Learning Homology of Manifolds With L_p Balls
ASC Report

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1 Abstract:

The goal of the research project was to adapt the results of Niyogi, Smale, Weinberger [1], in which sampling is used to learn the homology of an unknown manifold, to L_1 and L_∞ balls rather than Euclidean balls. We attempted multiple approaches and ended up with a partial result.

2 Motivation and Background:

To begin discussing the project it is important to be aware of two simplicial complexes which can be formed from any metric space. The first is the Rips complex, which can be defined on a sample of points $\bar{x}$ in $\mathbb{R}^k$ with a given radius ϵ. It is defined by forming an n simplex with every set A of $n+1$ points for which any pair of points $x_i, x_j \in A$ we have $B_\epsilon(x_i) \cap B_\epsilon(x_j) \neq \emptyset$. The second is the Čech complex, which is defined by forming an n simplex for every set A of $n+1$ points for which $\bigcap_{x \in A} B_\epsilon(x) \neq \emptyset$.

These simplicial complexes are commonly used in approximating shapes. A key difference between the two is their computational complexity. Note that the Rips complex is completely defined by its 1-skeleton, whereas the Čech complex is not. This means that the Rips complex has a much lower computational complexity than the Čech, as one need only store the 1-skeleton of the Rips complex, from which the rest can be faithfully reconstructed. The Čech complex does however have a major benefit for shape approximation over the Rips complex. That advantage is the nerve lemma.

Definition 2.1. The nerve of a cover $\{U_i\}_{i \in S}$ is a simplicial complex defined by placing an an n simplex for every collection of $n+1$ sets such that their intersection is non-empty.

Note that the nerve of the collection of balls $\bigcup_{x \in \bar{x}} B_\epsilon(x)$ will be exactly the Čech complex for $\bar{x}$.

Theorem 2.2. (The Nerve Lemma) If $\mathcal{U}$ is a finite collection of open contractible subsets of a topological space X, with all non-empty sub-collections of $\mathcal{U}$ contractible, then $\mathcal{N}(\mathcal{U})$, the nerve of $\mathcal{U}$ is homotopic to the union $\bigcup_{n=1}^{N} U_n$.

This result can be leveraged by considering the union $\bigcup_{x \in \bar{X}} B_\epsilon(x)$ of (convex) balls of radius ϵ that cover our manifold $\mathcal{M}$. It allows us to study the homology of the Čech complex to learn the homology of our unknown manifold $\mathcal{M}$.

The motivation for proving similar results to [1] for L_1 and L_∞ metrics stemmed from the fact the Čech $\mathcal{C}(\bar{x})$ and Rips $\mathcal{R}(\bar{x})$ complexes agree up to the 2-skeleton in the L_1 metric and are exactly the same in the L_∞ metric. To prove this we will use a definition and theorem by Olof Hanner [2]. Such a result means that we may sample a manifold and then consider the much less memory intensive Rips complex with respect to either the L_1 or L_∞ metrics, knowing that it will agree with the Čech complex either partially or fully. This allows us to get the fidelity of the Čech complex white retaining the lower computational complexity of the Rips complex.

We begin with the following notational conventions:

- $B^p_\epsilon(x) = \{y \,|\, d_p(y, x) < \epsilon\}$.
- $\mathcal{R}^n_\epsilon(\bar{x})$ is the n skeleton of the Rips complex of $\bar{x}$ with radius ϵ.
- $\mathcal{C}^n_\epsilon(\bar{x})$ is the n skeleton of the Čech complex of $\bar{x}$ with radius ϵ.

Definition 2.3. Recall the definition of a polytope K. Define its intersection number $I(K)$ to be the smallest positive integer m for which there exist m vectors $u_1, \cdots, u_m$ such that:

1. $(K + u_l) \cap (K + u_k) \neq \emptyset, \quad \forall\, 1 \leq l \leq k \leq m$
2. $\displaystyle\bigcap_{i=1}^{m} (K + u_i) = \emptyset$

If no such integer exists then we define $I(K) = \infty$

Definition 2.4. Two polytopes $J, K \subset \mathbb{R}^k$ are affinely equivalent if there exists a $v \in \mathbb{R}^k$ such that $J + v = K$.

Definition 2.5. A face of an n polytope K is a $n - 1$ polytope that is part of the boundary of K.

Theorem 2.6. (Olof Hanner 1956 [2]) For K a polytope we have the following facts about $I(K)$:

1. $I(K) = 3, 4$ or ∞
2. $I(K) > 3$ if and only if all of the following three conditions hold:
   - K is a convex polyhedron.
   - K is centrally symmetric
   - For any two disjoint faces L_1 and L_2 of K there are two distinct parallel planes Π_1 and Π_2 such that $L_1 \subset \Pi_1$ and $L_2 \subset \Pi_2$.
3. $I(K) = \infty$ if and only if K is a parallelepiped.
4. For each dimension n there exist only finitely many affinely non-equivalent polyhedra K such that $I(K) > 3$.

Theorem 2.7. Given a point cloud sample $\bar{x} \subset \mathbb{R}^k$ the Čech $\mathcal{C}_\epsilon(\bar{x})$ and Rips $\mathcal{R}_\epsilon(\bar{x})$ complexes agree up to the 2 skeleton in the L_1 metric and are exactly the same in the L_∞ metric, for a fixed radius ϵ.

Proof. Let $\Delta \in \mathcal{C}_\epsilon(\bar{x})$ be an n simplex in the Čech complex of $\bar{x}$ in either the L_1 or L_∞ metric. Therefore $\bigcap_{i=1}^n B_\epsilon(x_i) \neq \emptyset$ in particular for any $x_i, x_j \in \bar{x}$ we have $B_\epsilon(x_i) \cap B_\epsilon(x_j) \supset \bigcap_{i=1}^n B_\epsilon(x_i) \supsetneq \emptyset$. Therefore we have $\Delta \in \mathcal{R}_\epsilon(\bar{x})$. Since Δ was arbitrary we have $\mathcal{C}_\epsilon(\bar{x}) \subset \mathcal{R}_\epsilon(\bar{x})$ in both metrics.

Now seperately consider the L_1 metric, note that $I(B_\epsilon^1(x)) = 4$. Therefore any simplex $\Delta \in \mathcal{R}_\epsilon^2(\bar{x})$, determined by less than four points with pairwise intersection of their L_1 radius ϵ balls must have mutual intersection of their epsilon balls by the definition of intersection number. Therefore $\mathcal{R}_\epsilon^2(\bar{x}) \subset \mathcal{C}_\epsilon^2(\bar{x})$ and so we see for the L_1 metric we have $\mathcal{R}_\epsilon^2(\bar{x}) = \mathcal{C}_\epsilon^2(\bar{x})$.

Now for the L_∞ metric note that $B_\epsilon^\infty(x)$ is a parallelepiped and so $I(B_\epsilon^\infty(x)) = \infty$ by theorem 2.6 [2]. Consider an n simplex $\Delta \in \mathcal{R}_\epsilon(\bar{x})$. This simplex is defined by n points whose L_∞ radius ϵ balls pairwise intersect. However since $I(B_\epsilon^\infty(x)) = \infty$ we must have mutual intersection. Therefore $\Delta \in \mathcal{C}_\epsilon(\bar{x})$ and so $\mathcal{R}_\epsilon(\bar{x}) \subset \mathcal{C}_\epsilon(\bar{x})$. This means that in the L_∞ metric we have $\mathcal{R}_\epsilon(\bar{x}) = \mathcal{C}_\epsilon(\bar{x})$. $\square$

Remark 2.8. This means that in the L_∞ case we can use the Čech complex of $\bar{x}$ to study the homology and homotopy groups of $\mathcal{M}$, while in the L_1 case we can only study the Čech complex to tell us things about the 0th and 1st homology groups and fundamental group of $\mathcal{M}$, as these rely only on the 2-skeleton of the complex.

Remark 2.9. We can demonstrate that the Rips and Čech need only match in the L_1 metric up to the 2-skeleton by considering the following case in $\mathbb{R}^3$. $\bar{x} = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$ with $\mathcal{R}_\epsilon^3(\bar{x}) \neq \mathcal{C}_\epsilon^3(\bar{x})$. This shows that in the L_1 metric the best we can hope for is $\mathcal{R}_\epsilon^2(\bar{x}) = \mathcal{C}_\epsilon^2(\bar{x})$.

3 The Naive Approach:

Our naive first attempt to find conditions on ϵ/τ such that U deformation retracts to $\mathcal{M}$ was to attempt to modify the approach of Smale [1] to L_1 and L_∞ metrics. In the course of this approach to the problem I learned that the situation is much more complex since we lose many degrees of symmetry in the ϵ balls, which were reducing the number of factors we had to consider in the $p = 2$ case. This indicated that a higher powered approach may be necessary.

First some definitions from the original paper:

- $\mathcal{M} \subset \mathbb{R}^k$ is a compact Riemannian manifold.
- $B_\epsilon^p(x) = \{y \mid d_p(y,x) < \epsilon\}$.
- $Tub(\mathcal{M}) := \{x | d_2(x, \mathcal{M}) < \tau\}$
- $\pi_0 : Tub(\mathcal{M}) \to \mathcal{M},\ \pi(x) := \underset{m \in \mathcal{M}}{\arg\min}\, d(x,m)$
- $\pi := \pi_0|_U$
- $\overline{vp} = \{\lambda p + (1-\lambda)v \mid 0 \leq \lambda \leq 1\}$

- $U := \bigcup_{x\in\bar{x}} B^2_\epsilon(x)$
- $\pi^{-1}(p) := \bigcup_{x\in\bar{x}} B^2_\epsilon(x) \cap T_p^\perp \cap B^2_\tau(p)$
- $st(p) := \bigcup_{\{x\in\bar{x}|x\in B^2_\epsilon(p)\}} B^2_\epsilon(x) \cap T_p^\perp \cap B^2_\tau(p)$

Adapting the last two sets to the L_1 case we get:

$$\pi^{-1}(p) := \bigcup_{x\in\bar{x}} B^1_\epsilon(x) \cap T_p^\perp \cap B^1_\tau(p)$$

$$st(p) := \bigcup_{\{x\in\bar{x}|x\in B^1_\epsilon(p)\}} B^1_\epsilon(x) \cap T_p^\perp \cap B^1_\tau(p)$$

and for the L_∞ case:

$$\pi^{-1}(p) := \bigcup_{x\in\bar{x}} B^\infty_\epsilon(x) \cap T_p^\perp \cap B^\infty_\tau(p)$$

$$st(p) := \bigcup_{\{x\in\bar{x}|x\in B^\infty_\epsilon(p)\}} B^\infty_\epsilon(x) \cap T_p^\perp \cap B^\infty_\tau(p)$$

Theorem 3.1. *$st(p)$ is star shaped around p in both the L_1 and L_∞ cases.*

Proof. For the sake of this proof we subdue the superscript on $B_\epsilon(x)$ to allow for the generality of both $B^1_\epsilon(x)$ and $B^\infty_\epsilon(x)$.

Let $v \in st(p)$. $\overline{vp} \subset T_p^\perp \cap B_\tau(p)$ by the fact that $v, p \in T_p^\perp \cap B_\tau(p)$ and convexity of both $T_p^\perp$ and $B_\tau(p)$. Since $v \in st(p)$, there exists $x \in \bar{x}$ such that $v \in B_\epsilon(x)$ and $x \in B_\epsilon(p)$. Equivalently $v, p \in B_\epsilon(x)$, so by convexity of $B_\epsilon(x)$ we have $\overline{vp} \subset B_\epsilon(x)$, specifically $\overline{vp} \subset \bigcup_{\{x\in\bar{x}|x\in B_\epsilon(p)\}} B_\epsilon(x)$, hence $\overline{vp} \subset st(p)$. Since v was arbitrary we conclude that $st(p)$ is star shaped around p for both the L_1 and L_∞ cases. □

The idea to proceed was to work with inequalities, deriving a contradiction in order to find conditions on ϵ/τ such that $\pi^{-1}(p) = st(p)$. This equality would allow us to use the straight line homotopy $F(x,t) = tx + (1-t)\pi(x)$. We followed the inequalities through using the basic relation between the L_1 and L_∞ to the L_2 metrics:

$$\frac{1}{\sqrt{k}} d_1(x,y) \leq d(x,y) \leq d_1(x,y)$$

$$d_\infty(x,y) \leq d(x,y) \leq \sqrt{k} d_\infty(x,y)$$

These simple inequalities are too coarse as the final requirements were either complex for $k > 4$ (In the L_1 case we had a $\sqrt{\frac{1}{k} - \frac{1}{4}}$ term, resulting in a complex ϵ/τ ratio when ϵ and τ should both be positive real numbers), or simply not there (In the L_∞ case we required a quadratic in ϵ/τ to be greater than 0 when $0 < \epsilon/\tau < r$ for some $r \in \mathbb{R}$, however at 0 it had a negative value, making the task impossible by virtue of continuity. The only exception to this was the $k = 1$ case). To give a sense of the type of proofs involved there is an example of a lemma included below.

Remark 3.2. I have recently noticed that the proof is invalid because we have used a bound on $|\theta|$ that may not necessarily hold in the L_∞ case (or the L_1 case). I have decided to leave the proof in as it displays well this approach to the problem.

Lemma 3.3. Let $v \in T_p^\perp \cap B_\tau^\infty(p)$. If there is a $q \in M$ such that $d_\infty(v,q) < \epsilon$ while $d_\infty(p,q) \geq \epsilon$ then
$d(v,p) < \frac{\epsilon}{2} + \frac{\sqrt{k}\epsilon^2}{2\tau} + \epsilon\sqrt{\frac{\epsilon^2}{4\tau^2} + k - 1}$.

Proof. We will need the inequality:

$$\begin{aligned} d(p,q) &\leq d(p,v) + d(v,q) \\ &\leq d(p,v) + d_\infty(v,q)\sqrt{k} \\ &< \tau + \epsilon\sqrt{k} \end{aligned}$$

Thus $\epsilon \leq d(p,q) < \tau + \epsilon\sqrt{k}$. Also note that $|\theta| < \arcsin\frac{\epsilon}{2\tau}$, as the following diagram illustrates.

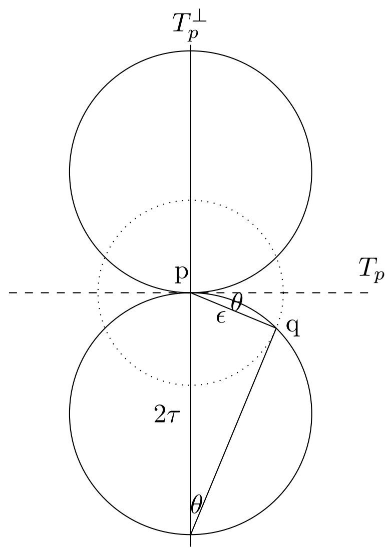


Figure 1: A diagram showing why $|\theta| \leq \arcsin\frac{\epsilon}{2\tau}$ when $k = 2$.

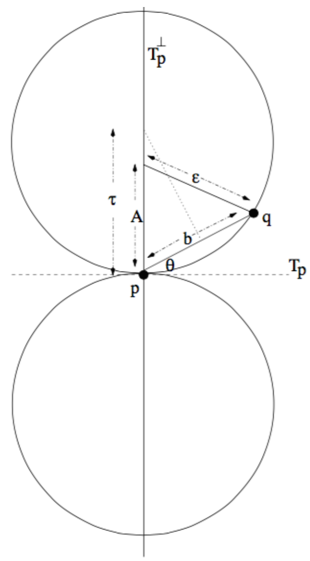


Figure 2: A picture showing the worst case, the circles are for the $p = 2$ case, however the expression for A is always valid.

Now:

$$
\begin{aligned}
d(v,p) &= d(p,q)\sin\theta + \sqrt{d(v,q)^2 - d(p,q)^2\cos^2\theta} \quad (= A) && \text{from Figure 2} \\
&= d(p,q)\sin\theta + \sqrt{d(v,q)^2 + d(p,q)^2(\sin^2\theta - 1)} \\
&< d(p,q)\sin\theta + \sqrt{k\epsilon^2 + d(p,q)^2(\sin^2\theta - 1)} && d(v,q)^2 \leq k d_\infty(v,q)^2 < k\epsilon^2 \\
&\leq d(p,q)\frac{\epsilon}{2\tau} + \sqrt{k\epsilon^2 + d(p,q)^2(\frac{\epsilon^2}{4\tau^2} - 1)} && |\theta| < \arcsin\frac{\epsilon}{2\tau} \\
&\leq (\tau + \epsilon\sqrt{k})\frac{\epsilon}{2\tau} + \sqrt{k\epsilon^2 + \epsilon^2(\frac{\epsilon^2}{4\tau^2} - 1)} && \epsilon \leq d(p,q) < \tau + \epsilon\sqrt{k} \\
&= \frac{\epsilon}{2} + \frac{\sqrt{k}\epsilon^2}{2\tau} + \epsilon\sqrt{\frac{\epsilon^2}{4\tau^2} + k - 1}
\end{aligned}
$$

□

4 Investigating $1 < p < \infty$

The lack of results from the previous attempt led us to investigate the situation in L_p metrics and see if what we found would be applicable to the cases of interest. We suspected that strictly convex balls would make generalising the methods easier.

We start with the following definitions:

- $\mathcal{M} \subset \mathbb{R}^k$ is a compact manifold
- $d(y, \mathcal{M}) = \inf_{m \in \mathcal{M}} d(y, m)$
- $\Gamma(y, \mathcal{M}) = \{m \in \mathcal{M} \,|\, d(y, m) = d(y, \mathcal{M})\}$

The first issue we looked into was the definition of τ, the condition number of the manifold. In [1] it is defined as:

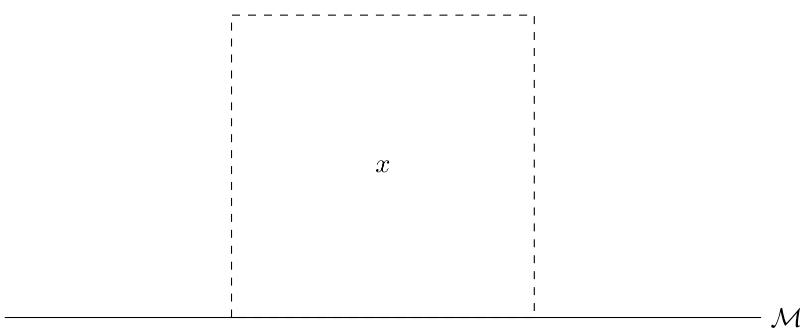
$$\tau = \sup\{r \,|\, \text{for all } x \text{ such that } d(x, \mathcal{M}) < r, \Gamma(x, \mathcal{M}) \text{ is a single point}\}$$

We define $\tau := \infty$ if the set is unbounded and $\tau := 0$ if the set is empty. For the L_∞ and L_1 cases any section of $\mathcal{M}$ that runs parallel to any face of the unit ball will result in a condition number of 0, see the diagram below for an illustration.

To counteract this we considered the following definition:

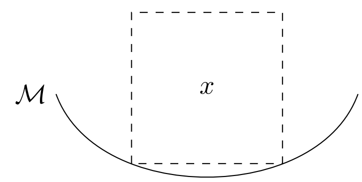
$$\hat{\tau} = \sup\{r \,|\, \text{for all } x \text{ such that } d(x, \mathcal{M}) < r, \Gamma(x, \mathcal{M}) \text{ is contractible}\}$$

Again defining $\hat{\tau} := \infty$ if the set is unbounded and $\tau := 0$ if the set is empty. We can see immediately that $\hat{\tau} \geq \tau$ since if $\Gamma(x, \mathcal{M})$ is a point it must be contactible. We can also see that $\hat{\tau} \neq \tau$ for L_1 or L_∞ by way of the following counterexample.



By picking x as close to or as far away from $\mathcal{M}$ as we like we see that while $\tau = 0$ for this $\mathcal{M}$, $\hat{\tau} = \infty$. Note that this can represent either the L_1 or L_∞ case when $k = 2$.

We planned to show that $\hat{\tau} = \tau$ for $1 < p < \infty$, in order to justify its use for $p = 1$ or ∞. It was left open whether this was the case when we realised that even this was a very limiting condition number. Consider a curved section of the manifold (below):



By taking x to be lower and lower and noting that $\Gamma(x, \mathcal{M})$ is not contractible for any of these x, we see that a simple section like this would force the condition number of $\mathcal{M}$ to be 0.

At this point we decided to set the definition of the condition number aside and focus on showing a form of the result for $1 < p < \infty$. This proved very difficult as there were many things special to the L_2 metric (such as standard trigonometry) that facilitated the result and made it unclear how at all to proceed with the general $1 < p < \infty$ case. Ultimately it is still unclear if this approach is viable.

4.1 Future Directions

A future direction of inquiry regarding this approach would be to investigate whether, for a fixed $\mathcal{M}$, any of the following limits exist, where τ_p is τ when found using the L_p metric, and likewise for $\hat{\tau}_p$.

$$\lim_{p\to\infty} \tau_p, \lim_{p\to 1^+} \tau_p, \lim_{p\to\infty} \hat{\tau}_p \text{ or } \lim_{p\to 1^+} \hat{\tau}_p$$

and furthermore whether they agree with $\tau_\infty, \tau_1, \hat{\tau}_\infty$ or $\hat{\tau}_1$.

5 The Final Attempt

The final method of approach we used was a way to show that $\mathcal{M}(\epsilon)$ is homotopy equivalent to $\mathcal{M}(\rho)$ for all ϵ and ρ such that $0 < \rho < \epsilon < \tau$ in the case of the L_∞ metric. This is in a sense half of the problem we wanted to solve initially, and relies on a conjecture.

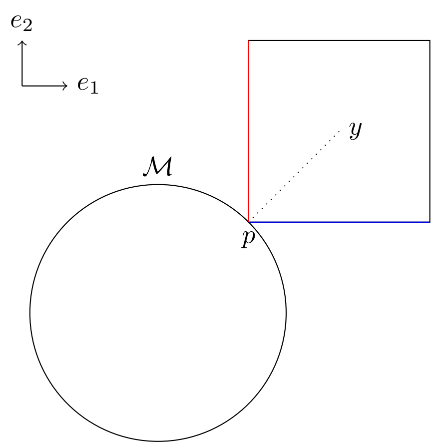
Firsly let us introduce the following definitions and notation:

Throughout this section any vector v is of the form $v = (v_1, \cdots, v_k)$.

- $\mathcal{M} \subset \mathbb{R}^k$ is a compact manifold
- $B_\delta(x) = \{y \mid d(x, y) < \delta\}$
- $\mathcal{M}(\epsilon) = \{x \mid d(x, \mathcal{M}) \leq \epsilon\}$

- $\mathcal{M}(< \epsilon) = \{x \mid d(x, \mathcal{M}) < \epsilon\}$
- $d(y, \mathcal{M}) = \inf_{m \in \mathcal{M}} d(y, m)$
- $\Gamma(y, \mathcal{M}) = \{m \in \mathcal{M} \mid d(y, m) = d(y, \mathcal{M})\}$
- $F(\pm e_i)$ is the $k-1$ dimensional face of $B_{d(y,\mathcal{M})}(y)$ containing $y \pm d(y, \mathcal{M}) e_i$.
- $\tau = \sup\{r \mid \text{for all } x \text{ such that } d(x, \mathcal{M}) < r \text{ we have } \Gamma(x, \mathcal{M}) \subset \overline{F(e_i)} \text{ for some } i\}$
- $Q(y) = \{x \neq 0 \mid x_i \geq 0$ if $\overline{F(e_i)} \supset \Gamma(y, M)$ and $x_i \leq 0$ if $\overline{F(-e_i)} \supset \Gamma(y, M)$ and there exists either an e_j such that $\overline{F(e_j)} \supset \Gamma(y, M)$ and $x_i > 0$ or a $-e_j$ such that $\overline{F(-e_j)} \supset \Gamma(y, M)$ and $x_i < 0\}$

For clarity see the diagram below:



In the diagram above we have $\Gamma(y, \mathcal{M}) = \{p\}$. The red side of the square is $\overline{F(-e_1)}$, and the blue is $\overline{F(-e_2)}$.

As such we have $Q(y) = \{x = (x_1, x_2) \neq (0, 0) \mid x_1 \leq 0 \text{ and } x_2 \leq 0\}$.

Conjecture 5.1. Suppose $\epsilon < \tau$, then for all $y \in \mathcal{M}(\epsilon) \setminus \mathcal{M}$ there exists a $\delta > 0$ and a v such that $v \in Q(x)$ for all $x \in B_\delta(y)$.

Lemma 5.2. If $v, w \in Q(y)$ then $v + w \in Q(y)$

Proof. Suppose $v, w \in Q(y)$ then for every e_i such that $\overline{F(e_i)} \supset \Gamma(y, \mathcal{M})$ we have $v_i + w_i \geq 0$ and similarly for $-e_i$. By picking a suitable e_j from the coordinates of v we have an e_j such that $\overline{F(e_j)} \supset \Gamma(y, \mathcal{M})$ and $v_j + w_j < 0$ (or respectively for some $-e_j$). Therefore $v + w \in Q(y)$. □

Lemma 5.3. Suppose for some $0 < \rho < \epsilon$ there exists a smooth vector field V over $\mathcal{M}(\epsilon) \setminus \mathcal{M}(< \rho)$ such that $V(y) \in Q(y)$ for all $y \in \mathcal{M}(\epsilon) \setminus \mathcal{M}(< \rho)$. Then $\mathcal{M}(\epsilon)$ is homotopy equivalent to $\mathcal{M}(\rho)$.

Proof. The flow is clearly continuous by the fact that V is smooth. Pick a $y \in \mathcal{M}(\epsilon) \setminus \mathcal{M}(\rho)$, either $V(y)$ directly decreases the distance to $\mathcal{M}$ or decreases the number of co-ordinates determining the distance to $\mathcal{M}$ without increasing the distance to $\mathcal{M}$. Since there are only finitely many co-ordinates this means that the flow never leaves $\mathcal{M}(\epsilon) \setminus \mathcal{M}(\rho)$ until it reaches a point on $\mathcal{M}(\rho)$.

Now consider $r : \mathcal{M}(\epsilon) \to \mathcal{M}(\rho)$ that sends $y \in \mathcal{M}(\epsilon) \setminus \mathcal{M}(\rho)$ to it's destination via the vector flow along V and sends $y \in \mathcal{M}(\rho)$ to y. Also consider $i : \mathcal{M}(\rho) \to \mathcal{M}(\epsilon), i(y) = y$. $r \circ i \sim id_{\mathcal{M}(\rho)}$ clearly and $i \circ r \sim id_{\mathcal{M}(\epsilon)}$ is given by the homotopy defined by flowing along V. Hence $\mathcal{M}(\epsilon)$ is homotopy equivalent to $\mathcal{M}(\rho)$. □

Theorem 5.4. $\mathcal{M}(\epsilon)$ is homotopy equivalent to $\mathcal{M}(\rho)$ for all $0 < \rho < \epsilon < \tau$.

Proof. Since $\epsilon < \tau$ we know that $Q(y)$ is non-empty. By our conjecture we have a cover of $\mathcal{M}(\epsilon) \setminus \mathcal{M}(< \rho)$ such that each set U_i in the cover has a vector v such that $v \in Q(y)$ for all $y \in U_i$. Now by the compactness of $\mathcal{M}(\epsilon) \setminus \mathcal{M}(< \rho)$ there is a finite subcover $U = \{U_i\}_{i \in I}$ such that for each subset U_i there exists a vector v_i such that $v_i \in Q(y)$ for all $y \in U_i$. Now consider a smooth partition of unity $\Phi = \{\phi_i\}_{i \in I}$ such that for each ϕ_i we have $\text{supp}(\phi_i) \subset U_i$. Now consider the vector field:

$$V(y) = \sum_{i \in I} \phi_i(y) v_i$$

V is smooth as it is the sum of finitely many smooth vector fields, and by the closure of $Q(y)$ under addition we have that $V(y) \in Q(y)$ for all $y \in \mathcal{M}(\epsilon) \setminus \mathcal{M}(< \rho)$. So by the previous lemma we have that $\mathcal{M}(\epsilon)$ is homotopy equivalent to $\mathcal{M}(\rho)$. □

References

[1] P. Niyogi, S. Smale, S. Weinberger. Finding the Homology of Submanifolds with High Confidence from Random Samples. Discrete & Computational Geometry. 13 September 2004.

[2] O. Hanner. Intersections of Translates of Convex Bodies. Mathematica Scandinavica. 18 September 1956.