THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

TERM 1, 2021

MATH3161/MATH5165
Optimization

(1) TIME ALLOWED – THREE (3) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 3

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) START EACH QUESTION ON A NEW PAGE.

ALL STUDENTS MAY ATTEMPT ALL QUESTIONS. MARKS GAINED ON ANY QUESTION WILL BE COUNTED. GRADES OF PASS AND CREDIT CAN BE GAINED BY SATISFACTORY PERFORMANCE ON UNSTARRED QUESTIONS.
GRADES OF DISTINCTION AND HIGH DISTINCTION WILL REQUIRE SATISFACTORY PERFORMANCE ON ALL QUESTIONS, INCLUDING STARRED QUESTIONS

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

YOU ARE TO COMPLETE THE TEST UNDER STANDARD EXAM CONDITIONS, WITH HANDWRITTEN SOLUTIONS.

YOU WILL THEN SUBMIT ONE OR MORE FILES CONTAINING YOUR SOLUTIONS. **ONE PDF FILE IS PREFERRED**.

MAKE SURE YOU SUBMIT ALL YOUR ANSWERS.

ONE OF THE SUBMITTED FILES MUST INCLUDE A PHOTOGRAPH OF YOUR **STUDENT ID CARD** WITH THE **SIGNED**, HANDWRITTEN STATEMENT: "**I declare that this submission is entirely my own original work.**"

YOU CAN DELETE AND/OR RELOAD FILES UNTIL THE DEADLINE.

Start a new page clearly marked Question 1

1. [23 marks]

i) Consider the optimization problem with the convex objective function

$$f(\mathbf{x}) = 2x_1^2 - 4x_1x_2 + 3x_2^2 - 4x_2 + 8,$$

the convex constraint functions

$$\begin{aligned} c_1(\mathbf{x}) &= -x_1 + x_2^2 - 1, \\ c_2(\mathbf{x}) &= x_1 + x_2 - 1, \end{aligned}$$

and the feasible regions

$$\begin{aligned} \Omega &= \left\{ \mathbf{x} \in \mathbb{R}^2 \, : \, c_1(\mathbf{x}) \leq 0, \quad c_2(\mathbf{x}) \leq 0 \right\}, \\ \Omega_E &= \left\{ \mathbf{x} \in \mathbb{R}^2 \, : \, c_1(\mathbf{x}) = 0, \quad c_2(\mathbf{x}) \leq 0 \right\}. \end{aligned}$$

a) Consider minimizing $f(\mathbf{x})$ over the **convex region** Ω, and the point $\mathbf{x}^* = \left[\frac{1}{3}, \quad \frac{2}{3} \right]^T$.
   A) Show that $\mathbf{x}^*$ is a constrained stationary point.
   B) Explain why $\mathbf{x}^*$ is the global minimizer of f over Ω.

b) Consider optimizing $f(\mathbf{x})$ over the **non-convex region** Ω_E, and the point $\widehat{\mathbf{x}} = [-1, \quad 0]^T$.
   A) Show that $\widehat{\mathbf{x}}$ is a regular feasible point.
   B) Show that $\widehat{\mathbf{x}}$ is a constrained stationary point.
   C) Calculate the reduced Hessian of the Lagrangian function at $\widehat{\mathbf{x}}$, and determine, if possible, the nature (strict local minimizer, strict local maximizer, neither) of $\widehat{\mathbf{x}}$.

ii) Consider the following **convex optimization** problem

$$(\mathrm{P}_2) \qquad \begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & \mathbf{x}^T\mathbf{x} - 2\mathbf{d}^T\mathbf{x} + \mathbf{d}^T\mathbf{d} \\ \text{s.t.} \quad & \mathbf{a}^T\mathbf{x} - b \leq 0, \end{aligned}$$

where $n \geq 1$, $\mathbf{a}, \mathbf{d} \in \mathbb{R}^n$, $b \in \mathbb{R}$, $\mathbf{a}^T\mathbf{d} - b > 0$, $\mathbf{a} \neq \mathbf{0}$ and $\|\mathbf{a}\|_2 = \sqrt{\mathbf{a}^T\mathbf{a}}$.

a) Formulate the penalty function problem for (P_2) with parameter μ.

b) Verify that, for each $\mu > 0$, the penalty function problem of part a) has a stationary point $\mathbf{x}(\mu) = \mathbf{d} - \left(\dfrac{\mu(\mathbf{a}^T\mathbf{d} - b)}{1 + \mu\|\mathbf{a}\|_2^2} \right) \mathbf{a}$.

c) Explain why $\mathbf{x}(\mu)$, for $\mu > 0$, is a global minimizer of the penalty function problem of part a).

d) Find the limit point $\mathbf{x}^*$ of the vectors $\mathbf{x}(\mu)$ as $\mu \to \infty$.

e) Verify that the limit point $\mathbf{x}^*$ is a constrained stationary point for (P_2).

f) Show that the limit point $\mathbf{x}^*$ is a global minimizer for (P_2).

Start a new page clearly marked Question 2

2. [25 marks]

i) Consider minimizing a quadratic function f on $\mathbb{R}^2$ with the starting point $\mathbf{x}^{(1)} = \left[-\frac{1}{2}, -\frac{1}{2}\right]^T$ where

$$\nabla f(\mathbf{x}) = \left[\, 4x_1 - 3x_2 + 1, \quad -3x_1 + 6x_2 + 1 \,\right]^T.$$

a) Calculate the Newton direction $\mathbf{s}_N^{(1)}$ for $f(\mathbf{x})$ at $\mathbf{x}^{(1)}$.

b) Show that the next Newton iterate $\mathbf{x}^{(2)}$ is the unique global minimizer for $f(\mathbf{x})$.

ii) Consider minimizing the strictly convex quadratic function

$$q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T G \mathbf{x} + \mathbf{d}^T\mathbf{x} + c$$

where G is an $(n \times n)$ symmetric **positive definite** constant matrix, $\mathbf{d}$ is a constant $n \times 1$ vector and c is a scalar. Let $\mathbf{x}^{(1)}$ be the starting point, where $\mathbf{g}(\mathbf{x}^{(1)}) \neq \mathbf{0}$, $\mathbf{x}^{(1)} \neq \mathbf{x}^*$ and $\mathbf{x}^*$ is the minimizer of $q(\mathbf{x})$ and $\mathbf{g}(\mathbf{x}) = \nabla q(\mathbf{x})$.

a) Consider applying the steepest descent method with exact line searches to the quadratic function $q(\mathbf{x})$. Suppose that $\mathbf{x}^{(1)}$ can be expressed as $\mathbf{x}^{(1)} = \mathbf{x}^* + \mu\mathbf{v}$, where $\mu \in \mathbb{R}$, $\mu \neq 0$ and $\mathbf{v}$ is an eigenvector of G with eigenvalue λ.

A) Show that the initial steepest descent direction is $\mathbf{s}_D^{(1)} = -\mu\lambda\mathbf{v}$.

B) **Given that** $\mathbf{g}(\mathbf{x}^{(1)} + \alpha\mathbf{s}_D^{(1)})^T\mathbf{s}_D^{(1)} = -\lambda^2(1 - \alpha\lambda)\|\mathbf{v}\|^2$, find the step length $\alpha^{(1)}$.

C) Show that the method terminates in one iteration.

D)* Hence or otherwise show that if $G = \gamma I$ then the method terminates in one iteration for **any** $\mathbf{x}^{(1)}$, where I is the $(n \times n)$ identity matrix and $\gamma > 0$ is a scalar.

b) Consider applying the conjugate gradient method with exact line searches to $q(\mathbf{x})$.

A) Show that the search direction $\mathbf{s}^{(k)}$ of the conjugate gradient method at $\mathbf{x}^{(k)}$ is a descent direction, where $\mathbf{g}(\mathbf{x}^{(k)}) \neq \mathbf{0}$ and $k \geq 2$.

B) What is the greatest number of iterations necessary to minimize $q(\mathbf{x})$? [*Do not perform any iterations, but give reasons for your answer.*]

iii) Consider the following quadratic optimization problem

$$(\mathrm{P}_3) \qquad \min_{\mathbf{x}\in\mathbb{R}^n} \ \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} \qquad \text{s.t.} \ \ \mathbf{x}^T\mathbf{x} - r \leq 0,$$

where A is a symmetric $(n \times n)$ constant matrix, $\mathbf{b}$ is a constant $n \times 1$ vector and $r > 0$ is a scalar. You are **given that** $\mathbf{x}^* \in \mathbb{R}^n$, $\lambda^* \in \mathbb{R}$ and the following conditions hold:

$$(A + \lambda^* I)\mathbf{x}^* + \mathbf{b} = 0, \quad {\mathbf{x}^*}^T\mathbf{x}^* - r \leq 0, \quad \lambda^*({\mathbf{x}^*}^T\mathbf{x}^* - r) = 0, \quad (A + \lambda^* I) \succeq 0, \quad \lambda^* \geq 0,$$

where I is the $(n \times n)$ identity matrix and $(A + \lambda^* I) \succeq 0$ means that the matrix $(A + \lambda^* I)$ is a positive semi-definite matrix. Note that the matrix A is **not** assumed to be positive semi-definite.

a) Write down the Lagrangian function $L(\mathbf{x}, \lambda^*)$ of (P_3) with the given λ^*.

b) Write down the gradient $\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda^*)$ and Hessian $\nabla^2_{\mathbf{x}} L(\mathbf{x}, \lambda^*)$ of $L(\mathbf{x}, \lambda^*)$.

c) Show that the Lagrangian function $L(\mathbf{x}, \lambda^*)$ is a convex function of $\mathbf{x}$ on $\mathbb{R}^n$.

d) Show that $\mathbf{x}^*$ is a global minimizer of the Lagrangian function $L(\mathbf{x}, \lambda^*)$ on $\mathbb{R}^n$.

e)* Hence or otherwise show that $\mathbf{x}^*$ is a global minimizer of (P_3).

Start a new page clearly marked Question 3

3. [**12 marks**]Consider the optimal control problem

$$\begin{aligned} \min \quad & \int_0^T (2x_1^2 + 2u_1x_1 + u_1^2)\, dt \\ \text{s.t} \quad & \dot{x}_1 = x_1 + u_1, \\ & x_1(0) = 2,\ x_1(T) = 1, \end{aligned}$$

where $u_1 \in \mathcal{U}_u$, the unrestricted control set, and $T > 0$ is free.

i) Write down the Hamiltonian function H for this problem. [You may assume the problem is normal and set the co-state variable $z_0 = -1$.]

ii) Assume that a solution exists for the optimal control problem. Applying the Pontryagin Maximum Principle conditions, find the optimal control $u_1^*(t)$, the optimal state $x_1^*(t)$ and the optimal time T^*.

END OF EXAMINATION