6305AFE
Advanced Microeconomics

Assignment 1: Problem Solving Assignment

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(1) The following is a utility function of a particular consumer.

$\boldsymbol{U = (x_1 + 7)(x_2 + 5)}$

(a) Derive the consumer's first order optimality conditions for utility maximisation and the corresponding demand functions (Marshallian) for x_1 and x_2.

$\max(u) = (x_1 + 7)(x_2 + 5)$

subject to: $M = P_1 x_1 + P_2 x_2$

Therefore, the Lagrange function is:

$\mathcal{L} = (x_1 + 7)(x_2 + 5) + \lambda(M - P_1 x_1 - P_2 x_2)$

The first order optimality conditions for utility maximisation can be derived with respect to x_1, x_2 and λ.

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 + 5 - \lambda P_1 \quad \ldots = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1 + 7 - \lambda P_2 \quad \ldots = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - P_1 x_1 - P_2 x_2 \quad \ldots = 0 \qquad (3)$$

The corresponding demand functions for x_1 and x_2 can then be derived from the first order optimality conditions.

$$\frac{\lambda P_1}{\lambda P_2} = \frac{x_2 + 5}{x_1 + 7}$$

$$\frac{P_1}{P_2}(x_1 + 7) = x_2 + 5$$

$$x_2 = \frac{P_1}{P_2}(x_1 + 7) - 5 \qquad (4)$$

To obtain corresponding demand functions, Equation 4 must be substituted into Equation 3.

$$M - P_1 x_1 - P_2\left(\frac{P_1}{P_2}(x_1 + 7) - 5\right) \quad \ldots = 0$$

$$M - P_1 x_1 - P_1 x_1 - 7P_1 + 5P_2 \quad \ldots = 0$$

$$M - 7P_1 + 5P_2 = 2P_1 x_1$$

Therefore, the Marshallian demand function for x_1 is:

$$x_1 = \frac{M - 7P_1 + 5P_2}{2P_1} \qquad (5)$$

Therefore, the Marshallian demand function for x_2 is:

$$x_2 = \frac{M + 7P_1 - 5P_2}{2P_2} \qquad (6)$$

(b) Demonstrate, using a numerical example, that the homogeneity property holds for the demand functions derived in (a). Explain, intuitively, why this property holds.

Numerical values have been chosen to assist in displaying that the homogeneity property holds for the demand functions derived in (a). These values are substituted into the Marshallian demand functions.

$P_1 = 2$	$P_2 = 5$	$M = 51$
$P_1 = 4$	$P_2 = 10$	$M = 102$

The Marshallian demand function for x_1 is:

$$x_1 = \frac{M - 7P_1 + 5P_2}{2P_1}$$

Substitute $P_1 = 2$, $P_2 = 5$ and $M = 51$ in the demand function:

$$x_1 = \frac{51 - 7(2) + 5(5)}{2(2)}$$

$$x_1 = 15.5$$

Substitute $P_1 = 4$, $P_2 = 10$ and $M = 102$ in the demand function:

$$x_1 = \frac{102 - 7(4) + 5(10)}{2(4)}$$

$$x_1 = 15.5$$

The Marshallian demand function for x_2 is:

$$x_2 = \frac{M + 7P_1 - 5P_2}{2P_2}$$

Substitute $P_1 = 2$, $P_2 = 5$ and $M = 51$ in the demand function:

$$x_2 = \frac{51 + 7(2) - 5(5)}{2(5)}$$

$$x_2 = 4$$

Substitute $P_1 = 4$, $P_2 = 10$ and $M = 102$ in the demand function:

$$x_2 = \frac{102 + 7(4) - 5(10)}{2(10)}$$

$$x_2 = 4$$

Intuitively, this property holds because when the values are scaled by double, the outcomes of x_1 and x_2 remain constant. The transformation of the variables remains, which verifies that the homogeneity property holds.

(c) Find a function for λ as a function of income and interpret it as the marginal utility of income.

The indirect utility function is defined as:

$$U = \left[\left(\frac{M - 7P_1 + 5P_2}{2P_1}\right) + 7\right]\left[\left(\frac{M+7P_1 - 5P_2}{2P_2}\right) + 5\right]$$

$$U = \left(\frac{M - 7P_1 + 5P_2 + 14P_1}{2P_1}\right)\left(\frac{M+7P_1 - 5P_2 + 10P_2}{2P_2}\right)$$

$$U = \left(\frac{M + 5P_2 + 7P_1}{2P_1}\right)\left(\frac{M+7P_1 - 5P_2}{2P_2}\right)$$

$$U = \frac{(M + 5P_2 + 7P_1)^2}{4P_1P_2}$$

Use the chain rule to differentiate with respect to M.

$$\frac{\partial U}{\partial M} = \frac{2(M + 5P_2 + 7P_1)}{4P_1P_2}$$

The marginal utility of income is:

$$\frac{\partial U}{\partial M} = \frac{M + 5P_2 + 7P_1}{2P_1P_2}$$

Find a function for λ as a function of income:

$$\lambda = \frac{x_1 + 7}{P_2} = \frac{x_2 + 5}{P_1}$$

$$\lambda = \frac{\left(\frac{M - 7P_1 + 5P_2}{2P_1} + 7\right)}{P_2} = \frac{\left(\frac{M - 5P_2 + 7P_1}{2P_2} + 5\right)}{P_1}$$

$$\lambda = \frac{M - 7P_1 + 5P_2 + 14P_1}{2P_1P_2} = \frac{M - 5P_2 + 7P_1 + 10P_2}{2P_2P_1}$$

$$\lambda = \frac{M + 7P_1 + 5P_2}{2P_1P_2} = \frac{M + 5P_2 + 7P_1}{2P_2P_1}$$

$$\lambda = \frac{\partial U}{\partial I} = \frac{M + 7P_1 + 5P_2}{2P_1P_2}$$

Therefore, λ is equal to the marginal utility of income.

(2) **Jay drinks two types of wine, and has the following utility function:**

$$\boldsymbol{U(W_F, W_C) = W_F{}^{1/2} + W_C{}^{1/2}}$$

where W_F is French Bordeaux wine and W_C is California varietal wine. Prices of two types of wine are P_F and P_C and Jay's income is M.

(a) Derive the following functions for Jay:

(i) The Marshallian demand functions

$$max\, U\,(W_F, W_C) = W_F{}^{\frac{1}{2}} + W_C{}^{\frac{1}{2}}$$

subject to: $M = P_F W_F + P_C W_C$

Therefore, the Lagrange function is:

$$\mathcal{L} = W_F{}^{\frac{1}{2}} + W_C{}^{\frac{1}{2}} + \lambda\,(M - P_F W_F - P_C W_C)$$

The first order optimality conditions for utility maximisation can be derived with respect to W_F, W_c and λ.

$$\frac{\partial \mathcal{L}}{\partial W_F} = \frac{1}{2} W_F{}^{-\frac{1}{2}} - \lambda P_F \qquad ... = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial W_C} = \frac{1}{2} W_C{}^{-\frac{1}{2}} - \lambda P_C \qquad ... = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - P_F W_F - P_C W_C \qquad ... = 0 \qquad (3)$$

Set Equation 1 equal to Equation 2.

$$\frac{\frac{1}{2} W_F{}^{-\frac{1}{2}}}{\frac{1}{2} W_C{}^{-\frac{1}{2}}} = \frac{\lambda P_F}{\lambda P_C}$$

$$\frac{W_F{}^{-\frac{1}{2}}}{W_C{}^{-\frac{1}{2}}} = \frac{P_F}{P_C}$$

$$\frac{W_C{}^{\frac{1}{2}}}{W_F{}^{\frac{1}{2}}} = \frac{P_F}{P_C}$$

$$P_C W_C{}^{\frac{1}{2}} = P_F W_F{}^{\frac{1}{2}}$$

$$W_F{}^{\frac{1}{2}} = \frac{P_C W_C{}^{\frac{1}{2}}}{P_F}$$

Therefore, W_F is:

$$W_F = \left(\frac{P_C}{P_F}\right)^2 W_C \qquad (4)$$

Therefore, W_C is:

$$W_C = \left(\frac{P_F}{P_C}\right)^2 W_F \qquad (5)$$

To obtain corresponding demand functions, Equation 4 must be substituted into Equation 3.

$$M = P_F\left[\left(\frac{P_C}{P_F}\right)^2 W_C\right] + P_C W_C$$

$$M = P_F\left[\frac{{P_C}^2}{{P_F}^2} W_C\right] + P_C W_C$$

$$M = \frac{{P_C}^2}{P_F} W_C + P_C W_C$$

$$M = W_C\left(\frac{{P_C}^2}{P_F} + P_C\right)$$

$$W_C = \frac{M}{\left(\frac{{P_C}^2}{P_F} + P_C\right)}$$

Therefore, the Marshallian demand function for W_C is:

$$W_C = \frac{M}{P_C\left(\frac{P_C}{P_F} + 1\right)}$$

To obtain corresponding demand functions, Equation 5 must be substituted into Equation 3. Therefore, the Marshallian demand function for W_F is:

$$W_F = \frac{M}{P_F\left(\frac{P_F}{P_C} + 1\right)}$$

(ii) The indirect utility function

Substitute the demand functions for W_C and W_F derived in Question (a)(i) into the utility function. Therefore, the indirect utility function is:

$$U = \left(\frac{M}{P_F\left(\frac{P_F}{P_C} + 1\right)}\right)^{1/2} + \left(\frac{M}{P_C\left(\frac{P_C}{P_F} + 1\right)}\right)^{1/2}$$

(iii) The Hicksian demand functions

$min\,M \;=\; P_F W_F +\; P_C W_C$

subject to: $W_F^{\frac{1}{2}} + W_C^{\frac{1}{2}}$

Therefore, the Lagrange function is:

$\mathcal{L} =\; P_F W_F -\; P_C W_C + \lambda(\,\mu - W_F{}^{1/2} -\; W_C{}^{1/2}\,)$

The first order optimality conditions for utility maximisation can be derived with respect to W_F, W_c and λ.

$$\frac{\partial \mathcal{L}}{\partial W_F} = P_F - \frac{1}{2}\lambda W_F{}^{-\frac{1}{2}} \qquad \ldots = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial W_C} = P_C - \frac{1}{2}\lambda W_C{}^{-\frac{1}{2}} \qquad \ldots = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \;\mu - W_F{}^{1/2} -\; W_C{}^{1/2} \qquad \ldots = 0 \qquad (3)$$

Divide Equation 1 by Equation 2.

$$\frac{P_F}{P_c} = \frac{\frac{1}{2}\lambda W_F{}^{-\frac{1}{2}}}{\frac{1}{2}\lambda W_C{}^{-\frac{1}{2}}}$$

$$\frac{P_F}{P_c} = \frac{W_F{}^{-\frac{1}{2}}}{W_C{}^{-\frac{1}{2}}}$$

$$\frac{P_F}{P_c} = \frac{W_C{}^{\frac{1}{2}}}{W_F{}^{\frac{1}{2}}}$$

$$P_F W_F{}^{1/2} \;=\; P_c W_c{}^{1/2}$$

$$W_F{}^{1/2} = \frac{P_C W_C{}^{1/2}}{P_F} \qquad (4)$$

$$W_F = \left(\frac{P_c}{P_F}\right)^2 W_c \qquad (5)$$

$$\text{Therefore, } W_c = \left(\frac{P_F}{P_C}\right)^2 W_F \qquad (6)$$

Substitute Equation 4 into Equation 3.

$$\mu = \left(\frac{P_c}{P_F}\right) W_C^{\ 1/2} + W_C^{\ 1/2}$$

$$\mu = \left(1 + \frac{P_c}{P_F}\right) W_C^{\ 1/2}$$

$$\mu = \left(\frac{P_F}{P_F} + \frac{P_c}{P_F}\right) W_C^{\ 1/2}$$

$$\mu = \left(\frac{P_F + P_c}{P_F}\right) W_C^{\ 1/2}$$

$$\mu P_F = (P_F + P_c) W_C^{\ 1/2}$$

$$W_C^{\ 1/2} = \frac{\mu \ P_F}{P_F + P_c}$$

Therefore, the Hicksian demand function for W_C is:

$$W_C = \frac{(\mu \ P_F)^2}{(P_F + P_c)^2}$$

Therefore, the Hicksian demand function for W_F is:

$$W_F = \frac{(\mu \ P_C)^2}{(P_F + P_c)^2}$$

(iv) The expenditure function

$$M = P_F W_F + P_C W_C$$

$$M = P_F \left(\frac{(\mu \ P_C)^2}{(P_F + P_c)^2}\right) + P_C \left(\frac{(\mu \ P_F)^2}{(P_F + P_c)^2}\right)$$

$$M = \frac{P_F \mu^2 P_C^{\ 2}}{(P_F + P_c)^2} + \frac{P_C \mu^2 P_F^{\ 2}}{(P_F + P_c)^2}$$

$$M(P_F + P_c)^2 = P_F \mu^2 P_C^{\ 2} + P_C \mu^2 P_F^{\ 2}$$

$$M(P_F + P_c)^2 = \mu^2 P_F P_C (P_F + P_C)$$

Let a = $P_F + P_c$

$$Ma^2 = u^2 \ P_F P_c \times a$$

$$Ma = u^2 \ P_F P_c$$

$$M(P_F + P_c\ ) = u^2 \ P_F P_c$$

Therefore, the expenditure function is:

$$M = \frac{u^2\ P_F P_c}{(P_F + P_c)}$$

(b) State Roy's identity and verify it from a(i) and a(ii).

Roy's identity is a method of deriving the Marshallian demand function of a good for some consumer from the indirect utility function of that consumer. Therefore, deriving the indirect utility function in Question a(ii) will result in deriving the corresponding Marshallian demand functions from Question a(i).

Therefore, the indirect utility function obtained from Question a(ii) is:

$$U = \left(\frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}\right)^{1/2} + \left(\frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}\right)^{1/2}$$

This can also be written as:

$$U = \sqrt{\frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}} + \sqrt{\frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}}$$

To find W_F, take the negative partial derivative with respect to P_F and divide by the partial derivative with respect to M.

$$W_F = \frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}$$

W_F can be simplified further:

$$W_F = \frac{MP_C}{P_F P_C + {P_F}^2}$$

Solve for $\frac{\partial U}{\partial P_F}$

$$\frac{\partial U}{\partial P_F} = \frac{M}{2{P_F}^2\left(\frac{P_C}{P_F}+1\right)^2\sqrt{\frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}}} + \frac{-\frac{M}{{P_F}^2\left(\frac{P_F}{P_C}+1\right)} - \frac{M}{P_C P_F\left(\frac{P_F}{P_C}+1\right)^2}}{2\sqrt{\frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}}}$$

$$\frac{\partial U}{\partial P_F} = -\frac{1}{2}\left(\frac{M}{(P_C + P_F)^2\sqrt{\frac{MP_F}{P_C(P_C+P_F)}}} - \frac{(P_C+2P_F)\left(\frac{MP_C}{P_F(P_C+P_F)}\right)^{\frac{3}{2}}}{MP_C}\right)$$

Solve for $\frac{\partial \mathrm{U}}{\partial \mathrm{M}}$

$$\frac{\partial U}{\partial M} = \frac{1}{2P_C\left(\frac{P_C}{P_F}+1\right)\sqrt{\frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}}} + \frac{1}{2P_F\left(\frac{P_F}{P_C}+1\right)\sqrt{\frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}}}$$

$$\frac{\partial U}{\partial M} = \frac{\sqrt{\frac{MP_F}{P_C(P_C+P_F)}} + \sqrt{\frac{MP_C}{P_F(P_C+P_F)}}}{2M}$$

To find W_F:

$$W_F = -\frac{\frac{\partial U}{\partial P_F}}{\frac{\partial U}{\partial M}}$$

$$W_F = -\frac{\frac{M}{(P_C + P_F)^2\sqrt{\frac{MP_F}{P_C(P_C+P_F)}}} - \frac{(P_C+2P_F)\left(\frac{MP_C}{P_F(P_C+P_F)}\right)^{\frac{3}{2}}}{MP_C}}{2\left(\frac{\sqrt{\frac{MP_F}{P_C(P_C+P_F)}} + \sqrt{\frac{MP_C}{P_F(P_C+P_F)}}}{2M}\right)}$$

$$W_F = -\frac{M\left(\frac{\sqrt{M}\sqrt{P_C}}{\sqrt{P_F}(P_C+P_F)^{\frac{3}{2}}} - \frac{\sqrt{M}\sqrt{P_C}(P_C+2P_F)}{P_F^{\frac{3}{2}}(P_C+P_F)^{\frac{3}{2}}}\right)}{\frac{\sqrt{M}\sqrt{P_C}}{\sqrt{P_F}\sqrt{P_C+P_F}} + \frac{\sqrt{M}\sqrt{P_F}}{\sqrt{P_C}\sqrt{P_C+P_F}}}$$

Therefore, W_F equals:

$$W_F = \frac{MP_C}{P_CP_F + {P_F}^2}$$

To find W_C, take the negative partial derivative with respect to P_C and divide by the partial derivative with respect to M.

$$U = \sqrt{\frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}} + \sqrt{\frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}}$$

$$W_C = \frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}$$

W_C can be simplified further:

$$W_C = \frac{MP_F}{P_F P_C + {P_C}^2}$$

Solve for $\frac{\partial U}{\partial P_C}$

$$\frac{\partial U}{\partial P_C} = -\frac{1}{2}\left(\frac{M}{(P_C+P_F)^2\sqrt{\frac{MP_C}{P_C P_F + {P_F}^2}}} - \frac{(2P_C+P_F)\left(\frac{MP_F}{P_C(P_C+P_F)}\right)^{\frac{3}{2}}}{MP_F}\right)$$

Solve for $\frac{\partial U}{\partial M}$

$$\frac{\partial U}{\partial M} = \frac{\sqrt{\frac{M}{P_C(P_C+P_F)}} + \sqrt{\frac{M}{P_F(P_C+P_F)}}}{2M}$$

To find W_C:

$$W_C = -\frac{\frac{\partial U}{\partial P_C}}{\frac{\partial U}{\partial M}}$$

$$W_C = -\frac{1}{2}\frac{\left(\frac{M}{(P_C+P_F)^2\sqrt{\frac{MP_C}{P_C P_F + {P_F}^2}}} - \frac{(2P_C+P_F)\left(\frac{MP_F}{P_C(P_C+P_F)}\right)^{\frac{3}{2}}}{MP_F}\right)}{\frac{\sqrt{\frac{MP_F}{P_C(P_C+P_F)}} + \sqrt{\frac{MP_C}{P_F(P_C+P_F)}}}{2M}}$$

$$W_C = -\frac{M\left(\frac{\sqrt{M}\sqrt{P_C P_F + {P_F}^2}}{\sqrt{P_C}(P_C + P_F)^2} - \frac{\sqrt{M}\sqrt{P_F}(2P_C + P_F)}{{P_C}^{\frac{3}{2}}(P_C + P_F)^{\frac{3}{2}}}\right)}{\frac{\sqrt{M}\sqrt{P_C}}{\sqrt{P_F}\sqrt{P_C + P_F}} + \frac{\sqrt{M}\sqrt{P_F}}{\sqrt{P_C}\sqrt{P_C + P_F}}}$$

$$W_C = -\frac{-2M{P_C}^2 P_F + P_C\left(M\sqrt{P_F}\sqrt{P_C + P_F}\sqrt{P_F(P_C + P_F)} - 3M{P_F}^2\right) - M{P_F}^3}{P_C(P_C + P_F)^3}$$

$$W_C = -M\left(\frac{\sqrt{P_F}\sqrt{P_F(P_C + P_F)}}{(P_C + P_F)^{\frac{5}{2}}} + \frac{P_C}{(P_C + P_F)^2} - \frac{1}{P_C}\right)$$

$$W_C = -M\left(\frac{P_C}{(P_C + P_F)^2} + \frac{P_F}{(P_C + P_F)^2} - \frac{1}{P_C}\right)$$

Therefore, W_C equals:

$$W_C = \frac{MP_F}{P_C P_F + {P_C}^2}$$

Consequently, Roy's identity holds true as W_C and W_F derived from the utility function equal the Marshallian demand functions derived in Question 2a(i).

(c) Jay has \$256 to spend to establish a small wine collection. He enjoys two vintages in particular: an expensive 1981 French Bordeaux at \$24 per bottle and a less expensive 1993 California varietal wine priced at \$8. How much of each wine should he purchase if his utility is characterised by the following function? (Hint: use the Marshallian demand functions derived in (a)(i)).

Therefore, M=\$256, P_F=\$24, and P_C=\$8. Substitute these values in the Marshallian demand function for W_C and W_F derived in Question (a)(i).

$$W_C = \frac{M}{P_C\left(\frac{P_C}{P_F} + 1\right)}$$

$$W_C = \frac{256}{8\left(\frac{8}{24} + 1\right)}$$

$$W_C = 24$$

$$W_F = \frac{M}{P_F\left(\frac{P_F}{P_C} + 1\right)}$$

$$W_F = \frac{256}{24\left(\frac{24}{8}+1\right)}$$

$$W_F = 2\tfrac{2}{3}$$

Therefore, given a budget constraint of \$256, Jay should purchase 24 units of the California and $2\frac{2}{3}$ units of the French Bordeaux wine.

(d) Now assume: when Dr Jay arrived at the wine store, he discovered that the price of the 1981 French Bordeaux had fallen to \$8 a bottle. If the price of the California wine remains stable at \$8 per bottle, how much of each wine should our friend purchase to maximise utility under these altered conditions? (Hint: once again use the Marshallian demand functions derived in (a)(i)).

Therefore, M=\$256, P_F=\$8, and P_C=\$8. Substitute these values in the Marshallian demand function for W_C and W_F derived in Question (a)(i).

$$W_C = \frac{M}{P_C\left(\frac{P_C}{P_F}+1\right)}$$

$$W_C = \frac{256}{8\left(\frac{8}{8}+1\right)}$$

$$W_C = 16$$

$$W_F = \frac{M}{P_F\left(\frac{P_F}{P_C}+1\right)}$$

$$W_F = \frac{256}{8\left(\frac{8}{8}+1\right)}$$

$$W_F = 16$$

Therefore, given a budget constraint of \$256, Jay should purchase 16 units of the California and 16 units of the French Bordeaux wine.

(3) A production function of a firm is given below.

$$\boldsymbol{Q = L^{1/2} + K^{1/2}}$$

where Q = output, L = labour unit (units per hour), and K = capital input (units per hour).

(a) Find the marginal product of labour and capital.

$$MP_L = \frac{\partial Q}{\partial L}$$

$$MP_L = \frac{1}{2}L^{-\frac{1}{2}}$$

$$MP_K = \frac{\partial Q}{\partial K}$$

$$MP_K = \frac{1}{2}K^{-\frac{1}{2}}$$

(b) Find the marginal rate of technical substitution (MRTS).

$$MRTS = \frac{MP_L}{MP_K}$$

$$MRTS = \frac{\frac{1}{2}L^{-\frac{1}{2}}}{\frac{1}{2}K^{-\frac{1}{2}}}$$

$$MRTS = \frac{L^{-\frac{1}{2}}}{K^{-\frac{1}{2}}}$$

Therefore, the marginal rate of technical substitution is:

$$MRTS = \frac{K^{\frac{1}{2}}}{L^{\frac{1}{2}}}$$

(c) Does this production function demonstrate increasing, decreasing or constant returns to scale?

$$Q = L^{1/2} + K^{1/2}$$

$$\alpha Q = (\alpha L)^{1/2} + (\alpha K)^{1/2}$$

$$\alpha Q = (\alpha^{1/2}(L^{1/2} + K^{1/2}))$$

$$\alpha Q = \alpha^{1/2}(Q)$$

Therefore, as $\alpha^{1/2} < 1$, the production function exhibits decreasing returns to scale.

(d) Are the isoquants convex or concave? Briefly explain your answer.

$$MRTS = \frac{K^{\frac{1}{2}}}{L^{\frac{1}{2}}} \qquad \longrightarrow \qquad MRTS = K^{1/2}L^{-1/2}$$

To determine if the isoquants are convex or concave, differentiate MRTS with respect to *L*.

$$\frac{\partial MRTS}{\partial L} = -\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{3}{2}} < 0 \qquad or \qquad = -\frac{1}{2}\frac{K^{\frac{1}{2}}}{L^{\frac{3}{2}}} < 0$$

The isoquants are convex because $\frac{\partial MRTS}{\partial L} < 0$. Therefore, this demonstrates diminishing returns to scale.

(4) A firm uses two inputs, L and K, to produce one output Q. The prices of the two inputs are the output are *w* and *r* and *p* respectively. The production function is:

$$\boldsymbol{Q = (L+2)^{1/4}(K+3)^{1/4}}$$

(a) Solve the firm's profit maximisation problem to find:

(i) The unconditional input-demand functions.

$maximise\ \pi = pQ - wL - rK$

$maximise\ \pi = p(L+2)^{1/4}(K+3)^{1/4} - wL - rK$

The first order conditions are derived with respect to *L* and *K*.

$$\frac{\partial \pi}{\partial L} = \frac{1}{4}p(L+2)^{-\frac{3}{4}}(K+3)^{\frac{1}{4}} - w \qquad \ldots = 0 \qquad (1)$$

$$\frac{\partial \pi}{\partial K} = \frac{1}{4}p(L+2)^{\frac{1}{4}}(K+3)^{-\frac{3}{4}} - r \qquad \ldots = 0 \qquad (2)$$

From Equation 1.

$$\frac{1}{4}p(L+2)^{-\frac{3}{4}}(K+3)^{\frac{1}{4}} = w$$

$$p = \frac{w}{\frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4}} \qquad (1i)$$

From Equation 2.

$$\frac{1}{4}p(L+2)^{\frac{1}{4}}(K+3)^{-\frac{3}{4}} = r$$

$$p = \frac{r}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}} \qquad (2i)$$

Set Equation *1i* equal to Equation $2i$.

$$\frac{w}{\frac{1}{4}(L+2)^{-\frac{3}{4}}(K+3)^{\frac{1}{4}}} = \frac{r}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}}$$

$$\frac{w}{r} = \frac{\frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4}}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}}$$

$$\frac{w}{r} = \frac{(K+3)}{(L+2)}$$

$$K+3=\frac{w}{r}(L+2)$$

$$K=\frac{w}{r}(L+2)-3 \qquad (3)$$

Substitute Equation 3 into Equation 2.

$$\frac{1}{4}p(L+2)^{\frac{1}{4}}(\frac{w}{r}(L+2)-3+3)^{-\frac{3}{4}}-r \qquad ...=0$$

$$\frac{1}{4}p(L+2)^{\frac{1}{4}}\left(\frac{w}{r}(L+2)\right)^{-\frac{3}{4}}=r$$

$$\frac{1}{4}p(L+2)^{\frac{1}{4}}\frac{w^{-\frac{3}{4}}}{r^{-\frac{3}{4}}}(L+2)^{-\frac{3}{4}}=r$$

$$\frac{1}{4}p\frac{w^{-3/4}}{r^{-3/4}}(L+2)^{-1/2}=r$$

$$(L+2)^{-1/2}=\frac{r}{\frac{1}{4}p\frac{w^{-3/4}}{r^{-3/4}}}$$

$$(L+2)^{-1/2}=\frac{4r}{\frac{pw^{-3/4}}{r^{-3/4}}}$$

$$(L+2)^{-1/2}=\frac{4r^{1/4}}{pw^{-3/4}}$$

$$(L+2)^{1/2}=\frac{pw^{-3/4}}{4r^{1/4}}$$

$$(L+2)=\frac{p^2w^{-3/2}}{16r^{1/2}}$$

Thus, L^* is the unconditional input-demand function for L.

$$L^*=\frac{p^2}{16w^{3/2}r^{1/2}}-2$$

Substitute L^* into Equation 3.

$$K^*=\frac{w}{r}\left(\frac{p^2}{16w^{3/2}r^{1/2}}-2+2\right)-3$$

$$K^*=\frac{w}{r}\left(\frac{p^2}{16w^{3/2}r^{1/2}}\right)-3$$

Thus, K^* is the unconditional input-demand function for K.

$$K^* = \frac{p^2}{16w^{1/2}r^{3/2}} - 3$$

(ii) The supply function.

$$Q = (L+2)^{1/4}(K+3)^{1/4}$$

Substitute L^* and K^* into the original production function.

$$Q^* = \left(\frac{p^2}{16w^{3/2}r^{1/2}} - 2 + 2\right)^{1/4}\left(\frac{p^2}{16w^{1/2}r^{3/2}} - 3 + 3\right)^{1/4}$$

$$Q^* = \left(\frac{p^2}{16w^{3/2}r^{1/2}}\right)^{1/4}\left(\frac{p^2}{16w^{1/2}r^{3/2}}\right)^{1/4}$$

$$Q^* = \left(\frac{p^{1/2}}{16^{1/4}w^{3/8}r^{1/8}}\right) \times \left(\frac{p^{1/2}}{16^{1/4}w^{1/8}r^{3/8}}\right)$$

Therefore, the supply function is:

$$Q^* = \frac{p}{4w^{1/2}r^{1/2}}$$

(iii) The profit function.

$$maximise\ \pi = pQ - wL - rK$$

$$\pi^* = p\left(\frac{p}{4w^{1/2}r^{1/2}}\right) - w\left(\frac{p^2}{16w^{3/2}r^{1/2}} - 2\right) - r\left(\frac{p^2}{16w^{1/2}r^{3/2}} - 3\right)$$

$$\pi^* = p\left(\frac{p}{4w^{1/2}r^{1/2}}\right) - w\left(\frac{p^2}{16w^{3/2}r^{1/2}} - 2 \cdot \frac{16w^{3/2}r^{1/2}}{16w^{3/2}r^{1/2}}\right) - r\left(\frac{p^2}{16w^{1/2}r^{3/2}} - 3 \cdot \frac{16w^{3/2}r^{1/2}}{16w^{3/2}r^{1/2}}\right)$$

$$\pi^* = \frac{p^2}{4w^{1/2}r^{1/2}} - w\left(\frac{p^2}{16w^{3/2}r^{1/2}} - \frac{32w^{3/2}r^{1/2}}{16w^{3/2}r^{1/2}}\right) - r\left(\frac{p^2}{16w^{1/2}r^{3/2}} - \frac{48w^{1/2}r^{3/2}}{16w^{1/2}r^{3/2}}\right)$$

$$\pi^* = \frac{p^2}{4w^{1/2}r^{1/2}} - w\left(\frac{p^2 - 32w^{3/2}r^{1/2}}{16w^{3/2}r^{1/2}}\right) - r\left(\frac{p^2 - 48w^{1/2}r^{3/2}}{16w^{1/2}r^{3/2}}\right)$$

$$\pi^* = \frac{p^2}{4w^{1/2}r^{1/2}} - \left(\frac{p^2 - 32w^{3/2}r^{1/2}}{16w^{1/2}r^{1/2}}\right) - \left(\frac{p^2 - 48w^{1/2}r^{3/2}}{16w^{1/2}r^{1/2}}\right)$$

Finding a common denominator.

$$\pi^* = \frac{4p^2}{16w^{1/2}r^{1/2}} - \frac{p^2 - 32w^{3/2}r^{1/2}}{16w^{1/2}r^{1/2}} - \frac{p^2 - 48w^{1/2}r^{3/2}}{16w^{1/2}r^{1/2}}$$

$$\pi^* = \frac{4p^2 - p^2 + 32w^{3/2}r^{1/2} - p^2 + 48w^{1/2}r^{3/2}}{16w^{1/2}r^{1/2}}$$

$$\pi^* = \frac{2p^2 + 32w^{3/2}r^{1/2} + 48w^{1/2}r^{3/2}}{16w^{1/2}r^{1/2}}$$

$$\pi^* = \frac{2p^2}{16w^{1/2}r^{1/2}} + \frac{16w^{1/2}r^{1/2}(2w + 3r)}{16w^{1/2}r^{1/2}}$$

Therefore, the profit function is:

$$\pi^* = \frac{p^2}{8w^{1/2}r^{1/2}} + 2w + 3r$$

(b) Verify that the derivatives of the profit function with respect to output price yields the supply function (i.e. verify part of Hotelling's lemma).

Hotelling's lemma is obtained through the partial derivate of the profit function with respect to p.

$$\frac{\partial \pi^*(w, r, p)}{\partial p} = Q^*(w, r, p)$$

Using the profit function obtained in Question a(iii).

$$\pi^* = \frac{p^2}{8w^{1/2}r^{1/2}} + 2w + 3r$$

$$\frac{\partial \pi^*}{\partial p} = \frac{2p}{8w^{1/2}r^{1/2}}$$

$$\frac{\partial \pi^*}{\partial p} = \frac{p}{4w^{1/2}r^{1/2}} = Q^*(w, r, p)$$

Therefore, Hotelling's lemma holds for these equations because the supply function can be derived from the profit function.

(c) Find the firm's conditional input demand functions and its cost function.

$min(c) = wL + rK$

subject to: $Q = (L + 2)^{1/4}(K + 3)^{1/4}$

Therefore, the Lagrange function is:

$\mathcal{L} = wL + rK + \lambda(q - (L + 2)^{1/4}(K + 3)^{1/4}$

The first order optimality conditions can be derived with respect to L, K and λ.

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \frac{1}{4}(L + 2)^{-3/4}(K + 3)^{1/4} \qquad ... = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4} \qquad \ldots = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q - (L+2)^{\frac{1}{4}}(K+3)^{\frac{1}{4}} \qquad \ldots = 0 \qquad (3)$$

From Equation 1.

$$\lambda \frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4} = w$$

$$\lambda = \frac{w}{\frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4}} \qquad (1i)$$

From Equation 2.

$$\lambda \frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4} = r$$

$$\lambda = \frac{r}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}} \qquad (2i)$$

Set Equation *1i* equal to Equation *2i*.

$$\frac{w}{\frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4}} = \frac{r}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}}$$

$$\frac{w}{r} = \frac{\frac{1}{4}(L+2)^{-3/4}(K+3)^{1/4}}{\frac{1}{4}(L+2)^{1/4}(K+3)^{-3/4}}$$

$$\frac{w}{r} = \frac{K+3}{L+2}$$

$$K + 3 = \frac{w}{r}(L+2)$$

$$K = \frac{w}{r}(L+2) - 3 \qquad (4)$$

Substitute Equation 4 into Equation 3.

$$q - (L+2)^{1/4}\left(\frac{w}{r}(L+2) - 3 + 3\right)^{\frac{1}{4}} = 0$$

$$q - (L+2)^{1/4}\left(\frac{w}{r}(L+2)\right)^{\frac{1}{4}} = 0$$

$$q - (L+2)^{1/4}\frac{w^{1/4}}{r^{1/4}}(L+2)^{\frac{1}{4}} = 0$$

$$\frac{w^{\frac{1}{4}}}{r^{\frac{1}{4}}}(L+2)^{1/2} = q$$

$$(L+2)^{1/2} = \frac{r^{\frac{1}{4}}}{w^{\frac{1}{4}}}q$$

$$(L+2) = \frac{r^{1/2}}{w^{1/2}}q^2$$

Therefore, the conditional factor demand function for *L* is:

$$L^* = \frac{r^{1/2}}{w^{1/2}}q^2 - 2$$

Substituting L^* into Equation 4.

$$K^* = \frac{w}{r}\left(\frac{r^{1/2}}{w^{1/2}}q^2 - 2 + 2\right) - 3$$

$$K^* = \frac{w}{r}\left(\frac{r^{1/2}}{w^{1/2}}q^2\right) - 3$$

Therefore, the conditional factor demand function for *K* is:

$$K^* = \frac{w^{1/2}}{r^{1/2}}q^2 - 3$$

The cost function $C^*(w,r,q)$ can be derived by substituting the conditional factor demand functions for $L(L^*(w,r,q))$ and $K(K^*(w,r,q))$ into the objective function for the firm's cost minimisation problem.

$$C = wL + rK$$

$$C^* = w\left(\frac{r^{1/2}}{w^{1/2}}q^2 - 2\right) + r\left(\frac{w^{1/2}}{r^{1/2}}q^2 - 3\right)$$

$$C^* = w\left(\frac{r^{1/2}}{w^{1/2}}q^2 - \frac{2w^{1/2}}{w^{1/2}}\right) + r\left(\frac{w^{1/2}}{r^{1/2}}q^2 - \frac{3r^{1/2}}{r^{1/2}}\right)$$

$$C^* = w\left(\frac{r^{1/2}q^2 - 2w^{1/2}}{w^{1/2}}\right) + r\left(\frac{w^{1/2}q^2 - 3r^{1/2}}{r^{1/2}}\right)$$

$$C^* = w^{1/2}(r^{1/2}q^2 - 2w^{1/2}) + r^{1/2}(w^{1/2}q^2 - 3r^{1/2})$$

$$C^* = w^{1/2}r^{1/2}q^2 - 2w + w^{1/2}r^{1/2}q^2 - 3r$$

Therefore, the cost function is:

$$C^* = 2w^{1/2}r^{1/2}q^2 - 2w - 3r$$

(5) Consider the following production function

$$\boldsymbol{x = \frac{10KL}{K+L}}$$

and let w=\$1, r=\$4 and $\bar{K}$=10 units in the short run.

(a) Derive the firm's short-run supply function.

$SRTC = wL + rK$

Substitute w=\$1, r=\$4 and $\bar{K}$=10 into the short-run total cost.

$$SRTC = (1)L + 4(10)$$

$$SRTC = L + 40 \qquad (1)$$

Rearrange the function to find L.

$$x = \frac{10(10)L}{10+L} = \frac{100L}{10+L}$$

$$x(10+L) = 100L$$

$$100L = 10x + xL$$

$$L = \frac{10x}{100} + \frac{xL}{100}$$

$$L = \frac{1}{10}x + \frac{xL}{100}$$

$$L\left(1 - \frac{1}{100}x\right) = \frac{1}{10}x$$

$$L = \frac{\frac{1}{10}x}{\left(1 - \frac{1}{100}x\right)}$$

$$L = \frac{\frac{1}{10}x}{\frac{1}{10}\left(10 - \frac{1}{10}x\right)}$$

$$L = \frac{x}{10 - \frac{1}{10}x} \qquad (2)$$

Substitute Equation 2 into Equation 1.

$$SRTC = L + 40$$

$$SRTC = \frac{x}{10 - \frac{1}{10}x} + 40$$

$$SRTC = x\left(10 - \frac{1}{10}x\right)^{-1} + 40$$

Take the derivative of SRTC with respect to x.

$$\frac{\partial SRTC}{\partial x} = SRMC = \frac{1}{10}x\left(10 - \frac{1}{10}x\right)^{-2} + \left(10 - \frac{1}{10}x\right)^{-1}$$

$$SRMC = \frac{x}{10\left(10 - \frac{1}{10}x\right)^2} + \frac{1}{\left(10 - \frac{1}{10}x\right)}$$

Finding a common denominator.

$$SRMC = \frac{x}{10\left(10 - \frac{1}{10}x\right)^2} + \frac{10\left(10 - \frac{1}{10}x\right)}{10\left(10 - \frac{1}{10}x\right)^2}$$

$$SRMC = \frac{x + 100 - x}{10\left(10 - \frac{1}{10}x\right)^2}$$

$$P_x = SRMC \qquad therefore\ P_x = \frac{x + 100 - x}{10\left(10 - \frac{1}{10}x\right)^2}$$

$$P_x = \frac{10}{\left(10 - \frac{1}{10}x\right)^2}$$

Rearrange the SRMC to solve for x.

$$P_x \cdot 10\left(10 - \frac{1}{10}x\right)^2 = x + 100 - x$$

$$10P_x\left(10 - \frac{1}{10}x\right)^2 = 100$$

$$\frac{10P_x\left(10 - \frac{1}{10}x\right)^2}{10P_x} = \frac{100}{10P_x}$$

$$\left(10 - \frac{1}{10}x\right)^2 = \frac{10}{P_x}$$

$$10 - \frac{1}{10}x = \left(\frac{10}{P_x}\right)^{1/2}$$

$$-\frac{1}{10}x = \left(\frac{10}{P_x}\right)^{1/2} - 10$$

$$-\frac{1}{10}x = \frac{10^{1/2}}{P_x^{\ 1/2}} - 10$$

$$x = \frac{-31.6227766}{P_x^{\ 1/2}} + 100$$

Therefore, short-run supply function is:

$$x = 100 - 31.6227766P_x^{\ -1/2}$$

For completeness, this function can also be written as:

$$x = 100 - \frac{10\sqrt{10}}{\sqrt{P_x}}$$

$$x = 10\left(10 - \frac{\sqrt{10}}{\sqrt{P_x}}\right)$$

(b) Suppose these is a technological change that increases output to
$$\boldsymbol{x = \frac{20KL}{K + L}}$$
Find the new short-run supply function.

$$SRTC = wL + rK$$

Substitute w=\$1, r=\$4 and $\overline{K}$=10 into the short-run total cost.

$$SRTC = (1)L + 4(10)$$

$$SRTC = L + 40 \qquad (1)$$

Rearrange the function to find L.

$$x = \frac{20(10)L}{10 + L} = \frac{200L}{10 + L}$$

$$x(10 + L) = 200L$$

$$L = \frac{10x}{200} + \frac{xL}{200}$$

$$L = \frac{1}{20}x + \frac{xL}{200}$$

$$L\left(1-\frac{1}{200}x\right)=\frac{1}{20}x$$

$$L=\frac{\frac{1}{20}x}{\left(1-\frac{1}{200}x\right)}$$

$$L=\frac{\frac{1}{20}x}{\frac{1}{20}\left(20-\frac{1}{10}x\right)}$$

$$L=\frac{x}{20-\frac{1}{10}x} \qquad (2)$$

Substitute Equation 2 into Equation 1.

$$SRTC=L+40$$

$$SRTC=\frac{x}{20-\frac{1}{10}x}+40$$

Take the derivative of SRTC with respect to x (quotient rule)

$$\begin{aligned} Let\ a &= x \\ b &= 20-\tfrac{1}{10}x \end{aligned}$$

$$SRTC'=\frac{a'b-ab'}{b^2}$$

$$\frac{\partial SRTC}{\partial x}=SRMC=\frac{1\left(20-\frac{1}{10}x\right)-x\left(-\frac{1}{10}\right)}{\left(20-\frac{1}{10}x\right)}$$

$$SRMC=20-\frac{1}{10}x+\frac{1}{10}x$$

$$SRMC=\frac{20}{\left(20-\frac{1}{10}x\right)^2}$$

$$P_x=SRMC \qquad therefore\ P_x=\frac{20}{\left(20-\frac{1}{10}x\right)^2}$$

Rearrange the SRMC to solve for x.

$$P_x\left(20-\frac{1}{10}x\right)^2=20$$

$$\left(20 - \frac{1}{10}x\right)^2 = \frac{20}{P_x}$$

$$\left(20 - \frac{1}{10}x\right) = \sqrt{\frac{20}{P_x}}$$

$$-\frac{1}{10}x = \frac{\sqrt{20}}{\sqrt{P_x}} - 20$$

$$x = \frac{-10\sqrt{20}}{\sqrt{P}_x} + 200$$

$$x = 200 - \frac{-10\sqrt{20}}{\sqrt{P_x}}$$

$$x = 10\left(20 - \frac{\sqrt{20}}{\sqrt{P_x}}\right)$$

$$x = 10\left(20 - \frac{2\sqrt{5}}{\sqrt{P_x}}\right)$$

Therefore, short-run supply function is:

$$x = 20\left(10 - \frac{\sqrt{5}}{\sqrt{P_x}}\right)$$

$$x = 200 - 44.7213595{P_x}^{-1/2}$$

(c) Is the technology change neutral or biased?

A technological change is an improvement in technology that enables a firm to produce more output from a given set of inputs. Technological changes that hold the marginal rates of technical substitution unchanged for every input combination is a neutral technological change. In contrast, when the marginal rates of technical substitution changes, this is referred to as a biased technological change. A neutral technological change is a positive monotonic transformation of a production function. This raises the amount of production represented by each isoquant without adjusting the marginal rates of technical substitution. Therefore, the change in technology between Question 5(a) and Question 5(b) is a **neutral** technological change because MRTS remains constant between the functions. This is demonstrated below.

$$\boldsymbol{x = \frac{10KL}{K + L}}$$

$$Let\ a = 10KL\ and\ b = L + K$$

$$MP_L = \frac{\partial x}{\partial L} = \frac{10K(K + L) - 1\times(10KL)}{(K + L)^2}$$

$$MP_L = \frac{10K^2 + 10KL - 10KL}{(K+L)^2}$$

$$MP_L = \frac{10K^2}{(K+L)^2}$$

$$MP_K = \frac{\partial x}{\partial K} = \frac{10L(K+L) - 1\times(10KL)}{(K+L)^2}$$

$$MP_K = \frac{10L^2}{(K+L)^2}$$

Substitute MP_L and MP_K to find MRTS.

$$MRTS = \frac{MP_L}{MP_K} = \frac{\frac{\partial x}{\partial L}}{\frac{\partial x}{\partial K}} = \frac{\left(\frac{10K^2}{(K+L)^2}\right)}{\left(\frac{10L^2}{(K+L)^2}\right)}$$

$$= \frac{10K^2}{(K+L)^2} \div \frac{10L^2}{(K+L)^2}$$

$$= \frac{10K^2 \times (K+L)^2}{(K+L)^2 \times 10L^2}$$

Therefore, the MRTS is:

$$= \frac{K^2}{L^2}$$

$$\boldsymbol{x = \frac{20KL}{K+L}}$$

Taking the above calculations, MRTS can simply be solved as:

$$MRTS = \frac{\left(\frac{20K^2}{(K+L)^2}\right)}{\left(\frac{20L^2}{(K+L)^2}\right)}$$

$$MRTS = \frac{20K^2}{(K+L)^2} \times \frac{(K+L)^2}{20L^2}$$

Therefore, the MRTS is:

$$= \frac{K^2}{L^2}$$

Therefore, the change in technology between Question 5(a) and Question 5(b) is verified as neutral because the MRTS remains constant.

(d) Now, return to the original production function. What is the short-run shut-down point? Explain your answer briefly.

The original production function is:

$$x = \frac{10KL}{K + L}$$

$$SRVC = wL$$

Substitute $w = 1$ and $L = \frac{x}{10 - \frac{1}{10}x}$

$$SRVC = 1 \times \frac{x}{\left(10 - \frac{1}{10}x\right)}$$

Derive SRAVC:

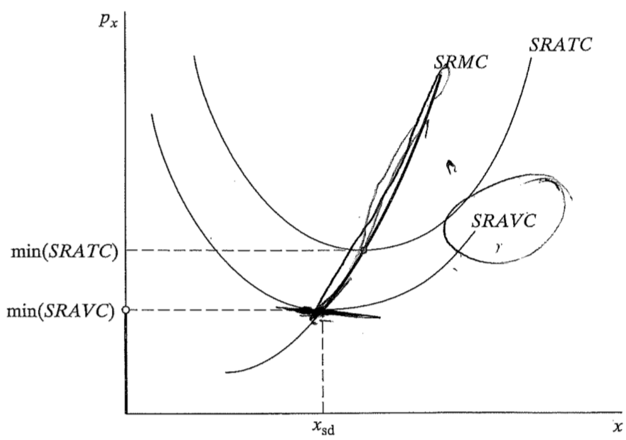
$$SRAVC = \frac{SRVC}{x}$$

$$SRAVC = \frac{\frac{x}{10 - \frac{1}{10}x}}{x}$$

$$SRAVC = \frac{1}{\left(10 - \frac{1}{10}x\right)} \qquad (1)$$

$$SRMC = \frac{10}{\left(10 - \frac{1}{10}x\right)^2} \quad from\ Question\ 5(a) \qquad (2)$$

The short-run shut-down point occurs where $SRMC = SRAVC$ (displayed in diagram).



(Supply curve is bold)

Therefore, let Equation 1 equal Equation 2:

$$\frac{10}{\left(10-\frac{1}{10}x\right)^2}=\frac{1}{\left(10-\frac{1}{10}x\right)}$$

$$10=\frac{\left(10-\frac{1}{10}x\right)^2}{\left(10-\frac{1}{10}x\right)}$$

$$10=\left(10-\frac{1}{10}x\right)$$

$$10=\frac{1}{10}x+10$$

$$\frac{1}{10}x=0$$

Therefore, this verifies the short-run shut-down point.

$$x=0$$

This result is verified via the quadratic equation.

$$\frac{10}{\left(10-\frac{1}{10}x\right)^2}=\frac{1}{\left(10-\frac{1}{10}x\right)}$$

$$10\left(10-\frac{1}{10}x\right)=\left(10-\frac{1}{10}x\right)^2$$

$$100-x=10^2-\frac{1}{10}x\times 10-10\times\frac{1}{10}x+\frac{1}{100}x^2$$

$$100-x=100-2x+\frac{1}{100}x^2$$

$$0=-x+\frac{1}{100}x^2$$

Input into the quadratic equation.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$Let\ a=\frac{1}{100},b=1,c=0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\frac{1}{100}0}}{2\left(\frac{1}{100}\right)}$$

$$x = \frac{1 \pm -1}{\frac{1}{50}}$$

$$x = \frac{2}{\frac{1}{50}} or \frac{0}{\frac{1}{50}}$$

This verifies that short-run shut-down point where *SRMC=SRAVC*.

(e) Using the original production function let the wage vary and the price equal $5 per unit. Derive the marginal revenue product and average revenue production function and the short-run demand function for labour.

Firstly, derive the MP_L.

$$MP_L = \frac{\partial x}{\partial L} \quad \rightarrow \quad x = \frac{10KL}{K + L}$$

Using the quotient rule:

$$MP_L = 10K\frac{\partial x}{\partial L}\left(\frac{L}{K + L}\right)$$

$$\frac{\partial x}{\partial L}(L) = 1$$

$$\frac{\partial x}{\partial L}(L + K) = 1$$

$$MP_L = 10K\frac{1 \cdot (K + L) - 1 \cdot L}{(K + L)^2}$$

$$MP_L = 10K\frac{K + L - L}{(K + L)^2}$$

Therefore, MP_L is equal to:

$$MP_L = \frac{10K^2}{(K + L)^2}$$

To find marginal revenue product of labour:

$$MRP_L = P_x \cdot MP_{L,}$$

$$MRP_L = 5 \cdot \frac{10K^2}{(K + L)^2}$$

$$MRP_L = \frac{50K^2}{(K+L)^2}$$

To find AP_L:

$$AP_L = \frac{x}{L}$$

$$AP_L = \frac{\frac{10KL}{K+L}}{L} \qquad \rightarrow \qquad \frac{10KL(K+L)^{-1}}{L}$$

$$AP_L = 10K(K+L)^{-1}$$

$$AP_L = \frac{10K}{K+L}$$

To find average revenue production function:

$$ARP_L = P_x \cdot AP_L$$

$$ARP_L = 5 \cdot \frac{10K}{K+L}$$

$$ARP_L = \frac{50K}{K+L}$$

Derive the short-run demand function for labour.

$$\pi = P_x x(L,K) - wL - rK$$

$$\frac{\partial \pi}{\partial L} = P_x \frac{\partial x}{\partial L} - w = 0$$

$$w = P_x \cdot MP_L$$

$$w = MRP_L$$

$$w = \frac{50K^2}{(K+L)^2}$$

Substitute $\overline{K} = 10$.

$$w = \frac{50(10)^2}{(10+L)^2}$$

$$w(10+L)^2 = 5000$$

$$(10+L)^2 = \frac{5000}{w}$$

$$10 + L = \frac{\sqrt{5000}}{\sqrt{w}}$$

$$L = \frac{\sqrt{5000}}{\sqrt{w}} - 10$$

Therefore, the short-run demand function for labour is:

$$L = 70.71067812w^{-1/2} - 10$$

(6) A farm produces corn using the following production function.
$\boldsymbol{q = (L^{1/2}K^{1/2})^{4/3}}$

where *L* and *K* are the amounts of labour and capital used. Let w_t and w_k denote the prices of labour and capital respectively. Find the cost function c(w,q)(and in the process find the conditional factors demands L(w,q) and K(w,q)).

$TC = wL + rK$

$max(q) \ = L^{1/2}K^{1/2})^{4/3}$

subject to: $q = (L^{1/2}K^{1/2})^{4/3}$

Therefore, the Lagrange function is:

$\mathcal{L} = wL + rK + \lambda[q - (L^{1/2}K^{1/2})^{4/3}]$

The first order optimality conditions can be derived with respect to L, K and λ.

$$\frac{\partial \mathcal{L}}{\partial L} = w - \frac{2}{3}L^{-\frac{1}{2}} \cdot \lambda \left(\left(L^{1/2} + K^{1/2}\right)^{\frac{1}{3}} \right) \quad ... = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \frac{2}{3}K^{-\frac{1}{2}} \cdot \lambda \left(\left(L^{1/2} + K^{1/2}\right)^{\frac{1}{3}} \right) \quad ... = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = q - (L^{1/2} + K^{1/2})^{4/3} \quad ... = 0 \qquad (3)$$

Divide Equation 1 by Equation 2.

$$\frac{w}{r} = \frac{\frac{2}{3}L^{-\frac{1}{2}} \cdot \lambda \left(\left(L^{1/2} + K^{1/2}\right)^{\frac{1}{3}} \right)}{\frac{2}{3}K^{-\frac{1}{2}} \cdot \lambda \left(\left(L^{1/2} + K^{1/2}\right)^{\frac{1}{3}} \right)}$$

$$\frac{w}{r} = \frac{L^{-1/2}}{K^{-1/2}} = \frac{K^{1/2}}{L^{1/2}}$$

$$L^{1/2} = \frac{K^{1/2} \ \cdot \ r}{w} \qquad (4)$$

$$K^{1/2} = \frac{L^{1/2} \ \cdot \ w}{r} \qquad (5)$$

Substitute Equation 4 into Equation 3.

$$q - \left(\frac{K^{1/2} \cdot r}{w} + K^{1/2}\right)^{4/3} = 0$$

$$q = \left(\frac{K^{1/2} \cdot r}{w} + K^{1/2}\right)^{4/3}$$

$$q^{3/4} = \left(\frac{K^{1/2} \cdot r}{w} + K^{1/2}\right)^{4/3 \times 3/4}$$

$$q^{3/4} = K^{1/2}\left(1 + \frac{r}{w}\right)$$

$$\frac{q^{3/4}}{\left(1 + \frac{r}{w}\right)} = K^{1/2}$$

Therefore, the conditional factor demand function for K is:

$$K = \frac{q^{3/2}}{\left(1 + \frac{r}{w}\right)^2}$$

Substitute Equation 5 into Equation 3.

$$q - \left(L^{1/2} + \frac{L^{1/2} \cdot w}{r}\right)^{4/3} = 0$$

$$q = \left(L^{1/2} + \frac{L^{1/2} \cdot w}{r}\right)^{4/3}$$

$$q^{3/4} = \left(L^{1/2} + \frac{L^{1/2} \cdot w}{r}\right)$$

$$q^{3/4} = L^{1/2}\left(1 + \frac{w}{r}\right)$$

$$\frac{q^{3/4}}{\left(1 + \frac{w}{r}\right)} = L^{1/2}$$

Therefore, the conditional factor demand function for L is:

$$L = \frac{q^{3/2}}{\left(1 + \frac{w}{r}\right)^2}$$

To find the cost function, substitute *K* and *L* into the total cost function.

$$TC = wL + rK$$

$$TC = w\left[\frac{q^{3/2}}{\left(1+\frac{w}{r}\right)^2}\right] + r\left[\frac{q^{3/2}}{\left(1+\frac{r}{w}\right)^2}\right]$$

$$TC = \frac{wq^{3/2}}{\left(1+\frac{w}{r}\right)^2} + \frac{rq^{3/2}}{\left(1+\frac{r}{w}\right)^2}$$

$$TC = \frac{wq^{3/2}}{\left(\frac{r+w}{r}\right)^2} + \frac{rq^{3/2}}{\left(\frac{w+r}{w}\right)^2}$$

$$TC = \frac{wq^{3/2}}{\left(\frac{r+w}{r^2}\right)^2} + \frac{rq^{3/2}}{\left(\frac{w+r}{w^2}\right)^2}$$

$$TC = \frac{r^2 \cdot w \cdot q^{3/2}}{(r+w)^2} + \frac{w^2 \cdot r \cdot q^{3/2}}{(w+r)^2}$$

$$TC = \frac{r^2 \cdot w \cdot q^{3/2} + w^2 \cdot r \cdot q^{3/2}}{(r+w)^2}$$

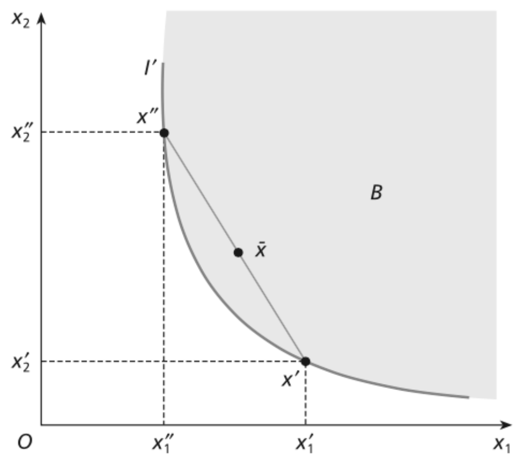
$$TC = \frac{r \cdot q^{3/2} \cdot w(r+w)}{(r+w)^2}$$

Therefore, the cost function is:

$$TC = \frac{r \cdot q^{3/2} \cdot w}{(w+r)}$$

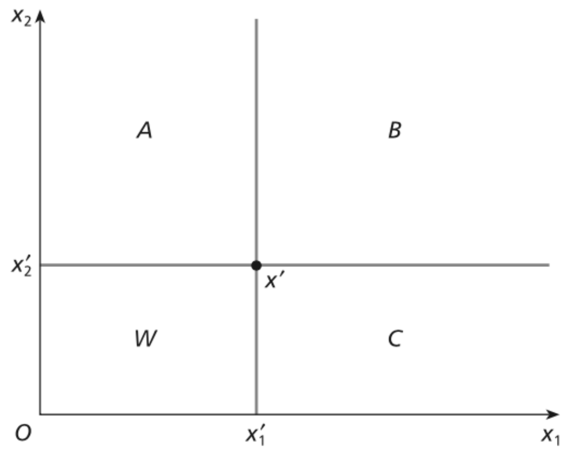
(7) We referred to the following assumptions when developing the theory of a consumer.
(i) strict convexity and (ii) non-satiation. Explain, with the aid of diagrams where necessary;
(a) What is meant by each of these assumptions; and

Strict convexity assumes that given two different consumption bundles on the same indifference curve, any point on the line connecting these two points (excluding the points themselves) will be on a higher indifference curve. The diagram below displays that the consumer always prefers a combination of two consumption bundles, which are indifferent to each other, to either one of the separate bundles (therefore $\bar{x}$ is preferred to x′ and x′′). Furthermore, if we move the consumer along the indifference curve leftward from point x′, reducing the quantity of x_1 by small, equal amounts, we have to compensate to remain on the indifference curve by giving larger increments of x_2 (Gravelle & Rees, 2004). Strict convexity suggests that the smaller the amount of x_1 and the greater the amount of x_2 held by the consumer, the more valuable are marginal changes in x_1 relative to marginal changes in x_2. Therefore, the MRS is decreasing if indifference curves are strictly convex. Finally, strict convexity is not necessary to make an indifference curve, but without it, it is assumed that the two goods are perfect substitutes, which is improbable (Dean, 2016).



In contrast, non-satiation assumes that a consumer always prefers more of a good to less, which results in a higher level of utility. The assumption ascertains a relationship between the quantities of goods in a bundle and its place in preference ordering - the more of each good it contains the better (Gravelle & Rees, 2004). This assumption is true no matter how large the amounts of the goods in the consumption bundle.

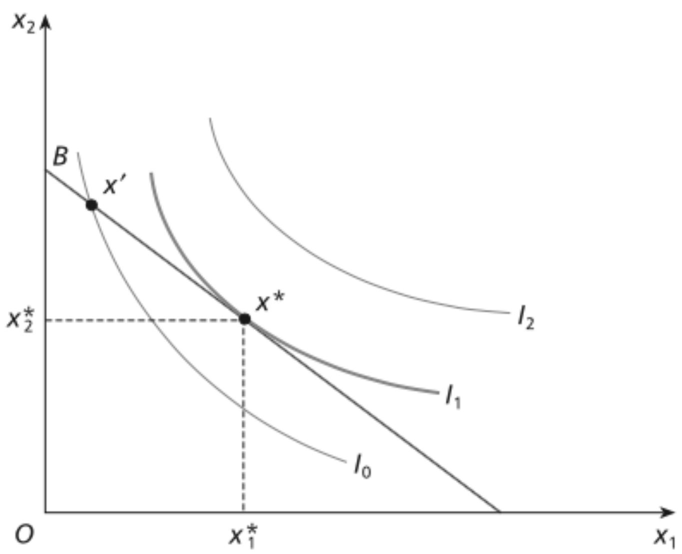
The assumption has two important consequences for the nature of indifference sets, which can be expressed geometrically and is displayed in the diagram below. Assume x_1 and x_2 are goods, and $x' = (x_1', x_2')$ is a consumption bundle. Non-satiation assumes that all bundles in the area *B* must be preferred to x′, and all points in the area *W* must be inferior to x′. Furthermore, points in the indifference set for x′ must lie in areas *A* and *C*. Therefore, we can only move between bundles in the indifference set if we substitute or trade off goods – giving more of one good must require revoking some of the other good in order to maintain the indifference set. Finally, an implication of this assumption is that it suggests there are no 'bad' goods, for example waste, which one would prefer to have less off (Dean, 2016). Furthermore, non-satiation assumes that the consumer is never satiated by any good.



(b) Why each of these assumptions is necessary to derive the results that the consumer maximised his/her utility where his/her budget line is tangential to the highest possible indifference curve.

These assumptions are essential to derive the result that the consumer has maximised their utility tangent to the budget constraint on the highest possible indifference curve. The strict convexity assumption represents that given any combination of indifference curves for a consumer, there will only be one tangency point. Tangency only arises when an indifference curve is strictly convex and the budget constraint is linear, therefore demonstrating the importance of this assumption. Therefore, a linear indifference curve will touch a linear budget constraint at one point, and this results in only one of the goods being consumed - corner solution. Therefore, it is crucial for the strict convexity assumption to hold to ensure the consumer maximises their utility given a consumption bundle.

The non-satiation assumption states that bundles on a higher indifference curve are preferred to those on a lower indifference curve. Therefore, the best consumption bundle for any consumer is the one on the highest possible indifference curve. A higher indifference curve represents a higher level of utility for the consumer. Non-satiation states that a consumer will never reach the 'top' as they continually seek to attain more. However, given the tangent budget constraint, the highest possible indifference curve is generated. The figure below displays a tangency solution where the optimal bundle x^* is on the highest attainable indifference curve I_1, which is tangent to the budget line. At this point, the consumer consumes a combination of both goods (Gravelle & Rees, 2004). Finally, at the optimal point, the slope of the indifference curve is equal to the slope of the budget line. Therefore, these assumptions are critical when developing consumer choice because they help derive the result for maximising consumer utility given a budget constraint.



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