Representation Theory of Compact Groups

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1 Preface

This essay can be viewed as an exploration of the representation theory of compact groups, motivated in part by the representation theory of finite groups. We will often refer back to what happens in the development of the theory in the finite group case. We will use compactness as our main tool to see how much of the theory we can salvage (Spoiler: a lot)

For reasons of space I will introduce definitions by bolding words **like this**, rather than making a separate heading.

2 Background

2.1 Topological Groups and Haar Measure

Throughout let G be a compact topological group and μ be the right normalised Haar measure often referred to as *the* Haar measure on G.

It is one of my favourite results that in the case of a compact group the right Haar measure gets left invariance for free. I will give a proof sketch because I find the proof quite pleasing.

Theorem 1. *For a compact group the right Haar measure is the left Haar measure.*

Proof. We define a measure $\mu_g(S) := \mu(gS)$. This measure is also a right Haar measure on G and so must be a scalar multiple of μ. Define $\Delta : G \to \mathbb{R}^\times_{>0}$ by $\mu_g = \Delta(g)\mu$. This defines a continuous homomorphism, so the image of Δ must be a compact subgroup of $\mathbb{R}^\times_{>0}$. The only such subgroup is $\{1\}$ and so we conclude that for all Borel sets S we have $\mu(gS) =: \mu_g(S) = \mu(S)$ □

2.2 Representations

In the finite group case we simply defined a representation as a homomorphism $\pi : G \to \mathrm{GL}_n(\mathbb{C})$. To deal with the topology on G and the added complexities of no longer having a finite group we define a **(unitary) representation** of G in a hilbert space $\mathcal{H}_\pi$ to be a homomorphism $\pi : G \to U(\mathcal{H}_\pi)$ that is continuous in the **strong operator topology**, which means the map $x \mapsto \pi(x)u$ is continuous for all $u \in \mathcal{H}_\pi$. Landing in $U(\mathcal{H}_\pi)$ means that $\pi(x)^{-1} = \pi(x^{-1}) = \pi(x)^*$.

We define the notation $\mathcal{C}(\pi, \rho)$ to mean the **space of intertwining operators** with $\mathcal{C}(\pi) = \mathcal{C}(\pi, \pi)$.

We say that two representations π_1 and π_2 are **(unitarily) equivalent**, or **isomorphic** if there exists an invertible unitary operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ in $\mathcal{C}(\pi_1, \pi_2)$.

If your memories of studying representation theory of finite groups are a little faded the unitary requirement might seem a little strange, but it isn't. If you don't remember, it is likely because for any representation π of a finite group the inner product can be averaged over the group, making it G-invariant. This makes π a unitary representation with respect to the new inner product.

One can do a similar thing with compact groups. Given a continuous homomorphism $\pi : G \to \mathrm{GL}(\mathcal{H}_\pi)$ and an inner product $\langle \cdot, \cdot \rangle_0$ one can define a new inner product via:

$$\langle u, v \rangle := \int \langle \pi(x)u, \pi(x)v \rangle_0 \, dx$$

Since the measure of G is 1 this new bilinear form is in fact an inner product (compactness is necessary to ensure positive definiteness).

So the unitary representations of G actually encompass all of the representations of G in a Hilbert space. As such I will often suppress the word unitary, though all representations from now on will be.

If one notes that a finite group with the discrete topology has the normalised counting measure as it's normalised Haar measure, then the integration we have done turns into a sum. The *exact same sum* used in the finite group case.

So we can assume unitary without actually losing any generality, but why should we? The answer is twofold. Firstly, if our representation is unitary and we find a closed invariant subspace W, then $W^{\perp}$ is also invariant and so $\mathcal{H}_{\pi} = W \oplus W^{\perp}$. This tells us via induction that any finite dimensional representation decomposes into a direct sum of irreducible representations.

The second reason is that we get Schur's lemma for unitary representations:

Theorem 2. *(Schur's Lemma)*

a A unitary representation of G is irreducible if and only if $\mathcal{C}(\pi)$ is exactly the scalar multiples of the identity.

b If π_1 and π_2 both irreducible and are not equivalent then $\mathcal{C}(\pi_1, \pi_2) = \{0\}$

The proof in this generality uses a bit more machinery but is the same as the finite dimensional case in spirit.

A common theme to look out for that will come up time and time again is starting with a non G-equivariant operator and using the Haar measure to average over the group and end up with something G-equivariant, then applying Schur's lemma to say that this new averaged operator is a scalar multiple of the identity.

The next subsection makes sense of what it means to "average operators over a group".

2.3 Integrating operators

We start with vector valued integrals. We take a "weak" approach to the problem, using linear functionals to reduce everything to the case of scalar functions. Let $f : G \to \mathcal{H}$, we say that f is **weakly integrable** if $\phi \circ f \in L^1(G)$ for all $\phi \in \mathcal{H}^*$. Suppose there exists a $v \in \mathcal{H}$ such that for all $\phi \in \mathcal{H}^*$:

$$\phi(v) = \int_G \phi \circ f(x)\, dx$$

This v is unique if it exists because functionals separate points. We then define v to be **the integral of** f and write $v = \int f(x)\, dx$.

A natural question to ask is whether bounded linear operators commute with integrals. This is the case and we will use it many times with little to no comment after this section.

Suppose $f : G \to \mathcal{H}$ is weakly integrable, $v = \int f d\mu$ exists and $T \in B(\mathcal{H}, \mathcal{H}')$ is a map between hilbert spaces. Now for any $\phi \in (\mathcal{H}')^*$ we have that $\phi \circ T \in \mathcal{H}^*$, so weak integrability of f implies weak integrability of $T \circ f$. Furthermore we have by the existence of the integral of f that:

$$\phi \circ T \left[\int f(x)\, dx \right] = \int \phi \circ T \circ f(x)\, dx$$

If one stares at this for long enough, one realises that we have bootstrapped from the integral of f existing to the fact that $\int T \circ f d\mu$ exists and the fact that:

$$T \int f(x)\, dx = \int T \circ f(x)\, dx$$

So we may pass bounded linear operators through the integral as we please. I will not go over the proof that integrals for weakly integrable functions exist, but rest assured that in our circumstances (a Haar measure on a compact group mapping into a Hilbert space) we have more than enough to guarantee the existence of integrals. I think of this construction the same way I think of many constructions category theoretically. We don't really care that much about the specifics of the construction beyond the fact that it exists. What we truly care about is its defining properties.

We promised to make sense of integrals of operators, which we have not done yet. If we have a family of operators $T_x \in B(\mathcal{H}, \mathcal{H}')$ for each $x \in G$ such that the map $x \mapsto T_x(v)$ is continuous for all $v \in \mathcal{H}$ (note that this is exactly the continuity condition imposed for representations of G!) then we can make sense of the average of this family of operators over G:

$$\tilde{T} := \int T_x\, dx$$

by determining values pointwise:

$$\tilde{T}(v) = \int T_x(v)\, dx$$

Since G is compact continuity of $x \mapsto T_x(v)$ implies weak integrability, so this integral actually exists.

3 General Theory on Compact Groups

Now that we have covered enough background, we can begin to imitate the development of representation theory in the compact group case. In particular the decomposition of any representation into a direct sum of irreducibles. However to even get started on that problem we need to solve an issue that is immediate in the finite group case.

3.1 Irreducible Representations Are Finite Dimensional

The following result may seem surprising. It certainly surprised me when I heard it, but it is one of the many ways in which compact makes a good substitute for finite.

Theorem 3. *All irreducible representations of a compact group G are finite dimensional*

Proof. I will give a sketch. Let π be a unitary representation of G in $\mathcal{H}_\pi$. Fix a unit vector $u \in \mathcal{H}_\pi$ and define the following operator on $\mathcal{H}_\pi$:

$$Tv = \int \langle v, \pi(x)u \rangle \pi(x)u \, dx$$

Think of this as the average over G of the projections of v onto the line spanned by $\pi(x)u$. This operator turns out to be positive, non-zero, compact and in $\mathcal{C}(\pi)$. Since π is irreducible by Schur's lemma $T = cI$ for some $c \neq 0$. However non-zero scalar multiples of the identity are compact if and only if $\dim(\mathcal{H}_\pi) < \infty$ $\square$

Crucial here is the ability to average over the group using the Haar measure to cook up an operator that gives us the result we want. This is much more work than the finite group case. In the finite group case one can simply take a vector and then take the span of its orbit under G. Since G is finite this gives a finite dimensional invariant subspace of $\mathcal{H}_\pi$, so every irreducible representation is finite dimensional.

3.2 Decomposition of Representations

The next thing to talk about is decomposing unitary representations of G into irreducibles.

Theorem 4. *If G is compact, then every unitary representation of G is a direct sum of irreducible representations.*

Proof. The key thing to note here is that, given a representation π of G, the operator T defined in the previous section does not need π to be irreducible to be a positive (hence self-adjoint), non-zero, compact intertwining operator for π. Therefore by the *spectral theorem* we know that $\mathcal{H}_\pi$ has an orthonormal eigenbasis for T. Therefore there must exist an eigenspace of T for some $\lambda \neq 0$, and this space must be finite dimensional by the compactness of T. Furthermore since $T \in \mathcal{C}(\pi)$ this subspace is G-invariant. Now if π is a finite dimensional representation we can use induction to quickly conclude that it decomposes into a direct sum of irreducible representations.

If π is infinite dimensional a Zorn's lemma argument allows us to find a maximal family of $\{\mathcal{M}_\alpha\}$ of mutually orthogonal irreducible invariant subspaces. Then if $\mathcal{H}_\pi$ does *not* decompose into a direct sum $\mathcal{H}_\pi = \bigoplus_\alpha \mathcal{M}_\alpha$ then the perp space $(\bigoplus_\alpha \mathcal{M}_\alpha)^\perp$ is a non-zero subrepresentation. We can apply the initial argument to to find an irreducible invariant subspace, contradicting the maximality of $\{\mathcal{M}_\alpha\}$. Therefore in fact $\mathcal{H}_\pi = \bigoplus_\alpha \mathcal{M}_\alpha$, as required. $\square$

Now as a matter of notation we define $\widehat{G}$ to be **the set of irreducible representations of G up to unitary equivalence**. Hence $[\pi] \in \widehat{G}$ will be a useful shorthand when choosing an irreducible representation of G.

4 Peter-Weyl Theorem and the Regular Representation

4.1 What We Are Trying To Generalise

Now that we have covered enough of the general theory I will state the main theorem for finite groups that we are looking to generalise by the end of the essay.

Theorem 5. *Let G be a finite group and $\{(\pi_i, V_i)\} = \widehat{G}$ its set of irreducible representations up to isomorphism. Then if we consider the regular representation given by G acting on its group algebra $k[G]$ then as representations:*

$$k[G] \cong \bigoplus_i \dim(V_i) V_i \cong \bigoplus_i \mathrm{End}(V_i)$$

The first problem we have is in replacing $k[G]$. It turns out that the correct generalisation for **the (right) regular representation** is $L^2(G)$, with G-action given by right-shift operators (with some minor adjustments one can do everything we cover in this essay using the left shift operators instead). $[R_x f](y) := f(yx)$. This is indeed a unitary action because for any $x \in G$ and $f, g \in L^2(G)$:

$$\langle R_x f, g\rangle = \int f(yx)\overline{g(y)}\,dy = \int f(y)\overline{g(yx^{-1})}\,dy = \langle f, R_{x^{-1}} g\rangle$$

Hence $R_x^{-1} = R_{x^{-1}} = (R_x)^*$. We crucially use the right invariance of the Haar measure on G.

Philosophically the group algebra does not suffice anymore because it does not care about the topology on G. If we want our representations to be continuous then we will somehow have to take the topology into account. $L^2(G)$ does this because the Haar measure is dependent on the topology on G. On another level, we can think of elements of $k[G]$ as functions on G, where the coefficient of e_g is the value of the function at $g \in G$.

4.2 Matrix elements

We now introduce a specific class of functions on G, which will be key in understanding the structure of the regular representation $L^2(G)$. Note quickly that since G is compact any continuous function on G is bounded and hence $C(G) \subset L^2(G)$ and is a dense subspace.

Let $[\pi] \in G$, then define $\pi_{u,v} : G \to \mathbb{C}$, $\pi_{u,v}(x) := \langle \pi(x)u, v\rangle$. We will call functions of this form **matrix entries**. The motivation for this terminology comes from the fact that if one picks a basis for $\mathcal{H}_\pi$ containing u and v then this function actually *is* one of the matrix entries of $\pi(x)$ when expressed with respect to this basis. If we fix an orthonormal basis $\{e_j\}$ then we define $\pi_{ij}(x) := \pi_{e_i,e_j}(x)$. We denote **the linear span of such elements** $\mathcal{E}_\pi \subset C(G) \subset L^2(G)$.

The astute reader will be asking at this point whether this subspace $\mathcal{E}_\pi$ is well defined given that we chose a representative of $[\pi] \in \widehat{G}$. We have implied this by our notation, and it turns out to be the case:

Proposition 1. *The subspace $\mathcal{E}_\pi$ is independent of unitary equivalence. It is invariant under the action of the right regular representation. Furthermore if $\dim \mathcal{H}_\pi = d_\pi$ then $\dim \mathcal{E}_\pi \le d_\pi^2$.*

Proof. Let $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ be a unitary equivalence, then $T^{-1}\pi'(x)T = \pi(x)$ for all $x \in G$. If $\pi_{u,v} \in \mathcal{E}_\pi$ then:

$$\pi_{u,v}(x) = \langle \pi(x)u, v\rangle = \langle T^{-1}\pi'(x)Tu, v\rangle = \langle \pi'(x)Tu, Tv\rangle = \pi'_{Tu,Tv}(x) \in \mathcal{E}_{\pi'}$$

Therefore $\mathcal{E}_\pi \subset \mathcal{E}_{\pi'}$, so by symmetry $\mathcal{E}_\pi = \mathcal{E}_{\pi'}$.

Again given $\pi_{u,v} \in \mathcal{E}_\pi$ consider $R_y \pi_{u,v}$:

$$[R_y \pi_{u,v}](x) = \pi_{u,v}(xy^{-1}) = \langle \pi(xy^{-1})u, v\rangle = \langle \pi(x)\pi(y^{-1})u, v\rangle = \pi_{\pi(y^{-1})u,v}(x) \in \mathcal{E}_\pi$$

Lastly if we choose an orthonormal basis $\{e_i : 1 \le i \le d_\pi\}$ for $\mathcal{H}_\pi$ then clearly $\mathcal{E}_\pi \subset \operatorname{span}\{\pi_{ij} : 1 \le i, j \le d_\pi\}$ which has dimension at most d_π^2, as required. □

From now we will define the notation $d_\pi := \dim(\mathcal{H}_\pi)$ whenever $[\pi] \in \widehat{G}$.

The matrix elements for the irreducible representations will form an orthonormal basis for $L^2(G)$. The next steps we will take in showing this will be the Schur orthogonality relations. I will give a sketch of the proof.

Theorem 6. *Let $[\pi] \ne [\pi'] \in \widehat{G}$, and consider $\mathcal{E}_\pi$ and $\mathcal{E}_{\pi'}$ as elements of $L^2(G)$, then:*

a $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$

b If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$ then $\{\sqrt{d_\pi}\pi_{ij} : 1 \le i, j \le d_\pi\}$ is an orthonormal basis for $\mathcal{E}_\pi$.

Proof. The proof hinges on the following construction. Let $A : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ be any map, then define:

$$\tilde{A} := \int \pi'(x^{-1})A\pi(x)\,dx$$

Using the translation invariance of the Haar measure and the fact that bounded linear operators commute with integrals we get that $\tilde{A}\pi(y) = \pi'(y)\tilde{A}$ for all $y \in G$, hence $\tilde{A} \in \mathcal{C}(\pi, \pi')$.

Now let $v \in \mathcal{H}_\pi$ and $v' \in \mathcal{H}_{\pi'}$ then we can define A by $Au := \langle u, v\rangle v'$, a rank 1 linear map. Then consider that:

$$\begin{aligned}\langle \tilde{A}u, u'\rangle &= \int \langle \pi'(x^{-1})A\pi(x)u, u'\rangle\,dx = \int \langle \pi(x)u, v\rangle\langle v', \pi'(x)u'\rangle\,dx \\ &= \int \pi_{u,v}(x)\overline{\pi'_{u',v'}(x)}\,dx = \langle \pi_{u,v}, \pi'_{u',v'}\rangle\end{aligned}$$

Now if $[\pi] \neq [\pi']$ then we have that $\tilde{A} = 0$ by Schur's lemma so the above quantity is zero. This proves that $\mathcal{E}_\pi \perp \mathcal{E}_{\pi'}$.

Part b. can be proved by applying Schur's lemma to see that $\tilde{A} = cI$ taking $u = e_i$, $u' = e_{i'}$, $v = e_j$ and $v' = e_{j'}$ and arguing that $\operatorname{Tr} \tilde{A} = \operatorname{Tr} A$ (taking the trace is a linear functional on $\operatorname{End} \mathcal{H}_\pi$ so commutes with the integrals) to solve for the constant c. □

This proof is another textbook case of using the Haar measure on a compact group to average (integrate) your way from something not G-invariant to something G-invariant.

I was actually unaware of this result for the case of finite groups, but this proof actually proves it for the finite group case if we turn the finite group into a compact group by giving it the discrete topology and then noting that the Haar measure is the normalised counting measure.

4.3 Break it Down

So far we have found subspaces $\mathcal{E}_\pi \subset L^2(G)$ for each $[\pi] \in \widehat{G}$. It is now natural to ask what the breakdown of this finite dimensional subrepresentation of $L^2(G)$ into irreducibles looks like. The answer is extremely nice, and could possibly have been guessed easily enough by someone reading along keeping one eye firmly on our destination.

Theorem 7. *Suppose π is irreducible. For $i = 1, ..., d_\pi$ define $\mathcal{R}_i := span\{\pi_{i1}, ..., \pi_{id_\pi}\}$ (the span of the entries in the i-th row). Then $\mathcal{R}_i$ is invariant under the right regular representation and $R|_{\mathcal{R}_i}$ is equivalent to π via:*

$$\sum_j c_j e_j \mapsto \sum_j c_j \pi_{ij}$$

Proof. Calculating the action of x on both representations we get:

$$\pi(x)\left(\sum_j c_j e_j\right) = \sum_{j,k} \pi_{kj}(x) c_j e_k$$

Now by noting that $\pi(yx) = \pi(y)\pi(x)$ we have that:

$$\left[R_x\left(\sum_j c_j \pi_{ij}\right)\right](y) = \sum_j c_j \pi_{ij}(yx) = \sum_j c_j \sum_k \pi_{ik}(y)\pi_{kj}(x) = \sum_{j,k} \pi_{kj}(x) c_j \pi_{ik}(y)$$

After this calculation we now know that $\mathcal{R}_i$ is a closed subrepresentation of $\mathcal{E}_\pi$. Comparing the two calculations we see that $R|_{\mathcal{R}_i}$ is equivalent to π via the indicated map. □

We have now answered the question of the decomposition of $\mathcal{E}_\pi$ into irreducible representations.

4.4 The Grand Finale

We are rapidly closing in on our destination. Define $\mathcal{E}$ to be the linear span of all the $\mathcal{E}_\pi$ for all $[\pi] \in G$.

I will give an indication of how this final piece we need to state the Peter-Weyl Theorem can be proven

Theorem 8. *$\mathcal{E}$ is dense in $C(G)$ for the uniform norm, and therefore dense in $L^2(G)$.*

One approach to this is to leverage the Gelfand-Raikov theorem from abstract Fourier Analysis, which states that the irreducible unitary representations of G separate points. Pairing this with the Stone-Weierstrass theorem we note that $\mathcal{E}$ is an algebra (pointwise multiplication can be achieved via the tensor product of representations), closed under complex conjugation (one can consider the induced representation on the dual of $\mathcal{H}_\pi$), contains constants (due to the trivial representation) and separates points (by the Gelfand-Raikov theorem). Therefore by the Stone-Weierstrass theorem $\mathcal{E}$ is dense in $C(G)$.

This result however predates the Gelfand-Raikov theorem. There is a proof method using convolutions, Hilbert-Schmidt operators, the spectral theorem and approximations to the identity to show that any $f \in L^2(G)$ orthogonal to $\mathcal{E}$ must itself vanish. Showing that $\mathcal{E}^\perp = 0$ and thus that $\mathcal{E}$ is dense.

Finally let's state the total of what we have learnt in one theorem.

Theorem 9. *(Peter-Weyl Theorem) Let G be a compact group. Then $\mathcal{E}$ is uniformly dense in $C(G)$, and $L^2(G)$ decomposes as $\bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$.*

$$\{\sqrt{d_\pi}\pi_{ij} : 1 \leq i, j \leq d_\pi, [\pi] \in \widehat{G}\}$$

is an orthonormal basis for $L^2(G)$, and for each $\mathcal{E}_\pi$ the span of the entries of a given row are a subrepresentation isomorphic to π.

References

[1] A course in Abstract Harmonic Analysis by G. Folland

[2] Hilbert's fifth problem and related topics by Terence Tao

[3] Representation Theory of Compact Groups Michael Ruzhansky and Ville Turunen