Fourier Analysis on Number Fields: A Summary of Some Theory

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November 1, 2020

Contents

1 Preface

This is a document I have written up as a summary of some of the content covered in a reading course I took with Uri Onn focusing on building the knowledge towards understanding Tate's Thesis.

I have tried to arrange the theory in a way that makes sense to me, so that I can remember it. As such this document contains barely any proofs. Proofs I have included are likely just because I find them fun or interesting.

For the same reason I tend not to worry very much about the specifics of some of the analysis that crops up. Where possible I will sometimes give some vague intuition for why the analysis really should work out the way it does.

Thoughout I will try to make notes of what the story of representations of topological groups looks like in some interesting easier cases such as the compact or finite group case.

I often do not state definitions precisely, fully accurately, or at all, but I do hope that I have preserved the spirit of the material with my ramblings.

This document is written by myself alone, but I spent time discussing the content with Martin Skilleter and Yuzheng Yan.

2 Rep Theory of Locally Compact Groups

2.1 The Measure Algebra of a Locally Compact Group

We will start our story by asking about what sort of structure we can give to the set of complex Radon measures $M(G)$ on a topological group G. This set is already a complex vectorspace, with a norm given by the total variation of a given measure.

Definition 1. A Radon measure μ is a measure defined on the Borel sets of a topological space X with the properties your intuition desires, specifically:

- $\mu(K) < \infty$ for $K \subset X$ compact.
- Borel subsets are well approximated by open sets containing them (outer regular)
- Open subsets are well approximated by compact subsets (inner regular)

This definition should not be memorised (in my opinion). The Radon measures should just be thought of as the reasonable measures on topological spaces, where the measure is compatible with the topology.

Another way of thinking of the complex Radon measures of a space X is via a formulation of the Riesz-Markov theorem, which identifies Radon measures (or measures that are almost as nice) with linear functionals on $C_0(X)$, the set of continuous, compactly supported complex valued functions on X.

This will allow us to define our next piece of additional structure on $M(G)$. For any $\mu, \nu \in M(G)$ define $\mu * \nu$ such that:

$$\int \phi \, d(\mu * \nu) = \int \int \phi \, d\mu d\nu, \qquad \forall \phi \in C_0(G)$$

This definition is justified by the preceding paragraph because $\phi \mapsto \int \int \phi d\mu d\nu$ is a bounded linear functional on $C_0(G)$.

This gives us an associative multiplication $*$ on $M(G)$. It will be commutative if G is. We also have a canonical involution on $M(G)$ defined by:

$$\mu^*(E) := \overline{\mu(E^{-1})}$$

These operations interact nicely together to give $M(G)$ the structure of a Banach $*$-algebra.

This algebra is quite unwieldly and unintuitive, so we tend to identify a copy of $L^1(G)$ inside it via $f \mapsto f dx$, where dx represents integration with respect to the left Haar measure. Under this identification we end up with the following additional structure on $L^1(G)$:

- $f * g(x) = \int f(y) g(y^{-1}x) dy$
- $f^*(x) = \overline{f(x^{-1})} \cdot \Delta(x^{-1})$

Where Δ is the modular function, which essentially measures the impact on the measure of sets when multiplying on the right (since the left Haar-measure need not be right translation invariant). This $L^1(G)$ is what we consider to be the correct analogue of the group algebra $\mathbb{C}[G]$ in the finite group case.

2.2 Group Algebra? What group Algebra?

A natural question to ask at this point is why $L^1(G)$ should be the object to replace $\mathbb{C}[G]$ in this story when one progresses away from finite groups.

Although one answer is given in the following subsections as to why this is the case I don't think it fully explains the philosophical question. One might think that the naive construction of $\mathbb{C}[G]$ might be the correct one even when G need not be finite. In discussing this with my friends we came up with many various answers and observations as to why this is *not* the correct analogue:

- If we consider the free vectorspace on G as functions on G via the identification $g \mapsto \delta_g$ we note that since elements of $\mathbb{C}[G]$ are finite linear combinations of these functions that any element of $\mathbb{C}[G]$ corresponds to a function that is 0 almost everywhere. If we are to view these as living in any space like $L^1(G)$ then they will all be quotiented to the zero function, so this structure is now uninteresting

- If we somehow tried to define a $\mathbb{C}[G]$ that keeps track of infinitely many non-zero values then we end up having issues if we try to define the multiplication

- $\mathbb{C}[G]$ would be infinite dimensional, so the natural topology one might like to put on it makes this vectorspace not locally compact

- $\mathbb{C}[G]$ is a cyclic representation generated by δ_e

All these points go some way towards answering this question, but the best answer that I came up with and have how settled on is the following:

$\mathbb{C}[G]$ is independent of the topology on G, so if we care about representations that interact with the topology on G in any way this can not be the correct construction.

2.3 This New Algebra is so Different! Or is it?

In this part I will take a moment to flesh out a little bit how we can consider $\mathbb{C}[G]$ in the finite group case as $L^1(G)$. Let G be a finite group. This group is a compact topological group with respect to the discrete topology. Then any function on G can be represented as:

$$f = \sum_{g \in G} a_g g \in \mathbb{C}[G]$$

Where $f(x) = a_x \in \mathbb{C}$. Since the normalised Haar measure (which we will denote dx or dg) on a finite group is just the counting measure divided by $|G|$

(to make the measure of the whole group 1). Integrating a function f with respect to the Haar measure gives:

$$\int_G f\, dg = \frac{1}{|G|}\sum_{g\in G} a_g = \frac{1}{|G|}\sum_{g\in G} f(g)$$

What does multiplication look like if we use this notation? Let:

$$a = \sum_{g\in G} a_g g, \quad b = \sum_{g\in G} b_g g$$

Using the typical multiplication law on $\mathbb{C}[G]$ we get:

$$ab = \sum_{g,h\in G} a_g b_h gh$$

If we now view this equality as functions on G we can evaluate the left hand side at some $x \in G$:

$$\begin{aligned}
ab(x) &= \sum_{gh=x} a_g b_h \\
&= \sum_{gh=x} a_g b_{g^{-1}x} \\
&= \sum_{g\in G} a_g b_{g^{-1}x} && \text{since we sum over } g = xh^{-1}, h \in G \\
&= \int_G a_g b_{g^{-1}x} dg \\
&= \int_G a(g) b(g^{-1}x) dg \\
&=: (a * b)(x)
\end{aligned}$$

So multiplication is in fact given by convolution if we view these elements as functions on G.

The involution induced on $\mathbb{C}[G]$ that does not make much of an appearance (as far as I know) in the finite dimensional theory would look like:

$$a^* = \sum_{g\in G} \overline{a_{g^{-1}}} g$$

So we can move forward with our minds at ease knowing that this new definition of the group algebra is actually compatible with what we already know to be the group algebra in the finite case.

2.4 Unitary Representations and Why We Like Them

In the following description of the theory we restrict attention to unitary representations of groups, so some explanation is in order to explain why we do this.

As far as I can tell we actually still *do* care about non-unitary representations, but they do not satisfy certain nice properties, which makes them harder to study. Here are some nice properties of unitary representations.

Theorem 1. *Suppose π is a unitary representation of G in $\mathcal{H}$ and $W \subset \mathcal{H}$ is a closed G-invariant subspace, then $W^\perp$ is G invariant and $\mathcal{H}$ decomposes as $\mathcal{H} = W \oplus W^\perp$*

Proof. Let $u \in W^\perp$ and $w \in W$:

$$\langle \pi(g)u, w\rangle = \langle u, \pi(g^{-1})w\rangle = 0$$

Therefore $W^\perp$ is also G-invariant and $\mathcal{H}$ decomposes as $W \oplus W^\perp$ because W is a closed subspace. $\square$

The above proof is extremely general, and the result is actually generally false for non-unitary representations.

Definition 2. A representation π is cyclic if it is equal to the closed linear span of $\{\pi(g)u : g \in G\}$ for some $u \in \mathcal{H}$.

It is useful to note that any irreducible representation is automatically cyclic, and that we have (via a Zorn's lemma argument) that any unitary representation decomposes into a direct sum of orthogonal cyclic representations.

One observation one can make using cyclic representations is that all irreducible representations of finite groups are finite dimensional. Indeed the cyclic representation corresponding to any vector will give you a finite dimensional G-invariant subspace.

We also get the familiar form of Schur's lemma, though one must use the spectral theory of normal operators on a Hilbert space to prove it. Again here we use the unitarity of our representation in a very big way.

One could reasonably ask if all representations of a locally compact group are unitary, and on the face of it this seems like it might work. Given a representation $\pi : G \to GL(\mathcal{H})$ we can define an inner product on $\mathcal{H}$ by:

$$\langle u, v\rangle_\pi := \int_G \langle \pi(g)u, \pi(g)v\rangle \, dg$$

This is certainly a G-invariant bilinear form due to translation invariance of the Haar measure. The problem comes when trying to prove non-degeneracy. One needs that the measure of the whole group is finite. So in particular, this shows the for *compact* groups that all (continuous) representations are in fact unitary.

2.5 New Dog, Same Old Tricks

In the finite group case we have a bijection:

$$\{\text{(unitary) representations } \pi : G \to U(H)\} \longleftrightarrow \{\text{ring homomorphisms } \pi : \mathbb{C}[G] \to \operatorname{End}(H)\}$$

A few words are in order to explain this. Unitary is in brackets because all representations of finite groups are unitary as per the previous section. We think of H as a Hilbert space because it is a finite dimensional complex vectorspace, and we can equip it with a suitably averaged inner product. We abuse notation by calling the objects in both sets π, but we accept this because one map is literally the restriction of the other.

The bijection is given in the finite case by simply extending your map linearly, or by restriction. Note that the "extension linearly" can be thought of as an integral with respect to the Haar measure:

$$\pi(f) = \pi(\sum_{g \in G} f(g)g) := \sum_{g \in G} f(g)\pi(g) \text{"} = \text{"} \int_G f(g)\pi(g)dg$$

While this works fine because the integral is a finite sum, the analysis will become a lot more involved in the topological group case.

We have actually suppressed a piece of information that becomes useful when the theory gets harder, namely the involution we have on $\mathbb{C}[G]$. This involution is in fact preserved. Recall that since π is a unitary representation we have:

$$\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$$

where * here denotes that adjoint linear map. To check that this map is in fact a morphism of $*$-algebras, or in other words a (Banach) $*$-representation we take $f \in \mathbb{C}[G]$ and compute:

$$\begin{aligned}
(\pi(f))^* &= (\sum_{g \in G} f(g)\pi(g))^* \\
&= \sum_{g \in G} (f(g)\pi(g))^* \\
&= \sum_{g \in G} \overline{f(g)}\pi(g)^* \\
&= \sum_{g \in G} \overline{f(g)}\pi(g^{-1}) \\
&= \sum_{g \in G} \overline{f(g^{-1})}\pi(g) \\
&= \pi(f^*)
\end{aligned}$$

This entire discussion begs the question of whether something similar might work out for the topological group case. In fact, if G is a locally compact group we have a similar bijection between:

$$\{\text{(continuous) unitary representations } \pi : G \to U(H_\pi)\}$$

and

$$\{\text{non-degenerate *-representations } \pi : L^1(G) \to \mathrm{End}\,(H_\pi)\}$$

Again a few words are in order to parse what is going on here. We now restrict to representations that interact well with the topology on G. Likewise we consider only the bounded linear operators on H_π. H_π can now be any complex Hilbert space, not just finite dimensional. We have added a "non-degenerate" condition:

Definition 3. A $*$-representation π of $L^1(G)$ in H is non-degenerate if for all $0 \neq u \in H$ there exists some $f \in L^1(G)$ such that $\pi(f)u \neq 0$

Basically, the only vector that is killed by everything in the image of $L^1(G)$ is the 0 vector.

It's best to just think of this as a technical condition. I believe that one could simply quotient by the subspace of vectors that are simultaneously annihilated by the image of $L^1(G)$ to create a non-degenerate $*$-representation from any arbitrary $*$-representation.

I will give a proof sketch of how the bijection is created, though if one looks carefully enough at the finite group case one could probably guess correctly.

Proof. Given a unitary representation of G in a Hilbert space H one can take any $f \in L^1(G)$ and "integrate over the group" to get an operator:

$$\pi(f) := \int_G f(g)\pi(g)\,dg$$

this has to be interpreted in the correct way. That way is to define this operator weakly, in the sense that we define:

$$\langle \pi(f)u, v\rangle = \int_G f(g)\langle \pi(g)u, v\rangle\,dg \quad \forall\, u, v \in H$$

There is a lot of analysis to check here. For one we have pulled a bounded operator through an integral. We need to check that this operator is in fact a bounded operator on H. I will ignore all these points because I do not find talking about or proving them particularly helpful for me remembering the story of the content.

If one ignores the possible analytic problems it is quite easy after staring at this definition to convince oneself that it should preserve the Banach $*$-algebra structure on $L^1(G)$.

How do we go the other way? In the finite case it was easy, you simply restricted to the canonical subset of G living inside $\mathbb{C}[G]$, but in this case all of the delta functions have been quotiented to 0. The substitute for this is that you approximate the δ_x function by functions supported on smaller and smaller open neighbourhoods of x, then define this limit to be $\pi(x)$. You have to do a fair bit of work to show that this defines a unitary representation, and that the unitary representation you get is unique. $\square$

The last part of this proof sketch (if it is even reasonable ot call it that) can tell us why we don't worry about non-degeneracy in the finite group case. It is because we have the δ_e function living inside our group algebra, and so clearly the representation of $\mathbb{C}[G]$ is non-degenerate.

2.6 So This Bijection. How Good is it Really?

In short, very good. If we look at the finite group case for a second. Let $\pi : G \to U(\mathcal{H})$ be a unitary representation of a finite group. If we follow the bijection laid out in the previous section we get two subsets of $\mathrm{End}\,(H)$, namely $\pi(G)$ and $\pi(\mathbb{C}[G])$. Since $\pi(\mathbb{C}[G])$ is simply taking the linear span of the operators $\pi(G)$ we have the following easy observations:

- $T \in \mathrm{End}\,(\mathcal{H})$ is G-invariant iff it commutes with every operator of $\pi(\mathbb{C}[G])$
- $W \subset \mathcal{H}$ is G-invariant iff it is invariant under every operator of $\pi(\mathbb{C}[G])$

However if we pass to the locally compact analogue we have two seemingly completely different sets of operators $\pi(G)$ and $\pi(L^1(G))$, provided that G is not discrete. However, magnificently we get the completely analogous result:

Theorem 2. *If π is a unitary representation of G in $\mathcal{H}$ then:*

- $T \in \mathrm{End}\,(\mathcal{H})$ *is G-invariant iff it commutes with every operator of* $\pi(L^1(G))$
- $W \subset \mathcal{H}$ *is G-invariant iff it is invariant under every operator of* $\pi(L^1(G))$

The proof uses Von Neumann algebras, which I must admit I am quite ignorant of. My intuition for why this result might be true is that we can approximate the point mass function δ_x as well as we like using functions in $L^1(G)$, which will give us functions in $\pi(L^1(G))$ approximating $\pi(x)$. Commuting with and being invariant under operators would hopefully be something stable under the correct formulation of limits. Conversely everthing in $\pi(L^1(G))$ is the integral of functions in $\pi(G)$, so if there is some justice in the world, that should mean that commuting with $\pi(G)$ means you commute with $\pi(L^1(G))$.

Essentially either side of this bijection keeps track of both the automorphisms of a representation and all of the invariant subspaces, which is great.

2.7 Functions of Positive Type

The next part in the development of the theory has us relating certain functionals on $L^1(G)$ to partucular types of representations we are interested it.

First recall that we can identify the dual of $L^1(G)$ with $L^\infty(G)$ via integration against a bounded function on G. I believe I may be avoiding some of the subtleties of the measure theoretic requirement that make this true, but the Haar measure is a fairly nice measure, so I hope this does work out. If it does not work out then we at least have a subspace of the dual we are working with here.

Now we introduce a new player to the game:

Definition 4. A **function of positive type** is a function $\phi \in L^\infty(G)$ such that:

$$\int (f^* * f)\phi \, d\lambda \geq 0 \, \forall f \in L^1(G)$$

Where $d\lambda$ denotes integration with respect to the Haar measure on G.

I find this definition somewhat unintuitive, hopefully motivation for defining it will follow as the theory reveals itself.

An alternative charaterisation of a function of positive type that I found useful, and can be discovered by changing the order of integration and adjusting some variables is:

$$\int\int f(x)\overline{f(y)}\phi(y^{-1}x)\,dx\,dy$$

The first indication that there might be a relationship between representations and these functionals is the following result, of which I will give a proof sketch, because I find the proof quite pleasing.

Proposition 1. *If π is a unitary representation of G and $u \in \mathcal{H}$, then:*

$$\phi(x) := \langle \pi(x)u, u\rangle$$

is a function of positive type.

Proof.

$$\begin{aligned}
\int\int f(x)\overline{f(y)}\phi(y^{-1}x)\,dx\,dy &= \int\int f(x)\overline{f(y)}\langle \pi(y^{-1}x)u, u\rangle\,dx\,dy \\
&= \int\int \langle f(x)\pi(x)u, f(y)\pi(y)u\rangle\,dx\,dy \\
&= \langle \int f(x)\pi(x)u\,dx, \int f(y)\pi(y)u\,dy\rangle \\
&= \langle \pi(f)u, \pi(f)u\rangle \\
&= \|\pi(f)u\|^2 \geq 0
\end{aligned}$$

So ϕ is a function of positive type □

Note that we use the unitarity of the representation in the second line. Of course there is something to be said regarding why I can treat the integrals the way I have, but I did say that I was just giving a proof sketch after all.

So we see here that a unitary representation gives rise to a function of positive type. Do we dare hope to get a bijection? Well, not as stated, but we can slightly alter our statement to make it so.

Note that the function of positive type we get is only dependent on the cyclic representation generated by u. So maybe we get a bijection to just the cyclic representations? Yes!

Theorem 3. *There is a bijection between functions of positive type on a locally compact topological group G and cyclic (unitary) representations of G.*

I will not prove this bijection, but I will give a sketch of how one can go from a function of positive type to a representation of G.

Proof. Given a function of positive type ϕ, one can define a positive semi-definite Hermetian form $L^1(G)$ by:

$$\langle f, g\rangle_\phi = \int\int f(x)\overline{g(y)}\phi(y^{-1}x)\,dx\,dy$$

Now if we quotient $L^1(G)$ by the minimal subspace N that makes this an inner product, we can then take the Hilbert space completion to turn it into a Hilbert space. We denote this new space $H_\phi = \overline{L^1(G)/N}$. This is the space on which G will act to give a representation, but how?

By doing a calculation we can see that $\langle L_x f, L_x g\rangle_\phi = \langle f, g\rangle_\phi$. Therefore G acts on $L^1(G)$ unitarily by left shift operators. Since this action is unitary it will descend to the quotient $L^1(G)/N$, from which it can be uniquely extended to H_ϕ.

It even turns out that doing this procedure one ends up with a cyclic representation, though one must again take approximations to the identity to find the cyclic vector.

It is also some work to show that these operations are mutually inverse (up to isomorohism). $\square$

2.8 Functions of Positive Type in the Finite Case

To be honest I am still trying to untangle in my head how this story goes in the finite group case, and am unsure whether this is a story I have fully seen before or not.

I know since everything is finite we can identify $L^1(G) = L^\infty(G) = \mathbb{C}[G]$.

I have identified a function of positive type in this case to be $\phi(e) = 1$ and 0 elsewhere, perhaps walking through this example will shed some light on what is going on.

Example 1. Indeed let $f \in L^1(G)$:

$$\begin{aligned}\int\int f(x)\overline{f(y)}\phi(y^{-1}x)\,dx\,dy &= \sum_{x\in G}\sum_{y\in G} f(x)\overline{f(y)}\phi(y^{-1}x)\\ &= \sum_{y^{-1}x=e} f(x)\overline{f(y)}\\ &= \sum_{g\in G} f(g)\overline{f(g)} \geq 0\end{aligned}$$

so $\phi = \delta_e$ is indeed of positive type.

The bilinear form it defines on $\mathbb{C}[G]$ is then:

$$\langle f, g\rangle := \int\int f(x)\overline{f(y)}\phi(y^{-1}x)\,dx\,dy = \sum_{x\in G} f(x)\overline{g(x)}$$

A very familiar form indeed. Exactly what you would get if you defined $G \subset \mathbb{C}[G]$ to be an orthonormal basis. Therefore this form is already an inner product, and is already a Hilbert space because it is finite dimensional. Therefore we have recovered the regular representation (up to maybe an inverse) where G acts on $\mathbb{C}[G]$ by multiplication on the left. This representation is indeed unitary because the action of any element of G simply permutes the orthonormal basis $G \subset \mathbb{C}[G]$. It is also clearly cyclic.

This example I found strangely enlightening, and gives some intuition that if our positive type functions were somehow "less boring" we would end up with more interesting quotients of $\mathbb{C}[G]$.

So, to say something a little more generally, I believe I actually have seen this story before. Consider the representation theory of S_n. It turns out that every irreducible representation of S_n can be described as $\mathbb{C}[S_n]a_T b_T$, where a_T, b_T are dependent on a tableu of size n, a_T on the rows, b_T on the columns. We can view this as the quotient of $\mathbb{C}[S_n]$ by some subspace (specifically, a left ideal of $\mathbb{C}[S_n]$). We don't have to worry about the analysis because everything is finite dimensional. S_n acts on the left by multiplication, or "left translation" if you prefer. I suspect that this quotient must be the correct subspace to quotient by to make the bilinear form defined by some function of positive type non-degenerate.

It turns out that the coeffiecient of the identity for $a_T b_T$ is 1, essentially because the only way to have a permutation preserving the rows composed with a permutation preserving the columns that together gives the identity is to take the identity permutation both times. I believe this must have something to do with the fact that $\|\phi\|_\infty = \phi(1)$ for functions of positive type, and that extreme points of the set of positive functions with norm 1 corespond to irreducible representations, as we will see in the following sections. This is not currently fully correct as I have not established the connection with the choice of function of positive type and the subspace of $\mathbb{C}[G]$ that one must quotient by.

I will finish the exploration of this part of the theory for the case of finite groups with a fun corollary of the general theory:

Corollary 1. *A cyclic representation of a finite group G must have degree $\leq |G|$.*

Proof. By the general theory this cyclic representation $\pi \cong \pi_\phi$ as representations, where ϕ is the function of positive type $x \mapsto \langle \pi(x)u, u\rangle$. But this means that:

$$\dim \mathcal{H}_\pi = \dim \mathbb{C}[G]/N_\phi \leq \dim \mathbb{C}[G] = |G|$$

□

2.9 OK, But What About the Irreducible Representations?

In general irreducible representations are cyclic, but not conversely. Therefore perhaps one might hope that we could narrow our attention from functions of positive type to some smaller subset of functionals on $L^1(G)$ to get a bijection onto the *irreducible* representations of G.

We will need to fix some notation to state the following results in a precise way:

Definition 5. A **convex cone** of a real or complex vectorspace is a subset that is closed under linear combinations with positive coefficients

Definition 6. Let $\mathcal{P} \subset L^1(G)$ be the set of all functions of positive type.

Definition 7. Let $\mathcal{P}_1 := \{\phi \in \mathcal{P} : \|\phi\|_\infty = \phi(1) = 1\}$ and $\mathcal{P}_0 = \{\phi \in \mathcal{P} : \|\phi\|_\infty = \phi(1) \leq 1\}$.

Definition 8. Let $C \subset X$ be a convex set in a Banach space X, then $\mathcal{E}(C)$ is called the set of **extreme points of** C. It is defined to be all the points of C not contained in an open line segment joining points of C.

To give some intuition for the following definitions note that a convex cone always contains $\{0\}$ in it's closure, and in finite dimensions they truly look like cones.

The extreme points of a (finite volume) literal cone would be the tip and the perimeter of the circle at the base of the cone.

$\mathcal{P}$ is a convex cone. $\mathcal{P}_1$ and $\mathcal{P}_0$ are bounded convex sets.

We are now ready to state a beautiful theorem that I will not prove.

Theorem 4. *If $\phi \in \mathcal{P}_1$ then the cyclic representation corresponding to ϕ is irreducible if and only if $\phi \in \mathcal{E}(\mathcal{P}_1)$.*

The proof I find somewhat technical and unenlightening. Schur's lemma makes a notable appearance in proving the forwards direction, so we have well and truly used the fact that our representations are unitary a lot to get to this point.

Note that we don't actually lose any information by restricting to functions of positive type with norm 1. Scaling a function of positive type will simply lead to an appropriately scaled inner product on $L^1(G)$, leading eventually to an isomorphic representation.

One might naturally ask why we care about this bijection. The best answer I can come up with is that it now allows us to apply tools from functional analysis to representation theory, since we have packaged all the information about cyclic representations of G into a bounded convex subset of a Banach space X

Right here I would love some geometric argument for why there are only finitely many irreducible representations of a finite group, using the finite dimensionality of $\mathbb{C}[G]$, but I don't believe that the argument goes through in an enlightening way unfortunately...

From here there follows a story about different topologies one might put on $\mathcal{P}_1$ being equivalent, in particular the weak* topology one gets by viewing $\mathcal{P}_1 \subset L^\infty(G) \subset (L^1(G))^*$ and the topology that corresponds to uniform convergence on compact subsets of $L^1(G)$. Big cornerstone theorems in functional analysis like Banach-Alaoglu and Krein-Milman make an appearance. An argument about the density of compactly supported functions of positive type inside multiple different function spaces all comes together to prove a striking result called the Gelfand-Raikov theorem:

Theorem 5. *If G is anly locally compact group, then the irreducible unitary representations of G separate points on G.*

There is a lot one could say about this result. The result itself provides motivation for generalising representation theory to this context, since we now know that the representation theory must distinguish between all elements of the group.

For the case of G compact we have that all representations are finite dimensional, so we can consider the functions that pick matrix entries of a given representation. These still separate points and are all all elements of $C(G, \mathbb{C})$. The Stone-Weierstrass theorem then tells us that theses functions are dense!

The result for finite groups is already known, since we can always take the regular representation of G (which is faithful) and decompose it into its irreducible components.

For commutative groups this tells us that the characters of the group separate points. This I suppose is a nice segway into our next section.

3 Representation Theory of Locally Compact Abelian Groups and Pontrjagin Duality

For this section we restrict our attention to locally compact abelian groups.

3.1 Why the Dual Group

In the following parts there is a huge focus on the dual of a group G, defined as a group to be the set of homomorhisms from $G \to S^1$, with multiplication defined pointwise. I thought I would take a small section here to talk about why we take S^1 to play this role as opposed to any other locally compact abelian group.

We wish to study representations of an abelian group G, in particular the irreducible ones. Schur's lemma tells us that all of the irreducible representations of an abelian group are one dimensional, so we can write our theoretical irrep as $\chi : G \to GL(\mathbb{C})$. Now we can easily identify $GL(\mathbb{C}) \cong \mathbb{C}^\times$. However, recall that we are still restricting to unitary representations, the natural inner product on $\mathbb{C}$ is of course $\langle z, w\rangle = z\overline{w}$. If we require that $\chi(g)$ defines a unitary representation then we find:

$$\overline{\chi(g)} = \chi(g)^* = \chi(g)^{-1}$$

Therefore $|\chi(g)|^2 = 1$, so $\chi(g) \in S^1$. Since g was arbitrary we can in fact write:

$$\chi : G \to S^1$$

So really we are secretly studying the representation theory of locally compact abelian groups.

3.2 The Dual Group

We begin with a definition, and follow up with why it is in fact a reasonable definition.

Definition 9. For a locally compact abelian group G define the locally compact group, named its **dual group** to be $\hat{G} := \{\chi : G \to S^1\}$ under pointwise multiplication and the topology corresponding to uniform convergence on compact sets.

Ok, but now $\hat{G}$ can also be considered as a subset of $(L^1(G))^*$ by considering these characters as representations in their own right and using the bijection between unitary representations and non-degenerate *-representations of $L^1(G)$. Specifically for $\xi \in \hat{G}$:

$$\xi(f) = \int \langle \xi, x\rangle f(x)\, dx$$

So ξ defines a multiplicative functional on $L^1(G)$. An argument shows that in fact all multiplicative functionals arise in this way!

So why don't we put the weak* topology on this group instead? The answer is that we do, the topologies just happen to coincide because of the same functional analysis story that occurred just before we reduced to the case of abelian groups. That same story tells us that if we view this inside $L^\infty(G)$ that $\hat{G} \cup \{0\}$ is compact, from which we see that $\hat{G}$ is indeed locally compact (I believe we need some very mild separation axiom for the topological group, but let's not worry about that).

I will state a couple results that I like regarding what happens when our group is compact, and then we will move on to actually calculating these things.

Proposition 2. *If G is compact then $\hat{G}$ is orthonormal in $L^2(G)$.*

We need to take a second to parse this statement. Since G is compact we may take the Haar measure of G to be 1. Therefore $L^\infty(G) \subset L^2(G)$. The natural inner product on $L^2(G)$ is of course integration of the product of functions, where one of them is the complex conjugate. I think of this as the correct generalisation of the orthogonality of characters of finite groups to locally compact abelian groups, and if we restrict to finite abelian groups it is exactly what we get.

Proposition 3. *If G is compact then $\hat{G}$ is dicrete and vice versa.*

This is a neat result and will provide us with a sanity check when we begin our calculations.

3.3 But What Is the Dual Group Actually?

I believe that from here the best way to get a feel for the content is to state a few facts, and then use them to calculate dual groups for familiar groups we already know. I will state a proposition that is rather easy to prove because it will aid us in our calculations. I probably don't need to calculate so many dual groups, but the calculations are just so fun.

Proposition 4. *The dual group distributes over finite products $G_1 \hat{\times} G_2 \cong \hat{G}_1 \times \hat{G}_2$*

In the following examples I will only properly justify the isomorphisms as groups, because I don't want to worry about the topology too much.

Example 2. $\hat{\mathbb{R}} \cong \mathbb{R}$

Proof. Take a character $\chi : \mathbb{R} \to S^1$. Since $\mathbb{R}$ is simply connected we can lift this to the universal covering space of S^1, namely $\mathbb{R}$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp(2\pi i\cdot)"] \\ \mathbb{R} \arrow[r, "\chi"'] \arrow[ur, dashed, "\phi"] & S^1 \end{tikzcd}$$

But now the lifted map ϕ is a continuous additive map from $\mathbb{R}$ to $\mathbb{R}$, so by the very first assignment question in algebra 3 (!!) we know that $\phi(x) = cx$ for some $c \in \mathbb{R}$.

Therefore we can now read off the composition to get:

$$\chi_c(x) = \exp(2\pi icx)$$

We use the subscript to denote which character it is. Now note that:

$$\chi_c \cdot \chi_d = \chi_{c+d}$$

Therefore, at least as a group without a topology the characters of $\mathbb{R}$ look like $\mathbb{R}$ $\square$

Example 3. $\hat{S}^1 \cong \mathbb{Z}$

Proof. We identify S^1 with $\mathbb{R}/\mathbb{Z}$. Therefore the characters of S^1 are in bijection with the characters of $\mathbb{R}$ which are trivial on $\mathbb{Z}$. In other words we require:

$$1 = \chi_c(1) =: \exp(2\pi ic)$$

This is only the case for $c \in \mathbb{Z}$. Therefore we have an identification $\hat{S}^1 \cong \mathbb{Z}$ $\square$

Example 4. $\hat{\mathbb{Z}} \cong S^1$

Proof. $\mathbb{Z}$ is the free abelian group on 1 generator and equipped with the discrete topology, therefore all maps out of $\mathbb{Z}$ are continuous. A character of $\mathbb{Z}$ is determined exactly by $\chi(1)$, where $\chi(1)$ can be anything in S^1, and again if we denote $\chi_\alpha(n) = \alpha^n$ then:

$$\chi_\alpha \chi_\beta = \chi_{\alpha\beta}$$

so as a group $\hat{Z} \cong S^1$. $\square$

Example 5. $\hat{\mathbb{Z}}/n \cong \mathbb{Z}/n$

Proof. From the $\mathbb{Z}$ case we know the characters of $\mathbb{Z}/n$ are the characters with n in their kernel. Therefore they are exactly the χ_α such that $\alpha^n = 1$. Therefore the character group is isomorphic to the n-th roots of unity (since all roots of unity have norm 1). This is isomorphic to $\mathbb{Z}/n$ so we are done. $\square$

Example 6. $\hat{\mathbb{R}}^\times_{>0} \cong \mathbb{R}^\times_{>0}$

Proof. All we have to do to prove this one is note that $(\mathbb{R}, +) \cong \mathbb{R}^\times_{>0}$ via the maps ln and exp. We know that these maps are continuous because we learnt how to differentiate these maps in high school (!). $\square$

Example 7. $\hat{\mathbb{R}}^\times \cong \mathbb{R}^\times$

Proof. We just need to note that:

$$\hat{\mathbb{R}}^\times \cong \{1, -1\} \times \mathbb{R}^\times_{>0} \cong \mathbb{Z}/2 \times \mathbb{R}^\times_{>0}$$

One gets this product decomposition by considering the absolute value function. Then since we have written $\mathbb{R}^\times$ as a product of self dual groups it must be self dual! $\square$

Example 8. $\hat{\mathbb{C}}^\times \cong \mathbb{Z} \times \mathbb{R}^\times_{>0}$

Proof. We consider the isomorphism given by writing non-zero complex numbers in polar form. We get:

$$\mathbb{C}^\times \cong \mathbb{R}^\times_{>0} \times S^1$$

where the first factor keeps track of the modulus, while the second factor keeps track of the argument of the complex number (hence is isomorphic to S^1). One checks that this map is in fact a homeomorphism and then we are ready to use the stated proposition:

$$\hat{\mathbb{C}}^\times \cong \widehat{\mathbb{R}^\times_{>0} \times S^1} \cong \hat{\mathbb{R}}^\times_{>0} \times \hat{S}^1 \cong \mathbb{R}^\times_{>0} \times \mathbb{Z}$$

So we have found the dual group we were searching for. $\square$

Example 9. $\hat{\mathbb{Q}}_p \cong \mathbb{Q}_p$. This is a bit more involved to prove, but I thought I should put it in here because I suspect the result will be very important going forward. We will prove this in the next section.

To sanity check ourselves somewhat we note that the duals of compact spaces we calculated were discrete, and vice versa.

3.4 Fourier Analysis in Generality

In this section we see a big payoff for all of the abstract Banach algebra theory developed in the early chapters of [FOL], including the Gelfand transform.

Note that $L^1(G)$ is a commutative Banach $*$-algebra because G is commutative, we identify $\hat{G}$ with the space of multiplicative functionals, aka the spectrum of $L^1(G)$ via for $\xi \in \hat{G}$:

$$f \mapsto \overline{\xi}(f) = \int \overline{\langle \xi, x \rangle} f(x)\, dx$$

Now with this identification of the spectrum we are ready to write down the Gelfand transform. In this setup we call it the **Fourier transform** $\mathcal{F} : L^1(G) \to C_0(\hat{G})$:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int \overline{\langle \xi, x \rangle} f(x)\, dx$$

All of the abstract spectral theory done in earlier parts of the book makes the proof of the following very quick:

Theorem 6. *The Fourier transform is a norm-decreasing $*$-homomorphism from $L^1(G)$ to $C_0(\hat{G})$, and it's range is a dense subspace in $C_0(\hat{G})$*

Recall that we initially defined the group algebra $L^1(G)$ as a subalgebra of the measure algebra $M(G)$ on G. We can play a similar Fourier transform game with the dual group $\hat{G}$. Let $\mu \in M(\hat{G})$, then define a bounded continuous function ϕ_μ on G by:

$$\phi_\mu(x) = \int_{\hat{G}} \langle x, \xi \rangle d\mu(\xi)$$

This ends up being a linear, norm-decreasing injection $M(\hat{G}) \to L^\infty(G)$.

Again we ask, maybe we can restrict to an interesting subset of bounded functions on G to make this a bijection and indeed Bochner's theorem tells us among other things that there is a bijection between $M(\hat{G})$ and the linear span of $\mathcal{P}(G)$, the functions of positive type make another appearance!

If we do some more work, involving showing that certain measures are equal then we get our first rendition of the Fourier inversion formula:

Theorem 7. *(Fourier Inversion I) If f is in the span of $\mathcal{P}(G)$ and $L^1(G)$, and also $\hat{f} \in L^1(\hat{G})$ then for the Haar measure on $\hat{G}$ we get:*

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$$

But this is not well phrased as stated because the Haar measure is only well defined up to scaling. To make sense of this we introduce the **dual measure** on $\hat{G}$ to be the Haar measure on the dual that makes this true, after having fixed a Haar measure on G.

My favourite result relating to these is that if G is a discrete topological group equipped with the counting measure then $\hat{G}$ is compact and the dual measure is simply the Haar measure ensuring that $|G| = 1$.

One of the parts that I think makes this theory hard to understand is the generality of it. If we apply this to $\mathbb{R}$ we recover standard Fourier analysis on $\mathbb{R}$. If we apply this to a finite group $\mathbb{Z}/n$ we recover the discrete Fourier transform, and applying it to the duality pair $\mathbb{Z}$ and S^1 we get a strong statement about the structure of functions on S^1. Each of these one could spend a lot of time trying to understand individually.

The last result from this section I wish to mention is the Plancherel Theorem:

Theorem 8. *Plancherel's Theorem The Fourier transform on $L^1(G) \cap L^2(G)$ extends to a unitary isomorphism $L^2(G) \cong L^2(\hat{G})$.*

I find one of the most surprising applications of this to $\mathbb{Z}$ and S^1. $L^2(\mathbb{Z})$ are just sequences indexed by $\mathbb{Z}$ that are square integrable. Under the Fourier transform these become the coefficients of the characters of S^1, $x \mapsto x^n$. So every periodic square integrable function can be well approximated by Laurent polynomials.

3.5 Pontrjagin Duality (at last!)

We begin by stating the theorem of Pontrjagin duality.

Theorem 9. *Pontrjagin Duality Any locally compact abelian group G is canonically isomorphic to its double dual.*

The proof is a lot of keeping track of the various topologies to make sure that the canonical map $G \to \hat{\hat{G}}$ is indeed a homeomorphism.

The map is the typical map into the double dual of a space, evaluation at an element. I find it particularly pleasing that this means that the map is natural, once one figures out the correct induced map on the double duals.

This serves as a nice sanity check for all the dual groups we calculated in the dual groups sections, and gives us a nice description of the dual of $\mathbb{Z} \times \mathbb{R}^{\times}_{>0}$ as $\mathbb{C}^{\times}$.

Once we have this result we can prove a particularly clean statement of Fourier Inversion.

Theorem 10. *(Fourier Inversion II) If $f \in L^1(G)$, and $\hat{f} \in L^1(\hat{G})$ then for the dual Haar measure on $\hat{G}$ we get:*

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$$

almost everywhere, and if f is continuous then this relation holds everywhere.

The right way to view this statement is as the claim that $f(x) = \hat{\hat{f}}(x^{-1})$, where the inverse x is an artifact of our choice of identification of $\hat{G}$ with the spectrum of $L^1(G)$.

The last thing I want to talk about is the form of Hahn-Banach theorem for locally compact abelian groups, which I found very interesting:

Corollary 2. *If H is a closed subgroup of G, where G is commutative and locally compact, then any character defined on H extends to a character on G.*

I find it cool to witness this theorem in different special cases.

One can take subgroups of $\mathbb{Z}$, which must be of the form $n\mathbb{Z}$. clearly this is isomorphic to $\mathbb{Z}$, so we know any character on $n\mathbb{Z}$ is defined by $\chi(n) = \alpha$. We can extend this character by picking any n-th root of α, to which we will send 1. This works because $\mathbb{C}^{\times}$ is closed under daking nth roots in $\mathbb{C}$.

One can take $\mathbb{Z} \subset \mathbb{R}$ to see that every continuous homomorphism $\mathbb{Z} \to S^1$ extends to a map from $\mathbb{R}$.

The example I find most surprising is taking $\mathbb{R}^{\times}_{>0} \subset \mathbb{C}^{\times}$. Any character on the positive real axis must extend to a character on all of $\mathbb{C}^{\times}$!

4 p-adic groups

4.1 Introduction

This next section was written after the midsemester break. I thought it was best to keep the whole summary in one document. At this point in the course we deviated from following a strict textbook and started focusing on learning some of the concepts left that we needed for Tate's thesis. As such it is a little more ad hoc and flows a little less than some of the previous discourse, but hopefully it is still helpful.

4.2 p-adic groups

We take this section to calculate the duals of $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times$. I will give a proof sketch for $\mathbb{Q}_p$. For $\mathbb{Q}_p^\times$ I will state the answer and write out the progress we made on it ourselves, as this is likely to be the mathematics that I rememeber for this part of the course.

Example 10. $\hat{\mathbb{Q}}_p \cong \mathbb{Q}_p$

Proof. We begin by finding a single character, which we call χ_1. We define this character by:

$$\chi_1\left(\sum_{i=-N}^{\infty} a_i p^i\right) := \exp\left(2\pi i \sum_{i=-N}^{-1} a_i p^i\right)$$

We may view this as a composition:

$$\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}[1/p]/\mathbb{Z} \overset{\exp(2\pi i\cdot -)}{\to} S^1$$

The second map is well defined because exp is periodic in $\mathbb{Z}$. Writing it as a composition like this shows that the map is continuous. So we have found a continuous character. We can now define χ_t for any $t \in \mathbb{Q}_p$ by:

$$\chi_t(x) := \chi_1(tx)$$

Now the hard part starts, we have to show that every character arises as χ_t for some $t \in \mathbb{Q}_p$.

The first step is to take advantage of a "tension" going on between profinite groups and smooth groups. I am not defining these terms because this is nothing formal, just a guiding principle. The open normal subgroups around the identity in a profinite group form a base for the topology, in contrast to Lie groups like S^1 where we can take an open subset around the identity with no non-trivial subgroups inside it.

Doing this let $\chi : \mathbb{Q}_p \to S^1$ be a character. We take the preimage of a small neighbourhood of $1 \in S^1$. This must contain some ball $B(0, p^{-k}) \subset \mathbb{Q}_p$, but since this subset is a subgroup we must have that $\chi(B(0, p^{-k})) = \{0\}$.

This ball inside $\mathbb{Q}_p$ will be the starting point for constructing the $t \in \mathbb{Q}_p$ that realises χ.

The next insight is to notice that χ is fully determined by where it sends the powers of p. If p^{-k} is the largest power of k on which χ is non-trivial we have that:

$$\chi(p^{-k})^p = \chi(p^{-k+1}) = 1$$

from which $\chi(p^{-k})$ is a non 1 p-th root of unity. That is:

$$\chi(p^{-k}) = \exp(2\pi i c_{-k} p^{-1})$$

for $c_{-k} = 1, 2, ..., p-1$. This will be the lowest non-zero power of p in t. The rest follows by an induction argument, but I think that the most important thing to take away is the explicit identification:

$$t \mapsto \chi_t$$

which we will use. $\square$

Example 11.

$$\hat{\mathbb{Q}}_p^\times \cong \begin{cases} S^1 \times \hat{\mathbb{Z}}_p \times \mathbb{Z}/(p-1), & p \neq 2 \\ S^1 \times \hat{\mathbb{Z}}_p \times \mathbb{Z}/2 & p = 2 \end{cases}$$

Proof. I will not prove this result, but I will sketch the progress we made towards it in our discussion sessions.

The first thing to know about the statement is that $\mathbb{Z}_p$ can be identified as the Prufer group, the group of p^k roots of unity in S^1 for any $k \in \mathbb{N}$.

The first step we took was finding a product decomposition:

$$\mathbb{Q}_p \cong \mathbb{Z} \times \mathbb{Z}_p^\times$$

The $\mathbb{Z}$ factor keeps track of the lowest non-zero power of p in the p-adic rational, while $\mathbb{Z}_p^\times$ are all of the elements of $\mathbb{Q}_p$ that start with a non-zero constant value.

Therefore we have reduced to finding $\hat{\mathbb{Z}}_p^\times$. We can note via a small subgroup argument that any given character is trivial on a neighbourhood of the form $1 + p^k\mathbb{Z}_p$ for some $k \in \mathbb{N}$.

The other form of progress we made was by noticing that $\mathrm{Hom}\,(-, S^1)$ is a contravariant autoequivalence of the category of locally compact abelian groups. In particular it sends limits to colimits, and vice versa. Turning colimits into limits is very reasonable it expect since maps out of a colimit are "easy" to define. The other direction only holds in this case because of Pontryagin duality:

$$\begin{aligned} \mathrm{Hom}\,(\varprojlim A_i, S^1) &\cong \mathrm{Hom}\,(\varprojlim \mathrm{Hom}\,(\mathrm{Hom}\,(A_i, S^1), S^1), S^1) \\ &\cong \mathrm{Hom}\,(\mathrm{Hom}\,(\varinjlim \mathrm{Hom}\,(A_i, S^1), S^1), S^1) \\ &\cong \varinjlim \mathrm{Hom}\,(A_i, S^1) \end{aligned}$$

Where we used one direction + pontryagin duality to get the other.

With this in mind we tried to calculate $\hat{\mathbb{Z}}_p^\times$:

$$\begin{aligned}\hat{\mathbb{Z}}_p^\times &\cong \operatorname{Hom}(\varprojlim(\mathbb{Z}/p^k)^\times, S^1)\\ &\cong \varinjlim \operatorname{Hom}((\mathbb{Z}/p^k)^\times, S^1)\\ &\cong \varinjlim \operatorname{Hom}(\mathbb{Z}/p^{k-1} \times \mathbb{Z}/(p-1), S^1)\end{aligned}$$

The first isomorphism is the definition of the dual group, the second definition is the fact we just derived, and the third is a general fact about the group of units of $\mathbb{Z}/p^k$ when $p \neq 2$.

At this point it starts looking plausible that the result is true, but we had too much trouble figuring out which system this actually is taking the colimit over to be confident of our proof. $\square$

5 Schwartz-Bruhat functions

5.1 Why do we want these functions?

This next little bit will be dedicated to the Schwartz-Bruhart functions on $\mathbb{R}$ and $\mathbb{Q}_p$. As far as I can tell this is the problem that this particular class of functions are introduced to solve.

Suppose that we have the locally compact abelian group $\mathbb{R}$ and we have the character $\chi_1(x) = \exp(2\pi i x)$. This character is a continuous complex valued function on $\mathbb{R}$, we would like to be able to take it's Fourer transform, but it's fourier transform "wants" to be the point mass at 1. We want the class of functions we consider to be closed under the fourier transform, so we have two options, either allow the point mass at 1, in which case we will be going to a space of distributions or measures on $\mathbb{R}$. Or we disallow χ_1. This restricted class of functions are going to be what we call the Schwartz-Bruhat functions on $\mathbb{R}$ and $\mathbb{Q}_p$. They have certain properties and also a neat characterisation in each field.

5.2 The Real Case

Definition 10.

$$S(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}, \mathbb{C}) : \forall\, \alpha, \beta \in \mathbb{N}, \sup_{x\in\mathbb{R}} |x|^\alpha |\partial^\beta f(x)| < \infty\}$$

Intuitively think of these as functions that decay at infinity faster than any polynomial. A typical example is $f(x) = e^{-x^2}$. One can see pretty quickly from the definition that these functions form a sub-algebra of $C^\infty(\mathbb{R}, \mathbb{C})$ (with pointwise multiplication). They are also closed under differentiation and multiplication by x^n.

Note that these functions are quite restrictive, they do not include the identity, and the only constant function they include is the zero function.

These are closed under a few other operations that we care about:

- Fourier Transform
- Convolution

It will prove a lot of work to show that the Schwartz functions are closed under Fourier transform, but once one shows this we can use that $S(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and the Plancherel theorem to say that the fourier transform gives an automorphism of $S(\mathbb{R})$ (I think this is correct. We only need that the fourier transform is surjective for this application anyway). Then the formula:

$$\widehat{f * g}(\xi_t) = \hat{f}(\xi_t)\hat{g}(\xi_t)$$

Will tell us that $S(\mathbb{R})$ is also closed under convolution. Recall that we identify $\mathbb{R}$ with $\hat{\mathbb{R}}$ via $t \mapsto \xi_t$ and $\xi_t(x) = e^{2\pi i t x}$.

The way we prove that $S(\mathbb{R})$ is closed under Fourier transform is via a pair of formulas relating multiplication by x to differentiation via the Fourier transform. Let us begin:

$$\begin{aligned}
\widehat{\frac{df}{dx}}(\xi_t) &= \int_{\mathbb{R}} \overline{\xi_t(x)}\frac{df}{dx}(x)\,dx \\
&= \int_{\mathbb{R}} e^{-2\pi itx}\frac{df}{dx}(x)\,dx \\
&= \left[e^{-2\pi itx}f(x)\right]_{-\infty}^{\infty} + \int_{\mathbb{R}}(2\pi it)e^{-2\pi itx}f(x)dx \quad \text{Integration by parts} \\
&= 0 + 2\pi it\hat{f}(\xi_t)
\end{aligned}$$

The last line has that zero term because f is a Schwartz function. Hence we have the following:

$$\widehat{\frac{df}{dx}} = 2\pi it\hat{f}$$

A "dual" calculation involving pulling a differentiation through an integral yields:

$$\widehat{xf} = \frac{i}{2\pi}\frac{d\hat{f}}{dt}$$

there may be a factor of i wrong here but it is irrelevant for the purposes of what we wish to show. Note that the 2π factors coming out here are an artifact of our choice of identification of $\mathbb{R}$ with it's dual group.

Using these formulas we get that if $f \in S(\mathbb{R})$:

$$\begin{aligned}
\sup_{t\in\mathbb{R}}|t^\alpha\partial^\beta\hat{f}(t)| &= \sup_{t\in\mathbb{R}}|t^\alpha(2\pi i)^\beta\widehat{t^\beta f}(t)| \\
&= \sup_{t\in\mathbb{R}}|(2\pi i)^{\beta-\alpha}\widehat{\frac{d^\alpha(t^\beta f)}{dt}}(t)| \\
&= (2\pi)^{\beta-\alpha}\sup_{t\in\mathbb{R}}|\widehat{\frac{d^\alpha(t^\beta f)}{dt}}(t)| \\
&\leq (2\pi)^{\beta-\alpha}\|\frac{d^\alpha(t^\beta f)}{dt}\|_{L^1}
\end{aligned}$$

Now since we have established that $S(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$, closed under multiplication by t and differentiation we have shown that this quanitity is bounded.

A slogan for this proof I think would go something along the lines of "the Fourier transform allows differentiation and multipication by x to interact".

There are certainly details here missing but hopefully there is enough that if my future self comes back and is interested then they can iron out all the remaining kinks.

5.3 The p-adic Case

The definition of the Schwartz-Bruhat functions on $\mathbb{Q}_p$ take a very different form. I will omit many details here because we went so in depth with the real case.

Definition 11.

$$\begin{aligned} S(\mathbb{Q}_p) &:= \{f \in C(\mathbb{Q}_p, \mathbb{C}) : f \text{ is locally constant and compactly supported}\} \\ &= \{f = \sum_i c_i \mathbb{1}_{a_i + p^{k_i}\mathbb{Z}_p} | c_i \in \mathbb{C},\, a_i \in \mathbb{Q}_p,\, k_i \in \mathbb{Z}\} \end{aligned}$$

where the sum in i is finite.

Just a quick thing to note is that $\mathbb{Q}_p$ is totally disconnected, so the restriction locally constant is not as restrictive as it sounds if you are thinking in a Euclidean way. The locally constant functions on $\mathbb{Q}_p$ separate points, if it helps with intuition.

One can clearly see that this is family closed under addition and scalar multiplication. Pointwise multiplication is equally clear.

To check that the space is closed under convolution it is enough to check that the convolution of two such indicator functions is in $S(\mathbb{Q}_p)$.

These also happen to be closed under Fourier transform, though at the minute it is not clear to me why this is the case.

5.4 But They Look so Different?

These two different classifications feel completely different to one another. We asked Uri why locally constant compactly supported is the thing to replace smooth and decaying at infinity in the real case.

Uri gave an answer I don't think I fully understood at the time or fully remember now, but basically the answer comes down to representation theory again. The functions you get by taking matrix entries of representations give you smooth functions in the real case. In the $\mathbb{Q}_p$ case any character is trivial outside of a compact neighbourhood of 0, so this gives some vague reason why compact support comes into play. Locally constant is still a mystery to me though.

5.5 Some Important Calculations

Here is a fun fact, did you know that $f(x) := e^{-x^2}$ is a fixed point under the additive fourier transform on $\mathbb{R}$? The proof is generally a calculation using integration by parts, but I have found a cute trick that will allow us to show it without having to remember how to do integration by parts. Start with the following observation:

$$f'(x) = -2\pi x e^{-\pi x^2} = -2\pi x f(x)$$

Now taking the fourier transform of both sides allows us to apply the formulas we derived about the fourier transform and how it related multiplication by x and differentiation:

$$\begin{aligned}
\widehat{f'} &= -2\pi \widehat{xf(x)} \\
2\pi it\hat{f} &= -2\pi \frac{i}{2\pi}\frac{d\hat{f}}{dt} \\
-2\pi t\hat{f} &= \frac{d\hat{f}}{dt} \\
-\pi t^2 &= \ln(\hat{f})
\end{aligned}$$

We throw caution to the wind here regarding domains of definition and solve the differential equation using separation of variables. Taking exp of both sides we get:

$$\widehat{f}(t) = e^{-\pi t^2} = f(t)$$

Thus we have successfully demonstrated that f is a fixed point under the additive fourier transform on $\mathbb{R}$.

There is also a fixed point in the $\mathbb{Q}_p$ case. $\mathbb{1}_{\mathbb{Z}_p}$ is a fixed point under the fourier transform on $\mathbb{Q}_p$. These fixed points will play a big role in our grand finale, so pay attention.

We will use the computation of the dual of $\mathbb{Q}_p$ found in a previous chapter. All integrals are with respect to the normalised Haar measure on $\mathbb{Q}_p$ making $|\mathbb{Z}_p| = 1$.

$$\begin{aligned}
\widehat{\mathbb{1}_{\mathbb{Z}_p}}(\xi_t) &:= \int_{\mathbb{Q}_p} \overline{\xi_t(x)} \mathbb{1}_{\mathbb{Z}_p} \, dx \\
&= \int_{\mathbb{Z}_p} \overline{\xi_t(x)} dx
\end{aligned}$$

At this point if $t \in \mathbb{Z}_p$ we know that $xt \in \mathbb{Z}_p$ and therefore:

$$\begin{aligned}
\int_{\mathbb{Z}_p} \overline{\xi_t(x)} dx &= \int_{\mathbb{Z}_p} \overline{\xi_1(tx)} dx \\
&= \int_{\mathbb{Z}_p} \overline{\exp(0)} dx \\
&= \int_{\mathbb{Z}_p} dx \\
&= |\mathbb{Z}_p| \\
&= 1
\end{aligned}$$

recall the definition of ξ_1 from a previous section.

For the case where $t \notin \mathbb{Z}_p$ we will need a lemma, but dont worry, it's a fun one.

Lemma 1. *Let $\xi : K \to \mathbb{C}^\times$ be a continuous group homomorphism on a compact topological group K, then:*

$$\int_K \xi(x)\, dx = \begin{cases} |K| & , \quad \text{if } \xi = 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Proof. The first case is obvious, for the second case:

$$\begin{aligned} \int_K \xi(x)dx &= \int_K \xi(xy)d(xy) \\ &= \int_K \xi(x)\xi(y)d(x) \quad \text{Because } K \text{ is compact hence unimodular} \\ &= \xi(y)\int_K \xi(x)dx \end{aligned}$$

Therefore if $\xi(y) \neq 1$ then $\int_K \xi(x)dx = 0$, as required. □

One should regard this as another instance of the orthogonality of characters for compact topological groups.

But now we note that $\xi_t|_{\mathbb{Z}_p} = 1$ iff $t \in \mathbb{Z}_p$. Hence if $t \notin \mathbb{Z}_p$ then:

$$\int_{\mathbb{Z}_p} \overline{\xi_t(x)}dx = 0$$

because $\bar{\cdot} \circ \xi_t$ is a character on $\mathbb{Z}_p$, which is compact.

Therefore we have succesfully shown that $\widehat{\mathbb{1}_{\mathbb{Z}_p}} = \mathbb{1}_{\mathbb{Z}_p}$, as required.

Rememeber this fact for later as it wil rear its head in the final chapter.

6 Adeles and Ideles

6.1 Introduction

This section consists of my notes from the week we spent getting to know the Adeles and Ideles. I will describe various properties of these objects. They will be used as an organisational tool for the grand finale.

6.2 The Adeles

Definition 12. We define the adeles first as a set, then discuss various properties:

$$\mathbb{A}_{\mathbb{Q}} := \{(r_p)_p \in \mathbb{R} \times \prod_{p\text{: prime}} \mathbb{Q}_p : r_p \in \mathbb{Z}_p \text{ almost always}\}$$

Here we use almost always to mean "for all but finitely many". Basically this is the product of all the completions of $\mathbb{Q}$, where only finitely many of the co-ordinates have negative powers of p.

This is clearly a ring, and thus an abelian group under addition. It also has many other properties, we will go over them one by one.

6.2.1 The Topology

We define the topology by defining sets of the following form to be open:

$$U := \prod_{p \leq \infty} U_p$$

Where $U_p \subset \mathbb{Q}_p$ ($\mathbb{Q}_\infty = \mathbb{R}$) is an open subset, and for all but finitely many factors $U_p = \mathbb{Z}_p$. This makes the adeles a topological ring.

This also makes the adeles locally compact. Given $(r_p)_p \in \mathbb{A}_{\mathbb{Q}}$ take $\mathbb{Z}_p$ whenever $r_p \in \mathbb{Z}_p$ and pick some compact neighbourhood in the finitely many other co-ordinates using local compactness of each of the factors. You have just described a compact neighbourhood of $(r_p)_p$.

6.2.2 The Measure

Now that we have a locally compact abelian group it is natural to ask what the Haar measure looks like. It is as natural as you could hope, just take the product of all the Haar measures on the factors and restrict your measure to the adeles. By convention we choose to scale our measure such that:

$$\left|[0,1] \times \prod_{p<\infty} \mathbb{Z}_p\right| = 1$$

Now that we have a measure it begs the question of how to integrate functions $\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$.

I only know how to integrate so called simple integrable functions, which are functions of the form:

$$f = \prod_{p \leq \infty} f_p : \mathbb{A}_{\mathbb{Q}} \to \mathbb{C}$$

where $f_p : \mathbb{Q}_p \to \mathbb{C}$ and $f_p = \mathbb{1}_{\mathbb{Z}_p}$ almost always. This condition here ensures that this seemingly infinite product in fact always has only finitely many terms not equal to 1 (because we take the restricted product as our domain).

After a few moments' thought the integral of such a function is what it ought to be:

$$\int_{\mathbb{A}_{\mathbb{Q}}} f(r)dr := \prod_{p \leq \infty} \int_{\mathbb{Q}_p} f_p(x)dx$$

Again this infinite looking product is always in fact finite because of how we have restricted the functions we are regarding, and how we have normalised the measure on the adeles.

This class of functions looks pretty restrictive, but we will see that all characters are in fact functions of this form.

6.2.3 The Characters

First given $(s_p)_p \in \mathbb{A}_{\mathbb{Q}}$ consider the following character:

$$\prod_{p \leq \infty} \xi_{s_p} : \mathbb{A}_{\mathbb{Q}} \to S^1$$

This is actually well defined because $\xi_t|_{\mathbb{Z}_p}$ is 1 $\iff$ $t \in \mathbb{Z}_p$ so the infinite product:

$$\prod_{p \leq \infty} \xi_{s_p}(r_p)$$

has only finitely many non 1 terms if $r_p \in \mathbb{A}_{\mathbb{Q}}$. This is actually a homomorphism because everything is abelian.

We will now show that all characters are of this form. Suppose:

$$\xi : \mathbb{A}_{\mathbb{Q}} \to S^1$$

is a character, by taking a neighbourhood of 1 containing only the trivial subgroup in the target. The preimage is an open set and thus contains a set of the form:

$$\prod_{p \leq \infty} U_p$$

where $U_p = \mathbb{Z}_p$ almost always. Since $\mathbb{Z}_p$ is a subgroup this tells us that ξ is trivial on $\mathbb{Z}_p$ almost always.

We know that the composition $\mathbb{Q}_p \to \mathbb{A}_{\mathbb{Q}} \to S^1$ is a character of $\mathbb{Q}_p$ and thus is equal to ξ_{s_p} for some $s_p \in \mathbb{Q}_p$. The above tell us that $s_p \in \mathbb{Z}_p$ almost always.

This allows us to run the following argument. Let $i_p : \mathbb{Q}_p \to \mathbb{A}_{\mathbb{Q}}$ be injections of the p-th co-ordinate. We then have:

$$\begin{aligned} \xi((r_p)_p) &= \xi(\sum_p i_p(r_p)) \\ &= \prod_p \xi(i_p(r_p)) \\ &= \prod_p \xi_{s_p}(r_p) \end{aligned}$$

The commutativity is justified by the fact that $s_p \in \mathbb{Z}_p$ almost always. This also tells us that $(s_p)_p \in \mathbb{A}_{\mathbb{Q}}$ and that ξ is indeed of the form described in the beginning of this subsection.

Therefore we have shown (modulo some topology) that $\widehat{\mathbb{A}_{\mathbb{Q}}} \cong \mathbb{A}_{\mathbb{Q}}$. We now have a satisfactory description of the characters.

6.2.4 The Fourier Transform

Once we have characters and have our hands on the dual group it is reasonable to ask what the fourier transform looks like. If you have picked up on the theme here you will have guessed that it is just the fourier transform in each factor. Here is the calculation: Let $f = \prod_p f_p$ be a simple integrable function:

$$\begin{aligned} \hat{f}(\xi_{(s_p)_p}) &= \int_{\mathbb{A}_{\mathbb{Q}}} \overline{\xi_{(s_p)_p}((r_p)_p)} \cdot f((r_p)_p) dr \\ &:= \prod_p \int_{\mathbb{Q}_p} \overline{\xi_{s_p}(r_p)} f_p(r_p) dr_p \\ &= \prod_p \widehat{f_p}(\xi_{s_p}) \end{aligned}$$

Now note that if we were to some reason take f_p to be fixed under the additive fourier transform on $\mathbb{Q}_p$ then the simple integrable function they form together would be a fixed point for the Fourier transform on $\mathbb{A}_{\mathbb{Q}}$.

6.2.5 The Rationals Live Inside, are They Lonely?

Yes, they are lonely. The rationals embed into the adeles diagonally:

$$\mathbb{Q} \to \mathbb{A}_{\mathbb{Q}}, r \mapsto (r, r, ...)$$

this map is a group homomorphism and actually lands in $\mathbb{A}_{\mathbb{Q}}$ because any rational has only finitely many prime factors in its denominator, so $r \in \mathbb{Z}_p$ almost always. The image of this map is a subgroup so to show that $\mathbb{Q}$ embeds discretely we need only show that $\{0\}$ is open in the subspace topology.

$$\mathbb{Q} \cap (-1, 1) \times \prod_{p<\infty} \mathbb{Z}_p = \{0\}$$

does the trick. To check split into the cases where m/n is an integer, and when it has a denominator.

6.3 The Ideles

We think of the ideles $\mathbb{I}_{\mathbb{Q}}$ as the group of units of $\mathbb{A}_{\mathbb{Q}}$:

$$\mathbb{I}_{\mathbb{Q}} = \{(x_p)_p \in \mathbb{R}^{\times} \times \prod_{p\text{: prime}} \mathbb{Q}_p^{\times} : x_p \in \mathbb{Z}_p^{\times} \text{ almost always}\} = \mathbb{A}_{\mathbb{Q}}^{\times}$$

This is clearly a group, and surely it is a topological group...

6.3.1 The Topology

Indeed we could put the subspace topology on the ideles, but instead we declare the following sets to be open:

$$\prod_{p \leq \infty} V_p$$

where $V_p \subset \mathbb{Q}_p^{\times}$ is open and equal to $\mathbb{Z}_p^{\times}$ almost always. This is in fact the topology induced from the following inclusion:

$$\mathbb{I}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}},\ x \mapsto (x, x^{-1})$$

which I suppose is the next most natural thing after the subspace topology. This is the case because the compact subgroup that we want in each factor is not $\mathbb{Z}_p - \{0\}$ but $\mathbb{Z}_p^{\times}$. Indeed note that:

$$\mathbb{Z}_p \setminus \{0\} \cap (\mathbb{Z}_p \setminus \{0\})^{-1} = \mathbb{Z}_p^{\times}$$

6.3.2 The Norm and the Rationals

One big difference between the topology on the adeles and the topology on the ideles is that the topology on the ideles is induced by a norm.

$$\|(x_p)_p\| := \prod_{p \leq \infty} \|x_p\|_p$$

This infinite product has only finitely many non 1 terms because of the requirement for $(x_p)_p$ to be in the ideles.

We have the restriction of the imbedding we talked about with the adeles:

$$\mathbb{Q}^{\times} \to \mathbb{I}_{\mathbb{Q}},\ x \mapsto (x, x, ...)$$

One may note that the norm of any element in $\mathbb{Q}^{\times}$ is 1 because the prime factors picked up in all of the prime factors will be exactly cancelled out by the norm on the real factor. This will be important for later, despite it seeming like a trivial fun fact.

6.3.3 The Rest of the Other things

Basically everything else on the ideles is the same as on the adeles, we can intergrate simple integrable functions, the measure is the product measure etc.

7 The Grand Finale

Here is where we pull everything that we have done together to talk about the zeta function and try to prove the analytic continuation. I will not prove the analytic continuation, because honestly I don't fully understand the proof. I will give a sketch and try to outline where the big stones of some of the theory we have covered come into play.

7.1 The Zeta Function, and Completed Zeta Functions

We first begin with the following definition of the Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This is a complex valued function well defined for when the real part of s is greater than 1. Our mission is to extend this definition meromorphically to the whole complex plane minus 0 and 1.

The first great insight is the formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - p^{-s}}$$

A very enlightening sketch can be seen by writing out each term in the product as a geometric series with common ratio p^{-s}. Then using the fact that every natural number has a unique decomposition as a product of primes to see that there is in fact a bijection in the terms on both sides. Of course this is not a rigorous proof, but honestly it is almost better than a rigorous proof because it really tells us why this *should* be true.

We define in this section the "completed" zeta function, which will make more sense after the next bit.

$$\tilde{\zeta}(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Where the Γ function is the extension of the factorial function via an integral:

$$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{x} dx$$

this is a complex valued function and this domain of definition works for the real part of z greater than 0.

7.2 Wait, You're Saying It's Just an Integral?

That's right. I'm going to write the completed zeta function as an integral over the ideles of a certain function. I will chose a function on each factor of the ideles and glue them together. First let's do the factors corresponding to honest primes. I will use $d^\times x$ to denote the Haar measure on $\mathbb{Q}_p^\times$ normalised such that $|\mathbb{Z}_p^\times| = 1$. (Here and in the calculations I won't differentiate between the p-adic integers and the p-adic integers without zero because they differ by a set of measure zero.)

$$\begin{aligned}
\int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x) \cdot |x|_p^s d^\times x &= \int_{\mathbb{Z}_p} |x|_p^s d^\times x \\
&= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} |x|_p^s d^\times x && \mathbb{Z}_p = \coprod_{k=0}^{\infty} p^k \mathbb{Z}_p^\times \\
&= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} p^{-ks} d^\times x && \text{definition of the } p \text{ norm} \\
&= \sum_{k=0}^{\infty} p^{-ks} \int_{p^k \mathbb{Z}_p^\times} d^\times x \\
&= \sum_{k=0}^{\infty} p^{-ks} |p^k \mathbb{Z}_p^\times| \\
&= \sum_{k=0}^{\infty} p^{-ks} |\mathbb{Z}_p^\times| && \text{multiplicative Haar measure} \\
&= \sum_{k=0}^{\infty} p^{-ks} && \text{choice of normalisation} \\
&= \frac{1}{1 - p^{-s}} && \text{geometric series}
\end{aligned}$$

So after seeing this magnificent calculation we can write:

$$\zeta(s) = \prod_{p<\infty} \int_{\mathbb{Q}_p^\times} \mathbb{1}_{\mathbb{Z}_p}(x) \cdot |x|_p^s d^\times x$$

You probably see where this is going in terms of the completed zeta function. Take the fixed point of the fourier transform $f(x) = e^{-\pi x^2}$:

$$\int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s d^\times x = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} dx$$

This equality comes from the definition of the multiplicative Haar measure on $\mathbb{R}^\times$ being $d^\times x = dx/|x|$. Now we make the following change of variables:

$$u = \pi x^2 \iff x = \sqrt{u/\pi}$$
$$du = 2\pi x dx \iff dx = du/2\sqrt{\pi u}$$

and our integral becomes:

$$\begin{aligned}
2 \int_0^\infty e^{-u} u^{s/2-1/2} \pi^{1/2-s/2} du/2\sqrt{\pi u} &= \pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} du \\
&= \pi^{-s/2} \Gamma(s/2)
\end{aligned}$$

So now we are able to put this all together to write:

$$\tilde{\zeta}(s) = \prod_{p\leq\infty} \int_{\mathbb{Q}_p^\times} f_p(x)|x|_p^s d^\times x$$

where $d^\times x$ is the relevant measure on $\mathbb{Q}_p^\times$ or $\mathbb{R}^\times$ and $f_p = \mathbb{1}_{\mathbb{Z}_p}$ if $p < \infty$ and $f_\infty(x) = e^{-\pi x^2}$. So if we let:

$$f := \prod_{p\leq\infty} f_p$$

then we can write:

$$\tilde{\zeta}(s) = \int_{\mathbb{I}_\mathbb{Q}} f(x)\|x\|_{\mathbb{I}_\mathbb{Q}}^s d^\times x$$

where here $d^\times x$ is the measure on the ideles.

Hence we have an integral over the ideles, as promised. Notice that we have chosen a function that is a fixed point under the *additive* fourier transform, though we are integrating over the ideles, which are inherently *multiplicative.* This will be crucial in our final sketch.

7.3 A Sketch of The Big Punchline

So we have written the completed zeta function as an integral over the ideles, and now we want to show that it satisfies the functional equation:

$$\tilde{\zeta}(s) = \tilde{\zeta}(1-s)$$

The proof, as best I understand takes a sufficiently nice function on the adeles, say, a function made out of Schwartz-Bruhat functions on each factor, and defines the following:

$$Z(f,s) = \int_{\mathbb{I}_\mathbb{Q}} f(x)|x|_{\mathbb{I}_\mathbb{Q}}^s d^\times x$$

We then use the norm on the ideles to write this as:

$$\int_0^\infty \int_{\mathbb{A}_t^\times} f(x)|x|_{\mathbb{I}_\mathbb{Q}}^s d^\times x\, dt/t := \int_0^\infty Z(f,s,t)\, dt/t$$

so we are integrating over the slices of the ideles of a given norm from 0 to ∞.

The next step is to exploit the self duality of the adeles and the Poisson summation formula (which I have not talked about. This is a detail to fill in if I am reading this in the future).

Notice that I said the adeles and not the ideles. That is not by accident, we apply Poisson summation to the additive structure on the adeles, but to do this we need to edit our slices a little bit. So define:

$$g(f,s;t) := Z(f,s,t) + f(0) \int_{\mathbb{A}_t^\times/\mathbb{Q}^\times} \|x\|^s d^\times x$$

As to what we mean here by $\mathbb{A}_t^\times/\mathbb{Q}^\times$ I can say that it is a compact abelian group, and since the norm of elements in $\mathbb{Q}^\times$ are 1 the function that is being integrated is actually well defined on this quotient. I can't say much more than this.

My understanding is that Poisson summation formula is a result for locally compact abelian groups, and makes use of the correspondence between subgroups of G and quotients of its dual. To apply this for $\mathbb{Q}$ we have to add in a term for zero, hence this augmentation of our slices.

The big result here that I will take as a black box and chalk up to Poisson summation is that:

$$g(f, s; t) = g(\hat{f}, 1-s; t^{-1})$$

From here a little lemma/observation is needed:

$$\int_{\mathbb{A}_t^\times/\mathbb{Q}^\times} \|x\|^s d^\times x = \int_{\mathbb{A}_t^\times/\mathbb{Q}^\times} t^s d^\times x = Vt^s$$

where V is the measure of $\mathbb{A}_t^\times/\mathbb{Q}^\times$ (finite because it is compact).

Now we are home free bar a few algebraic manipulations:

$$\begin{aligned}
\int_0^1 Z(f, s; t)\, dt/t &= \int_0^1 g(f, s, t)\, dt/t - f(0) \int_0^1 Vt^s\, dt/t \\
&= \int_0^1 g(\hat{f}, 1-s, t^{-1})\, dt/t - \frac{f(0)V}{s} \\
&= \int_0^1 Z(\hat{f}, 1-s, t^{-1})\, dt/t + \hat{f}(0) \int_0^1 V(t^{-1})^{1-s}\, dt/t - \frac{f(0)V}{s} \\
&= \int_1^\infty Z(\hat{f}, 1-s; t)\, dt/t - \frac{\hat{f}(0)V}{1-s} - \frac{f(0)V}{s}
\end{aligned}$$

using the general fact that $\int_0^1 f(t)dt/t = \int_1^\infty f(t)dt/t$ (check using high school maths, don't forget to track the endpoints)

Now this finally allows us to write:

$$\begin{aligned}
Z(f, s) &= \int_1^\infty Z(f, s; t)\, dt/t + \int_0^1 Z(f, s; t)\, dt/t \\
&= \int_1^\infty Z(f, s; t)\, dt/t + \int_1^\infty Z(\hat{f}, 1-s; t)\, dt/t - \frac{\hat{f}(0)V}{1-s} - \frac{f(0)V}{s}
\end{aligned}$$

Now comes the magic moment when we notice that this formula for $Z(f, s)$ is symmetric under $f \mapsto \hat{f}$ and $s \mapsto 1-s$.

Now we have to go all the way back and rememeber that:

$$\tilde{\zeta}(s) = Z(f, s)$$

for f consisting of $e^{-\pi x^2}$ and $\mathbb{1}_{\mathbb{Z}_p}$. In particular f is fixed under the additive fourier transform on $\mathbb{A}_\mathbb{Q}$, so we have:

$$\tilde{\zeta}(s) = Z(f, s) = Z(\hat{f}, 1-s) = Z(f, 1-s) = \tilde{\zeta}(1-s)$$

We have shown the functional equation for $\tilde{\zeta}$, as required.

I think the best way to think about this "proof" might be that self-duality of the adeles should allow Poisson summation to say something very strong, but to do that we need to move from $\mathbb{Q}^\times$ to $\mathbb{Q}$, hence the augmentation factor. Another part of the miracle is that the zeta function can be expressed in terms of such an idelic integral in the first place.

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