THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

MATH3161/MATH5165 — OPTIMIZATION – Term 1, 2021

Problem Sheet 9 Solutions – Optimal Control Problems

1. Consider the growth equation
$$\frac{dx}{dt} = tu(t)$$
with $x(0) = 0$, $x(1) = 1$ and with the cost function
$$J = \int_0^1 u(t)^2 \, dt$$

   (a) Show that $u^*(t) = 3t$ is a successful (feasible) control, with $x^* = t^3$ and $J^* = 3$ the corresponding trajectory and cost.

   (b) If $u(t)$ is any successful control, show that $\int_0^1 tu(t)dt = 1$.

   (c) Hence, show that u^* is the optimal control problem, i.e. show that u^* is the successful control that minimizes the cost.
$$\text{Hint: } \int_0^1 u(t)^2 \, dt = \int_0^1 \left[(u(t) - 3t)^2 + 6tu(t) - 9t^2\right] \, dt$$

Answer

(a) Let $u^*(t) = 3t, x^*(t) = t^3$ and $J^* = 3$. We now verify that u^* is a feasible control.
To see this, note that
$$\begin{aligned} &\dot{x}^*(t) = 3t^2 = t(3t) = tu^*(t) \\ &x^*(0) = 0^3 = 0, \quad x^*(1) = 1^3 = 1 \\ \text{and } &\int_0^1 (u^*(t))^2 \, dt = \int_0^1 9t^2 \, dt = [3t^3]\Big|_{t=0}^{t=1} = 3 = J^* \end{aligned}$$
We see that u^* is a feasible optimal control with corresponding trajectory x^* and cost J^*.

(b) For any feasible control $u(t)$, we have $\dot{x}(t) = tu(t)$ and $x(0) = 0$. This together with the Fundamental Theorem of Calculus gives us that $x(t) = \int_0^t su(s) \, ds$.
The initial condition $x(1) = 1$ gives
$$1 = x(1) = \int_0^1 su(s) \, ds$$
That is, $\int_0^1 tu(t) \, dt = 1$.

(c) Note that, for any feasible control $u(t)$,

$$
\begin{aligned}
J = \int_0^1 u(t)^2\,dt &= \int_0^1 [(u(t)-3t)^2 + 6tu(t) - 9t^2]\,dt \\
&= \int_0^1 (u(t)-3t)^2\,dt + 6\int_0^1 tu(t)\,dt - \int_0^1 9t^2\,dt \\
&= \int_0^1 (u(t)-3t)^2\,dt + 6 - [3t^3]\Big|_{t=0}^{t=1} \\
&= \int_0^1 (u(t)-3t)^2\,dt + 3 \\
&\geq 3 = \int_0^1 u^*(t)^2\,dt
\end{aligned}
$$

Hence, by definition, $u^*(t) = 3t$ is the optimal control.

□

2. Determine the optimal solution (if it exists) for the following problem

$$
\begin{aligned}
&\text{minimize} && \int_0^{t_1} \frac{1}{2}(k^2 + u_1^2)dt \\
&\text{subject to} && \dot{x}_1 = x_2, \dot{x}_2 = u_1, \\
&&& x_1(0) = X, x_2(0) = 0, \\
&&& x_1(t_1) = 0, x_2(t_1) = 0
\end{aligned}
$$

where $X > 0, k > 0, t_1$ is free and $u_1 \in \mathcal{U}_u$. Find the maximum size of the control used in the optimal solution.

Answer This is an autonomous problem with fixed target and free final time.

i) The Hamiltonian is

$$H(x_1, x_2, z_0, z_1, z_2, u_1) = z_0\left(\frac{1}{2}(k^2 + u_1^2)\right) + z_1x_2 + z_2u_1$$

Assuming the problem is normal, we obtain by PMP that $z_0 = -1$. So,

$$H(x_1, x_2, z_0, z_1, z_2, u_1) = -\frac{1}{2}(k^2 + u_1^2) + z_1x_2 + z_2u_1.$$

ii) The costate equation is $\dot{\hat{\mathbf{z}}} = -\frac{\partial H}{\partial \hat{\mathbf{x}}}$; that is,

$$\dot{z}_0 = -\frac{\partial H}{\partial x_0} = 0 \tag{1}$$

$$\dot{z}_1 = -\frac{\partial H}{\partial x_1} = 0 \tag{2}$$

$$\dot{z}_2 = -\frac{\partial H}{\partial x_2} = -z_1 \tag{3}$$

Now, (1) is satisfied with $z_0 = -1$; (2) gives $z_1 = \alpha$ (α is a constant); (3) shows $\dot{z}_2 = -\alpha$ and so $z_2 = -\alpha t + \beta$ (α, β are constants). So the solution to the system of costate equations is

$$z_0^* = -1, \quad z_1^* = \alpha, \quad z_2^* = -\alpha t + \beta.$$

iii) By PMP, u_1^* is a solution of $\max\limits_{u_1 \in \mathcal{U}_u} H(x_1^*, x_2^*, z_0^*, z_1^*, z_2^*, u_1)$. Note that

$$\begin{aligned}
&\max_{u_1 \in \mathcal{U}_u} H(x_1^*, x_2^*, z_0^*, z_1^*, z_2^*, u_1) \\
&= \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2}(k^2 + u_1^2) + z_1^* x_2^* + z_2^* u_1 \right] \\
&= -\frac{1}{2}k^2 + z_1^* x_2^* + \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2}u_1^2 + z_2^* u_1 \right] \\
&= -\frac{1}{2}k^2 + z_1^* x_2^* + \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2}(u_1 - z_2^*)^2 + \frac{(z_2^*)^2}{2} \right]
\end{aligned}$$

So $u_1^* = z_2^* = -\alpha t + \beta$.

iv) Since $\dot{x}_1^* = x_2^*$ and $\dot{x}_2^* = u_1^*$, it follows that

$$\begin{aligned}
\dot{x}_2^* &= u_1^* = -\alpha t + \beta; \\
x_2^* &= -\frac{\alpha}{2}t^2 + \beta t + \gamma; && (\gamma \text{ is constant}) \\
x_1^* &= -\frac{\alpha}{6}t^3 + \frac{\beta}{2}t^2 + \gamma t + \delta && (\delta \text{ is constant})
\end{aligned}$$

Now, the initial conditions, $x_1^*(0) = X$ and $x_2^*(0) = 0$, give

$$0 = x_2^*(0) = \gamma, \quad X = x_1^*(0) = \delta.$$

Also, $x_1^*(t_1) = 0$ and $x_2^*(t_1) = 0$ give us

$$-\frac{\alpha}{2}t_1^2 + \beta t_1 = 0 \tag{4}$$

$$X - \frac{\alpha}{6}t_1^3 + \frac{\beta}{2}t_1^2 = 0. \tag{5}$$

$$(4) \text{ and } t_1 > 0 \implies \quad \beta = \frac{\alpha}{2}t_1 \tag{6}$$

$$(5) \text{ and } (6) \implies \quad X - \frac{\alpha}{6}t_1^3 + \frac{\alpha}{4}t_1^3 = 0 \implies X = -\frac{\alpha}{12}t_1^3 \implies \alpha = -\frac{12X}{t_1^3}$$

And so $\beta = -\dfrac{6X}{t_1^2}$.

v) The final time t_1 is free and so by PMP

$$H(x_1^*, x_2^*, z_0^*, z_1^*, z_2^*, u_1^*) = 0.$$

Applying this condition at $t = t_1$, we get

$$\begin{aligned}
0 &= -\frac{1}{2}\left[k^2 + (u_1^*(t_1))^2\right] + z_1^*(t_1)x_2^*(t_1) + z_2^*(t_1)u_1^*(t_1) \\
&= -\frac{1}{2}\left[k^2 + \left(\frac{6X}{t_1^2}\right)^2\right] + 0 + \left(\frac{6X}{t_1^2}\right)^2 \\
&= -\frac{1}{2}k^2 + \frac{1}{2}\left(\frac{6X}{t_1^2}\right)^2.
\end{aligned}$$

So we have $t_1^2 = \frac{6X}{k}$ and $t_1 = \sqrt{\frac{6X}{k}}$. Therefore,

$$\begin{aligned}
u_1^*(t) &= \frac{12X}{t_1^3}t - \frac{6X}{t_1^2} \\
x_1^*(t) &= \frac{2X}{t_1^3}t^3 + \left(\frac{-3X}{t_1^2}\right)t^2 + X \\
x_2^*(t) &= \frac{6X}{t_1^3}t^2 + \left(-\frac{6X}{t_1^2}\right)t
\end{aligned}$$

The maximum size of the control used is $\max_{t\in[0,t_1]} |u_1^*(t)| = k$. □

3. A one-dimensional stable system returns to its equilibrium position in an infinite time, so a control is applied to speed up the restoration of equilibrium. The state equation and the cost function are given by

$$\dot{x}_1 = -x_1 + u_1, \quad J = \int_0^{t_1} (k + \frac{1}{2}u_1^2)\, dt$$

where k is a positive constant, t_1 is free and $u_1 \in \mathcal{U}_u$. Find the optimal solution (if it exists) for the initial state $x^0 = X$, where X is positive.

Answer This is an autonomous problem with fixed target and free final time.

i) The Hamiltonian is $H(x_1, z_0, z_1, u_1) = z_0(k + \frac{1}{2}u_1^2) + z_1(-x_1 + u_1)$. As the problem is normal, $z_0 = -1$. So,

$$H(x_1, z_0, z_1, u_1) = -(k + \frac{1}{2}u_1^2) + z_1(-x_1 + u_1)$$

ii) The costate equation is $\dot{\hat{\mathbf{z}}} = -\frac{\partial H}{\partial \hat{\mathbf{x}}}$, that is

$$\dot{z}_0 = -\frac{\partial H}{\partial x_0} = 0 \tag{1}$$
$$\dot{z}_1 = -\frac{\partial H}{\partial x_1} = z_1 \tag{2}$$

(1) is satisfied with $z_0 = -1$. Solving (2), we get $z_1 = \alpha e^t$ where α is a constant. So, a solution for the costate equation is $z_1^* = \alpha e^t$.

iii) By PMP, u_1^* is a solution of $\max_{u_1\in\mathcal{U}_u} H(x_1^*, z_0^*, z_1^*, u_1)$. Note that

$$\begin{aligned}
\max_{u_1\in\mathcal{U}_u} H(x_1^*, z_0^*, z_1^*, u_1) &= \max_{u_1\in\mathcal{U}_u}\left[-(k+\frac{1}{2}u_1^2)+z_1^*(-x_1^*+u_1)\right] \\
&= -k - z_1^*x_1^* + \max_{u_1\in\mathcal{U}_u}\left[-\frac{1}{2}u_1^2+z_1^*u_1\right] \\
&= -k - z_1^*x_z^* + \max_{u_1\in\mathcal{U}_u}\left[-\frac{1}{2}(u_1-z_1^*)^2+\frac{(z_1^*)^2}{2}\right].
\end{aligned}$$

So, $u_1^* = z_1^* = \alpha e^t$.

iv) Solving the state equation with $u_1 = u_1^*$,

$$\begin{aligned}
&\dot{x}_1^* + x_1^* = \alpha e^t \\
\iff\ & e^t\dot{x}_1^* + e^t x_1^* = \alpha e^{2t} \\
\iff\ & \frac{d}{dt}(e^t x_1^*) = \alpha e^{2t} \\
\iff\ & e^t x_1^* = \frac{\alpha}{2}e^{2t} + \beta \\
\iff\ & x_1^*(t) = \frac{\alpha}{2}e^t + \beta e^{-t}.
\end{aligned}$$

Now, the initial and the terminal conditions, $x_1(0) = X$ and $x_1(t_1) = 0$, give

$$X = x_1(0) = \frac{\alpha}{2} + \beta \tag{3}$$
$$0 = x_1(t_1) = \frac{\alpha}{2}e^{t_1} + \beta e^{-t_1} \tag{4}$$

$$(3)\times e^{t_1} - (4) \implies Xe^{t_1} = \beta(e^{t_1} - e^{-t_1}) \implies \beta = \frac{Xe^{t_1}}{e^{t_1}-e^{-t_1}} = \frac{X}{1-e^{-2t_1}} > 0$$

$$(3) \implies \alpha = 2(X-\beta) = 2X\left(1 - \frac{1}{1-e^{-2t_1}}\right) = \frac{2Xe^{-2t_1}}{1-e-2t_1} = \frac{-2X}{e^{2t_1}-1} = \frac{2X}{1-e^{2t_1}} < 0.$$

v) As the final time is free, $H(x_1^*, z_0^*, z_1^*, u_1^*) = 0$. Applying this at $t = t_1$, we have

$$\begin{aligned}
0 &= -\left[k + \frac{1}{2}(u_1^*)^2\right] + z_1^*(-x_1^* + u_1^*) \\
&= -\left[k + \frac{1}{2}\alpha^2 e^{2t_1}\right] + \alpha e_1^t\left(-\frac{\alpha}{2}e_1^t - \beta e^{-t_1} + \alpha e^{t_1}\right) \\
&= -k - \alpha\beta = -k - \alpha\left(X - \frac{\alpha}{2}\right).
\end{aligned}$$

This shows that

$$\frac{\alpha^2}{2} - \alpha X - k = 0 \iff \alpha^2 - 2\alpha X - 2k = 0 \iff (\alpha - X)^2 = X^2 + 2k$$

and $\alpha = X \pm \sqrt{X^2+2k}$. As $\alpha < 0$, $\alpha = X - \sqrt{X^2+2k}$ and $\beta = X - \frac{\alpha}{2} = \frac{X+\sqrt{X^2+2k}}{2}$. So

$$\begin{aligned} u_1^*(t) &= \alpha e^t = (X - \sqrt{X^2+2k})e^t \\ x_1^*(t) &= \frac{\alpha}{2}e^t + \beta e^{-t} = \frac{X-\sqrt{X^2+2k}}{2}e^t + \frac{X+\sqrt{X^2+2k}}{2}e^{-t} \end{aligned}$$

To find the optimal time t_1^*, we note that t_1^* satisfies

$$\frac{2X}{1-e^{2t_1^*}} = \alpha = X - \sqrt{X^2+2k} \implies e^{2t_1^*} = 1 - \frac{2X}{X-\sqrt{X^2+2k}}$$

Hence, $t_1^* = \ln\left(\frac{X+\sqrt{X^2+2k}}{\sqrt{2k}}\right)$.

□

4. Find the optimal control (if it exists) of the problem

$$\begin{aligned} &\text{minimize} && \int_0^1 \frac{1}{2}(3x_1^2 + u_1^2)\, dt \\ &\text{subject to} && \dot{x}_1 = -x_1 + u_1, \\ &&& x_1(0) = 0, x_1(1) = 2 \end{aligned}$$

where $u_1 \in \mathcal{U}_u$.

Answer This is an autonomous problem with a fixed target and fixed final time $t_1 = 1$.

i) The Hamiltonian $H(x_1, z_0, z_1, u_1) = z_0\left[\frac{1}{2}(3x_1^2+u_1^2)\right] + z_1(-x_1+u_1)$. As the problem is normal, $z_0 = -1$. So,

$$H(x_1, z_0, z_1, u_1) = -\frac{1}{2}(3x_1^2+u_1^2) + z_1(-x_1+u_1)$$

ii) The costate equation is $\dot{\hat{\mathbf{z}}} = -\frac{\partial H}{\partial \hat{\mathbf{x}}}$, that is,

$$\begin{aligned} \dot{z}_0 &= -\frac{\partial H}{\partial x_0} = 0 \\ \dot{z}_1 &= -\frac{\partial H}{\partial x_1} = 3x_1 + z_1 \end{aligned}$$

So a solution for the costate equation is

$$z_0^* = -1 \text{ and } z_1^* \text{ satisfying } \dot{z}_1^* = 3x_1^* + z_1^* \qquad (*)$$

iii) The PMP states that u_1^* is a solution of $\max\limits_{u_1 \in \mathcal{U}_u} H(x_1^*, z_0^*, z_1^*, u_1)$. So,

$$\begin{aligned}
\max_{u_1 \in \mathcal{U}_u} H(x_1^*, z_0^*, z_1^*, u_1) &= \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2} \left(3(x_1^*)^2 + u_1^2 \right) + z_1^*(-x_1^* + u_1) \right] \\
&= -\frac{3}{2}(x_1^*)^2 - z_1^* x_1^* + \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2} u_1^2 + z_1^* u_1 \right] \\
&= -\frac{3}{2}(x_1^*)^2 - z_1^* x_1^* + \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2} (u_1 - z_1^*)^2 + \frac{(z_1^*)^2}{2} \right]
\end{aligned}$$

The maximum of $H(x_1^*, z_0^*, z_1^*, u_1)$ is attained at $u_1^* = z_1^*$.

iv) Solving the state equation with $u_1 = u_1^*$, we have

$$\begin{aligned}
\ddot{x}_1^* &= -\dot{x}_1^* + \dot{z}_1^* \\
&= -\dot{x}_1^* + (3x_1^* + z_1^*) && ((*)) \\
&= -\dot{x}_1^* + (3x_1^* + u_1^*) && (u_1^* = z_1^*) \\
&= -\dot{x}_1^* + (3x_1^* + \dot{x}_1^* + x_1^*) && (\dot{x}_1^* = -x_1^* + u_1^*)
\end{aligned}$$

So $\ddot{x}_1^* - 4x_1^* = 0$. The characteristic equation is $\lambda^2 - 4 = 0$ and the roots are $\lambda_1 = 2$ and $\lambda_2 = -2$. The general solution to the second-order equation is $x_1^*(t) = \alpha e^{2t} + \beta e^{-2t}$. Since $x_1^*(0) = 0, x_1^*(1) = 2$, it follows that

$$\begin{aligned}
0 &= x_1^*(0) = \alpha + \beta \\
2 &= x_1^*(1) = \alpha e^2 + \beta e^{-2}.
\end{aligned}$$

These give

$$\begin{aligned}
\alpha &= \frac{2}{e^2 - e^{-2}} = \frac{1}{\sinh 2} \\
\beta &= -\alpha = -\frac{1}{\sinh 2}.
\end{aligned}$$

Hence, the optimal state and the optimal control are given by

$$\begin{aligned}
x_1^*(t) &= \frac{1}{\sinh 2}(e^{2t} - e^{-2t}) \\
u_1^*(t) &= \dot{x}_1^*(t) + x_1^*(t) = 3\alpha e^{2t} - \beta e^{-2t} = \frac{4\cosh 2t + 2\sinh 2t}{\sinh 2}.
\end{aligned}$$

□

5. Consider the optimal control problem

$$\begin{aligned}
&\text{minimize} && \int_0^1 \frac{1}{2} u_1^2 \, dt \\
&\text{subject to} && \dot{x}_1 = -t + u_1, \\
& && x_1(0) = 0, x_1(1) = 1
\end{aligned}$$

where $u_1 \in \mathcal{U}_u$, the unrestricted control set. Assume that this optimal control problem is normal and has a solution, find the optimal control.

Answer This is a non-autonomous problem with fixed target and fixed final time.

i) Let $x_2 = t$. Then $\dot{x}_2 = 1, x_2(0) = 0, x_2(1) = 1$ and the above problem can be rewritten as:

$$\begin{array}{ll} \text{minimize} & \displaystyle\int_0^1 \frac{1}{2}u_1^2 \, dt \\ \text{subject to} & \dot{x}_1 = -x_2 + u_1, \\ & \dot{x}_2 = 1, \\ & x_1(0) = 0, x_1(1) = 1, \\ & x_2(0) = 0, x_2(1) = 1, \\ & u_1 \in \mathcal{U}_u \end{array}$$

The Hamiltonian is $H(x_1, x_2, z_0, z_2, u_1) = z_0\left(\frac{1}{2}u_1^2\right) + z_1(-x_2 + u_1) + z_2$. As the problem is normal, $z_0 = -1$ and so,

$$H(x_1, x_2, z_0, z_1, z_2, u_1) = -\frac{1}{2}u_1^2 + z_1(-x_2 + u_1) + z_2.$$

ii) The costate equation is $\dot{\mathbf{z}} = -\frac{\partial H}{\partial \hat{\mathbf{x}}}$; that is,

$$\dot{z}_0 = -\frac{\partial H}{\partial x_0} = 0 \tag{1}$$

$$\dot{z}_1 = -\frac{\partial H}{\partial x_1} = 0 \implies z_1 = \alpha \tag{2}$$

$$\dot{z}_2 = -\frac{\partial H}{\partial x_2} = -z_1 \implies \dot{z}_2 = -\alpha \implies z_2 = -\alpha t + \beta. \tag{3}$$

The solution to the system of costate equations is

$$z_0^* = -1, \quad z_1^* = \alpha, \quad z_2^* = -\alpha t + \beta. \qquad (\alpha, \beta \text{ are constants})$$

iii) By PMP, u_1^* is a solution of $\max_{u_1 \in \mathcal{U}_u} H(x_1^*, x_2^*, z_0^*, z_1^*, z_2^*, u_1)$ and

$$\begin{aligned} \max_{u_1 \in \mathcal{U}_u} H(x_1^*, x_2^*, z_0^*, z_1^*, z_2^*, u_1) &= \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2}u_1^2 + z_1^*(-x_2^* + u_1) + z_2^*\right] \\ &= -z_1^* x_2^* + z_2^* + \max_{u_1 \in \mathcal{U}_u} \left[\frac{1}{2}u_1^2 + z_1^* u_1\right] \\ &= -z_1^* x_2^* + z_2^* + \max_{u_1 \in \mathcal{U}_u} \left[-\frac{1}{2}(u_1 - z_1^*)^2 + \frac{(z_1^*)^2}{2}\right]. \end{aligned}$$

So $u_1^* = z_1^* = \alpha$.

iv) Solving the state equation with $u_1 = u_1^*$, we get $\dot{x}_1^* = -x_2^* + u_1^* = -t + \alpha$, and so

$$x_1^*(t) = -\frac{t^2}{2} + \alpha t + \gamma. \qquad (\gamma \text{ is a constant})$$

The initial and terminal conditions, $x_1^*(0) = 0$ and $x_1^*(1) = 1$, give

$$\begin{aligned} 0 &= x_1^*(0) = \gamma \\ 1 &= x_1^*(1) = -\frac{1}{2} + \alpha,\ \alpha = \frac{3}{2} \end{aligned}$$

Hence,

$$x_1^*(t) = -\frac{t^2}{2} + \frac{3}{2}t, \quad x_2^*(t) = t, \quad u_1^*(t) = \frac{3}{2}, \forall t \in [0.1], \quad J^* = \frac{9}{8}.$$

$\square$