^+$ and $z \in \mathbb{C}^\times$ we define $l \cdot z = lz$.

This action is free as if we suppose for some $z \in \mathbb{C}^\times$ that $x \cdot z = x$ for all $x \in L^+_m$, since $L^+_m \simeq \mathbb{C}^\times$ the only possibility is that $z = 1$.

Since $L \to M$ is a line bundle by the local triviality condition we get for every $m \in M$ an open neighbourhood $U \subseteq M$ of m and a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times \mathbb{C}$. We saw that this diffeomorphism gives rise to local non-vanishing sections $\sigma\colon U \to L$. Given these sections define a diffeomorphism $\phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^\times$ by $\phi(y) = (\pi(y), y/\sigma(\pi(y)))$. Since $\sigma(\pi(y)) \neq 0$ and $\sigma(\pi(y)) \in L_{\pi(y)} \simeq \mathbb{C}$ we may regard $y/\sigma(\pi(y))$ as a $\mathbb{C}^\times$-valued function. Define $\psi\colon \pi^{-1}(U) \to \mathbb{C}^\times$ by $\psi(y) = y/\sigma(\pi(y))$ and let $z \in \mathbb{C}^\times$. Since $y \in L^+_m$ for some $m \in M$ then $y \cdot z \in L^+_m$ hence, $\pi(y) = \pi(y \cdot z)$. Therefore, $\psi(y \cdot z) = \psi(y) \cdot z$.

Let $O_x = \{y \in L^+ \mid y = x \cdot z \text{ for some } z \in \mathbb{C}^\times\}$ be an orbit by this group action and let $m = \pi(x)$ then, $O_x \subseteq L^+_m$. Now suppose y is in the same fibre as x then there exists a well defined complex number y/x such that $y = x \cdot (y/x)$ so, $L^+_m \subseteq O_x$. Hence, $L^+_m = O_x$ and so the orbit space consists of the sets of fibres L^+_m. So we have a smooth bijection $M \to L^+/\mathbb{C}^\times$. Therefore, $L^+ \to M$ is a $\mathbb{C}^\times$-bundle.

So for each complex line bundle $L \to M$ we can construct a principal $\mathbb{C}^\times$-bundle L^+. Given L^+ you can recover L as an *associated bundle* in the following way. Define the following space

$$L^+ \times^{\mathbb{C}^\times} \mathbb{C} := (L^+ \times \mathbb{C})/\mathbb{C}^\times$$

Where $\mathbb{C}^\times$ acts on $L^+ \times \mathbb{C}$ by $\lambda \cdot (x, z) = (\lambda^{-1} \cdot x, \lambda \cdot z)$. We can construct a line bundle $L^+ \times^{\mathbb{C}^\times} \mathbb{C} \to M$ in the following way. Define the projection $\pi^+\colon L^+ \times^{\mathbb{C}^\times} \mathbb{C} \to$

M by $\pi^+([(x,z)]) = \pi(x)$. This is well defined as if we take another representative (y,w) of $[(x,z)]$ we have $(y,w) = (\lambda^{-1}x, \lambda^{-1}z)$ for some $\lambda \in \mathbb{C}^\times$. Then $\pi^+([(y,w)]) = \pi^+([(\lambda^{-1}x, \lambda^{-1}z)]) = \pi(\lambda^{-1}x) = \pi(x)$. So projection is well defined and surjectivity follows from π being surjective.

On the fibres we have

$$(L^+ \times^{\mathbb{C}^\times} \mathbb{C})_m = \{[(x,z)] \in L^+ \times^{\mathbb{C}^\times} \mathbb{C} \mid \pi(x) = m\}.$$

We need these to be $\mathbb{C}$-vector spaces. Define the following addition on the fibres.

$$[(x,w)] + [(x,z)] = [(x, w+z)].$$

Since x and y are in the same fibre, they differ by some number $\lambda \in \mathbb{C}^\times$. So we can fix a point in our fibre and give all other points as a multiple of it. So we have to show this addition is well defined and independent of our fixed point.

Lets show this is well defined. Let $[(x', w')]$ and $[(x', z')]$ be representations for $[(x,w)]$ and $[(x,z)]$ respectively. Then

$$\begin{aligned}
[(x', w')] + [(x', z')] &= [(x, \lambda w')] + [(x, \lambda z')] \\
&= [(x, \lambda w' + \lambda z')] \\
&= [(x', (\lambda w' + \lambda z')\lambda^{-1})] \\
&= [(x', w' + z')]
\end{aligned}$$

For our fibres to be vector spaces we need our addition to commute. This is clear from how we have defined addition.

Define scalar multiplication in the following way.

$$\lambda[(x,w)] = [(x, \lambda w)] \text{ for } \lambda \in \mathbb{C}.$$

Then this is clearly well defined as if we have some other representative $[(x', w')] = [(x,w)]$ then we have

$$(x', w') = (\alpha^{-1}x, \alpha w) \text{ for some } \alpha \in \mathbb{C}^\times.$$

Therefore,

$$\begin{aligned}
\lambda[(x', w')] &= [(\alpha^{-1} \cdot x, \alpha\lambda w)] \\
&= [(x, \lambda w)] \\
&= \lambda[(x,w)]
\end{aligned}$$

If we let the zero vector be $[(x,0)]$ and define the additive inverse as $[(x,-z)]$ then clearly the fibres are complex vector spaces. Now we need that $\dim((L^+ \times^{\mathbb{C}^\times} \mathbb{C})_m) = 1$. This is clear as if we fix some point $x \in L^+_m$ then we have $\text{span}([x, i]) = (L^+ \times^{\mathbb{C}^\times} \mathbb{C})_m$. The final condition we need is local triviality.

A local trivialisation of this line bundle is given in ([11], p.g. 54) and therefore $L \times^{\mathbb{C}^\times} \mathbb{C}$ is a complex line bundle.

Now define the following maps

$$\begin{aligned} \phi_L &\colon L^+ \times^{\mathbb{C}^\times} \mathbb{C} \to L & [x, \lambda] &\mapsto \lambda \cdot x \\ \phi_L^{-1} &\colon L \to L^+ \times^{\mathbb{C}^\times} \mathbb{C} & y &\mapsto [x, \lambda] \text{ where } \lambda \cdot x = y \end{aligned}$$

Then we have

$$(\phi_L \circ \phi_L^{-1})(y) = \phi_L([x, \lambda]) = \lambda \cdot x = y.$$

Likewise

$$(\phi_L^{-1} \circ \phi_L)([y, u]) = \phi_L^{-1}(u \cdot y) = [x, \lambda] \text{ where } \lambda \cdot x = u \cdot y.$$

Then we have $u \cdot \lambda^{-1} \cdot [x, \lambda] = u \cdot [\lambda \cdot x, 1] = [u^{-1} \cdot \lambda \cdot x, w] = [y, w]$. So both maps are mutual inverses and hence, ϕ_L is an isomorphism. Therefore, $L^+ \times^{\mathbb{C}^\times} \mathbb{C} \simeq L$.

This construction is called the *contracted product* of $\mathbb{C}^\times$-manifolds L^+ and $\mathbb{C}$. More generally, if we have a $\mathbb{C}^\times$-bundle $P \to M$ we can construct a corresponding line bundle $L_P \to M$ by taking $L_P = (P \times \mathbb{C})/\mathbb{C}^\times$ where the group action is the same as above. So for every line bundle we can construct a $\mathbb{C}^\times$-bundle and vice versa.

From this we would hope that there is some correspondence between complex line bundles and $\mathbb{C}^\times$-bundles and there is. We can state this correspondence formally using category theory.

Fix a manifold M and let $\mathcal{C}_1$ be the category of complex line bundles over M with morphisms being line bundle isomorphisms. Let $\mathcal{C}_2$ be the category of principal $\mathbb{C}^\times$-bundles over M with morphisms being principal bundle isomorphisms.

The reason that $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories are as follows. Both $\mathcal{C}_1$ and $\mathcal{C}_2$ have a composition map as it is just regular composition of functions which is associative. The identity morphism in each case is just the identity map which clearly respects the identity laws. The objects and morphisms are stated above.

Lemma 4.7.5. *$F \colon \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(L) = L^+$ is a functor.*

Proof. Let $L \in \mathcal{C}_1$ then by definition of F we have $F(L) = L^+ \in \mathcal{C}_2$. Let $f \in \mathcal{C}_1(L_1, L_2)$ be a line bundle isomorphism. So $f \colon L_1 \to L_2$ is a diffeomorphism that is linear over the fibres. So if we let $F(f) = f$ by forgetting that f acts on zero vectors, we get an induced map $F(f)\colon L_1^+ \to L_2^+$. Since f is an isomorphism it preserves identity so f does not map a zero vector to a non-zero vector. Since the group action is dilation over the fibres and $f((L_1)_m) \subseteq (L_2)_m$ we have that $f(x \cdot z) = f(x) \cdot z$ for all $x \in L_1^+$ and $z \in \mathbb{C}^\times$ so $F(f)(x \cdot z) = F(f)(x) \cdot z$. So $F(f)$ is a $\mathbb{C}^\times$-bundle isomorphism. It is clear from how F is defined that $F(id_L) = id_{F(L)}$ and that F respects composition. So $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ is a functor. □

Theorem 4.7.6. *Given the functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(L) = L^+$. Then F is an equivalence of categories.*

Proof. Let $P \to M$ be a $\mathbb{C}^\times$-bundle then via the contracted product we get a line bundle $L_P \to M$. Then if we define a map $u\colon L_P^+ \to P$ by $u([(p,z)]) = p$ we get a $\mathbb{C}^\times$-bundle homomorphism and thus an isomorphism. So for all $P \in \mathcal{C}_2$ there exists an $L \in \mathcal{C}_1$ such that $F(L) \simeq P$. So F is essentially surjective on objects.

Let $f, g \in \mathcal{C}_1(L_1, L_2)$ and suppose $F(f) = F(g)$. Since we regard $F(f)$ as the same as f but with the zero vector removed we must have $f = g$. So F is faithful.

Lastly, we must show that F is full. Let $f \in \mathcal{C}_2(F(L_1), F(L_2))$ we want to show there exists some line bundle isomorphism $g \in \mathcal{C}_1(L_1, L_2)$ such that $F(g) = f$.

Given $f\colon L_1^+ \to L_2^+$ we can construct a line bundle isomorphism via the associated line bundles

$$\begin{aligned} f^+\colon L_1^+ \times^{\mathbb{C}^\times} \mathbb{C} &\to L_2^+ \times^{\mathbb{C}^\times} \mathbb{C} \\ [(x,z)] &\mapsto [f(x), z] \end{aligned}$$

Let's show this is an isomorphism. Firstly by definition we have that

$$f^+((L_1^+ \times^{\mathbb{C}^\times} \mathbb{C})_m) \subseteq (L_2^+ \times^{\mathbb{C}^\times} \mathbb{C})_m.$$

Let $[(x,u)], [(x,v)] \in (L_1^+ \times^{\mathbb{C}^\times} \mathbb{C})_m$ and $\lambda \in \mathbb{C}$. Since addition is independent of our fixed point, fix $x \in L_1^+$ and $f(x) \in L_2^+$. Then we have

$$\begin{aligned} f^+([(x,u)] + [(x,v)]) &= f^+([(x, u+v)]) \\ &= [(f(x), u+v)] \\ &= [(f(x), u)] + [(f(x), v)] \\ &= f^+([(x,u)]) + f^+([x,v]). \end{aligned}$$

For scalar multiplication we have

$$\begin{aligned} \alpha f^+([(x,u)]) &= \alpha[(f(x), u))] \\ &= [(f(x), \alpha u)] \\ &= f^+([(x, \alpha u)]) \\ &= f^+(\alpha[(x,u)]) \end{aligned}$$

Now we need f^+ to be a bijection. Suppose that $f^+([(x,u)]) = f^+([(y,v)])$. Then we have that $\alpha^{-1}f(x) = f(\alpha x) = f(y)$ and $\alpha u = v$ for some $\alpha \in \mathbb{C}^\times$. Since f is injective we have $\alpha^{-1}x = y$ and $\alpha u = v$. Therefore $[(x,u)] = [(y,v)]$. Now let $y \in L_2^+$ then since f is surjective there exists some $x \in L_1^+$ such that $f(x) = y$. So $[(y,z)] = [(f(x),z)] = f^+([(x,v)])$ and therefore, f^+ is a bijection.

Since $L^+ \times^{\mathbb{C}^\times} \mathbb{C} \simeq L$ we get the following commutative diagram.

$$\begin{tikzcd}
L_1^+ \times^{\mathbb{C}^\times} \mathbb{C} \arrow[r, "f^+"] \arrow[d, "\phi_{L_1}"'] & L_2^+ \times^{\mathbb{C}^\times} \mathbb{C} \arrow[d, "\phi_{L_2}"] \\
L_1 \arrow[r, "g"] & L_2
\end{tikzcd}$$

So we get

$$g = \phi_{L_2} \circ f^+ \circ \phi_{L_1}^{-1}.$$

Which is an isomorphism since the composition of isomorphisms is an isomorphism. Then since F is a functor we have

$$\begin{aligned}
F(g)(p) &= F(\phi_{L_2} \circ f^+ \circ \phi_{L_1}^{-1})(p) \\
&= (F(\phi_{L_2}) \circ F(f^+) \circ F(\phi_{L_2}^{-1}))(p) \\
&= (F(\phi_{L_2}) \circ F(f^+))([(x,\lambda)]) \\
&= F(\phi_{F_2})([(f(x),\lambda)]) \\
&= \lambda \cdot f(x) \\
&= f(\lambda \cdot x) \\
&= f(p).
\end{aligned}$$

So $F(g) = f$ and hence F is faithful. Therefore, F is an equivalence of categories. $\square$

The contracted product of $\mathbb{C}^\times$-manifolds shows that the examples we gave of complex line bundles can now be considered as principal $\mathbb{C}^\times$ bundles. So for each line bundle example we gave we need only to compute L^+.

Example 4.7.5. **The trivial $\mathbb{C}^\times$-bundle.**

Fibres of the $M \times \mathbb{C}$ are of the form $\{m\} \times \mathbb{C}$ and so the zero element of $(M \times \mathbb{C})_m$ is $(m,0)$. So define the zero section $\mathbf{0}\colon M \to M \times \mathbb{C}$ to be $\mathbf{0}(m) = (m,0)$. So

$$(M \times \mathbb{C})^+ = (M \times \mathbb{C}) \backslash \{(m,0) \mid m \in M\} = M \times \mathbb{C}^\times.$$

Example 4.7.6. **The tangent bundle to the sphere TS^2.**

Recall that for $x \in S^2$ the fibres are of the form $TS^2_x = \{x\} \times T_xS^2$. So then the zero element of the fibre would be $(x,0)$. So define the zero section $\mathbf{0}\colon S^2 \to TS^2$ to be

$\mathbf{0}(x) = (x, 0)$. Then

$$(TS^2)^+ = TS^2 \backslash \mathbf{0}(S^2) = TS^2 \backslash \{(x, 0) \mid x \in S^2\}.$$

Example 4.7.7. **The Hopf Bundle.**

Recall that the fibres are of the form $H_{[x]} = \{([x], \lambda x) \mid \lambda \in \mathbb{C}\}$. Then define the zero section $\mathbf{0}\colon \mathbb{C}P^1 \to H$ by $\mathbf{0}([x]) = ([x], 0)$. So $H^+ = H \backslash \mathbf{0}(H)$. Since we identify $[x]$ with a complex line, we can think of H^+ as the complex lines with the origin removed.

Chapter 5

Bundle Gerbes

Here we provide an overview of bundle gerbes by looking at the various constructions and geometry related to these objects. The overarching ideas we saw in looking at line bundles and $\mathbb{C}^\times$-bundles will again be present here in their own form. We saw that complex line bundles and $\mathbb{C}^\times$-bundles are characterised by degree-two cohomology and we will see that bundle gerbes are characterised by degree-three cohomology. So in a sense, we can regard bundle gerbes as a 'higher dimensional' version of a $\mathbb{C}^\times$-bundle.

5.1 Definitions

Here we provide the final definitions and minor results needed to define a bundle gerbe.

Definition 5.1.1. Let $f\colon X \to Y$ be a smooth map between differentiable manifolds. We say f is a *submersion* if for all $x \in X$ df_x is surjection.

Definition 5.1.2. [16] Let $f\colon X \to Y$ be a surjective submersion. Let $p \in \mathbb{N}$, the *p-fold fibre product* of f is a submanifold of X^p and is defined as

$$X^{[p]} = \{(x_1, \ldots, x_p) \in X^p \mid f(x_1) = \ldots = f(x_p)\}.$$

For $i = 1, \ldots, p$, we have projection maps $\pi_i\colon X^{[p]} \to X^{[p-1]}$ defined by

$$\pi_i(x_1, \ldots, x_p) = (x_1, \ldots, \hat{x}_i, \ldots, x_p)$$

Where $\hat{x}_i$ means remove the i-th component.

Definition 5.1.3. [16] Let $f\colon X \to M$ and $g\colon Y \to M$ be surjective submersions. The *fibre product* of X and Y over M is a submanifold of $X \times Y$ and is defined as

$$X \times_M Y = \{(x, y) \in X \times Y \mid f(x) = g(x)\}.$$

Definition 5.1.4. A *simplicial space* Δ is a sequence of spaces Δ_n for $n \in \mathbb{N}$ together with face maps $d_i \colon \Delta_n \to \Delta_{n-1}$ and degeneracy maps $s_i \colon \Delta_n \to \Delta_{n+1}$ for $i = 0, \dots, n$ such that the following identities hold where composition is defined.

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ for } i < j.$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ for } i \leq j.$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & i < j \\ id & i = j, i = j+1 \\ s_j \circ d_{i-1} & i > j+1 \end{cases}$$

Given the collection $\{Y^{[p]}\}$ define maps $s_i \colon Y^{[p]} \to Y^{[p+1]}$ in the following way

$$s_i(y_1, \dots, y_i, \dots, y_p) = (y_1, \dots, y_i, y_i, \dots, y_p)$$

Lemma 5.1.5. *The collection $\{Y^{[p]}\}$ with face maps $\pi_i \colon Y^{[p]} \to Y^{[p-1]}$ and degeneracy maps $s_i \colon Y^{[p]} \to Y^{[p+1]}$ given above is a simplicial space.*

Proof. Suppose $i < j$ then

$$\pi_i(\pi_j(y_1, \dots, y_p)) = (y_1, .., \hat{y}_i, .., \hat{y}_j, .., y_p).$$

Then $i \leq j - 1$ so

$$\pi_{j-1}(\pi_i(y_1, \dots, y_p)) = \pi_{j-1}(y_1, .., \hat{y}_i, .., y_p).$$

Since everything is shifted by one, removing y_j is the same as removing y_{j-1}. So,

$$\pi_{j-1}(y_1, .., \hat{y}_i, .., y_p) = (y_1, .., \hat{y}_i, .., \hat{y}_j, .., y_p).$$

Therefore, $\pi_i \circ \pi_j = \pi_{j-1} \circ \pi_i$.

Now suppose $i \leq j$.

$$\begin{aligned}(s_i \circ s_j)(y_1, \dots, y_j, \dots, y_p) &= s_i(y_1, \dots, y_j, y_j, \dots, y_p) \\ &= (y_1, \dots, y_i, y_i, \dots, y_j, y_j, \dots, y_p)\end{aligned}$$

Then the $j-$th element is at position $j + 1$ and j after applying s_j. Then after applying s_i, the elements at $j + 1$ and j are at $j + 2$ and $j + 1$. Since $i \leq j$ applying s_i shifts the j-th element to $j + 1$ and j. So then s_{j+1} shifts $j + 1$ and j to $j + 2$ and $j + 1$. Likewise for the elements in the i-th position. Therefore, $s_i \circ s_j = s_{j+1} \circ s_i$.

Now checking composition between the face and degeneracy maps. For $i = j$ clearly $\pi_i \circ s_i = id$ as you have the same element in both positions i and $i + 1$ and then you project out the i-th element. The same reasoning applies for $i = j + 1$.

Now we want $\pi_i \circ s_j = s_{j-1} \circ \pi_i$ for $i < j$. We have that

$$\begin{aligned}(\pi_i \circ s_j)(y_1, \ldots, y_p) &= \pi(y_1, \ldots, y_j, y_j, \ldots, y_p) \\ &= (y_1, \ldots, \hat{y}_i, \ldots, y_j, y_j, \ldots, y_p)\end{aligned}$$

However, applying π_i first shifts all the element down one position so the j-th element becomes the $(j-1)$-th element. So we then have

$$\begin{aligned}(s_{j-1} \circ \pi_i)(y_1, \ldots, y_p) &= (y_1, \ldots \hat{y}_i, \ldots, y_p) \\ &= (y_1, \ldots \hat{y}_i, \ldots, y_{j-1}, y_{j-1}, \ldots y_p) \\ &= (y_1, \ldots \hat{y}_i, \ldots, y_j, y_j, \ldots y_p) \\ &= (\pi_i \circ s_j)(y_1, \ldots, y_p)\end{aligned}$$

The same argument applies for $\pi_i \circ s_j = s_j \circ \pi_{i-1}$ for $i > j+1$. Therefore, $\{Y^{[p]}\}$ with π_i and s_i is a simplicial space. $\square$

Let $\Omega^q(Y^{[p]})$ be the space of differential q-forms on $Y^{[p]}$ and define the following map $\delta \colon \Omega^q(Y^{[p-1]}) \to \Omega^q(Y^{[p]})$ to be the alternating sum of pullbacks of the projection maps π_i.

$$\delta = \sum_{i=1}^{p} (-1)^{i-1} \pi_i^*. \tag{5.1.1}$$

Lemma 5.1.6. *Exterior derivative commutes with δ.*

Proof. Let $\omega \in \Omega^q(Y^{[p]})$. Then

$$\begin{aligned}\delta(d\omega) &= \sum_{i=1}^{p} (-1)^{i-1} \pi_i^*(d\omega) \\ &= \sum_{i=1}^{p} (-1)^{i-1} d(\pi_i^*(\omega)) \\ &= d(\sum_{i=1}^{p} (-1)^{i-1} \pi_i^*(\omega)) \\ &= d(\delta(\omega)).\end{aligned}$$

$\square$

Proposition 5.1.7. $\delta^2 = 0$

Proof. This is mainly due to fact (5.1.5) that $Y^{[p]}$ with the projection maps is a simplicial space. Let $\omega \in \Omega^q(Y^{[p]})$. Then

$$\delta^2(\omega) = \sum_{j=1}^{p+1} (-1)^{j-1} \pi_j^* (\sum_{i=1}^{p} (-1)^{i-1} \pi_i^*(\omega)).$$

So

$$\delta^2(\omega) = \sum_{j=1}^{p+1}\sum_{i=1}^{p}((-1)^{i+j-2})\pi_j^*(\pi_i^*(\omega)).$$

For $i < j$ we have

$$\begin{aligned}\pi_j^* \circ \pi_i^* &= (\pi_i \circ \pi_j)^* \\ &= (\pi_{j-1} \circ \pi_i)^* \\ &= \pi_i^* \circ \pi_{j-1}^*\end{aligned}$$

So for any term $\pi_j^* \circ \pi_i^*$ in the sum there is another term $\pi_{i+1}^* \circ \pi_j^* = \pi_j^* \circ \pi_i^*$ which cancels it. Therefore, $\delta^2 = 0$. $\square$

So we can form a cochain complex called the *fundamental complex* [16] .

$$0 \to \Omega^{p-1}(M) \xrightarrow{\pi^*} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \dots \tag{5.1.2}$$

Proposition 5.1.8. [16] *The fundamental complex is exact for all $q \geq 0$.*

Proof. We will start with the trivial case where we have $Y = M \times F$ with surjective submersion $\mathrm{pr}_1 \colon M \times F \to M$. Firstly, note that $Y^{[p]} = M \times F^p$. For convenience we will denote the collection of q-vectors $(X^1, \dots, X^q)$ by X and if we have a q-form $\omega \in \Omega^q(Y^{[p]})$ the action of ω on these vectors is

$$\omega(X^1, \dots, X^q) = \omega(X).$$

Suppose we have vectors tangent to F^{p+1} at $z = (z_1, \dots, z_{p+1})$. Then each X_i is a collection of vectors $(X_1, \dots, X_{p+1})$ and each X_j^i is a collection of q-vectors in $T_{z_j}(F)$.

Then we have

$$\begin{aligned}&\delta(\omega)(m, z)(\zeta, (X_1, \dots, X_{p+1})) \\ &= \sum_{i=1}^{p+1}(-1)^i\omega(m, z_1, \dots, \hat{z}_i, \dots, z_{p+1})((\zeta, (X_1, \dots, \hat{X}_i, \dots, X_{p+1}))) \\ &= \sum_{i=1}^{p}(-1)^i\omega(m, z_1, \dots, \hat{z}_i, \dots, z_{p+1})((\zeta, (X_1, \dots, \hat{X}_i, \dots, X_{p+1})) \\ &+ (-1)^{p+1}\omega(m, z_1, \dots, \hat{z}_{p+1})((\zeta, (X_1, \dots, \hat{X}_{p+1}))\end{aligned}$$

Where ζ is a q-tuple of vectors tangent to M at m. Now fix a point $z \in F$ and a q-tuple of vectors $X \in T_f(F)$. Define the following q-form $\rho \in \Omega^q(Y^{[p-1]})$ by

$$\rho(m, z_1, \dots, z_p)(\zeta, X_1, \dots, X_p) = \omega(m, z_1, \dots, z_p, z)(\zeta, X_1, \dots, X_p, X).$$

We want to show this sequence is exact so suppose that $\delta(\omega) = 0$. From the above calculation and the fact that we are supposing $\delta(\omega) = 0$ we have that

$$\begin{aligned}\delta(\rho) &= \sum_{i=1}^{p}(-1)^i\omega(m, z_1, \dots, \hat{z}_i, \dots, z)((\zeta, (X_1, \dots, \hat{X}_i, \dots, X)) \\ &= (-1)^{p+1}\omega(m, z_1, \dots, \hat{z})((\zeta, (X_1, \dots, \hat{X}))\end{aligned}$$

Therefore,

$$\delta(\rho) = (-1)^{p+1}\omega.$$

So the sequence is exact for the trivial case. Choose an open cover $\{U_\alpha\}$ of M over which the surjective submersion is trivial. Let ψ_α be a partition of unity subordinate to that cover. Let $Y_\alpha = \pi^{-1}(U_\alpha)$ and similarly define $(Y^{[p]})_\alpha$. For each projection map $Y^{[p]} \to M$ we can pull back ψ_α to get a partition of unity on $Y^{[p]}$ which we will denote by the same symbol.

Let $\omega \in \Omega^q(Y^{[p]})$. Then since Y is trivial over the open cover we have that $\omega\mid_{U_\alpha} = \delta(\rho_\alpha)$ for some ρ_α. So we have that

$$\omega = \sum_\alpha \psi_\alpha \delta(\rho_\alpha) = \delta(\sum_\alpha \psi_\alpha \rho_\alpha).$$

Let $\rho = \sum_\alpha \psi_\alpha \rho_\alpha$ so we have that $\omega = \delta(\rho)$. Therefore, the fundamental complex is exact. $\square$

If we have a map $g\colon Y^{[p-1]} \to \mathbb{C}^\times$ we define $\delta(g)\colon Y^{[p]} \to \mathbb{C}^\times$ by

$$\delta(g) = (g \circ \pi_1) - (g \circ \pi_2) + (g \circ \pi_3) + \ldots + (-1)^{p+1}(g \circ \pi_p). \tag{5.1.3}$$

Definition 5.1.9. If we have a $\mathbb{C}^\times$-bundle $P \to Y^{[p-1]}$ we define a new $\mathbb{C}^\times$-bundle $\delta(P) \to Y^{[p]}$ by

$$\delta(P) = \pi_1^{-1}(P) \otimes \pi_2^{-1}(P)^* \otimes \pi_3^{-1}(P) \otimes \ldots. \tag{5.1.4}$$

This product ends with $\pi_p^{-1}(P)$ if p is odd and $\pi_p^{-1}(P)^*$ if p is even. Note that $\delta(P)$ is a $\mathbb{C}^\times$-bundle because it is the contracted product of pullbacks of $\mathbb{C}^\times$-bundles.

Definition 5.1.10. If $s\colon Y^{[p-1]} \to P$ is a section of a $\mathbb{C}^\times$-bundle P. Then define $\delta(s)\colon Y^{[p]} \to \delta(P)$ by

$$\delta(s) = (s \circ \pi_1) \otimes (s^* \circ \pi_2) \otimes \ldots \tag{5.1.5}$$

Where the final term is $s^* \circ \pi_p$ if p is even and $s \circ \pi_p$ if p is odd.

Note that if we have a principal $\mathbb{C}^\times$-bundle $P \to Y^{[p-1]}$ with connection A then $\delta P \to Y^{[p]}$ has an induced connection δA given by

$$\pi_1^{-1}(A) \otimes \pi_2^{-1}(A)^* \otimes \pi_3^{-1}(A) \otimes \ldots$$

Where $\pi_i^{-1}(A)$ is the induced connection on the pullback bundle $\pi_i^{-1}(P)$.

Proposition 5.1.11. $\delta(\delta(P))$ *is canonically trivial.*

Proof. This essentially follows from example (4.7.3) that $P \otimes P^*$ is canonically trivial. Let $P \to Y^{[p-1]}$ be a $\mathbb{C}^\times$-bundle and suppose p is even. Then

$$\delta(P) = \pi_1^{-1}(P) \otimes \pi_2^{-1}(P)^* \otimes \ldots \otimes \pi_p^{-1}(P)^*.$$

So

$$\delta^2(P) = \pi_1^{-1}(\delta(P)) \otimes \pi_2^{-1}(\delta(P))^* \otimes \ldots \otimes \pi_{p+1}^{-1}(\delta(P)).$$

Let $(y_1, \ldots y_{p+1}) \in Y^{[p+1]}$. Then

$$\begin{aligned}
\delta^2(P)_{(y_1,\ldots y_{p+1})} &= \delta(P)_{(y_2,\ldots,y_{p+1})} \otimes \delta(P)^*_{(y_1,y_3,\ldots,y_{p+1})} \otimes \ldots \otimes \delta(P)_{(y_1,\ldots,y_p)} \\
&= (P_{(y_3,\ldots,y_{p+1})} \otimes \ldots \otimes P^*_{(y_2,\ldots y_p)}) \otimes (P_{(y_3,\ldots,y_{p+1})} \otimes \ldots \otimes P^*_{(y_1,y_3,\ldots y_p)})^* \\
&\otimes \ldots \otimes (P_{(y_2,\ldots,y_p)} \otimes \ldots \otimes P^*_{(y_1,\ldots y_{p-1})}).
\end{aligned}$$

Then after contracting like terms we have

$$\delta^2(P)_{(y_1,\ldots y_{p+1})} = (P \otimes P^*)_{(\hat{y_1},\ldots y_{p+1})} \otimes (P \otimes P^*)_{(y_1,\hat{y_2},\ldots y_p)} \otimes \ldots \otimes (P \otimes P^*)_{(y_1,\ldots,\hat{y_{p+1}})}.$$

Therefore, $\delta^2(P)$ is canonically trivial. For the case where p is odd the calculation is the same except the final term of $\delta(P)$ is $\pi_p^{-1}(P)$.

We can see from here that $\delta^2(P)$ has a canonical section $\delta s \colon Y^{[p+1]} \to \delta^2(P)$ defined by

$$\delta s(y_1, \ldots, y_{p+1}) = (p_{21\ldots p+1}, p^*_{21\ldots p+1}) \otimes (p_{13\ldots p+1}, p^*_{13\ldots p+1}) \otimes \ldots \otimes (p_{12\ldots p}, p^*_{12\ldots p}).$$

Where $(p_{21\ldots p+1}, p^*_{21\ldots p+1}) \in (P \otimes P^*)_{(y_2,\ldots,y_{p+1})}$ and so on. □

We denote the canonical section of $\delta^2(P)$ by 1.

5.2 Definition of a bundle gerbe

Definition 5.2.1. [16] A *bundle gerbe* is a triple of manifolds (P, Y, M), a surjective submersion $\pi \colon Y \to M$, a $\mathbb{C}^\times-$bundle $P \to Y^{[2]}$ and an isomorphism of $\mathbb{C}^\times-$bundles called the *bundle gerbe multiplication*

$$m \colon \pi_3^{-1}(P) \otimes \pi_1^{-1}(P) \to \pi_2^{-1}(P) \qquad (5.2.1)$$

which must be associative whenever triple products are defined. Meaning that if $P_{(y_1,y_2)}$ is the fibre of P over $Y^{[2]}$ then we have the following commutative diagram for all

$(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

$$\begin{array}{ccc} P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} & \longrightarrow & P_{(y_1,y_3)} \otimes P_{(y_3,y_4)} \\ \downarrow & & \downarrow \\ P_{(y_1,y_2)} \otimes P_{(y_2,y_4)} & \longrightarrow & P_{(y_1,y_4)} \end{array} \tag{5.2.2}$$

The maps π_i are the three projection maps $\pi_i \colon Y^{[3]} \to Y^{[2]}$. An important fact of the bundle gerbe multiplication is that at the level of fibres it induces an isomorphism

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \xrightarrow{\sim} P_{(y_1,y_3)} \text{ for } (y_1, y_2, y_3) \in Y^{[3]}. \tag{5.2.3}$$

The reason for this is as follows. Take $(y_1, y_2, y_3) \in Y^{[3]}$ then

$$\pi_3^{-1}(P) = \{((y_1, y_2, y_3), p) \in Y^{[3]} \times P \mid (y_1, y_2) = \pi_P(p)\}$$

So on the fibres

$$\pi_3^{-1}(P)_{(y_1,y_2,y_3)} = \{p \in P \mid \pi_P(p) = (y_1, y_2)\} = P_{(y_1,y_2)}$$

Likewise, $\pi_2^{-1}(P)_{(y_1,y_2,y_3)} = P_{(y_1,y_3)}$ and $\pi_1^{-1}(P)_{(y_1,y_2,y_3)} = P_{(y_2,y_3)}$. So by the multiplication we have an isomorphism

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \xrightarrow{\sim} P_{(y_1,y_3)}.$$

An easy way to picture a bundle gerbe is using the following diagram

$$\begin{tikzcd} P \arrow[d] & \\ Y^{[2]} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & Y \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Remark 5.2.1. As we have an equivalence of categories between $\mathbb{C}^\times$-bundles and complex line bundles we have an equivalent definition of bundle gerbes taking $P \to Y^{[2]}$ to be a complex line bundle.

5.3 Examples of bundle gerbes

Here we provide some important examples and constructions with bundle gerbes to demonstrate how the definition works.

Example 5.3.1. [16] **Trivial bundle gerbes**.

Let $R \to Y$ be a $\mathbb{C}^\times$-bundle and let $\delta(R) \to Y^{[2]}$ be the $\mathbb{C}^\times$-bundle as in definition (5.1.9). Then $\delta(R)$ has a natural bundle gerbe multiplication. Let $(y_1, y_2, y_3) \in Y^{[3]}$ then

$$\delta(R)_{(y_1,y_2)} \otimes \delta(R)_{(y_2,y_3)} \to \delta(R)_{(y_1,y_3)}$$

is defined in the following way. We have

$$\begin{aligned}\delta(R)_{(y_1,y_2)} &= \pi^{-1}(R)_{(y_1,y_2)} \otimes \pi_2^{-1}(R)^*_{(y_1,y_2)} \\ &= R_{y_2} \otimes R^*_{y_1}\end{aligned}$$

So then,

$$\delta(R)_{(y_1,y_2)} \otimes \delta(R)_{(y_2,y_3)} = R_{y_2} \otimes R^*_{y_1} \otimes R_{y_3} \otimes R^*_{y_2}$$

Then contracting R_{y_2} and $R^*_{y_2}$ we get

$$\delta(R)_{(y_1,y_2)} \otimes \delta(R)_{(y_2,y_3)} = R_{y_3} \otimes R^*_{y_1} = \delta(R)_{(y_1,y_3)}.$$

This multiplication is associative on $(y_1, y_2, y_3, y_4) \in Y^{[4]}$ as you just contract over pairs of indices that are the same. So after choosing a surjective submersion $\pi\colon Y \to M$ we have the following bundle gerbe.

$$\begin{tikzcd} \delta(R) \arrow[d] & \\ Y^{[2]} \arrow[r, shift left, "\pi_i"] \arrow[r, shift right] & Y \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Example 5.3.2. [16] **The lifting bundle gerbe.**

Let G be a Lie group and suppose we have a central extension

$$0 \to \mathbb{C}^\times \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \to 0.$$

Let $P \to M$ be a G-bundle. We can ask whether P *lifts* to a $\hat{G}$ bundle $\hat{P} \to M$. What this means is that is there a principal bundle $\hat{P} \to M$ and homomorphism $f\colon \hat{P} \to P$ such that $f(pg) = f(p)\pi(g)$ for all $p \in P$ and $g \in G$. One way of deciding whether P lifts to a $\hat{G}$-bundle is to take an open cover $\{U_\alpha\}$ of M such that it admits transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ and choose lifts $\hat{g}_{\alpha\beta}\colon U_{\alpha\beta} \to \hat{G}$ such that $\pi(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. However, $\hat{g}_{\alpha\beta}$ may not satisfy the cocycle condition and therefore the $\hat{g}_{\alpha\beta}$ would not be transition functions. What we can say is that there are some functions $\epsilon_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times$ such that

$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma} = \iota(\epsilon_{\alpha\beta\gamma})\hat{g}_{\alpha\gamma}$. Now since we have a central extension $\iota(\mathbb{C}^\times) \subseteq Z(\hat{\mathcal{G}})$. So we have

$$\begin{aligned}
(\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma})\hat{g}_{\gamma\delta} &= (\iota(\epsilon_{\alpha\beta\gamma})\hat{g}_{\alpha\gamma})\hat{g}_{\gamma\delta} \\
&= \iota(\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\gamma\delta})\hat{g}_{\alpha\delta} \\
(\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma})\hat{g}_{\gamma\delta} &= \hat{g}_{\alpha\beta}\iota(\epsilon_{\beta\gamma\delta})\hat{g}_{\beta\delta} \\
&= \iota(\epsilon_{\alpha\beta\delta}\epsilon_{\beta\gamma\delta})\hat{g}_{\alpha\delta}.
\end{aligned}$$

So, $\iota(\epsilon_{\alpha\beta\delta}\epsilon_{\beta\gamma\delta}) = \iota(\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\gamma\delta})$ and since ι is injective we have

$$\epsilon_{\alpha\beta\delta}\epsilon_{\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\gamma\delta}.$$

Therefore, $\epsilon_{\alpha\beta\gamma}$ defines a class in $H^2(M, \underline{\mathbb{C}^\times}) \simeq H^3(M, \mathbb{Z})$. The image of $[\epsilon_{\alpha\beta\gamma}]$ under this isomorphism is the obstruction to the bundle $P \to M$ having lift. What this means is that $[\epsilon_{\alpha\beta\gamma}]$ vanishes if and only if $P \to M$ has lift. This result is shown by Brylinski and McLaughlin in [2].

However, we can also view this problem through the language of bundle gerbes by constructing a bundle gerbe which is trivial if and only if P has lift. Let $P \to M$ be a G-bundle and consider the fibre product $P^{[2]} \rightrightarrows Y$. Then we can specify a difference map $\tau \colon P^{[2]} \to G$ by $p_1\tau(p_1, p_2) = p_2$. Recall that this gives a pre-G bundle.

$$\begin{tikzcd}
 & G \\
P^{[2]} \arrow[ur, "\tau"] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\
 & M
\end{tikzcd}$$

However, we would like a $\mathbb{C}^\times$-bundle over $P^{[2]}$. Recall that we can view the above central extension as a $\mathbb{C}^\times$-bundle $\hat{G} \to G$. Then using τ we can pullback $\hat{G}$ to obtain the following

$$\begin{tikzcd}
\tau^{-1}\hat{G} \arrow[d] & \\
P^{[2]} \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\
 & M
\end{tikzcd}$$

Where $\tau^{-1}\hat{G} = \{(y_1, y_2, \hat{g}) \in P^{[2]} \times \hat{G} \mid p(\hat{g}) = \tau(y_1, y_2)\}$. For this to be a bundle gerbe we require an associative multiplication

$$\tau^{-1}\hat{G}_{(y_1,y_2)} \otimes \tau^{-1}\hat{G}_{(y_2,y_3)} \to \tau^{-1}\hat{G}_{(y_1,y_3)}.$$

However, note that $\tau^{-1}\hat{G}_{(y_1,y_2)} = \hat{G}_{\tau(y_1,y_2)}$. If we let $\tau(y_1, y_2) = g$ and $\tau(y_2, y_3) = h$, then $\tau(y_1, y_2)\tau(y_2, y_3) = gh = \tau(y_1, y_3)$. So the above multiplication is essentially group multiplication which is associative and an isomorphism. Therefore, we have a bundle gerbe $(\tau^{-1}\hat{G}, P, M)$ called the *lifting bundle gerbe*. The reason for the name is that this bundle gerbe is trivial exactly when the G-bundle $P \to M$ has lift. We defer the proof of this fact to the coming section on the Dixmier-Douady class. This is example will be the focus of the next chapter.

Example 5.3.3. [17] **Local bundle gerbes.**

Let M be a differentiable manifold and $\mathcal{U} = \{U_\alpha\}$ be an open cover of M. Let $Y_\mathcal{U} = \{(x,\alpha) \mid x \in U_\alpha\}$ and define the projection to be $\pi(x,\alpha) = x$. Then $Y_\mathcal{U}^{[2]} = \{(x,\alpha,\beta) \mid x \in U_\alpha \cap U_\beta\}$ with maps $\pi_1(x,\alpha,\beta) = (x,\beta)$ and $\pi_2(x,\alpha,\beta) = (x,\alpha)$. Finally take the trivial $\mathbb{C}^\times$-bundle $Y_\mathcal{U}^{[2]} \times \mathbb{C}^\times$. Then looking at how multiplication should be defined we need an associative map

$$m\colon \pi_3^{-1}(Y_\mathcal{U}^{[2]} \times \mathbb{C}^\times) \otimes \pi_1^{-1}(Y_\mathcal{U}^{[2]} \times \mathbb{C}^\times) \to \pi_2^{-1}(Y_\mathcal{U}^{[2]} \times \mathbb{C}^\times).$$

Which on the fibre $(x,\alpha,\beta,\gamma) \in Y_\mathcal{U}^{[3]}$ looks like

$$\{(x,\alpha,\beta)\} \times \mathbb{C}^\times \otimes \{(x,\beta,\gamma)\} \times \mathbb{C}^\times \to \{(x,\alpha,\gamma)\} \times \mathbb{C}^\times.$$

So our multiplication must be of the form

$$((x,\alpha,\beta),z) \otimes ((x,\beta,\gamma),w)) \to ((x,\alpha,\gamma), g_{\alpha\beta\gamma}(x)zw).$$

Where $g_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times$ is the difference map of z and w. The multiplication is associative if and only if $g_{\alpha\beta\gamma}$ is a 2-cocycle. To see this, suppose firstly that $g_{\alpha\beta\gamma}$ is a 2-cocycle. What we want to show is that

$$(((x,\alpha,\beta),z)\otimes((x,\beta,\gamma),w))\otimes((x,\gamma,\delta),v) = ((x,\alpha,\beta),z)\otimes(((x,\beta,\gamma),w)\otimes((x,\gamma,\delta),v)).$$

Computing the left hand side we get

$$\begin{aligned}((x,\alpha,\delta), g_{\alpha\beta\gamma}(x)g_{\alpha\gamma\delta}(x)zwv) &= ((x,\alpha,\delta), g_{\beta\gamma\delta}(x)g_{\alpha\beta\gamma}(x)zwv)\\ &= ((x,\alpha,\beta),z) \otimes ((x,\beta,\delta), g_{\beta\gamma\delta}wv)\\ &= ((x,\alpha,\beta),z) \otimes (((x,\beta,\gamma),w) \otimes ((x,\gamma,\delta),v)).\end{aligned}$$

So the multiplication is associative. Now suppose that the multiplication is associative. From computing the left and right hand sides above we see that

$$(((x,\alpha,\beta),z) \otimes ((x,\beta,\gamma),w)) \otimes ((x,\gamma,\delta),v) = ((x,\alpha,\delta), g_{\alpha\beta\gamma}(x)g_{\alpha\gamma\delta}(x)zwv)$$

and

$$((x,\alpha,\beta),z) \otimes (((x,\beta,\gamma),w) \otimes ((x,\gamma,\delta),v)) = ((x,\alpha,\delta), g_{\alpha\beta\gamma}(x)g_{\beta\gamma\delta}(x)zwv).$$

Therefore,

$$g_{\alpha\beta\gamma}g_{\alpha\gamma\delta} = g_{\alpha\beta\gamma}g_{\beta\gamma\delta}$$

Therefore the functions form a 2-cocycle.

The same constructions that we had for $\mathbb{C}^\times$-bundles and line bundles are present for bundle gerbes.

Example 5.3.4. [17] **Pullback.**

Let $Y \to M$ be a surjective submersion and $f\colon N \to M$ be a smooth map. Then we can pullback $Y \to M$ by f which induces a map $\hat{f}\colon f^{-1}(Y) \to Y$ which covers f.

$$\begin{tikzcd} f^{-1}(Y) \arrow[r, "\hat{f}"] \arrow[d] & Y \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

The map $\hat{f}$ then induces a map $\hat{f}\colon f^{-1}(Y)^{[2]} \to Y^{[2]}$. So if we have a bundle gerbe (P, Y, M) we can pullback the $\mathbb{C}^\times$-bundle $P \to Y^{[2]}$ by $\hat{f}$ to get a $\mathbb{C}^\times$-bundle $(\hat{f})^{-1}(P) \to \hat{f}^{-1}(Y)^{[2]}$.

So we have

$$\begin{tikzcd} \hat{f}^{-1}(P) \arrow[d] & \\ \hat{f}^{-1}(Y)^{[2]} \arrow[r, shift left] \arrow[r, shift right] & f^{-1}(Y) \arrow[d] \\ & N \end{tikzcd}$$

Where the multiplication is induced using $\hat{f}$ to pull back the multiplication on (P, Y, M).

So we have a bundle gerbe $(\hat{f}^{-1}(P), f^{-1}(Y), N)$.

Example 5.3.5. [17] **Dual.**

If we have a bundle gerbe (P, Y, M) then the dual bundle gerbe $(P, Y, M)^*$ is defined to be (P^*, Y, M). This is immediately a bundle gerbe as the only thing that changes is the group action on the $\mathbb{C}^\times$-bundle P.

Example 5.3.6. [17] **Product.**

Let (P, Y, M) and (Q, X, M) be bundle gerbes. Take the fibre product $Y \times_M X$. Then take the $\mathbb{C}^\times$-bundle to be the contracted product $P \otimes Q \to (Y \times_M X)^{[2]}$. Which on the fibres takes the form $(P \otimes Q)_{((y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}$. If m_P and m_Q are bundle gerbe multiplications on P and Q respectively, we can form a multiplication on $P \otimes Q$ by taking $m_P \otimes m_Q$. The associativity follows from the associativity of m_P and m_Q. So $(P, Y, M) \otimes (Q, X, M) = (P \otimes Q, Y \times_M X, M)$.

5.4 Isomorphism of bundle gerbes

Definition 5.4.1. [22] Let (P, X, M) and (Q, Y, N) be bundle gerbes. A *bundle gerbe morphism* $\bar{f}\colon P \to Q$ is a triple $\bar{f} = (\hat{f}, f, f')$ of smooth maps $f'\colon M \to N$, $f\colon X \to Y$ covering f' and $\hat{f}\colon P \to Q$ covering the induced map $f^{[2]}\colon X^{[2]} \to Y^{[2]}$. Lastly, the map $\hat{f}$ is required to commute with the bundle gerbe multiplications on P and Q. This means that

$$\hat{f}(m_P(x \otimes y)) = m_Q(\hat{f}(x) \otimes \hat{f}(y)).$$

In the case when $M = N$ the a bundle gerbe morphism is a triple $(\hat{f}, f, id_M)$.

Definition 5.4.2. [22] Let (P, X, M) and (Q, Y, M) be bundle gerbes. A *bundle gerbe isomorphism* $\bar{f}\colon P \to Q$ is a bundle gerbe morphism $\bar{f} = (\hat{f}, f, id_M)$ such that $f\colon X \to Y$ is a diffeomorphism and $\hat{f}\colon P \to Q$ covering $f^{[2]}\colon X^{[2]} \to Y^{[2]}$ is a $\mathbb{C}^\times$-bundle isomorphism.

We denote isomorphic bundle gerbe by $(P, X, M) \simeq (Q, Y, M)$.

Definition 5.4.3. [22] Let (P, Y, M) be a bundle gerbe. We say (P, Y, M) is *trivial* if there exists some $\mathbb{C}^\times$-bundle $R \to Y$ such that $(P, Y, M) \simeq (\delta(R), Y, M)$. This choice of the $\mathbb{C}^\times$-bundle $R \to Y$ is called a *trivialisation* of P.

Recall that a $\mathbb{C}^\times$-bundle is trivial if it has a global section. A similar thing is true for a bundle gerbe.

Proposition 5.4.4. [17] *Let (P, Y, M) be a bundle gerbe and suppose $\pi\colon Y \to M$ has a global section $s\colon M \to Y$. Then (P, Y, M) is trivial.*

Proof. Define the map $f\colon Y \to Y^{[2]}$ by $f(y) = (y, s(\pi(y)))$. Then let R be the pullback of P by f. So $R = f^{-1}(P) \to Y$. Let $(y_1, y_2) \in Y^{[2]}$. Then

$$\begin{aligned}
\delta(R)_{(y_1,y_2)} &\simeq R_{y_2} \otimes R^*_{y_1} \\
&= P_{(y_2,s(\pi(y_2)))} \otimes P^*_{(y_1,s(\pi(y_1)))} \\
&= P_{(y_2,s(\pi(y_2)))} \otimes P^*_{(y_1,s(\pi(y_2)))} \\
&= P_{(y_2,s(\pi(y_2)))} \otimes P_{(s(\pi(y_2)),y_1)} \\
&= P_{(y_1,y_2)}
\end{aligned}$$

It is clear that this isomorphism preserves the bundle gerbe multiplication so $(P, Y, M) \simeq (\delta(R), Y, M)$. Therefore, (P, Y, M) is trivial. □

The ideas in the proof will be used again when proving a bundle gerbe is trivial if its Dixmier-Douady class is zero.

Definition 5.4.5. [17] Two bundle gerbes (P, Y, M) and (Q, X, M) are said to be *stably isomorphic* if $(P, Y, M)^* \otimes (Q, X, M)$ is trivial. Such a choice of trivialisation is called a *stable isomorphism* from (P, Y, M) to (Q, X, M).

We will see later that stable isomorphism is the correct notion of equivalence for bundle gerbes.

5.5 Connections on a bundle gerbe

Let (P, Y, M) be a bundle gerbe. Since $P \to Y^{[2]}$ is a $\mathbb{C}^\times$-bundle it has a connection one-form which we will call $A \in \Omega^1(P)$ valued in $\mathbb{C}^\times$.

Definition 5.5.1. [16] A $\mathbb{C}^\times$-bundle connection A is a *bundle gerbe connection* if it respects the bundle gerbe multiplication. This means that the induced connection by $\pi_2^{-1}(P)$ is the same as as image of the induced connection by $\pi_3^{-1}(P) \otimes \pi_1^{-1}(P)$.

$$m((\pi_3^{-1}(A) \otimes \pi_1^{-1}(A)) = \pi_2^{-1}(A).$$

We can give this definition in an alternative way by talking about the bundle gerbe multiplication as a section of $\delta(P) \to Y^{[3]}$ instead. We will continue to view connections this way as it will make existence of connections and the discussion of curving easier.

Let us rephrase the definition of curvature in this way. Let $P \to Y^{[2]}$ be a $\mathbb{C}^\times$-bundle and take $\delta(P) \to Y^{[3]}$ to be as defined above. Then the bundle gerbe multiplication gives rise to the following section $s\colon Y^{[3]} \to \delta(P)$. Let $(y_1, y_2, y_3) \in Y^{[3]}$, $p_{12} \in P_{(y_1,y_2)}$ and $p_{23} \in P_{(y_2,y_3)}$. Define $s\colon Y^{[3]} \to \delta(P)$ by

$$s(y_1, y_2, y_3) = p_{23} \otimes m(p_{12}, p_{23})^* \otimes p_{12}. \tag{5.5.1}$$

This is a section of $\delta(P)$ as if we look at the fibre

$$\delta(P)_{(y_1,y_2,y_3)} = P_{(y_2,y_3)} \otimes P^*_{(y_1,y_3)} \otimes P_{(y_1,y_2)}$$

which by definition of the multiplication is exactly where the section maps to. However, if we took any section $s\colon Y^{[3]} \to \delta(P)$ we get a map into

$$\pi_1^{-1}(P) \otimes \pi_2^{-1}(P)^* \otimes \pi_3^{-1}(P).$$

Which is exactly how the bundle gerbe multiplication is defined if we took the dual of $\pi_2^{-1}(P)^*$.

We will now see that the associativity condition on the multiplication is the same as requiring that $\delta(s)$ is the canonical section of $\delta(\delta(P))$ i.e. $\delta(s) = \underline{1}$.

Firstly, suppose we are given a multiplication which is not yet associative and define the section as in (5.5.1) and let $(y_1, y_2, y_3, y_4) \in Y^{[4]}$. Then

$$\begin{aligned}\delta(s)(y_1, y_2, y_3, y_4) &= s(y_2, y_3, y_4) \otimes s^*(y_1, y_3, y_4) \otimes s(y_1, y_2, y_4) \otimes s^*(y_1, y_2, y_3)\\ &= (p_{34} \otimes m^*(p_{23}, p_{34}) \otimes p_{23}) \otimes (p_{34} \otimes m^*(p_{13}, p_{34}) \otimes p_{13})^*\\ &\otimes (p_{24} \otimes m^*(p_{12}, p_{24}) \otimes p_{12}) \otimes (p_{23} \otimes m^*(p_{12}, p_{23}) \otimes p_{12})^*.\end{aligned}$$

Recall that the canonical section has the form

$$\underline{1}(y_1, y_2, y_3, y_4) = p_{23} \otimes p_{24}^* \otimes p_{23} \otimes p_{34}^* \otimes p_{14} \otimes p_{13}^* \otimes p_{24} \otimes p_{14}^* \otimes p_{12} \otimes p_{23}^* \otimes p_{13} \otimes p_{12}^*.$$

Where the choice of $p_{ij} \in P_{ij}$ determines the choice of $p_{ij}^* \in P_{ij}^*$. It is the same point just considered as an element of the dual. So $\delta(s) = \underline{1}$ if and only if $p_{14}^* = m^*(p_{13}, p_{34}) = m(p_{12}, p_{24}) = p_{14}$. However $m^*(p_{13}, p_{34}) = m(p_{12}, p_{24})$ if and only if $m^*(p_{13}, m(p_{32}, p_{24})) = m(m(p_{13}, p_{32}), p_{24})$. Which is exactly the requirement that the multiplication be associative. So $\delta(s) = \underline{1}$ if and only if the multiplication is associative.

Proposition 5.5.2. [17] *Let $A \in \Omega^1(P)$ be a $\mathbb{C}^\times$-bundle connection and $s\colon Y^{[3]} \to \delta(P)$ be the section defined by the multiplication. Then A is a bundle gerbe connection if $s^*(\delta(A)) = 0$. Meaning that $\delta(A)$ is flat with respect to s.*

Proof. Suppose A is a bundle gerbe connection then we have

$$\begin{aligned}s^*(\delta(A))_{(y_1,y_2,y_3)} &= \delta(A)_{(p_{12},m(p_{12},p_{23}),p_{23})}\\ &= \pi_1^{-1}(A)_{(p_{12},m(p_{12},p_{23}),p_{23})} \otimes \pi_2^{-1}(A)^*_{(p_{12},m(p_{12},p_{23}),p_{23})} \otimes \pi_3^{-1}(A)_{(p_{12},m(p_{12},p_{23}),p_{23})}\\ &= A_{p_{12}} \otimes -A_{m(p_{12},p_{23})} \otimes A_{p_{23}}\\ &= A_{p_{12}} \otimes -A_{p_{12}} \otimes -A_{p_{23}} \otimes A_{p_{23}}\\ &= 0\end{aligned}$$

Recall from example (4.7.3) that $P \otimes P^*$ is trivialisable with trivial connection $A \otimes A^*$. Meaning that $A \otimes A^*$ is flat for the canonical section. □

Using this viewpoint of a bundle gerbe connection we can prove the following.

Proposition 5.5.3. [17] *Every bundle gerbe (P, Y, M) has a bundle gerbe connection.*

Proof. Let A be a connection on P that is not necessarily a bundle gerbe connection. So $s^*(\delta(A)) \neq 0$. Firstly we have $\delta(s^*(\delta(A))) = (\delta s)^*(\delta\delta A)$ [23]. We then have that $(\delta s)^*(\delta\delta A) = 0$ because $\delta\delta(A)$ is flat with respect to $\delta(s) = \underline{1}$ as a section of the canonically trivial bundle $\delta^2 P$. So $\delta(s^*(\delta(A))) = 0$ and since $s^*(\delta(A)) \in \Omega^1(Y^{[3]})$ and using the fact that the fundamental sequence is exact, there must be some $a \in \Omega^1(Y^{[2]})$ such that $\delta(a) = s^*(\delta(A))$. So pulling back a by $\pi\colon P \to Y^{[2]}$ we get $\pi^*(a) \in \Omega^1(P)$. Therefore, $s^*(\delta(A - \pi^*(a))) = 0$ so $A - \pi^*(a)$ is a bundle gerbe connection. □

5.6 Curving of a bundle gerbe connection

Recall that for a principal $\mathbb{C}^\times$-bundle we have a connection which is a one-form and from that we get the curvature which is a two-form on the base space. The case for bundle gerbes is similar however we end up with a three-form on M called the three curvature. However, there is the problem of descending the one form on P down to a three form.

Suppose we have a bundle gerbe connection $A \in \Omega^1(P)$ and let dA be its curvature. The curvature dA descends to a two-form on $Y^{[2]}$ this means that there is a two-form $F \in \Omega^2(Y^{[2]})$ such that $\pi^*(F) = dA$. similarly given the section $s\colon Y^{[3]} \to \delta(P)$ we can pull down $\delta(dA)$ and get $\delta(F)$. So we have $s^*\delta(dA) = \delta(F)$.

F is a 2-form on $Y^{[2]}$ however we would like a three form on M. So what we first need is a two-form on Y which we can then descend down to a three-form on M. We have

$$\begin{aligned}\delta(F) &= s^*(\delta(dA))\\ &= s^*(d(\delta(A))\\ &= d(s^*(\delta(A)))\\ &= 0\end{aligned}$$

So by exactness of the fundamental complex there must be some $f \in \Omega^1(Y)$ such that $\delta(f) = F$. Such a choice of f is called a *curving* for A and the pair (A, f) is called a *connective structure* for (P, Y). Now we have that

$$\begin{aligned}\delta(df) &= d(\delta(f))\\ &= dF\\ &= 0\end{aligned}$$

So there must be some $\omega \in \Omega^3(M)$ such that $\pi^*(\omega) = df$. So

$$\begin{aligned}\pi^*(d\omega) &= d\pi^*(\omega)\\ &= d^2 f\\ &= 0\end{aligned}$$

Hence ω is a closed three-form on M called the *three-curvature* of the connective structure. Notice that the three-curvature is not unique as a different choice of curving will result in a different three-curvature.

We have the result from [16] that $[\omega/2\pi i]$ is a representative of the Dixmier-Douady class in real cohomology. We will discuss the Dixmier-Douady class in the next section.

Like with principal bundles we can provide a local description of connection and curvature.

Let (P, Y, M) be a bundle gerbe with connection and curving (A, f). Suppose we have an open cover $\{U_\alpha\}$ of M that has local sections $s_\alpha\colon U_\alpha \to Y$. Then we have local sections

$\sigma_{\alpha\beta}\colon P_{\alpha\beta} \to U_{\alpha\beta}$. Then define the following differential forms.

$$\begin{aligned}
A_{\alpha\beta} &= (s_\alpha, s_\beta)^*(A) \in \Omega^1(U_{\alpha\beta}) \\
f_\alpha &= s_\alpha^*(f) \in \Omega^2(U_\alpha) \\
\omega_\alpha &= df_\alpha
\end{aligned}$$

The differential forms A and f are related in the following way.

Proposition 5.6.1. [17] *The differential forms $A_{\alpha\beta}$ and f_α defined above satisfy the following.*

$$\begin{aligned}
A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} &= g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \\
f_\beta - f_\alpha &= df_{\alpha\beta}
\end{aligned}$$

We finish this section by providing examples of the connective data on some of the examples of bundle gerbes given above.

Example 5.6.1. **The trivial bundle gerbe**

Let $R \to Y$ be a $\mathbb{C}^\times$-bundle with connection A with curvature F and consider the trivial bundle gerbe $(\delta(R), Y, M)$. A bundle gerbe connection is given by $\delta(A)$ with curving $\delta(F)$ and trivial three-curvature.

It is clear from the bundle gerbe multiplication that

$$m(\pi_3^{-1}(\delta A) \otimes \pi_1^{-1}(\delta A)) = \pi_2^{-1}(\delta A).$$

So any connection on R gives rise to a bundle gerbe connection on the δR. Let $F = d\delta A$. Now since δ commutes with exterior derivative (lemma (5.1.6)) we have

$$\delta(dA) = d(\delta A) = F$$

So $f = dA$ is a curving for F. Now we want some three-from $\omega \in \Omega^3(M)$ such that $\pi^*(\omega) = df$. But since $f = dA$ we have $df = 0$. Hence, $\pi^*(\omega) = 0$. However, since the pullback of a surjective submersion is injective we have that $\omega = 0$.

Example 5.6.2. **Product**

Suppose we have bundle gerbes (P, Y, M) and (Q, X, M) with connective structures (A_P, f_P), (A_Q, f_Q) and three-curvatures ω_P and ω_Q.

Let $\pi_X\colon X \times_M Y \to X$ be the obvious projection and likewise define π_Y. Then $(P \otimes Q, Y \times_M X, M)$ has connective structure $(A_Q \otimes A_P, \pi_X^* f_Q + \pi_Y^* f_P)$ and three-curvature $\omega_P + \omega_Q$.

In the next chapter we focus on calculating the connective structure and three-curvature of a specific bundle gerbe called the basic bundle gerbe.

5.7 The Dixmier-Douady Class

As we saw with line bundles we could take an open cover of our base space and local sections to get transition functions which define a cocycle and hence a class in $H^2(M,\mathbb{Z})$ called the first chern class. We will now see a similar idea for bundle gerbes however we have 2-cocycles and a class in $H^3(M,\mathbb{Z})$.

Let (P, Y, M) be a bundle gerbe and choose an open cover $\{U_\alpha\}$ of M such that on each U_α we have local sections $s_\alpha \colon U_\alpha \to Y$. Then on the intersections we can define a map

$$(s_\alpha, s_\beta)\colon U_\alpha \cap U_\beta \to Y^{[2]}$$

by

$$(s_\alpha, s_\beta)(x) = (s_\alpha(x), s_\beta(x)).$$

The map $(s_\alpha, s_\beta)(x) \in Y^{[2]}$ as since s_α are sections of $Y \to M$ we have

$$\pi(s_\alpha(x)) = x = \pi(s_\beta(x)).$$

So by definition of $Y^{[2]}$ we have $(s_\alpha(x), s_\beta(x)) \in Y^{[2]}$.

Let $P_{\alpha\beta} = (s_\alpha, s_\beta)^{-1}P$ be the pullback of P by this map. If we let $\pi_P \colon P \to M$ be the projection of this $\mathbb{C}^\times$-bundle we have that

$$P_{\alpha\beta} = \{(m, p) \in U_{\alpha\beta} \times P \mid (s_\alpha, s_\beta)(m) = \pi_P(p)\}.$$

So fibres of this bundle look like

$$(P_{\alpha\beta})_m = \{p \in P \mid \pi_P(p) = (s_\alpha, s_\beta)(m)\} = P_{(s_\alpha,s_\beta)(m)}.$$

Since the pullback of a $\mathbb{C}^\times$-bundle is again a $\mathbb{C}^\times$-bundle we can choose sections $\sigma_{\alpha\beta} \colon U_\alpha \cap U_\beta \to P_{\alpha\beta}$. So for $x \in U_\alpha \cap U_\beta$ we have $\sigma_{\alpha\beta}(x) \in (P_{\alpha\beta})_x$ which implies $\sigma_{\alpha\beta}(x) \in P_{(s_\alpha,s_\beta)(x)}$. Let $x \in U_{\alpha\beta\gamma}$, so $(s_\alpha(x), s_\beta(x), s_\gamma(x)) \in Y^{[3]}$. Then by the bundle gerbe multiplication $m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x))) \in P_{(s_\alpha,s_\gamma)(x)}$. Since $\sigma_{\alpha\gamma}(x) \in P_{(s_\alpha,s_\gamma)(x)}$ We can consider the difference of $\sigma_{\alpha\gamma}(x)$ by $m(\sigma_{\alpha\beta}(x), \sigma_{\beta\gamma}(x))) = g_{\alpha\beta\gamma}(x)\sigma_{\alpha\gamma}(x)$, where $g_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times$.

Then we have that $g_{\alpha\beta\gamma}$ satisfies the Čech 2-cocycle condition,

$$g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1. \tag{5.7.1}$$

The reason for this is due to the associativity condition on the bundle gerbe multiplication. Consider the intersection $U_{\alpha\beta\gamma\delta}$, then we have

$$m(m(\sigma_{\alpha\beta}, \sigma_{\beta\gamma}), \sigma_{\gamma\delta}) = m(\sigma_{\alpha\beta}, m(\sigma_{\beta\gamma}, \sigma_{\gamma\delta})).$$

So

$$m(m(\sigma_{\alpha\beta}, \sigma_{\beta\gamma}), \sigma_{\gamma\delta}) = m(g_{\alpha\beta\gamma}\sigma_{\alpha\gamma}, \sigma_{\gamma\delta}) = g_{\alpha\beta\gamma}g_{\alpha\gamma\delta}\sigma_{\alpha\delta}$$

We also have that from the right hand side

$$m(\sigma_{\alpha\beta}, m(\sigma_{\beta\gamma}, \sigma_{\gamma\delta})) = m(\sigma_{\alpha\beta}, g_{\beta\gamma\delta}\sigma_{\beta\delta}) = g_{\alpha\beta\delta}g_{\beta\gamma\delta}\sigma_{\alpha\delta}.$$

So

$$g_{\alpha\beta\gamma}g_{\alpha\gamma\delta}\sigma_{\alpha\delta} = g_{\alpha\beta\delta}g_{\beta\gamma\delta}\sigma_{\alpha\delta}.$$

So,

$$g_{\alpha\beta\gamma}g_{\alpha\gamma\delta} = g_{\alpha\beta\delta}g_{\beta\gamma\delta}.$$

Therefore,

$$g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1.$$

It can be shown that this construction is independent of any choices made. Thus we have $[g_{\alpha\beta\gamma}] \in H^2(M, \underline{\mathbb{C}^\times})$. We again make use of another result from Brylinski.

Theorem 5.7.1. [1] $H^2(M, \underline{\mathbb{C}^\times}) \simeq H^3(M, \mathbb{Z})$

So under the above isomorphism the image of $[g_{\alpha\beta\gamma}]$ defines a class in $H^3(M, \mathbb{Z})$.

Definition 5.7.2. [16] The *Dixmier-Douady class* of a bundle gerbe (P, Y, M) is the image of the class $[g_{\alpha\beta\gamma}]$ under the isomorphism $H^2(M, \underline{\mathbb{C}^\times}) \simeq H^3(M, \mathbb{Z})$ and is denoted $DD(P, Y, M)$.

Like with the Chern class the Dixmier-Douady class respects the various operations of dual, pullback and addition.

Theorem 5.7.3. [17] *Let $f\colon N \to M$ be a smooth map and $(P, Y, M), (Q, X, M)$ be a bundle gerbes. Then the Dixmier-Douady class has the following properties.*

1. $DD((P, Y)^*) = -DD((P, Y))$
2. $DD((P, Y) \otimes (Q, X)) = DD((P, Y)) + DD((Q, X))$
3. $DD(f^{-1}(P, Y)) = f^*(DD((P, Y)))$

An important fact about the Dixmier-Douady class is that it is the obstruction to the bundle gerbe being trivial. This is analogous to the result that a principal $\mathbb{C}^\times$-bundle is trivial if and only if it has zero chern class.

Theorem 5.7.4. [16] *A bundle gerbe (R, Y, M) is trivial if and only if its Dixmier-Douady class is zero.*

Proof. Suppose (R, Y) is trivial. Then we can consider the Dixmier-Douady class of the trivial bundle gerbe $(\delta(R), Y)$. We will show that $DD(\delta(R), Y) = 0$ and hence any trivial bundle gerbe will have zero Dixmier-Douady class. Let $R \to Y$ be a $\mathbb{C}^\times$-bundle and choose a open cover $\{U_\alpha\}$ of M such that $\pi\colon Y \to M$ has local sections $s_\alpha\colon U_\alpha \to Y$. Then pullback R by s_α and take local sections $\eta_\alpha\colon U_\alpha \to s_\alpha^{-1}(R)$. Let $m \in U_\alpha$, then

$$s_\alpha^{-1}(R)_m = R_{s_\alpha(m)}.$$

So $\eta_\alpha(m) \in R_{s_\alpha(m)}$. Now pullback $\delta(R)$ by $(s_\alpha, s_\beta)\colon U_{\alpha\beta} \to Y^{[2]}$. Then for $m \in U_{\alpha\beta}$ we have

$$\begin{aligned}(s_{\alpha,\beta})^{-1}(\delta(R))_m &= \delta(R)_{(s_\alpha(m), s_\beta(m))}\\ &= R_{s_\beta(m)} \otimes R^*_{s_\alpha(m)}.\end{aligned}$$

So we can define local sections of $(s_\alpha, s_\beta)^{-1}(\delta(R))$ by $\sigma_{\alpha\beta} = \eta_\beta \otimes \eta_\alpha^*$. Then by the bundle gerbe multiplication $m(\sigma_{\alpha\beta}, \sigma_{\beta\gamma}) = g_{\alpha\beta\gamma}\sigma_{\alpha\gamma}$. However

$$\begin{aligned}m(\sigma_{\alpha\beta}, \sigma_{\beta\gamma}) &= \eta_\beta \otimes \eta_\alpha^* \otimes \eta_\gamma \otimes \eta_\beta^*\\ &= \eta_\gamma \otimes \eta_\alpha^*.\end{aligned}$$

Therefore $g_{\alpha\beta\gamma} = 1$ and so under the isomorphism $H^2(M, \underline{\mathbb{C}^\times}) \simeq H^3(M, \mathbb{Z})$ we have that $DD(\delta(R), Y) = 0$.

Now suppose we have a bundle gerbe with Dixmier-Douady class zero. Then if we take an open over $\{U_\alpha\}_{\alpha \in I}$ of M with sections $s_\alpha\colon U_\alpha \to Y$ then we can form sections $\sigma_{\alpha\beta}\colon U_{\alpha\beta} \to P$. These sections are related by some $g_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times$ which satisfy the 2-cocycle condition.

However, since $DD(P, Y) = 0$ there must be some $h_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^\times$ such that

$$g_{\alpha\beta\gamma} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta}.$$

If we replace $\sigma_{\alpha\beta}$ by $\sigma_{\alpha\beta}/h_{\alpha\beta}$ we can assume that $g_{\alpha\beta\gamma} = 1$. Now we want to construct some $\mathbb{C}^\times$-bundle $R \to Y$ and show that $\delta(R) \simeq P$ hence showing that P is trivial.

Let $Y_\alpha = \pi^{-1}(U_\alpha)$ and define $R_\alpha \to Y_\alpha$ fibre-wise by letting $(R_\alpha)_y = P_{(y, s_\alpha(\pi(y)))}$ for $y \in Y_\alpha$. We now need an isomorphism between $(R_\alpha)_y$ and $(R_\beta)_y$ for $y \in Y_\alpha \cap Y_\beta$. Firstly we have that $(y, s_\alpha(\pi(y)), s_\beta(\pi(y))) \in Y^{[3]}$. So by the bundle gerbe multiplication we have that

$$P_{(y, s_\alpha(\pi(y)))} = P_{(y, s_\beta(\pi(y)))} \otimes P_{(s_\alpha(\pi(y)), s_\beta(\pi(y)))}.$$

We also have that $\sigma_{\alpha\beta}(y) \in P_{(s_\alpha(\pi(y)), s_\beta(\pi(y)))}$. So the sections $\sigma_{\alpha\beta}$ give an isomorphism between R_α and R_β.

$$\begin{aligned}\chi_{\alpha\beta}\colon R_\alpha &\xrightarrow{\sim} R_\beta\\ x &\mapsto x \otimes \sigma_{\alpha\beta}\end{aligned}$$

Using the clutching construction we can form a $\mathbb{C}^\times$-bundle $R \to Y$ in the following way. Firstly take the disjoint union of R_α and define the following equivalence relation.

$$(\alpha, x) \sim (\beta, y) \iff \chi_{\alpha\beta}(x) = y.$$

The fact that this is an equivalence relation follows from the fact that $\sigma_{\alpha\beta}$ satisfy the cocycle condition.

Then define R to the disjoint union quotient the above equivalence relation.

$$R = (\bigsqcup_\alpha R_\alpha)/\sim$$

Therefore, by the clutching construction we have a $\mathbb{C}^\times$-bundle $R \to Y$. Lastly, we need to show that $(\delta(R), Y, M) \simeq (P, Y, M)$. Let $(y_1, y_2) \in Y^{[2]}$. Then we have that $\pi(y_1) = \pi(y_2)$. So $y_1, y_2 \in Y_\alpha$ for some $\alpha \in I$. So

$$\begin{aligned}
\delta(R)_{(y_1,y_2)} &\simeq R_{y_2} \otimes R^*_{y_1} \\
&= P_{(y_2, s_\alpha(\pi(y_2)))} \otimes P^*_{(y_1, s_\alpha(\pi(y_1)))} \\
&= P_{(y_2, s_\alpha(\pi(y_2)))} \otimes P^*_{(y_1, s_\alpha(\pi(y_2)))} \\
&= P_{(y_2, s_\alpha(\pi(y_2)))} \otimes P_{(s_\alpha(\pi(y_2)), y_1)} \\
&= P_{(y_1,y_2)}
\end{aligned}$$

The above isomorphism clearly preservers the bundle gerbe multiplications on P and δR. Hence, $\delta(R) \simeq P$ and therefore, P is trivial. □

Proposition 5.7.5. [17] *Two bundle gerbes (P, Y, M) and (Q, X, M) are stably isomorphic if and only if*

$$DD(P, Y, M) = DD(Q, X, M).$$

Proof. (P, Y, M) and (Q, X, M) are stably isomorphic if and only if $(P, Y, M)^* \otimes (Q, X, M)$ is trivial which is true if and only if

$$DD((P, Y, M)^* \otimes (Q, X, M)) = 0.$$

Then by theorem (5.7.3)

$$DD((P, Y, M)^* \otimes (Q, X, M)) = -DD(P, Y, M) + DD(Q, X, M)$$

Therefore

$$DD(P, Y, M) = DD(Q, X, M).$$

□

We saw with line bundles and $\mathbb{C}^\times$-bundles that their characteristic class, namely their first chern class classifies them up to isomorphism. The result above is the analogue for bundle gerbes. That is, the Dixmier-Douady class classifies bundle gerbes up to stable isomorphism. Therefore, stable isomorphism is a better notion of equivalence than just isomorphism.

We finish this section by providing a calculation of the Dixmier-Douady of a bundle gerbe.

Example 5.7.1. [17] **The lifting bundle gerbe**.

Let $(\tau^{-1}\hat{G}, Y, M)$ be a lifting bundle gerbe. In the construction of the Dixmier-Douady class we take an open cover $\{U_\alpha\}$ of M such that it admits sections $s_\alpha \colon U_\alpha \to Y$. This gives rise to a section $(s_\alpha, s_\beta) \colon U_\alpha \cap U_\beta \to Y^{[2]}$.

We then need to pullback $\tau^{-1}\hat{G}$ by (s_α, s_β) to get the bundle $(\tau^{-1}\hat{G})_{\alpha\beta} \to U_{\alpha\beta}$. We now require sections $\sigma_{\alpha\beta} \colon U_{\alpha\beta} \to (\tau^{-1}\hat{G})_{\alpha\beta}$ where $\sigma_{\alpha\beta}(x) \in \tau^{-1}\hat{G}_{(s_\alpha(x), s_\beta(x))}$. We then relate these sections by the bundle gerbe multiplication, but since the multiplication is just the group multiplication from $\hat{G}$ we have

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = g_{\alpha\beta\gamma}\sigma_{\alpha\gamma}.$$

For some $g_{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathbb{C}^\times$. Then a representative of the Dixmier-Douady class of the lifting bundle gerbe is the image of $g_{\alpha\beta\gamma}$ under the isomorphism $H^2(M, \underline{\mathbb{C}^\times}) \simeq H^3(M, \mathbb{Z})$.

Recall that since P is a G-bundle the sections s_α are related by transition functions $s_\alpha = g_{\alpha\beta}s_\beta$. Therefore $(\tau^{-1}\hat{G})_{\alpha\beta}$ consists of triples $(s_\alpha, s_\beta, \hat{g})$ where $p(\hat{g}) = g_{\alpha\beta}$. So the sections $\sigma_{\alpha\beta}$ are given in terms of the candidate transition functions $\hat{g}_{\alpha\beta}$. So rephrasing the above condition on the sections $\sigma_{\alpha\beta}$ in terms of candidate transition functions we have that

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}\hat{g}_{\alpha\gamma}.$$

Which is exactly the condition for obstructing a lift. So the Dixmier-Douady class of the lifting bundle gerbe is the obstruction to a G-bundle having lift. Therefore, the lifting bundle gerbe is trivial if and only if P lifts to a $\hat{G}$-bundle.

Chapter 6

The Basic Bundle Gerbe

In this final chapter we wish to bring together all the theory previously outlined and focus on a specific bundle gerbe and provide explicit formulas for the connective data and characteristic class. The bundle gerbe we will focus on will be the lifting bundle gerbe of the central extension of the loop group of based loops ΩG. This particular bundle gerbe is also known as the basic bundle gerbe.

6.1 Central Extensions

The following section is based on the work of Murray and Stevenson in ([14], [15]).

Recall that given a Lie group G and a central extension

$$\mathbb{C}^\times \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G.$$

Recall from example (4.7.4) that we can consider this as a principal $\mathbb{C}^\times$-bundle $\hat{G} \to G$. This means we can talk about the connection, curvature and isomorphisms of central extensions viewed as a $\mathbb{C}^\times$-bundles. We will denote the curvature and connection of this central extension by R and μ respectively.

Since we want $\hat{G}$ to be a group there must be a multiplication $M\colon \hat{G} \times \hat{G} \to \hat{G}$ which covers the multiplication on $m = d_1\colon G \times G \to G$. So we have the following commutative diagram.

$$\begin{tikzcd} \hat{G} \times \hat{G} \arrow[r, "M"] \arrow[d, "(\pi,\pi)"'] & \hat{G} \arrow[d, "\pi"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Since $\hat{G}$ is a central extension we must have $M(\hat{g}z, \hat{h}w) = M(\hat{g}, \hat{h})(zw)$ for any $\hat{g}, \hat{h} \in \hat{G}$ and $z, w \in \mathbb{C}^\times$. So in a similar way that the bundle gerbe multiplication gives rise to a section of $\delta(P)$ we get a section of $\delta(\hat{G})$ given by

$$s(g,h) = \hat{g} \otimes M(\hat{g},\hat{h})^* \otimes \hat{h}.$$

Where $\hat{g} \in \hat{G}_g$ and $\hat{h} \in \hat{G}_h$. From earlier results, $\delta(s)$ is a section of $\delta^2(\hat{G})$ and the associativity of M is equivalent to $\delta(s) = \underline{1}$. So a central extension gives rise to a $\mathbb{C}^\times$-bundle $\hat{G} \to G$ with a section s of $\delta(\hat{G}) \to G \times G$ such that $\delta(s) = \underline{1}$.

Let $\mu \in \Omega^1(\hat{G})$ be a connection on $\hat{G} \to G$. Then we have an induced connection $\delta\mu$ on $\delta(\hat{G}) \to G \times G$. Since this bundle is trivial we can pullback $\delta\mu$ by the section s. So let $\alpha = s^*(\delta\mu) \in \Omega^1(G \times G)$. Now we have that $\delta\alpha = \delta(s^*(\delta\mu)) = (\delta s)^*(\delta\delta\mu) = 0$. Let $R = d\mu$ be the curvature of μ. So we have $d\alpha = d(s^*(\delta\mu)) = s^*(d\delta\mu) = s^*(\delta d\mu) = \delta R$.

So the central extension gives rise to a closed integral two-form $R \in \Omega^2(G)$ and a one-form $\alpha \in \Omega^1(G \times G)$ such that $d\alpha = \delta R$ and $\delta\alpha = 0$. Let $\Gamma(G)$ be the set of pairs of differential forms $(R, \alpha) \in \Omega^2(G) \times \Omega^1(G \times G)$ which satisfy the above conditions.

Let $C(G)$ be the set of isomorphism classes of central extensions of G. By isomorphisms of central extensions we mean isomorphisms of the central extensions as principal bundles and groups. Therefore we have a map

$$C(G) \to \Gamma(G).$$

We would now like an inverse to this map. What this means is given a pair of differential forms $(R, \alpha) \in \Gamma(G)$ we would like to construct a central extension of G. From now on we will assume that G is simply connected.

Suppose we have a pair of differential forms $(R, \alpha) \in \Omega^2(G) \times \Omega^1(G \times G)$ such that R is closed and integral, $d\alpha = \delta R$ and $\delta\alpha = 0$.

It is a well know fact that given a 2-form $R \in \Omega^2(G)$ there exists a $\mathbb{C}^\times$-bundle $P \to G$ with connection $a \in \Omega^1(P)$ such that $da = R$. We also have the fact from ([11], p.g. 92) if $Q \to G$ is a $\mathbb{C}^\times$-bundle over a simply connected space with a flat connection A. Then $Q \to G$ is trivial and there exists a section $s\colon G \to Q$ such that $s^*(A) = 0$.

Since $d\alpha = \delta R$ we have that $\delta a - \pi^*\alpha$ is a flat connection on $\delta P \to G \times G$. So there is a section $s\colon G \times G \to \delta P$ such that $s^*(\delta a) = \alpha$. Given this section we can define a multiplication as before

$$s(p,q) = p \otimes M(p,q)^* \otimes q.$$

Now we have that

$$\delta(s)^*(\delta\delta a) = \delta(s^*(\delta a)) = \delta\alpha = 0.$$

For the canonical section $\underline{1}$ we have

$$\underline{1}^*(\delta\delta a) = 0.$$

Therefore, $\delta(s)$ and 1 differ by some constant $z \in \mathbb{C}^\times$. So we have

$$M(M(\hat{g}, \hat{h}), \hat{k}) = zM(\hat{g}, M(\hat{h}, \hat{k})).$$

For all $\hat{g}, \hat{h}, \hat{k} \in \hat{G}$. Now choose $\hat{g} \in \hat{G}_e$ where $e \in G$ is the identity. Then we have that $M(\hat{g}, \hat{g}) \in \hat{G}_e$. So there must be some $w \in \mathbb{C}^\times$ such that $M(\hat{g}, \hat{g}) = \hat{g}w$. So letting $\hat{h} = \hat{k} = \hat{g} \in \hat{G}_e$ we have that

$$M(M(\hat{g}, \hat{g}), \hat{g}) = zM(\hat{g}, M(\hat{g}, \hat{g})).$$

Therefore, $\hat{g}w^2 = \hat{g}w^2 z$ and so $z = 1$. Hence, $\delta(s) = 1$. So from (R, α) we have a $\mathbb{C}^\times$-bundle $P \to G$ with a section s of $\delta(P) \to G \times G$ such that $\delta(s) = 1$. Which is exactly what it means to have a central extension of G when phrased in the language of $\mathbb{C}^\times$-bundles.

6.2 The central extension of the loop group

Let G be a compact, simple and simply connected Lie group and $\mathfrak{g}$ its Lie algebra. We would now like apply the above theory to the specific example of the central extension of the loop group.

Definition 6.2.1. [20] *The loop group* of G denoted LG is the group of smooth paths in G with the same start and end points.

$$LG = \{\gamma \colon [0, 2\pi] \to G \mid \gamma(0) = \gamma(2\pi)\}.$$

We also have the subgroup of LG which consists of loops in G which start at the identity. We denote this group by ΩG. This group is sometimes referred to as the based loop group.

$$\Omega G = \{\gamma \colon [0, 2\pi] \to G \mid e = \gamma(0) = \gamma(2\pi)\}$$

This group has a well known central extension [20].

$$\mathbb{C}^\times \xrightarrow{\iota} \widehat{LG} \xrightarrow{\pi} LG$$

As a vector space $\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}^\times$ where the Lie bracket is defined by

$$[(X, z), (Y, w)] = ([X, Y], R(X, Y)).$$

Here, $[\gamma, \eta]$ is the Lie bracket on the $L\mathfrak{g}$.

We also have a well known differential form $R \colon L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}^\times$ given in ([20], p.g. 39) by Pressley and Segal.

$$R_\gamma(\gamma X, \gamma Y) = \frac{i}{4\pi}\int_{S^1} \langle \Theta_\gamma(\gamma X), \partial\Theta_\gamma(\gamma Y)\rangle d\theta$$

Where $\langle\ ,\ \rangle$ is a symmetric invariant form on the Lie algebra $L\mathfrak{g}$ and Θ is the Maurer-Cartan form on LG, that is $\Theta_\gamma(\gamma X) = X$. Lastly ∂ is the derivative with respect to θ. So we can simplify the above equation to be

$$R_\gamma(\gamma X, \gamma Y) = \frac{i}{4\pi}\int_{S^1} \langle X, \partial Y\rangle d\theta$$

For the above bracket to be define a Lie algebra on $\widehat{L\mathfrak{g}}$ we require skew-symmetry and the Jacobi identity to hold. Skew symmetry is clear from integration by parts.

The Jacobi identity holds as the inner product is invariant under the bracket on $L\mathfrak{g}$ [20]. Meaning that

$$\langle [\gamma, \eta], \mu\rangle = \langle \gamma, [\eta, \mu]\rangle.$$

The point of the differential form R is that it is precisely the two form we need to reconstruct the central extension. Now we need a one form α on $LG \times LG$ such that $\delta(R) = d\alpha$. This one-form is provided by Murray and Stevenson in ([14], p.g. 548).

$$\alpha(\gamma, \eta)(\gamma X_1, \eta X_2) = \frac{i}{2\pi}\int_{S^1} \langle X_1, Z_2\rangle d\theta$$

Where

$$\begin{aligned} Z\colon LG &\to L\mathfrak{g} \\ \gamma &\mapsto \partial\gamma\gamma^{-1} \end{aligned}$$

and Z_2 is shorthand for $\pi_1^* Z$. Meaning that $Z_2(\gamma, \eta) = \partial\eta\eta^{-1}$.

So what we need to check is that $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$. Hence we have the following proposition.

Proposition 6.2.2. [14] *The pair of differential forms (R, α) satisfy $\delta(R) = d\alpha$ and $\delta(\alpha) = 0$*

Proof. Let's firstly calculate δR. Note that

$$\delta R = \pi_1^* R - m^* R + \pi_2^* R$$

Where $m\colon LG \times LG \to LG$ is multiplication of loops in LG and π_i is the regular projection of the i-th factor. We have that

$$\pi_i^* R = \int_{S^1} \langle \pi_i^*\Theta, \partial\pi_i^*\Theta\rangle d\theta.$$

So we need to calculate m^*R. Let $\gamma, \eta \in LG$ and $X_1, Y_1, X_2, Y_2 \in L\mathfrak{g}$. Then we have that

$$(m_*)_{(\gamma,\eta)}(\gamma X_1, \eta X_2) = \gamma\eta(\mathrm{ad}_{\eta^{-1}} X_1 + X_2).$$

So

$$\begin{aligned}
(m^*\Theta)_{(\gamma,\eta)}(\gamma X_1, \eta X_2) &= (\Theta m_*)_(\gamma,\eta)(\gamma X_1, \eta X_2) \\
&= \Theta(\gamma\eta(\mathrm{ad}_{\eta^{-1}}(X_1) + X_2) \\
&= \mathrm{ad}_{\eta^{-1}} X_1 + X_2 \\
&= \mathrm{ad}_{\eta^{-1}} \Theta_\gamma(\gamma X_1) + \Theta_\eta(\eta X_2)
\end{aligned}$$

Therefore,

$$m^*\Theta_{(\gamma,\eta)} = \mathrm{ad}_{\eta^{-1}}\Theta_\gamma + \Theta_\eta.$$

So we can now calculate m^*R as follows. Note that we have $\Theta_1 = \pi_2^*\Theta$ and likewise for Θ_2.

$$\begin{aligned}
m^*R &= \frac{i}{4\pi}\int_{S^1} \langle m^*\Theta, \partial m^*\Theta\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_{\eta^{-1}}\Theta_1 + \Theta_2, \partial(\mathrm{ad}_{\eta^{-1}}(\Theta_1) + \Theta_2)\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial(\mathrm{ad}_{\eta^{-1}}(\Theta_1))\rangle + \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial\Theta_2\rangle + \langle \Theta_2, \partial(\mathrm{ad}_{\eta^{-1}}(\Theta_1))\rangle + \langle \Theta_2, \partial\Theta_2\rangle d\theta
\end{aligned}$$

We make use of the following identities

$$\partial\Theta = \mathrm{ad}_{\gamma^{-1}}(d(\partial\gamma\gamma^{-1}))$$

and

$$\partial(\mathrm{ad}_{\gamma^{-1}}(X)) = \mathrm{ad}_{\gamma^{-1}}([X, \partial\gamma\gamma^{-1}]) + \mathrm{ad}_{\gamma^{-1}}(\partial X).$$

So the above becomes

$$\begin{aligned}
\frac{i}{4\pi}\int_{S^1} &\langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \mathrm{ad}_{\eta^{-1}}([\Theta_1, Z_2])\rangle + \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \mathrm{ad}_{\eta^{-1}}\partial\Theta_1\rangle + \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial\Theta_2\rangle \\
&+ \langle \Theta_2, \partial(\mathrm{ad}_{\eta^{-1}}(\Theta_1))\rangle + \langle \Theta_2, \partial\Theta_2\rangle d\theta
\end{aligned}$$

Applying integration by parts we get the following

$$\begin{aligned}
\int_{S^1} \langle \Theta_2, \partial\mathrm{ad}_{\eta^{-1}}(\Theta_1)\rangle d\theta &= \Theta_2\mathrm{ad}_{\eta^{-1}}(\Theta_1)\Big|_0^{2\pi} - \int_{S^2} \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta)\rangle d\theta \\
&= 0 - \int_{S^1} \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta)\rangle d\theta \\
&= -\int_{S^1} \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta)\rangle d\theta
\end{aligned}$$

So the above then becomes

$$\frac{i}{4\pi}\int_{S^1}\langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \mathrm{ad}_{\eta^{-1}}([\Theta_1, Z_2])\rangle + \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \mathrm{ad}_{\eta^{-1}}\partial\Theta_1\rangle + \langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial\Theta_2\rangle$$
$$- \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta)\rangle + \langle \Theta_2, \partial\Theta_2\rangle d\theta$$

Since the inner product is ad-invariant we can simplify further so m^*R can be given as

$$m^*R = \frac{i}{4\pi}\int_{S^1}\langle \Theta_1, [\Theta_1, Z_2]\rangle + \langle \Theta_1, \partial\Theta_1\rangle - \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta_1)\rangle$$
$$- \langle \partial\Theta_2, \mathrm{ad}_{\eta^{-1}}(\Theta)\rangle + \langle \Theta_2, \partial\Theta_2\rangle d\theta$$

Now we can calculate δR as follows

$$\begin{aligned}
\delta(R) &= \frac{i}{4\pi}\int_{S^1}\langle \Theta_1, \partial\Theta_1\rangle + \langle \Theta_2, \partial\Theta_2\rangle d\theta - m^*R \\
&= \frac{i}{4\pi}\int_{S^1} -\langle \Theta_1, [\Theta_1, Z_2]\rangle - 2\langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial\Theta_2\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} -\langle [\Theta_1, \Theta_1], Z_2\rangle - 2\langle \mathrm{ad}_{\eta^{-1}}(\Theta_1), \partial\Theta_2\rangle d\theta \\
&= \frac{i}{2\pi}\int_{S^1} -\frac{1}{2}\langle [\Theta_1, \Theta_1], Z_2\rangle - \langle \Theta_1, \partial\mathrm{ad}_{\eta^{-1}}(\Theta_2)\rangle d\theta \\
&= \frac{i}{2\pi}\int_{S^1}\langle d\Theta_1, Z_2\rangle - \langle \Theta_1, \partial\mathrm{ad}_{\eta^{-1}}(\Theta_2)\rangle d\theta \\
&= \frac{i}{2\pi}\int_{S^1}\langle d\Theta_1, Z_2\rangle - \langle \Theta_1, dZ_2)\rangle d\theta
\end{aligned}$$

Calculating $d\alpha$ we get

$$\begin{aligned}
d\alpha &= \frac{i}{2\pi}\int_{S^1} d\langle \pi_2^*\Theta, \pi_1^*Z\rangle d\theta \\
&= \frac{i}{2\pi}\int_{S^1}\langle d\pi_2^*\Theta, \pi_1^*Z\rangle - \langle \pi_2^*\Theta, d\pi_1^*Z\rangle d\theta \\
&= \frac{i}{2\pi}\int_{S^1}\langle d\Theta_1, Z_2\rangle - \langle \Theta_1, dZ_2\rangle d\theta.
\end{aligned}$$

Therefore,

$$\delta(R) = d\alpha.$$

It can also be checked that $\delta(\alpha) = 0$. □

Now that we have the differential forms (R, α) we can reconstruct the central extension $\widehat{LG}$. The point of all this is to use the lifting bundle gerbe to provide a solution to the problem of given a LG-bundle P when can we lift it to a $\widehat{LG}$-bundle? To do this we need to calculate the Dixmier-Douady class of this bundle gerbe as it will be the obstruction to LG-bundle having lift. In the case of the loop group the Dixmier-Douady class is called the *string class* and is denoted $s(P)$.

6.3 The path fibration

Let PG be the space of smooth paths in G starting at the identity.

$$PG = \{\gamma \colon [0, 2\pi] \to G \mid \gamma(0) = e\}.$$

There is a smooth map $\mathrm{ev} \colon PG \to G$ which sends a path to its endpoint. This is called the path fibration and it is shown in [7] that $PG \to G$ is a principal ΩG-bundle.

Recall from lemma (5.1.5) that the collection $\{P^{[n]}\}$ with maps $\pi_i \colon P^{[n]} \to P^{[n+1]}$ is a simplicial space.

It is shown in [19] that we have a simplicial space $P \times \Omega G^n$ with face maps

$$d_k(p, g_1, \dots, g_n) = \begin{cases} (pg_1, \dots, g_n) & k = 0 \\ (p, g_1, \dots, g_k g_{k+1}, \dots, g_n) & k = 1, \dots n-1 \\ (p, g_1, \dots, g_{n-1}) & k = n \end{cases}$$

It also shown in [19] that $P^{[\bullet+1]}$ and $P \times \Omega G^{\bullet}$ are isomorphic. Recall the following setup of the lifting bundle gerbe.

$$\begin{tikzcd} \tau^{-1}\hat{G} \arrow[d] & \\ P^{[2]} \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\ & M \end{tikzcd}$$

However for the following calculations we will make use of the fact that $P^{[\bullet+1]} \simeq P \times \Omega G^{[\bullet]}$. Recall that we began this chapter with discussing the central extension on LG however from now on will will focus on the subgroup ΩG and provide the obstruction to the ΩG-bundle PG having a lift.

6.4 A connection for the basic bundle gerbe

We would like to solve the problem of lifting the ΩG-bundle $PG \to G$ to an $\widehat{\Omega G}$-bundle $\hat{PG} \to G$. As before, the obstruction to the problem is the bundle gerbe $(\tau^{-1}\widehat{\Omega G}, PG, G)$.

Let (α, R) be the pair of differential forms as in section (6.2). Since R is an integral 2-form there exists a principal bundle $L \to G$ with connection μ such that $R = d\mu$. So we have the following diagram. Note that $\tau_n \colon P \times G^n \to G^n$ just projects off the P element.

$$\begin{tikzcd}
 & L \arrow[d, "\pi"] & \\
\tau^{-1}L \arrow[dr] & G & \\
 & P \times G \arrow[u, "\tau"] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\
 & & M
\end{tikzcd}$$

The natural idea for a connection on $\tau^{-1}L$ would be to pullback the connection μ by τ. What we require for this to be a bundle gerbe connection is that $s^{-1}(\delta(\tau^{-1}\mu)) = 0$. Where s is the section on the trivial bundle $\delta L \to G^2$.

If we then use the δ map from the fundamental complex to pull everything back we then have

$$\begin{tikzcd}
 & \delta L \arrow[d] & L \arrow[d, "\pi"] & \\
\delta\tau^{-1}L \arrow[dr] & G^2 \arrow[u, bend right, "s"'] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G & \\
 & P \times G^2 \arrow[u, "\tau_2"] \arrow[ul, bend left, "s"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P \times G \arrow[u, "\tau"] \arrow[r, shift left] \arrow[r, shift right] & P
\end{tikzcd}$$

What we then have is $s^{-1}\delta(\tau^{-1}\mu) = \tau_2^{-1}\alpha$. Which is not identically zero as clearly α is not always zero. Therefore, $\tau^{-1}\mu$ is not a bundle gerbe connection. However, recall that α satisfies $\delta\alpha = 0$. Hence we have that

$$\delta(\tau_2^{-1}\alpha) = \tau_3^{-1}\delta\alpha = 0.$$

Then by the fact that the fundamental complex is exact there must exist some one-form $\epsilon \in \Omega^1(P \times G)$ such that $\delta\epsilon = \tau_2^{-1}\alpha$. Or if we allow some abuse of notation we have $\delta\epsilon = \alpha$.

So if we define a new one-form $\mu - \pi^{-1}\epsilon$ and repeat this process we have that

$$s^{-1}(\delta(\mu - \pi^{-1}\epsilon)) = 0$$

and therefore $\mu - \pi^{-1}\epsilon$ is a bundle gerbe connection.

So if we have a connection μ whose curvature is R to make μ into a bundle gerbe connection we require an ϵ such that $\delta(\epsilon) = \alpha$.

The formulae below are due to Roberts and Vozzo in [21].

Proposition 6.4.1. [21] *The 1-form $\nabla := \mu - \pi^*\epsilon$ where at $(p,\gamma) \in PG \times \Omega G$, we have*

$$\epsilon_{(p,\gamma)} := \frac{i}{2\pi}\int_{S^1} \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta$$

is a bundle gerbe connection for the basic bundle gerbe.

Proof. From the above discussion all that we need to check is that $\delta(\epsilon) = \alpha$. Let's calculate $\delta(\epsilon)_{(p,\gamma,\eta)}$ for $(p,\gamma,\eta) \in PG \times \Omega G^2$.

$$\begin{aligned}\delta(\epsilon)_{(p,\gamma,\eta)} &= \epsilon_{(p\gamma,\eta)} - \epsilon_{(p,\gamma\eta)} + \epsilon_{(p,\gamma)} \\ &= \frac{i}{2\pi}\int_{S^1} \langle \Theta_{p\gamma}, \partial\eta\eta^{-1}\rangle - \langle \Theta_p, \partial\gamma\eta(\gamma\eta)^{-1}\rangle + \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta \\ &= \frac{i}{2\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p + \Theta_\gamma, \partial\eta\eta^{-1}\rangle - \langle \Theta_p, \partial\gamma\eta(\gamma\eta)^{-1}\rangle + \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta\end{aligned}$$

We now make use of the following identities

$$\partial\gamma^{-1} = -\gamma^{-1}(\partial\gamma)\gamma^{-1}$$

and

$$\partial(\gamma\eta)(\gamma\eta)^{-1} = \partial\gamma\gamma^{-1} + \mathrm{ad}_\gamma\partial\eta\eta^{-1}.$$

So the above then becomes

$$\begin{aligned}&= \frac{i}{2\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p + \Theta_\gamma, \partial\eta\eta^{-1}\rangle - \langle \Theta_p, \partial\gamma\gamma^{-1} + \mathrm{ad}_\gamma\partial\eta\eta^{-1}\rangle + \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta \\ &= \frac{i}{2\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\eta\eta^{-1}\rangle + \langle \Theta_\gamma, \partial\eta\eta^{-1}\rangle - \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle - \langle \Theta_p, \mathrm{ad}_\gamma\partial\eta\eta^{-1}\rangle + \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta \\ &= \frac{i}{2\pi}\int_{S^1} \langle \Theta_p, \mathrm{ad}_\gamma\partial\eta\eta^{-1}\rangle + \langle \Theta_\gamma, \partial\eta\eta^{-1}\rangle - \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle - \langle \Theta_p, \mathrm{ad}_\gamma\partial\eta\eta^{-1}\rangle + \langle \Theta_p, \partial\gamma\gamma^{-1}\rangle d\theta \\ &= \frac{i}{2\pi}\int_{S^1} \langle \Theta_\gamma, \partial\eta\eta^{-1}\rangle d\theta \\ &= \alpha\end{aligned}$$

Therefore, $\mu - \pi^{-1}\epsilon$ is a bundle gerbe connection. □

6.5 A curving for the basic bundle gerbe

Now that we have a connection for the basic bundle gerbe the last part we need to calculate a three-curvature is a curving.

Proposition 6.5.1. [21] *A curving for the connection in* (6.4.1) *is given by*

$$B_p := \frac{i}{4\pi}\int_{S^1} \langle \Theta_p, \partial\Theta_p\rangle d\theta.$$

Proof. Recall that the curving is a differential form such that $\delta(B) = F$ and that μ was chosen such that its curvature is R. So what we need so check is

$$\delta(B)(p,\gamma) = B_{p\gamma} - B_p = R_\gamma - d\epsilon_{(p,\gamma)}.$$

Let's calculate $B_{p\gamma} - B_p$.

$$\begin{aligned}
B_{p\gamma} - B_p &= \frac{i}{4\pi}\int_{S^1} \langle \Theta_{p\gamma}, \partial\Theta_{p\gamma}\rangle - \langle \Theta_p, \partial\Theta_p\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p + \Theta_\gamma, \partial(\mathrm{ad}_{\gamma^{-1}}\Theta_p + \Theta_\gamma)\rangle - \langle \Theta_p, \partial\Theta_p\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\mathrm{ad}_{\gamma^{-1}}\Theta_p\rangle + \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\Theta_\gamma\rangle + \langle \Theta_\gamma, \partial\mathrm{ad}_{\gamma^{-1}}\Theta_p\rangle + \langle \Theta_\gamma, \partial\Theta_\gamma\rangle \\
&- \langle \Theta_p, \partial\Theta_p\rangle d\theta \\
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\mathrm{ad}_{\gamma^{-1}}\Theta_p\rangle + \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\Theta_\gamma\rangle + \langle \Theta_\gamma, \partial\mathrm{ad}_{\gamma^{-1}}\Theta_p\rangle - \langle \Theta_p, \partial\Theta_p\rangle d\theta
\end{aligned}$$

Making use of the following identity

$$\partial\mathrm{ad}_{\gamma^{-1}}\Theta_p = \mathrm{ad}_{\gamma^{-1}}(\partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}])$$

and the fact the the inner product is ad-invariant we have

$$\begin{aligned}
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} \langle \Theta_p, \partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}]\rangle + \langle \mathrm{ad}_{\gamma^{-1}}\Theta_p, \partial\Theta_\gamma\rangle + \langle \Theta_\gamma, \mathrm{ad}_{\gamma^{-1}}(\partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}])\rangle \\
&- \langle \Theta_p, \partial\Theta_p\rangle d\theta \\
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} \langle \Theta_p, \partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}]\rangle + \langle \Theta_p, \mathrm{ad}_\gamma\partial\Theta_\gamma\rangle + \langle \Theta_\gamma, \mathrm{ad}_{\gamma^{-1}}(\partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}])\rangle \\
&- \langle \Theta_p, \partial\Theta_p\rangle d\theta \\
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} \langle \Theta_p, [\Theta_p, \partial\gamma\gamma^{-1}]\rangle + \langle \Theta_p, \mathrm{ad}_\gamma\partial\Theta_\gamma\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}]\rangle d\theta \\
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} \langle [\Theta_p, \Theta_p], \partial\gamma\gamma^{-1}\rangle + \langle \Theta_p, \mathrm{ad}_\gamma\partial\Theta_\gamma\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p + [\Theta_p, \partial\gamma\gamma^{-1}]\rangle d\theta \\
&= R_\gamma + \frac{i}{4\pi}\int_{S^1} -2\langle d\Theta_p, \partial\gamma\gamma^{-1}\rangle + 2\langle \Theta_p, d\partial\gamma\gamma^{-1}\rangle + \langle -\mathrm{ad}_\gamma\Theta_\gamma, [\partial\gamma\gamma^{-1}, \Theta_p]\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)} + \frac{i}{4\pi}\int_{S^1} \langle -\Theta_p, [\mathrm{ad}_\gamma\Theta_\gamma, \partial\gamma\gamma^{-1}]\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)} + \frac{i}{4\pi}\int_{S^1} \langle -\Theta_p, \mathrm{ad}_\gamma\partial\Theta_\gamma - \partial\mathrm{ad}_\gamma\Theta_\gamma\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)} + \frac{i}{4\pi}\int_{S^1} \langle -\Theta_p, \mathrm{ad}_\gamma\partial\Theta_\gamma - [\mathrm{ad}_\gamma\Theta_\gamma, \partial\gamma\gamma^{-1}]\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)} + \frac{i}{4\pi}\int_{S^1} \langle \partial\mathrm{ad}_\gamma\Theta_\gamma, \Theta_p\rangle + \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)} + \left[\mathrm{ad}_\gamma\Theta_\gamma\Theta_p\right]_0^{2\pi} - \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta + \frac{i}{4\pi}\int_{S^1} \langle \mathrm{ad}_\gamma\Theta_\gamma, \partial\Theta_p\rangle d\theta \\
&= R_\gamma - d\epsilon_{(p,\gamma)}.
\end{aligned}$$

Therefore, B is a curving for the bundle gerbe connection given above. □

6.6 The string class

Now that we have the connection and curving of this bundle gerbe we can calculate its three-curvature and thus have a representative of its Dixmier-Douady class in real cohomology.

Recall that the three-curvature is a differential 3-form such that $\pi^*\omega = dB$. For this particular bundle gerbe $\pi = \mathrm{ev}_{2\pi}$.

Proposition 6.6.1. [21] *The three-curvature of B is given by the standard 3-form on G*

$$\omega = \frac{i}{24\pi}\langle[\Theta_G, \Theta_G], \Theta_G\rangle$$

Proof. What needs to be checked is that $\mathrm{ev}_{2\pi}^*(\omega) = dB$.

$$\begin{aligned}
dB_p &= \frac{i}{4\pi}\int_{S^1} d\langle\Theta_p, \partial\Theta_p\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} \langle d\Theta_p, \partial\Theta_p\rangle - \langle\Theta_p, \partial d\Theta_p\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} -\frac{1}{2}\langle[\Theta_p, \Theta_p], \partial\Theta_p\rangle + \frac{1}{2}\langle\Theta_p, \partial[\Theta_p, \Theta_p]\rangle d\theta \\
&= \frac{i}{4\pi}\int_{S^1} -\frac{1}{2}\langle[\Theta_p, \Theta_p], \partial\Theta_p\rangle + \frac{1}{2}(\langle\Theta_p, [\partial\Theta_p, \Theta_p]\rangle + \langle\Theta_p, [\Theta_p, \partial\Theta_p]\rangle)d\theta \\
&= \frac{i}{8\pi}\int_{S^1} \langle[\Theta_p, \Theta_p], \partial\Theta_p\rangle d\theta.
\end{aligned}$$

Since Θ is valued in $L\mathfrak{g}$ evaluating $\langle[\Theta_{p(2\pi)}, \Theta_{p(2\pi)}], \Theta_{p(2\pi)}\rangle$ is zero. Then by the fundamental theorem of calculus we have

$$\int_{S^1} \partial\langle[\Theta_p, \Theta_p], \Theta_p\rangle d\theta = \left[\langle[\Theta_p, \Theta_p], \Theta_p\rangle\right]_0^{2\pi} = \langle[\Theta_{p(2\pi)}, \Theta_{p(2\pi)}], \Theta_{p(2\pi)}\rangle.$$

Then expanding

$$\begin{aligned}
&\int_{S^1} \partial\langle[\Theta_p, \Theta_p], \Theta_p\rangle d\theta \\
&= \int_{S^1} \langle[\partial\Theta_p, \Theta_p], \Theta_p\rangle + \langle[\Theta_p, \partial\Theta_p], \Theta_p\rangle + \langle[\Theta_p, \Theta_p], \partial\Theta_p\rangle d\theta \\
&= 3\int_{S^1} \langle[\Theta_p, \Theta_p], \partial\Theta_p\rangle d\theta.
\end{aligned}$$

Therefore,

$$dB_p = \frac{i}{24\pi}\langle[\Theta_{p(2\pi)}, \Theta_{p(2\pi)}], \Theta_{p(2\pi)}\rangle$$

□

Hence, we have a curvature 3-form for our connective structure which when divided by $2\pi i$ gives a representative for the Dixmier-Douady class of the basic bundle gerbe.

Bibliography

[1] J.-L. Brylinski. Loop spaces, characteristic classes and geometric quantization, volume 107 of Progress in Mathematics. Birkhauser Boston Inc., Boston, MA, 1993.

[2] J.-L. Brylinski and D. A. McLaughlin. The geometry of degree-four characteristic classes and of line bundles on loop spaces. I. Duke Math. J., 75(3):603–638, 1994.

[3] R. Bott and L. W. Tu. Differential forms in algebraic topology, volume 82 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1982.

[4] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray, and D. Stevenson. Twisted K-theory and K-theory of bundle gerbes. Comm. Math. Phys., 228(1):17–45, 2002

[5] A. L. Carey, S. Johnson, M. K. Murray, D. Stevenson, and B.-L. Wang. Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories. Comm. Math. Phys., 259(3):577–613, 2005.

[6] A. L. Carey, J. Mickelsson, and M. K. Murray. Bundle gerbes applied to quantum field theory. Rev. Math. Phys., 12(1):65–90, 2000

[7] A. L. Carey and M. K. Murray. String structures and the path fibration of a group. Comm. Math. Phys., 141(3):441–452, 1991.

[8] Erdmann, Karin, and Mark J. Wildon. Introduction to Lie Algebras. Springer Undergraduate Mathematics Series. London: Springer, 2006.

[9] Isham, C. J. Modern Differential Geometry for Physicists. World Scientific Lecture Notes in Physics, v. 32. Singapore; Teaneck, NJ: World Scientific, 1989.

[10] S. Johnson. Constructions with Bundle Gerbes. PhD thesis, The University of Adelaide, 2002.

[11] S. Kobayashi and K. Nomizu. Foundations of differential geometry. Vol I. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1963.

[12] J. M. Lee. Introduction to Smooth Manifolds, volume 218 of Graduate Texts in Mathematics. Springer, New York, second edition, 2013.

[13] T. Leinster. Basic Category Theory. Cambridge Studies in Advanced Mathematics, Vol. 143, Cambridge University Press, 2014.

[14] M. K. Murray and D. Stevenson. Higgs fields, bundle gerbes and string structures.Comm. Math. Phys., 243(3):541–555, 2003

[15] M. K. Murray and D. Stevenson. Yet another construction of the central extension of the loop group. In Geometric analysis and applications (Canberra, 2000), volume 39 of Proc. Centre Math. Appl. Austral. Nat. Univ., pages 194–200. Austral. Nat. Univ.,Canberra, 2001.

[16] M. K. Murray, Bundle gerbes, J. London Math. Soc. (2) **54** (1996), no. 2, 403–416.

[17] M. K. Murray. An introduction to bundle gerbes. In The Many Facets of Geometry, pages 237–260. Oxford Univ. Press, Oxford, 2010

[18] M. K. Murray. Line Bundles. The University of Adelaide, 1996.

[19] M. K. Murray, D. M. Roberts, D. Stevenson, and R. F. Vozzo. Equivariant bundle gerbes. Adv. Theor. Math. Phys., 21(4):921–975, 2017.

[20] A. Pressley and G. Segal.Loop groups. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[21] D.M. Roberts, R.F. Vozzo. Rigid Models for 2-Gerbes (In preparation).

[22] D. Stevenson. The geometry of bundle gerbes. PhD thesis, The University of Adelaide, 2000.

[23] R. F. Vozzo. Loop Groups, Higgs Fields and Generalised String Classes. PhD thesis, The University of Adelaide, 2009.

[24] Woodhouse, N. M. J. Geometric Quantization. 2nd ed. Oxford Mathematical Monographs. Oxford : New York: Clarendon Press; Oxford University Press, 1992.