Final Exam, 2021 – Solutions

Q1) a) Let $f(x) = 2x_1^2 - 4x_1x_2 + 3x_2^2 - 4x_2 + 8$

A) $\Omega = \{x \in \mathbb{R}^2 \mid c_1(x) \leq 0,\ c_2(x) \leq 0\}$.

$x^* = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. Then $\nabla f(x) = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 6x_2 - 4 \end{bmatrix}$ &

$\nabla c_1(x) = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix}$ & $\nabla c_2(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Substituting x^* into the constraints,

$c_1(x^*) = -1/3 + (2/3)^2 - 1 = -8/9 < 0$ &

$c_2(x^*) = 1/3 + 2/3 - 1 = 0 \leq 0$. So, x^* is a feasible point & $A(x^*) = \{2\}$, i.e. $c_2(x) \leq 0$ is active at x^*.

Solving $\nabla L(x^*, \lambda_2^*) = 0$ gives

$$\left.\begin{aligned} 4x_1^* - 4x_2^* + \lambda_2^* \cdot 1 &= 0 \\ -4x_1^* + 6x_2^* - 4 + \lambda_2^* \cdot 1 &= 0 \end{aligned}\right\} \Rightarrow \lambda_2^* = 4/3.$$

So, x^* is a constrained stationary point.

B) It is given that $f(x)$ is a convex function, $c_1(x)$ & $c_2(x)$ are also convex functions. So $\min\{f(x) \mid c_1(x) \leq 0, c_2(x) \leq 0\}$ is a convex optimization problem. Also $\lambda_2^* = 4/3 \geq 0$, for $i \in I \cap A(x^*) = \{2\}$. Hence x^* is a global minimizer of $f(x)$ over Ω by the KKT sufficiency Theorem.

b) Let $\Omega_E = \{x \in \mathbb{R}^2 \mid c_1(x) = 0,\ c_2(x) \le 0\}$.

& $\hat{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

A) Substituting $\hat{x}$ into the constraints

$c_1(\hat{x}) = -(-1) + 0 - 1 = 0$ & $c_2(\hat{x}) = -1 + 0 - 1 < 0$.

So, $\hat{x}$ is feasible & $\mathcal{A}(\hat{x}) = \{1\}$

$\{\nabla c_1(\hat{x})\} = \left\{ \begin{bmatrix} -1 \\ 2\hat{x}_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$ is linearly independent because $\begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence it is a regular point.

B) Solving $\nabla L(\hat{x}, \hat{\lambda}_1) = 0$,

$$\begin{bmatrix} 4\hat{x}_1 - 4\hat{x}_2 + \hat{\lambda}_1(-1) \\ -4\hat{x}_1 + 6\hat{x}_2 - 4 + \hat{\lambda}_1(2\hat{x}_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 - \hat{\lambda}_1 \\ 0 + \hat{\lambda}\, 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \hat{\lambda}_1 = -4.$$

So, $\hat{x}$ is a regular constrained stationary point.

c) The Hessian of the Lagrangian at $\hat{x}$ is

$$\nabla^2 L(\hat{x},\hat{\lambda}) = \begin{bmatrix} 4+\hat{\lambda}_1 0 & -4+\hat{\lambda}_1 0 \\ -4+\hat{\lambda}_1 0 & 6+\hat{\lambda}_1 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & -2 \end{bmatrix}, \text{ when } \hat{\lambda}_1 = -4.$$

Let $\hat{Z}$ be a 2×1 matrix such that $\hat{Z}^T A(\hat{x}) = 0$. Then, choose $\hat{z}$ as $\hat{Z} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so that

$$\hat{Z}^T A(\hat{x}) = [0 \;\; 1] \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0 \text{ \&}$$

rank $\hat{Z} = 1$.

Now, the reduced Hessian is

$$W_{\hat{Z}} = \hat{Z}^T \nabla^2 L(\hat{x},\hat{\lambda}) \hat{Z}$$

$$= [0 \;\; 1] \begin{bmatrix} 4 & -4 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2 < 0$$

So, $W_{\hat{Z}}$ is negative definite & $I \cap \mathcal{A}(\hat{x}) = \emptyset$.

Hence by the 2^{nd} order sufficient optimality conditions, $\hat{x}$ is a strict local maximizer of f over Ω_E.

ii) (P_2) $\min \ \underline{x}^T\underline{x} - 2\underline{d}^T\underline{x} + \underline{d}^T\underline{d}$

s.t. $\underline{a}^T\underline{x} - b \le 0$

a) The penalty function problem is

$$\min_{\underline{x} \in \mathbb{R}^n} x^T x - 2d^T x + d^T d + \mu\left[\max\{0, a^T x - b\}\right]^2, \quad \mu > 0$$

b) The gradient of the penalty function f_μ is given by

$$\nabla f_\mu(\underline{x}) = 2\underline{x} - 2\underline{d} + 2\mu \max\{0, a^T x - b\}\underline{a}.$$

Substituting x_μ into $\nabla f_\mu(x)$ gives

$$2x(\mu) - 2\underline{d} + 2\mu\max\{0\ a^T x(\mu) - b\},$$

where $a^T x(\mu) - b$

$$= a^T\left(\underline{d} - \frac{\mu(a^T d - b)}{1 + \mu\|a\|^2}\right)\underline{a} - b$$

$$\cancel{= (1 + \mu\|a\|^2) d - \mu}$$

$$= a^T d - \mu\frac{(a^T d - b)\|a\|^2}{1 + \mu\|a\|^2} - b$$

$$= \frac{(a^T d - b)(1 + \mu\|a\|^2) - \mu(a^T d - b)\|a\|^2}{1 + \mu\|a\|^2}$$

$$= \frac{(a^T d - b)}{1 + \mu\|a\|^2} > 0 \quad \text{as } \mu > 0 \ \& \ a^T d - b > 0.$$

So, $\max\{a^T x(\mu) - b, 0\} = \dfrac{a^T d - b}{1 + \mu \|a\|_2^2}$

Hence,

$2x(\mu) - 2d + 2\mu \max\{0, a^T x(\mu) - b\} \underset{\sim}{a}$

$= 2\left(d - \dfrac{\mu(a^T d - b)}{1+\mu\|a\|^2}\underset{\sim}{a}\right) - 2d + 2\mu\left(\dfrac{a^T d - b}{1+\mu\|a\|^2}\right)\underset{\sim}{a}$

$= 0$.

c) ~~It is~~ The point $x(\mu)$, $\mu > 0$ is a global minimizer for $f_\mu(x)$ because $f_\mu(x)$ is a convex function & $\nabla f_\mu(x(\mu)) = 0$.

Note that

$f_\mu(x) = f(x) + \mu h(x)$, where

$f(x) = x^T x - 2d^T x + d^T d$ is a (given) convex function (as $\nabla^2 f(x) = 2I$ which is positive definite constant matrix.).

The function $h(x)$ is a convex function as $\max\{0, \underset{\sim}{a}^T x - b\}$ is a convex function & $\mu \max\{0, a^T x - b\}^2$ is also a convex function

d) $\lim_{\mu \to \infty} x(\mu) = \lim_{\mu \to \infty} \underline{d} - \left(\frac{\mu (a^T d - b)}{1 + \mu \|\underline{a}\|^2} \right) \underline{a}$

$$= \underline{d} - \lim_{\mu \to \infty} \frac{a^T d - b}{1/\mu + \|a\|^2} \underline{a}$$

$$= \underline{d} - (a^T d - b) \frac{\underline{a}}{\|a\|^2} = x^*.$$

e) Substituting x^* into the constraint $a^T x - b$, we see that

$$\underline{a}^T x^* - b = a^T \left(\underline{d} - (a^T d - b) \frac{\underline{a}}{\|a\|^2} \right) - b$$
$$= 0.$$

So, x^* is feasible & $A(x^*) = \{1\}$. (i.e. the constraint is active.

Solving $\nabla f(x^*) + \lambda_1^* \nabla c_1(x^*) = 0.$

$$2x^* - 2\underline{d} + \lambda_1^* \underline{a} = 0$$

$$2 \left(- (a^T d - b) \frac{\underline{a}}{\|a\|^2} \right) + \lambda_1^* \underline{a} = 0$$

(as $a \neq 0$)

$$\lambda_1^* = \frac{2 (a^T d - b)}{\|a\|^2} > 0.$$

So, it is a constrained stationary point.

f) It is a global minimizer because $\lambda_i^* \geq 0$, for $i \in I \cap A(x^*) = \{1\}$ & it is a convex optimization problem

Q2) i) $\nabla f(x) = [4x_1 - 3x_2 + 1 \quad -3x_1 + 6x_2 + 1]^T$, $\underline{x}^{(1)} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}$.

Then,

$$G(x) = \nabla^2 f(\underline{x}) = \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix}.$$

ii) The Newton direction is given by $s_N^{(1)}$ where

$$G(x^{(1)}) s_N^{(1)} = -g(x^{(1)})$$

$$\begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} s_{N_1}^{(1)} \\ s_{N_2}^{(1)} \end{bmatrix} = -\begin{bmatrix} 4(-1/2) - 3(-1/2) + 1 \\ -3(-1/2) + 6(-1/2) + 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -2 + 3/2 + 1 \\ 3/2 - 3 + 1 \end{bmatrix}$$

$$= -\begin{bmatrix} +1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

$$\begin{bmatrix} s_N^{(1)} \\ s_{N_2}^{(1)} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -3/30 \\ 1/30 \end{bmatrix} = \begin{bmatrix} -1/10 \\ 1/30 \end{bmatrix}$$

Next iterate is

$$x^{(2)} = x^{(1)} + s_N^{(1)} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix} + \begin{bmatrix} -1/10 \\ 1/30 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -7/15 \end{bmatrix}$$

Substituting $x^{(2)}$ into $g(x) = \nabla f(x)$,

$$\nabla f(x^{(2)}) = \begin{bmatrix} 4\left(-\frac{3}{5}\right) - 3\left(-\frac{7}{15}\right) + 1 \\ -3\left(-\frac{3}{5}\right) + 6\left(-\frac{7}{15}\right) + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$G = \nabla^2 f(x) = \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix}$ is positive definite as $\Delta_1 = 4 > 0$, $\Delta_2 = 15 > 0$ for all x. So, f is a strictly convex function. Hence the stationary point is a global minimizer.

11) A) $s_D^{(1)} = -g^{(1)} = -(Gx^{(1)} + d)$

$= -G(x^* + \mu v) - d$

$= -Gx^* - \mu Gv - d$

$= -(Gx^* + d) - \lambda \mu v$

$= 0 - \lambda \mu v$ $\quad$ ($\because \nabla q(x^*) = 0$, x^* is a minimizer)

$= -\lambda \mu \underline{v}$.

B) $\alpha^{(1)}$ satisfies the line search condition $g(x^{(1)} + \alpha^{(1)} s^{(1)})^T s^{(1)} = 0$

$l'(\alpha) = -\lambda^2 (1 - \alpha^{(1)} \lambda) \|v\|^2 = 0$

Steplength $= \alpha^{(1)} = 1/\lambda$ $\quad$ ($\because \|v\|^2 > 0$ & $\lambda > 0$.

Moreover, $l''(\alpha) = +\lambda^3 \|v\|^2 > 0$,

where $l(\alpha) = f(x^{(1)} + \alpha s^{(1)})$.

$\alpha^{(1)}$ is a minimizer of $l(\alpha)$,

c) The new point is

$\underline{x}^{(2)} = \underline{x}^{(1)} + \alpha^{(1)} \underline{s}_D$

$= x^* + \mu v + \frac{1}{\lambda}(-\lambda \mu \underline{v})$

$= x^*$

So the method terminates in one iteration.

Method 1.

$x^{(1)} \neq x^*$

D) Let $x^{(1)}$ be an arbitrary starting point.

Then $x^{(1)}$ can be written as

$$x^{(1)} = x^* + \mu\left(\frac{x^{(1)} - x^*}{\mu}\right),$$

where $\underline{v} = \frac{x^{(1)} - x^*}{\mu} \neq 0$. So, $\underline{v}$ is the eigenvector of $G = \sigma \underline{I}$ as

$G\underline{v} = \sigma \underline{I}\, \underline{v} = \sigma \underline{v}$. By part c) method terminates as $G = \sigma \underline{I}$ is positive definite.

Method 2: Let $G = \sigma I$. Then eigenvalues of G are $\lambda_i = \sigma$, $i = 1, 2 \ldots n$. So, $\lambda_{max} = \lambda_{min} = \sigma$.

So the condition number of G is $\kappa(G) = 1$.

steepest descent generate sequence $\{x^k\}$

s.t $\|x^{k+1} - x^*\| \leq \frac{\kappa(G) - 1}{\kappa(G) + 1} \|x^{(k)} - x^*\|$

In particular, $\|x^{(2)} - x^*\| \leq \left(\frac{1-1}{1+1}\right) \|x^{(1)} - x^*\| = 0$

Thus, $x^{(2)} = x^*$. Hence the method terminates in one iteration.

b) A) The search direction $s^{(k)}$, $k \geq 2$, of the conjugate gradient method with exact line searches is given by

$$\underline{s}^{(k)} = -\underline{g}^{(k)} + \beta^{(k)} \underline{s}^{(k-1)},$$

where $\beta^{(k)}$ is a scalar.

$$\underline{g}^{(k)T} \underline{s}^{(k)} = -\underline{g}^{(k)T} \underline{g}^{(k)} + \beta^{(k)} \underline{g}^{(k)T} \underline{s}^{(k-1)}$$

Exact line search condition at $\underline{x}^{(k-1)}$ is

$$\underline{g}(\underline{x}^{(k-1)} + \alpha^{(k-1)} \underline{s}^{(k-1)})^T \underline{s}^{(k-1)} = 0,$$

where $\underline{x}^{(k)} = \underline{x}^{(k-1)} + \alpha^{(k-1)} \underline{s}^{(k-1)}$.

So, $\underline{g}(\underline{x}^{(k)})^T \underline{s}^{(k-1)} = 0$.

Hence, $\underline{g}^{(k)T} \underline{s}^{(k)} = -\|\underline{g}^{(k)}\|^2 < 0$ & $\underline{s}^{(k)}$ is a descent direction

B) The greatest number of iterations is n which is the number of variables of $q(\underline{x})$.

The reason is that the maximum number of linear independent directions in $\mathbb{R}^n$ is n & conjugate gradient method of $q(x)$ generates linearly independent search directions.

Q2) iii) $\min_{x \in \mathbb{R}^n} \underline{x}^T A \underline{x} + 2b^T \underline{x}$

s.t. $\underline{x}^T \underline{x} - r \leq 0$.

It is given that

$$\boxed{\begin{array}{l}(A + \lambda^* I)\underline{x}^* + \underline{b} = 0, \ \underline{x}^{*T} \underline{x}^* - r \leq 0 \\ \lambda^* (\underline{x}^{*T} \underline{x}^* - r) = 0, \ (A + \lambda^* I) \succeq 0, \ \lambda^* \geq 0.\end{array}}$$

a) $L(x, \lambda^*) = \underline{x}^T A \underline{x} + 2b^T x + \lambda^* (\underline{x}^T \underline{x} - r)$

b) $\nabla L(x, \lambda^*) = 2A\underline{x} + 2\underline{b} + 2\lambda^* \underline{x}$

$= 2(A + \lambda^* I)\underline{x} + 2\underline{b}$

& $\nabla^2 L(x, \lambda^*) = 2(A + \lambda^* I)$

c) As $\nabla^2 L(x, \lambda^*) = 2(A + \lambda^* I) \succeq 0$ for all $\underline{x} \in \mathbb{R}^n$, $L(x, \lambda^*)$ is a convex function of $\underline{x}$ on $\mathbb{R}^n$.

d) By the first order characterization of convexity of $L(x, \lambda^*)$, we get

$L(x, \lambda^*) - L(x^*, \lambda^*) \geq \nabla L(x^*, \lambda^*)^T (x - x^*)$

Since, $\nabla L(x^*, \lambda^*) = 2[(A + \lambda^* I)x^* + b] = 0$,

$L(x, \lambda^*) \geq L(x^*, \lambda^*)$ for all $x \in \mathbb{R}^n$.

So, x^* is a global minimizer of $L(x, \lambda^*)$.

d) By d),

$$L(x, \lambda^*) \geq L(x^*, \lambda^*)$$

$$= x^{*T}Ax^* + 2b^Tx^* + \underbrace{\lambda^*(x^{*T}x^* - r)}_{0}$$

$$= x^{*T}Ax^* + 2b^Tx^*$$

Now, for each feasible x of the problem,

$x^Tx - r \leq 0$ and so $\lambda^*(x^Tx - r) \leq 0$,

as $\lambda^* \geq 0$.

Hence, for each feasible x,

$$\underline{x}^TA\underline{x} + 2b^T\underline{x} \geq \underline{x}^TA\underline{x} + \lambda^*(\underline{x}^T\underline{x} - r)$$

$$= L(x, \lambda^*)$$

$$\geq x^{*T}Ax^* + 2b^Tx^*.$$

~~Hence~~ So, x^* is a global minimizer.

Q3) $\min \int_0^T (2x_1^2 + 2u_1x_1 + u_1^2)\,dt$

s.t. $\dot{x}_1 = x_1 + u_1$

$x_1(0) = 2, \quad x_1(T) = 1,$

where $u_1 \in \mathcal{U}_u$ and $T > 0$ is free.

i) The Hamiltonian is

$$H = -(2x_1^2 + 2u_1x_1 + u_1^2) + z_1(x_1 + u_1)$$

The co-state equation is

$$\dot{z}_1 = -\frac{\partial H}{\partial x_1} = -(-4x_1 - 2u_1 + z_1)$$

$$= 4x_1 + 2u_1 - z_1 \qquad -(1)$$

Maximizing H as a function of u_1

$$\max_{u_1} -2x_1^{*2} - 2u_1x_1^* - u_1^2 + z_1^*x_1^* + z_1^*u_1$$

$$-2x_1^2 + z_1x_1 + \max_{u_1} -u_1^2 - 2u_1x_1^* + z_1^*u_1$$

$$-2x_1^2 + z_1x_1 + \max_{u_1} -\left[u_1 + \left(x_1^* - \frac{z_1^*}{2}\right)\right]^2 + \left(x_1^* - \frac{z_1^*}{2}\right)^2$$

$$-2x_1^{*2} + z_1^*x_1^* + \left(x_1^* - \frac{z_1^*}{2}\right)^2 + \max_{u_1} -\left[u_1 + \left(x_1 - \frac{z_1}{2}\right)\right]^2$$

$u_1^* = -x_1^* + \frac{z_1^*}{2}$ maximizes H.

Now, we solve the state equation

$$\dot{x}_1^* = x_1^* + u_1^* = x_1^* + \left(-x_1^* + z_1^*/2\right)$$

$$\dot{x}_1^* = z_1^*/2 \qquad \text{— (2)}$$

Solving (1) & (2),

$$\ddot{x}_1^* = \dot{z}_1^*/2 = \tfrac{1}{2}\left(4x_1^* + 2u_1^* - z_1^*\right)$$

$$= 2x_1^* + u_1^* - z_1^*/2$$

$$= 2x_1^* + \left(-x_1^* + z_1^*/2\right) - z_1^*/2$$

$$= x_1^*$$

$$\ddot{x}_1^* - x_1^* = 0$$

So, $x_1^*(t) = Ae^{t} + Be^{-t}$ is the solution, where A & B are arbitrary constants.

$$z_1^*(t) = 2\dot{x}_1^* = 2Ae^{t} - 2Be^{-t}. \qquad \text{— (3)}$$

Applying the initial & terminal conditions,

$$x_1^*(0) = 2 \Rightarrow A + B = 2 \qquad \text{— (4)}$$

$$x_1^*(T) = 1 \Rightarrow Ae^{T} + Be^{-T} = 1. \qquad \text{— (5)}$$

As T is free, $H=0$ along the optimal trajectory.

Applying this condition at $t=T$,

$$-2x_1^*(T)^2 - 2u_1^*(T)\,x_1(T) - u_1^*(T)^2 + z_1(T)\,x_1(T) + z_1^*(T)\,u_1^*(T) = 0.$$

Substituting for $u_1^*(T)$ as

$$u_1^*(T) = -x_1^*(T) + \frac{z_1^*(T)}{2}$$

$$= -1 + \frac{z_1^*(T)}{2},$$

~~i.e.~~ we get

$$-2\cdot 1 - 2\left(-1+\frac{z_1^*(T)}{2}\right)1 - \left(-1+\frac{z_1^*(T)}{2}\right)^2 + z_1^*(T)\,1 + z_1^*(T)\left(-1+\frac{z_1^*(T)}{2}\right) = 0$$

$$-2 + 2 - z_1^*(T) - 1 + z_1^*(T) - \frac{z_1^*(T)^2}{4} + z_1^*(T) - z_1^*(T) + \frac{z_1^*(T)^2}{2} = 0$$

$$+\frac{z_1^*(T)^2}{4} - 1 = 0$$

$$z_1^*(T) = \pm 2.$$

Case 1 $z_1^*(T) = 2$.

$$(3) \Rightarrow 2Ae^{T} - 2Be^{-T} = 2$$

$$Ae^{T} - Be^{-T} = 1$$

(5) $\Rightarrow Ae^{T} + Be^{-T} = 1$

$$2Be^{-T} = 1 \Rightarrow B = 0$$

(4) $\Rightarrow$ $A = 2$.

$$2e^{T} = 1 \Rightarrow e^{T} = 1/2 < 1$$

which is not possible as $T > 0$.

<u>case 2</u> : $z_1^*(T) = -2$.

(3) $\Rightarrow$ $Ae^{T} - Be^{-T} = -1$
(5) $\Rightarrow$ $Ae^{T} + Be^{-T} = 1$
$\Rightarrow 2Ae^{T} = 0$
$\boxed{A = 0}$.

(4) $\Rightarrow$ ~~Be~~ $\boxed{B = 2.}$

$$2e^{-T} = 1 \Rightarrow e^{-T} = 1/2$$

$$-T = \ln 1/2 = -\ln 2$$

$$T = \ln 2 > 0$$

Hence, $x_1^*(t) = 2e^{-t}$

$$u_1^*(t) = -x_1^* + z_1^*/2$$
$$= -2e^{-t} + 1/2(-2 \cdot 2e^{-t})$$
$$= -4e^{-t}$$

$$\& T^* = \ln 2.$$

Alternative approach

Applying $H|_{t=0} = 0$.

$$-2x_1(0)^2 - 2u_1(0)x_1(0) - u_1(0)^2 + z_1(0)x_1(0) + z_1(0)u_1(0) = 0$$

$$-4 + \frac{z_1(0)^2}{4} = 0$$

$$z_1(0) = \pm 4$$

Case 1

$z_1(0) = 4$: $\left.\begin{array}{l} 4 = 2A - 2B \\ 4 = 2A + 2B \end{array}\right\} \Rightarrow \begin{array}{l} A = 2 \\ B = 0 \end{array}$

$$2e^T = 1 \Rightarrow e^T = 1/2 < 1 \quad \text{as } T > 0$$

It is not possible.

Case 2: $z_1(0) = -4$

$\left.\begin{array}{l} -4 = 2A - 2B \\ 4 = 2A + 2B \end{array}\right\} \Rightarrow \begin{array}{l} A = 0 \\ B = 2. \end{array}$

So (5) $\Rightarrow$ $0e^T + 2e^{-T} = 1$

$$e^{-T} = 1/2$$

$$-T = \ln 1/2 = -\ln 2$$

$$T = \ln 2.$$

Hence $x_1^*(t) = 2e^{-t}$, $u_1^*(t) = -4e^{-t}$ &

$$T^* = \ln 2.$$