THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

TERM 1, 2019

MATH3161/MATH5165

Optimization

(1) TIME ALLOWED – Two hours

(2) TOTAL NUMBER OF QUESTIONS – 3

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) ALL STUDENTS MAY ATTEMPT ALL QUESTIONS. MARKS GAINED ON ANY QUESTION WILL BE COUNTED. GRADES OF PASS AND CREDIT CAN BE GAINED BY SATISFACTORY PERFORMANCE ON UNSTARRED QUESTIONS. GRADES OF DISTINCTION AND HIGH DISTINCTION WILL REQUIRE SATISFACTORY PERFORMANCE ON ALL QUESTIONS, INCLUDING STARRED QUESTIONS

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book marked Question 1

1. [**25 marks**] Consider minimizing the strictly convex quadratic function

$$q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T G\mathbf{x} + \mathbf{d}^T\mathbf{x} + c$$

where G is an $(n \times n)$ symmetric **positive definite** constant matrix, $\mathbf{d}$ is a constant $n \times 1$ vector and c is a scalar. Let $\mathbf{x}^{(1)}$ be the starting point, where $\nabla q(\mathbf{x}^{(1)}) \neq \mathbf{0}$, $\mathbf{x}^{(1)} \neq \mathbf{x}^*$ and $\mathbf{x}^*$ is the minimizer of $q(\mathbf{x})$.

i) Consider applying the **steepest descent** method with exact line searches to the quadratic function $q(\mathbf{x})$. Let $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ be the sequence generated by the steepest descent method with exact line searches.

   a) Write down the steepest descent direction $\mathbf{s}_D^{(k)}$ at $\mathbf{x}^{(k)}$.

   b) Confirm that $\mathbf{s}_D^{(k)}$ is a descent direction.

   c) Write down the line search condition that must be satisfied by the exact minimizer $\alpha^{(k)}$ of $\ell(\alpha) = q(\mathbf{x}^{(k)} + \alpha\mathbf{s}_D^{(k)})$.

   d) Hence or otherwise show that $(\mathbf{x}^{(k+2)} - \mathbf{x}^{(k+1)})^T(\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}) = 0$.

ii) It is given that the gradient of $q(\mathbf{x})$ is $\nabla q(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 + 4 \\ -x_1 + 5x_2 - 6 \end{bmatrix}$.

   a) Write down the Hessian matrix G of $q(\mathbf{x})$.

   b) Find the condition number of G.

   c) **Given** that $\|\mathbf{x}^{(1)} - \mathbf{x}^*\| \leq 1$, estimate the least number of iterations of the steepest descent method with exact line searches required to get $\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\| \leq 10^{-10}$.

   d) Consider applying **Newton's** method to the quadratic function $q(\mathbf{x})$ starting at $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

      ξ) Write down the Newton direction $\mathbf{s}_N^{(1)}$ at $\mathbf{x}^{(1)}$.

      η) Show that Newton's method terminates in one iteration.

iii) Consider applying a **conjugate gradient** method with exact line searches to $q(\mathbf{x})$ starting at $\mathbf{x}^{(1)}$.

   a) If the search directions $\mathbf{s}^{(1)}, \cdots, \mathbf{s}^{(n)}$ are non-zero and conjugate with respect to the $(n \times n)$ symmetric positive definite constant matrix G then show that the set $\{\mathbf{s}^{(1)}, \cdots, \mathbf{s}^{(n)}\}$ is linearly independent.

   b) Let $n = 5$. What is the **greatest** number of iterations necessary to minimize $q(\mathbf{x})$? [*Do not perform any iterations, but give reasons for your answer*].

Use a separate book marked Question 2

2. [**25 marks**]

i) Consider the following optimization problem

$$(P_1) \quad \min_{\mathbf{x}\in\mathbb{R}^3} \quad -x_1x_2x_3$$
$$\text{s.t.} \quad x_1 + 2x_2 + 2x_3 - 72 \le 0, \quad x_1 - 50 \le 0.$$

a) Formulate the penalty function problem for (P_1) with parameter μ.

b) Let $\mathbf{x}(\mu) = [x_1(\mu) \;\; x_2(\mu) \;\; x_3(\mu)]^T = [2a(\mu) \;\; a(\mu) \;\; a(\mu)]^T$,
where $a(\mu) = \dfrac{24}{1+\sqrt{1-4/\mu}}$ and $\mu > 4$.

ξ) **Given** that

$$x_1(\mu) + 2x_2(\mu) + 2x_3(\mu) - 72 = \frac{288}{\mu\big(1+\sqrt{1-4/\mu}\,\big)^2},$$

verify that $\mathbf{x}(\mu)$ is a stationary point of the penalty function problem of part a) for each $\mu > 4$.

η) Find the limit point $\mathbf{x}^*$ of the vectors $\mathbf{x}(\mu)$ as $\mu \to \infty$.

c) Verify that the limit point $\mathbf{x}^*$ is a feasible point for (P_1) and find the active constraint at $\mathbf{x}^*$.

d) Show that the limit point $\mathbf{x}^*$ is a constrained stationary point for (P_1).

e) Using second-order sufficient optimality conditions, show that $\mathbf{x}^*$ is a strict local minimizer for (P_1).

ii) Consider the following inequality constrained optimization problem

$$(P_2) \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \;\; \mathbf{x}^T\mathbf{x} - 2\mathbf{d}^T\mathbf{x}$$
$$\text{s.t.} \quad \mathbf{a}^T\mathbf{x} - b \le 0,$$

where $n > 1$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq 0$, $\|\mathbf{a}\|_2 = \sqrt{\mathbf{a}^T\mathbf{a}}$, $b \in \mathbb{R}$ and $\mathbf{d} \in \mathbb{R}^n$ with $\mathbf{a}^T\mathbf{d} - b > 0$.

a) Using the definition of a convex set, show that the feasible set

$$\Omega := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} - b \le 0\}$$

of (P_2) is a convex set.

b) Show that (P_2) is a convex optimization problem.

c) Write down the Karush-Kuhn-Tucker necessary optimality conditions at a minimizer $\mathbf{x}^*$ of (P_2).

d)* Hence or otherwise find the global minimizer of (P_2) and the minimum value of (P_2).

e)* Show that the Wolfe dual problem of (P_2) can be written in the form of the maximization problem

$$(\mathrm{D}_2) \qquad \max_{\lambda\in\mathbb{R},\; \lambda\ge 0} \; -\left(\frac{\|\mathbf{a}\|_2^2}{4}\right)\lambda^2 + (\mathbf{a}^T\mathbf{d} - b)\lambda - \mathbf{d}^T\mathbf{d}.$$

f)* Find the global maximizer of the Wolfe dual problem (D_2) and confirm that the minimum value of (P_2) = maximum value of (D_2).

Use a separate book marked Question 3

3. [**10 marks**] Consider the optimal control problem

$$\begin{aligned} \min_{u_1 \in \mathcal{U}_u} \quad & \int_0^{t_1} (x_1^2 + u_1^2)\, dt \\ \text{s.t.} \quad & \dot{x}_1 = -x_1 + u_1, \\ & x_1(0) = 1,\ x_1(t_1) = 2, \end{aligned}$$

where $\mathcal{U}_u$ is the unrestricted control set and $t_1 > 0$.

i) Write down the Hamiltonian function H for this problem. [You may assume the problem is normal.]

ii) Write down the differential equation for the co-state z_1.

iii) Assuming that t_1 is a fixed time and a solution exists, apply the Pontryagin Maximum Principle conditions to find the optimal control $u_1^*(t)$ and the optimal state $x_1^*(t)$.

END OF EXAMINATION