Algebraic Groups 2020 Solutions

Lachlan Potter

July 12, 2020

Contents

1 Preface

This is a document I wrote up mostly to check my understanding of the solutions to exercises posed in the 2020 spring semester course on algebraic groups, taught at ETH Zrich. As such I have often glossed over steps in proofs that I am already comfortable with and gone into a lot of detail for steps I was confused about.

2 Basic Notions Including Hopf Algebras

2.1 Exercise 1 (Problem 3.1.2 OV)

The closure of any subgroup of an algebraic group is an algebraic subgroup.

Solution:

Proof. Let $H < G$ be a subgroup. Then let $x \in G$. Note that:

$$x\cdot : G \to G,\, g \mapsto xg$$

is a homeomorphism and if $x \in H$ then it preserves H because H is a subgroup. Now using the fact that for continuous functions $f(\overline{X}) \subseteq \overline{f(X)}$, and for homeomorphisms we have equality. Therefore for all $x \in H$:

$$x\overline{H} = \overline{xH} = \overline{H}$$

Therefore $H\overline{H} = \bigcup_{x\in H} x\overline{H} = \bigcup_{x\in H} \overline{H} = \overline{H}$. Now since right multiplication is also a homeomorphism we have that for all $x \in \overline{H}$:

$$Hx \subset \overline{H}$$

Taking closure of both sides and using properties of homeomorphisms:

$$\overline{H}x = \overline{Hx} = \overline{H}$$

Therefore $\overline{H}$ is closed under multiplication and a closed subset of G, and therefore is an algebraic subgroup of G. $\square$

2.2 Exercise 2 (Problem 3.1.3 OV)

Any irreducible subgroup of an algebraic group épais (thick) in its closure is closed.

Solution:

Proof. Let $H < G$ be irreducible and contain an open subset of its closure $H \supseteq U \subseteq \overline{H}$. We claim that $H \supseteq \overline{H}$. This will tell us that H is indeed closed.

Note that if H is irreducible then so is $\overline{H}$, because if we have two open subsets V, W in $\overline{H}$ then $V \cap H, W \cap H$ are two non-empty (property of the closure of a set), opens in H, thus intersect. Therefore V and W have an intersection in $\overline{H}$.

Now since $\overline{H}$ is a subgroup by Exercise 1 $H \supseteq \langle U \rangle \subseteq \overline{H}$. I will show that $\langle U \rangle$ is closed in $\overline{H}$. Since $\overline{H}$ is irreducible, U, and thus $\langle U \rangle$ is dense in $\overline{H}$. This will tell us that the subgroup generated by U is closed and dense, and therefore equal to $\overline{H}$.

Note that:

$$\langle U \rangle = \{u_1^{\pm} \cdots u_n^{\pm} : u_i \in U\} = \bigcup_{u_1^{\pm}\cdots u_{n-1}^{\pm}} u_1^{\pm} \cdots u_{n-1}^{\pm}(U \cup U^{-1})$$

Therefore $\langle U \rangle$ is a union of open sets, hence open. Furthermore, since $\langle U \rangle$ is the complement of the union of its cosets, which are just $x\langle U \rangle$ for some $x \notin \langle U \rangle$, the image of $\langle U \rangle$ under a homeomorphism must be open. Therefore $\langle U \rangle$ is closed. Therefore $\langle U \rangle = \overline{H}$

Thus we conclude $H \supset \langle U \rangle = \overline{H}$. Therefore $H = \overline{H}$ is closed. $\square$

2.3 Exercise 3 (Problem 3.1.4 OV)

For subspaces of a vector space $W \subset U \subset V$ the group

$$GL(V; W, U) := \{A \in GL(V) : (A - E)U \subset W\}$$

Is algebraic.

Solution:

Proof. Pick a flag basis of $W \subset U \subset V$, then the subgroup can be cut out by the following:

$$A = \begin{bmatrix} GL(W) & * & * \\ 0 & E_{U/W} & * \\ 0 & 0 & GL(V/U) \end{bmatrix}$$

Which is a set of algebraic conditions on the entries of A. $\square$

2.4 Exercise 4 (Problem 3.1.7 OV)

Show that the subgroups:

$$\left\{ \exp t \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} : t \in \mathbb{C} \right\}, \left\{ \exp t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\} \subset GL_2(\mathbb{C})$$

Are Lie subgroups but not algebraic subgroups.

Solution:

Proof. Probably Lie subgroups because they are images of Lie sub-algebras of the corresponding Lie algebra under the exponential map, but let us focus on the non-algebraicity.

We will show that each group is not closed in the Zariski topology. The first subgroup is:

$$A := \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{it} \end{pmatrix} : t \in \mathbb{C} \right\}$$

I will show that the following matrix is in the closure of this set (while it is not in the set itself):

$$X := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $f \in k[GL_2(\mathbb{C})]$ such that f vanishes on A, we will show that f vanishes on X. If the variables in the entries of the matrix are a, b, c, d reading right-left, top-bottom. f has the form:

$$f(a,b,c,d) = \frac{1}{(ad-bc)^N} \sum_{m,n,k,l} C_{mnkl} a^m b^n c^k d^l$$

Since the determinant of all matrices in non-zero then by we may assume that $N = 0$ for the sake of checking if f vanishes on X.

Now since it vanishes on A we have the following identity:

$$f(e^t, 0, 0, e^{it}) = \sum_{i,j,k,l} C_{m00l} e^{mt} e^{ilt} = 0 \, \forall t \in \mathbb{C} \, (0^0 = 1 \text{ for sake of notation})$$

Now if we let $t = 2\pi j$ for any $j \in \mathbb{Z}$ we get:

$$f(e^{2\pi j}, 0, 0, 1) = \sum_{i,j,k,l} C_{m00l} (e^{2\pi j})^m = 0 \, \forall j \in \mathbb{Z} \, (0^0 = 1 \text{ for sake of notation})$$

This is a polynomial in a with infinitely many roots, therefore is 0. That is $f(a, 0, 0, 1) = 0$ for all $a \in \mathbb{C}$. In particular $f(2, 0, 0, 1) = 0$. Therefore since f was arbitrary $X \in \overline{A}$.

Now for the set:

$$B := \left\{ \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} : t \in \mathbb{C} \right\}$$

In a similar way to before suppose f vanishes on B, then we get the following identity:

$$f(e^t, te^t, 0, e^t) = \sum_{i,j,k,l} C_{m0kl} e^{mt} t^k e^{kt} e^{lt} = 0 \, \forall t \in \mathbb{C} \, (0^0 = 1 \text{ for sake of notation})$$

Now let $t = 2\pi ij$ for $j \in \mathbb{Z}$:

$$f(1, 2\pi ij, 0, 1) = \sum_{i,j,k,l} C_{m0kl} (2\pi ij)^k = 0 \, \forall j \in \mathbb{Z} \, (0^0 = 1 \text{ for sake of notation})$$

Therefore the poynomial $f(1, b, 0, 1)$ has infinitely many roots and so is 0. Therefore:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \overline{B} \setminus B$$

Therefore B is not closed and hence not an algebraic subgroup. □

2.5 Lemma: Connectedness and Irreducibility

For an algebraic group G, the irreducible components are the connected components.

Solution:

Proof. Suppose $G = X_1 \cup \cdots \cup X_n$ where the X_i are irreducible components. That is, they are maximal irreducible closed subsets of G. We may assume no redundancy, that is $X_i \not\subset \bigcup_{k \neq i} X_k$. To show that these are indeed the connected components we will show that they are disjoint. Suppose w.l.o.g. $x \in X_1 \cap X_2$. Then for all $g \in X_1$, $g = gx^{-1}x \in X_1 \cap gx^{-1}X_2$. Since multiplication acts on G via homeomorphism we know that the image of an irreducible component is an irreducible component. But this all together means that X_1 is redundant in the list of irreducible components. Therefore the X_i are in fact disjoint. We know each X_i is connected because it is irreducible. □

2.6 Exercise 5 (Problem 3.1.8 OV)

Any complex algebraic group connected in the real topology is irreducible.

Solution:

Proof. Suppose G is connected in the real topology then it is connected in the Zariski topology because the Zariski topology is coarser. In particular a disconnection of G in the Zariski topology gives a disconnection in the real topology. Since the connected components are the irreducible components we have that G is also irreducible. □

2.7 Exercise 6 (Problem 3-1 Milne)

Solution:

Proof. See Subhajit's solution. I do not want to TeX this one up. □

3 Representations and Jordan Decomposition

3.1 Exercise Optional

Let G be an algebraic group. Let $A := k[G]$ denote its coordinate ring, equipped with the right regular representation r_A of G. Let $g \in G$. Show that g is semisimple/unipotent iff $r_A(g)$ has semisimple/unipotent restriction to every finite dimensional subrepresentation of A. (Use proof technique given to show the existence and uniqueness of Jordan decompositions.)

Solution:

Proof. ($\Longleftarrow$): For any finite dimensional representation (r, V) of G we want to show that $r(g) = r(g)_s$. We know that $r_A(g)|_U = (r_A(g)|_U)_s$ for any finite dimensional subrepresentation $U \subset A := k[G]$.

Define the following maps $\lambda_W = r_W(g)$ and $\mu_W := r_W(g)_s$. Note that these maps satisfy the following:

- $\lambda_k = 1$
- $\lambda_{V\otimes W} = \lambda_V \otimes \lambda_W$
- For all morphisms $\phi : V \to W$ such that $\phi \circ \lambda_V = \lambda_W \circ \phi$

Define the following map of representations:

$$\Delta : V \to V_0 \otimes A,\ v \mapsto \sum_i v_i \otimes f_i$$

Where the f_i are chosen such that $r_V(h)v = \sum_i f_i(h)v_i$, where $\{v_i\}_i$ is a chosen basis for V. $V_0 = V$ but equipped with the trivial G-action.

We check that this is a morphism of representations:

$$\Delta(r_V(h)v) = \sum_i v_i \otimes g_i(h)$$

Where the g_i are such that $r_V(g)(r_V(h)v) = \sum_i g_i(g)v$

$$\begin{aligned} r_{V_0\otimes A}(h)(\Delta v) &= r_{V_0\otimes A}(h)(\sum_i v_i \otimes f_i) \\ &= \sum_i v_i \otimes (f_i \circ (\cdot h)) \end{aligned}$$

Now note that:

$$r_V(g)(r_V(h)v) = r_V(gh)v = \sum_i f_i(gh)v_i = \sum_i (f_i \circ (\cdot h))(g)v_i$$

Therefore we see that $f_i \circ (\cdot h) = g_i$ and therefore Δ is a morphism of representations.

Furthermore the map that evaluates the tensors are the identity is a left inverse, and therefore Δ is injective:

$$\epsilon(\Delta(v)) = \epsilon(\sum_i (v_i \otimes f_i)) = \sum_i f_i(1)v_i = r_V(1)v = v$$

We will take a moment now to show that the span of the f_i in A are finite. If we take the f_i generated by a finite basis of V then all othe f_i are linear combinations.

Now we not that finite dimensionality of V implies that Δ has finite dimensional image and so:

$$\begin{aligned}
\Delta \circ r_V(g) &= \Delta \circ \lambda_V \\
&= \lambda_{V_0 \otimes A} \circ \Delta && \text{property 3} \\
&= (\lambda_{V_0} \otimes \lambda_A) \circ \Delta && \text{property 2} \\
&= (1 \otimes r_A(g)) \circ \Delta && \text{property 1} \\
&= (1 \otimes r_A(g)|_{\text{Image}(\Delta)}) \circ \Delta && \text{span of } f_i \text{ which are finite dimensional} \\
&= (1 \otimes r_A(g)_s) \circ \Delta && \text{since composition only sees span of} f_i \\
&= (\mu_{V_0} \otimes \mu_A) \circ \Delta \\
&= \Delta \circ \mu_V \\
&= \Delta \circ r_V(g)_s
\end{aligned}$$

Then applying ϵ to both sides we see that $r_V(g) = r_V(g)_s$ for any finite dimensional rep of G. Therefore g is semisimple.

($\Longrightarrow$): If g is semi-simple/unipotent then it $r_V(g)$ is semisimple/unipotent for ALL finite dimensional representations, not just those that arise as restrictions of the right regular representation. □

3.2 Exercise 7 (Problem 3.2.1 OV)

Let A be a semisimple operator and suppose $U \subset V$ be a subspace invariant with respect to A. Show that $A|_U$ is semisimple and that there exists an invariant subspace complementary to U.

Solution:

Proof. Recall the fact that an operator is semi-simple iff its minimal polynomial splits and is separable in the field. Since we are working over algebraically closed fields it must split, but not necessarily be separable. $m_{A|_U} | m_A$ since any polynomial that vanishes on A must vanish on $A|_U$. Since A is semisimple its minimal polynomial is separable, and a factor af a separable polynomial is separable so $A|_U$ is semisimple. To find a complementary invariant subspace consider $A|_U, A|_{V/U}$, diagonalise the bases and pull back the second basis to get a basis of the complementary invariant subspace. □

3.3 Exercise 8 (Problem 3.2.2 OV)

Show that a family A of commuting semisimple linear operators can be simultaneously diagonalised.

Solution:

Proof. Consider $kA \subset \mathrm{End}(V)$. The span of this family A of operators. Pick a basis $\{g_1, \cdots, g_k\}$ for kA. This is finite dimensional because $\dim(kA) \leq \dim(\mathrm{End}(V)) = (\dim(V))^2 < \infty$. $\{g_1, \cdots, g_k\}$ are are pairwise commutative iff kA is pairwise commutative, iff A is pairwise commutative.

Therefore we need only prove this for the case of finitely many commuting operators. Suppose it is true for $\leq k-1$ operators (the base case of 0 or 1 operator is trivial).

Since all of the operators in our set are diagonaliseable split V into g_k eigenspaces. Since they commute all operators preserve this decomposition of V. Restricting our operators to each subspace we get that the g_k must be simultaneously diagonaliseable because it is a scalar operator, by induction the restriction of the other $k-1$ operators are all simultaneously diagonaliseable. Doing so for each eigenspace gives us simultaneous diagonaliseation of all k operators.

Therefore by induction the claim is true for finite subsets, and by the starting reduction it is then true for all pairwise commutative subsets. □

3.4 Exercise 9

Suppose k has characteristic zero. Show that each $g \in GL(V)$ of finite order is semisimple. Show that if g^n is semisimple then g is semisimple. Show that both of these results fail for k with positive characteristic.

Solution:

Proof. $g^n - 1 = 0$ implies that for any representation (r, V) we have $r(g)^n - 1 = 0$. Since k is algebraically closed and characteristic 0 we have that:

$$\gcd(x^n - 1, nx^{n-1}) = 1$$

Therefore $x^n - 1$ is separable. The minimal polynomial of $r(g)$ is a factor of it, therefore also separable. Therefore $r(g)$ is semisimple. Since it works for any finite rep g is semi-simple.

If $g^n \in GL(V)$ is semisimple then we have:

$$m_{g^n}(x) = \prod_{i=1}^{k}(x - \lambda_i), \qquad i \neq j \implies \lambda_i \neq \lambda_j$$

Since characteristic of k is zero we have n distinct n-th roots of λ_i. We know that $m_{g^n}(x^n)$ vanishes on g, and:

$$m_{g^n}(x^n) = \prod_{i=1}^{k}(x^n - \lambda_i), \qquad i \neq j \implies \lambda_i \neq \lambda_j$$

So the roots of this polynomial are all n-th roots of the λ_i. These roots must be distinct, otherwise we could raise two that are the same to the n-th power to conclude that $\lambda_i = \lambda_j$, which is false. Therefore m_g divides a separable split polynomial and so is sepreabale and split. □

3.5 Lemma: Hopf Algebra of Diagonaliseable Groups

For G a diagonaliseable group then $\mathcal{O}(G) \cong k[\mathfrak{X}(G)]$ as Hopf algebras, where on the right hand side $\Delta m := m \otimes m$.

Solution:

Proof. We know from lectures that any $f \in \mathcal{O}(G)$ can be written uniquely as a linear combination of characters in $\mathfrak{X}(G)$. Therefore these are clearly isomorphic as k-algebras and we need only check the compatibility of the Hopf algebra structure.

Since Δ is a linear map we need only show the compatibility of the isomorphism for a given character $(\chi : G \to \mathbb{G}_m) \in \mathcal{O}(G)$. Now note that $\chi(gh) = \chi(g)\chi(h)$ since it is a character, and therefore $\Delta\chi = \chi \otimes \chi$. Therefore the described Hopf algebra structure is indeed the one induced by $\mathcal{O}(G)$. □

3.6 Lemma: Character Group of Some Diagonaliseable Groups

$\mathfrak{X}(\mu_n) \cong \mathbb{Z}/n$ given $\mathrm{char}(k) \not| n$

Solution:

Proof. Due to the constraint on the characteristic we have:

$$k[\mathfrak{X}(\mu_n)] \cong \mathcal{O}(\mu_n) \cong k[t]/(t^n - 1) \cong k[\mathbb{Z}/n]$$

Where $t : \mu_n \to k$, $x \mapsto x$. Now applying the lemma proven in week 5 lecture 2 that states that for abelian groups $\mathrm{Hom}_{Hopf}(k[M_1], k[M_2]) \cong \mathrm{Hom}(M_1, M_2)$ given by restriction when the Hopf algebra structure is to take elements of M_i as group-like elements, i.e. $\Delta m = m \otimes m$.

Therefore an isomorphism of $k[\mathfrak{X}(\mu_n)] \cong k[\mathbb{Z}/n]$ as Hopf algebras induces via a standard abstract non-sense argument an isomorphism $\mathfrak{X}(\mu_n) \cong \mathbb{Z}/n$ □

3.7 Lemma: Products and Character Groups

For diagonaliseable groups G and H, $\mathfrak{X}(G \times H) \cong \mathfrak{X}(G) \times \mathfrak{X}(H)$

Solution:

Proof. We take an approach similar to the previous lemma:

$k[\mathfrak{X}(G\times H)] \cong \mathcal{O}(G\times H) \cong \mathcal{O}(G)\otimes\mathcal{O}(H) \cong k[\mathfrak{X}(G)]\otimes k[\mathfrak{X}(H)] \cong k[\mathfrak{X}(G)\times\mathfrak{X}(H)]$

And again leveraging the power of abstract non-sense we get $\mathfrak{X}(G \times H) \cong \mathfrak{X}(G) \times \mathfrak{X}(H)$.

The last isomorphism is visible when one writes out what a generic element on either side looks like. □

3.8 Exercise 10

Show that every connected diagonaliseable algebraic group is a torus.

Solution:

Proof. Recall from week 5 lecture 2 that every diagonaliseable algebraic group is isomorphic to a subgroup of a torus, and furthermore that $\mathcal{O}(G) \cong k[\mathfrak{X}(G)]$ as Hopf algebras where the structure on the right hand side is the unique Hopf algebra structure defined by $\Delta m : m \otimes m$ for all $m \in \mathfrak{X}(G)$.

Also recall that the functor $G \mapsto \mathfrak{X}(G)$ defines an antiequivalence of categories between diagonaliseable algebraic groups over k and finitely generated abelian groups without $\mathrm{char}(k)$ torsion. Now using the characterisation of finitely generated abelian groups we get that:

$$\mathfrak{X}(G) \cong \mathbb{Z}^n \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k$$

Now note that $\mathfrak{X}(\mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k}) \cong \mathfrak{X}(G)$ so applying the equivalence of categories we conclude that:

$$\mathbb{G}_m^n \times \mu_{n_1} \times \cdots \times \mu_{n_k} \cong G$$

The only n such that μ_n is connected is $n = 1$ and the product of a disconnected topologicaly space with any other space is till disconnected so we conclude that $n_1 = \cdots = n_k = 0$. Therefore $G \cong \mathbb{G}_m^n$ so G is a torus. □

3.9 Exercise 11

Let H be an algebraic subgroup of a torus T. Let $H^\perp := \{\chi \in \mathfrak{X}(T) : \chi|_H = 0\}$. Show that $H = \{t \in T : \chi(t) = 1 \,\forall\, \chi \in H^\perp\}$.

Solution:

Proof. Define $H' := \{t \in T : \chi(t) = 1 \,\forall\, \chi \in H^\perp\}$. Our goal is to show $H' = H$. $H' \supset H$ is clear since for all $h \in H$ we have $\chi(h) = 1$ for all $\chi \in H^\perp$.

Now note that H' is also a subgroup of T because for all $h, g \in H'$ and $\chi \in H^\perp$:

$$\chi(hg) = \chi(h)\chi(g) = 1$$

Therefore $hg \in H'$.

Since T is commutative H is a normal subgroup of T. Therefore using Proposition 4.3 in Szamuely we know that there exists a representation:

$$\rho : T \to GL(V)$$

such that $\ker(\rho) = H$. Since T is a torus we know by our theorem stating the equivalent definitions of diagonaliseable groups that this representation is diagonaliseable. Therefore we can pick a basis such that all $\rho(t)$ are diagonal matrices. Using this basis we now note that:

$$t \mapsto \rho(t)_{ii}$$

are all characters of T (being diagonal matrices now makes these group homomorphisms). Furthermore $\rho(h) = id$ for all $h \in H$ since $H \subset H' = \ker(\rho)$. Therefore $\rho(h)_{ii} = 1$ for all $h \in H$ so the characters $(t \mapsto \rho(t)_{ii}) \in H^{\perp}$. Now let $h' \in H'$. By the definition of H' applied to these characters in question we get that:

$$\rho(h')_{ii} = 1 \text{ for all } i$$

Combined with the fact that $\rho(h')$ is a diagonal matrix we conclude that $\rho(h') = id$. In other words:

$$h' \in \ker(\rho) = H \qquad \text{(by construction of } \rho)$$

Since $h \in H'$ was arbitrary we have that $H' \subset H$.

This allows us to conclude that $H' = H$, as required. □

3.10 Exercise 12

Let G be an algebraic group. For $g \in G$ define $G(g) := \overline{\langle g \rangle}$. Show that $G(g)$ is a commutative algebraic group. Show that $G(g)_s = G(g_s)$ and $G(g)_u = G(g_u)$. Deduce that multiplication defines an isomorphism $G(g) \cong G(g_s) \times G(g_u)$.

Solution:

Proof. Now consider the following regular function for all $y \in G(g)$:

$$f_y : G(g) \to G(g), f_y(x) := [x, y]$$

Note that $f_y^{-1}(1)$ is a closed subset of $G(g)$ and $\langle g \rangle \subset f_g^{-1}(1)$, therefore $G(g) \subset f_g^{-1}(1)$. Therefore $[g, G(g)] = [G(g), g]^{-1} = 1$. Therefore for all $x \in G(g)$:

$$\langle g \rangle \subset f_x^{-1}(1) \implies G(g) \in f_x^{-1}(1)$$

Therefore as we range over all $x \in G(g)$ we get that:

$$[G(g), G(g)] = 1$$

That is, $G(g)$ is a commutative algebraic group.

Now from the main result on commutative groups from week 7 lecture 1 we know the multiplication map $G(g)_s \times G(g)_u \to G$ is an isomorphism. We also know that $G(g)_s$ and $G(g)_u$ are closed algebraic subgroups. Therefore since $g_s \in G(g)_s$ we have that $G(g_s) \subset G(g)_s$, and the same for unipotent parts.

$$G(g) \cong G(g)_s \times G(g)_u \supset G(g_s) \times G(g_u)$$

But note that the right hand side is a closed subgroup of G containing $(g_s, g_u) = g$ under the identifications, since $G(g)$ is by design the smallest of all such subgroups we in fact have equality of sets on the right and so:

$$G(g) \cong G(g_s) \times G(g_u)$$

via the multiplication map. $\square$

3.11 Exercise 13

Prove that for any connected nilpotent algebraic group G, the subsets G_s and G_u are normal algebraic subgroups and that the multiplication map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.

Solution:

Proof. Since G is connected, nilpotent it is connected, solveable and so by Lie-Kolchin theorem G is trigonaliseable.

Let $r : G \to GL(V)$ be a faithful representation. Let $\dim(V) = n$, A_n be upper triangular $n \times n$ matrices and let D_n be diagonal $n \times n$ matrices.

Now we will prove some properties of G_s. Recall that in nilpoent groups semi-simple elements are central. In particular if g_s, h_s are semi-simple then since they commute $r(g_s)$ and $r(h_s)$ can be simultaneously diagonalised, and so $r(g_s h_s) = r(g_s)r(h_s)$ can be diagonalised. Therefore $g_s h_s \in G_s$. Therefore G_s is a subgroup.

Also $r(hgh^{-1}) = r(h)r(g)r(h)^{-1}$ can be diagonalised if and only if $r(g)$ can. Therefore G_s is in fact a normal subgroup.

Since G_s can be simultaneously diagonalised we have: $r(G_s) \subset D_n$. Clearly $r(G_s) \subset D_n \cap r(G)$, but note also that if $r(g) \in D_n$ then $r(g)$ is clearly diagonaliseable and so $g \in G_s$. We conclude that $r(g) \in r(G_s)$, giving us the reverse inclusion. So we have $r(G_s) = D_n \cap r(G)$. Therefore:

$$G_s = r^{-1}(D_n \cap r(G))$$

the preimage of a closed set. Therefore G_s is in fact a normal closed subgroup of G.

Now we will prove some facts about G_u. $G_u = \{g : (r(g) - 1)^n = 0\}$ by Cayley-Hamilton Theorem. Choosing a suitable basis of V we get that:

$$r(G_u) \subset r(G) \subset A_n$$

Since all eigenvalues of a unipotent element under r must be 1 we get that:

$$r(G_u) \subset B_n \cap r(G)$$

Where B_n are upper triangular $n \times n$ matrices with 1's on the diagonal. Similarly to before if $r(g) \in B_n$ it is unipotent, and so $g \in G_u$. Therefore in fact $r(G_u) = B_n \cap r(G)$, yielding:

$$G_u = r^{-1}(B_n \cap r(G))$$

So G_u is a closed subgroup of G, as it is the preimage of a closed subgroup.

G_u is a normal subgroup because for all $h \in G$ and $g_u \in G_u$ we have $(r(hg_uh^{-1}) - 1)^n = r(h)(r(g) - 1)^n r(h)^{-1} = r(h) \cdot 0 \cdot r(h)^{-1} = 0$. So G_u is also a normal closed subgroup of G.

I claim that the multiplication map $G_s \times G_u \xrightarrow{\phi} G$ is an isomorphism of algebraic groups. First to check it is a group homomorphism:

$$\phi((g_s, h_u)(g'_s, h'_u)) = \phi(g_s g'_s, h_u h'_u) = g_s g'_s h_u h'_u = g_s h_u g'_s h'_u = \phi(g_s, h_u)\phi(g'_s, h'_u)$$

Surjectivity comes from the Jordan decomposition of algebraic group elements and injectivity comes from the fact that G_s is central, so any decomposition of g into a semisimple and unipotent element commute with each other, and therefore must be the (unique) Jordan decomposition. Therefore ϕ is a group isomorphism.

It is algebraic because it is the restriction of the multiplication map $m : G \times G \to G$. To show that the inverse is regular its enough to show that $g \mapsto g_s$ is regular because $g_u = g_s^{-1} g$, so the projection onto the unipotent part would be regular too.

Now let $h : A_n \to A_n$ be the algebraic group morphism that sets all non-diagonal entries to 0. Then $g \mapsto g_s$ is simply $r^{-1} \circ h|_{r(G)} \circ r$, a regular function. Therefore ϕ is an isomorphism of algebraic groups, as required. $\square$

4 Projective Varieties and Borel Fixed Point Theorem

4.1 Results: Regular Functions Between Quasi-projective Varieties

Recall the definition of regular functions between quasi-projective varieties and prove the following:

- Recall the definition.

  Solution: $f : X \to Y$ is regular if for any open $U \subset Y$ and any h:

  $$h \in \mathcal{O}(U) := \bigcap_{p \in U} \mathcal{O}_{Y,p} \subset k(Y)$$

  Where $\mathcal{O}_{Y,p} = \{\frac{f}{g} \in k(Y) : g(p) \neq 0\}$ we have that $h \circ f \in \mathcal{O}(f^{-1}(U))$.

- Explain why it is enough to prove the condititon for an open cover of the domain.

  Solution: Let $f : X \to Y$ and $U \subset Y$, then $U \subset \bigcup_i U_i$ where $\{U_i\}$ is the open cover on which the condition holds. Thus by restriction the condition would hold if we can prove that the condition holding on two opens implies that it hold on their union.

  Now suppose for both U, V open subsets we have that if $h \in \mathcal{O}(U)$ then $h \circ f \in \mathcal{O}(f^{-1}(U))$ and likewise for V. Let $h \in \mathcal{O}(U \cup V)$. Then $h|_U \in \mathcal{O}(U)$ so by the condition $h \circ f \in \mathcal{O}(f^{-1}(U))$. Likewise $h \circ f \in \mathcal{O}(f^{-1}(V))$. Therefore:

  $$h \circ f \in \mathcal{O}(f^{-1}(U)) \cap \mathcal{O}(f^{-1}(V)) = \bigcap_{p \in f^{-1}(U)} \mathcal{O}_{Y,p} \cap \bigcap_{p \in f^{-1}(V)} \mathcal{O}_{Y,p}$$

  But the right hand side is exactly $\mathcal{O}(f^{-1}(U) \cup f^{-1}(V)) = \mathcal{O}(f^{-1}(U \cup V))$.

- Show that this new definition is consistent with the old definition when $f : X \to Y$ is between affine varieties.

  Solution: Clearly by taking Y as an open cover, if f is a polynomial in the co-ordinates of Y the any regular function on Y will pull back to a regular function on X.

  In the other direction. Again take Y as the open set, then let $h \in \mathcal{O}(Y)$ be projection onto the i-th co-ordinate. Then $h \circ f \in \mathcal{O}(X)$. Therefore the i-th co-ordinate of f is polynomial.

4.2 Exercise 14

(a). Write down a careful proof that the image of the Grassmanian $G_{n,m}$ under the Plücker embedding $G_{n,m} \hookrightarrow \mathbb{P}^N = \mathbb{P}(\wedge^m(k^n))$ is closed.

Solution:

Proof. First we will recall some definitions:

$$\wedge^m(k^n) = k\{e_{i_1} \wedge \cdots \wedge e_{i_m} : i_1 < \cdots < i_m\}$$

Where $\{e_i\}$ are a basis for k^n. the wedge is anti-symmetric and linear in each co-ordinate. Note that if we have two different bases $\{e_i\}$ and $\{f_i\}$ for a given m-dimensional subspace of k^n then we have that:

$$e_{i_1} \wedge \cdots \wedge e_{i_m} = \det(a_{ij}) f_{i_1} \wedge \cdots \wedge f_{i_m}$$

Where (a_{ij}) is the change of basis matric between them.

Therefore the map $p_m : G_{n,m} \to \mathbb{P}(\wedge^m(k^n))$ that sends a subspace $S \subset k^n$ to $w_1 \wedge \cdots \wedge w_m$ where $\{w_i\}$ form a basis for S is well defined.

To prove that the map is injective let S, S' be spaces in the same fiber and pick a basis of V such that $\{e_i\}$ and $\{e_{i+r}\}$ are respective bases for S and S'.

Then since they are in the same fiber we have that:

$$e_1 \wedge \cdots \wedge e_m = e_{r+1} \wedge \cdots \wedge e_{r+m}$$

Which happens exactly when $r = 0$, implying that $S = S'$.

Given an element $\omega \in \mathbb{P}(\wedge^m(k^n))$. Note that ω is in the image of p_d exactly when $\omega = w_1 \wedge \cdots \wedge w_m$ where $\{w_i\}$ forms a basis for some m dimensional subspace S.

I claim that this is in the image exactly when the map:

$$f_\omega : V \mapsto \wedge^{m+1}(k^n),\ v \mapsto v \wedge w_1 \wedge \cdots \wedge w_m$$

has a kernel of dimension m, and otherwise the kernel has dimension $< m$.

To see this, if $w_1 \wedge \cdots \wedge w_m \in \text{Image}\,(p_m)$ then v is in the kernel of f_ω exactly when $\{v, w_1, \cdots, w_m\}$ is linearly dependent which is exactly when $v \in k\{w_1, \cdots, w_m\} = S$. Conversely choose a basis $\{v_i\}$ of V that is an extension of the basis for the kernel of f_ω. Any element is a linear combination of wedges of m distinct v_i. If ω is not monomial then the vectors in the kernel of f_ω are those that appear in every monomial. Therefore there can be at most $m-1$ of these. This gives us that the dimension of the kernel of the map defined by ω is $< m$.

Therefore considering the matrix of f_ω we can imbed:

$$\mathbb{P}(\wedge^m(k^n)) \hookrightarrow \text{Hom}_k(V, \mathbb{P}(\wedge^{m+1}(k^n))), \omega \mapsto (v \mapsto v \wedge \omega)$$

and now the image of p_m are exactly the maps with kernel of dimension at least m, which is the same as the maps of ranks at most $\binom{n}{m+1} - m$, which is a polynomial condition on the minors. Therefore dit defines a closed set. $\square$

(b). Show that GL_n is irreducible.

Solution:

Proof. GL_n is an open subset of the irreducible $\mathbb{A}^{n^2}$. $\square$

(c). Show that the action of GL_n on $G_{n,m}$ is transitive.

Solution:

Proof. This is by virtue of the fact that there exists an invertible linear transformation taking any subspace to any other subspace of the same dimension. $\square$

(d). Verify that the action of GL_n on $\mathbb{P}^N$ is algebraic, and hence on $G_{n,m}$ is continuous. Deduce that the Plcker image of $G_{n,m}$ is irreducible.

Solution:

Proof. To check algebraicity of the action we need to check if the corresponding action map:

$$GL_n(k) \times \mathbb{P}(\wedge^m(k^n)) \to \mathbb{P}(\wedge^m(k^n)),\ (g, w_1 \wedge \cdots \wedge w_m) \mapsto gw_1 \wedge \cdots \wedge gw_m$$

is algebraic. It is because the output, once a basis is chosen, is polynomial in the entries of g.

Therefore if we pick $e_1 \wedge \cdots \wedge e_m$ then the restriction of the action map:

$$GL_n(k) \to G_{m,n},\ g \mapsto ge_1 \wedge \cdots \wedge ge_m$$

is surjective. Therefore $G_{m,n}$ is irreducible as it is the regular image of an irreducible space. $\square$

4.3 Results about Dimension of Algebraic Varieties:

briefly explain the following results:

1. If $\phi : X \to Y$ is surjective between quasi-projectives then $\dim(Y) \leq \dim(X)$

   Solution:

Proof. This works for any ϕ whose image is dense. We get a $\phi^* : k(Y) \to k(X)$ which implies the claimed inequality on the transcendence degrees, and thus the dimension. $\square$

2. $Y \subset X$ is a proper closed subvariety of X: quasi-projective, irreducible. Then $\dim(Y) < \dim(X)$.

   Solution:

   Proof. For the sake of dimensions we may assume Y is irreducible. Taking a typical chart we may assume that $Y \subset X$ are affine. Then we have $k[Y] \cong k[X]/P$ where P must be a prime ideal due to irreducibility. The result follows from pure commutative algebraic result:

   Let A be a finitely generated integral domain k-algebra, and let $0 \neq P \subset A$ be a prime ideal. Then A/P has transcendence degree strictly less that of A.

   This can be proven by taking a maximal algebraically independent set in the quotient, lifting it, adding a non-zero element of P and using the properties of the reconditions to show that the augmented set is algebraically independent. $\square$

3. Let X be an irreducible affine variety of dimension d. Suppose $0 \neq f \in k[X]$ such that f vanishes at some point in X. Show that $V(f) \cap X$ has dimension $d - 1$

   Solution: This follows from Krull's principal ideal theorem.

4. $\phi : X \to Y$ a morphism of quasi-projective varieties with dense image. Then for each point $P \in \text{Image}(\phi)$ we have $\dim(\phi) \geq \dim(X) - \dim(Y)$.

   Solution: We did not prove this but we will use it a lot in the rest of the course.

4.4 Important Theorems about Morphisms from Projective Varieties

We did not prove some of these but we will need to reference the results down the line.

1. $\phi : X \to Y$ with X projective variety and Y quasi-projective we have $\phi(X)$ is closed.

   Solution: $\Gamma_\phi = \{(x, \phi(x)) : x \in X\} \subset X \times Y$ is closed. Then $\partial_2(\Gamma_\phi) = \phi(X)$ is closed by the next result.

2. $X \times Y \to Y$ is a closed map when X is projective and Y is quasi-projective.

Solution: The proof idea is as follows. Observing the following we may assume $X = \mathbb{P}^n$:

$$X \times Y \to \mathbb{P}^n \times Y \to Y$$

because the first map is a closed map.

Covering Y by affines we may assume it is affine. Furthermore by considering the below we may assume $Y = k^n$:

$$\mathbb{P}^n \times Y \to \mathbb{P}^n \times k^n \to Y \to k^n$$

Since the first and last maps are closed.

Then take $Z \subset \mathbb{P}^n \times k^n$ and show that $k^n - p_2(k^n)$ is open.

3. If $\phi : X \to Y$ between quasi-projectives has dense image then in fact $\phi(X)$ contains an open subset of Y.

   Solution: We did not prove this.

4. (Special case of Borel Fixed Point Theorem): If G is a connected, solveable algebraic subgroup of $GL(V)$ for $V \neq 0$ and $X \subset \mathbb{P}(V)$ is a non-empty, G-stable subset. Then X indeed has a fixed point.

   Solution: The outline goes as follows: Induct on the dimension of V. For $\dim(V) \leq 2$ either the projectivisation is a single point or $\mathbb{P}^1$.

   If $X = \mathbb{P}^1$ then use Lie-Kolchin theorem and take the image of the 1-dimensional subspace.

   If X is on the contrary a finite set then leverage the connectedness of G.

   For the induction step use Lie-Kolchin again. If the image of the 1-dimensional subspace is in X then you are finished. If not then project through this point (linearly) to $\mathbb{P}(V/\langle v\rangle)$. Since G acts on $V, \langle v\rangle$ it also acts on $V/\langle v\rangle$. This projection is a G-invariant morphism and so the image of X is also closed.

   Now by induction there is a fixed point $[w] \in \mathbb{P}(V/\langle v\rangle)$. Now define $W = \langle v, w\rangle$. We know that $[v] \in X \cap \mathbb{P}(W)$. Therefore we meed the non-empty condition and we have that there exists a fixed point in the smaller set, therefore in X.

4.5 Exercise: The Image of an Algebraic Group Morphism:

Let $\phi : G \to H$ be an algebraic group morphism. Show using the above results that the image of ϕ is closed. In particular this tells us the the image of an algebraic group homomorphism is and algebraic group.

Solution:

Proof. First we will reduce to the case where G is irreducible. Suppose its true for G irreducible. Then as a set:

$$G = \coprod_i g_i G^0$$

Where g_i run over the finitely many coset representatives of G^0. Then we have that:

$$\phi(G) = \bigcup_i \phi(g_i G^0) = \bigcup_i \phi(g_i)\phi(G^0)$$

We have that $\phi(G^0)$ is irreducible because it is connected. Therefore $\phi(G)$ is the union of finitely many closed sets, thuse is closed.

Now for the irreducible case. Note that by exercise 1 $\overline{\phi(G)}$ is a subgroup of H. Note also that $\underline{\phi(G)}$ is irreducible. Now by result 3 just above $\phi(G)$ contins an open subset of $\overline{\phi(G)}$ because the image of $\phi : G \to \overline{(}\phi(G))$ is dense by construction. Therefore $\phi(G)$ is an irreducible subgroup of H that is thick in its closure, so by exercise 2 $\phi(G) = \overline{\phi(G)}$, that is, $\phi(G)$ is closed.

Therefore the image of G is indeed an algebraic subgroup of H. □

4.6 Results on Orbits of Actions of Algebraic Groups

Briefly explain the prrof method or importance of the following results:

- Let G be an affine algebraic group. Let G act on a quasi-projective variety X. Then the orbit of any point O_p is open in its closure.

  Solution: First note that O_p is the image of $G \to X$, $g \mapsto gp$, a map of quasi projectives. By making the co-domain $\overline{O_p}$ the image of this map is now dense and we can apply the theorem about dominant maps of qusiprojectives to say that there exists an open subset U of $\overline{O_p}$ such that $U \subset O_p$. Then we argue that $\mathcal{O}_p = \cup_{g\in G} gU$, which is open in $\overline{O_p}$.

- (Closed/Minimal Orbit Lemma): Given an action of an affine algebraic group G on a quasi-projective variety X. Any orbit of minimal dimension is closed.

  Solution: The dimension of orbits is only a priori defined due to the previous result. Reduce to the irreducible case. Take an orbit O and consider $\overline{O} - O$. Then use minimality.

4.7 Borel Fixed Point Theorem

If G is a connected solveable algebraic group acting on a projective variety X, then there exists a fixed point $x \in X$.

Solution: Reduce to the case where the action is transitive using the minimal orbit lemma.

Take the stabiliser of $p \in X$, G_p (closed subgroup) and use a refinement of the embeding theorem to find a representation $G \to GL(V)$ such that G_p is the stabiliser of a line L. Projectivise V and then note that there is a nice setup:

$$X \leftarrow Z \rightarrow Y$$

Where $Z = G(p, q)$ if q is the element corresponding to the line L and $Y = Gq$.

Now by the special case of the borel Fixed point theorem we reduce to showing that $Y \subset \mathbb{P}V$ is closed. Using the fact that projective varieties are universally closed we reduce to showing that $Z \subset X \times \mathbb{P}V$ is closed.

To show that use the minimal orbit lemma after using dimension results to show that $\dim(X) \leq \dim(\mathcal{O})$ and $\dim(X) = \dim(Z)$ for any orbit $\mathcal{O}$.

5 Homogeneous Spaces and Quotients

5.1 Lemma: The Commutator Subgroup

Show that the commutator subgroup $[G, G]$ of a connected algebraic group G is itself connected and algebraic.

Solution:

Proof. First note that:

$$[G, G] = \bigcup_i \phi_n(G^{2n})$$

Where:

$$\phi_n : G^{2n} \to G,\ (a_1, b_1, \cdots, a_n, b_n) \mapsto [a_1, b_1] \cdots [a_n, b_n]$$

This immediately tells us that $[G, G]$ is irreducible, as it is the increasing union of images of irreducible spaces. ϕ_n is a regular function for all n so now we define:

$$Z_n := \overline{\phi_n(G^{2n})}$$

This gives a chain of increasing closed subsets. This must terminate at some n because the dimension of G is finite.

Therefore $\overline{[G, G]} = Z_n = \overline{\text{Image}\,(\phi_n)}$. Now since ϕ_n is a morphism of quasi-projectives we know that its image contains an open subset $U \subset \overline{[G, G]}$. But since $[G, G] \supset \text{Image}\,(\phi_n)$ now $[G, G]$ is thick in its closure. Therefore by Exercise 2 it is closed, as required.

Therefore $[G, G]$ is an irreducbible algebraic subgroup. □

5.2 Generic Openness

Let $\phi : X \to Y$ be a morphism of quasi-projective varieties, then there exists a non-empty open $U \subset X$ such that $\phi|_U$ is an open map.

Solution: We do not prove this theorem but we use it to prove later in the course.

5.3 Homogeneous Spaces

- Define the category of homogeneous spaces for a fixed algebraic group G.

  Solution: A homogeneous space is a quasi-projective variety X equipped with a transitive G-action. A morphism of homogeneous spaces is a quasi-projective variety morphism that respects the group action.

- Prove the morphisms of homogeneous spaces are open maps.

  Solution:

Proof. We have by generic openess that if $\phi : X \to Y$ is a homogeneous space morphism then by generic openess $\phi|_U$ is an open map for some open $U \subset X$. Note now that $\phi|_{gU} = g \circ \phi|_U$ is also an open map and that the transitivity of the action implies that $X = \bigcup_{g \in G} gU$. Now let $V \subset X$ be an open subset:

$$\phi(V) = \phi(\bigcup_{g \in G} gU \cap V) = \bigcup_{g \in G} \phi(gU \cap V) = \bigcup_{g \in G} \phi|_{gU}(gU \cap V)$$

A union of open sets. Therefore ϕ is an open map. ☐

5.4 Quotients of Algebraic Groups

Define a quotient of an Algebraic Group.

Solution:

Proof. A quotient of an algebraic group G by a closed subgroup H is a pair (X, ρ) satisfying:

- X is a quasi-projective variety
- $\rho : G \to X$ is a morphism such that $\rho(hg) = \rho(g)$ for all $h \in H, g \in G$.

Such that the pair (X, ρ) is inital among all such pairs. ☐

6 Borel Subgroups and Maximal Tori

6.1 Relationship Between Borel and Parabolic Subgroups

State the definition of Borel and Parabolic subgroups.

Solution: A Borel sugroup is a maximal, connected, solveable algebraic subgroup.

A Parabolic subgroup is a subgroup such that G/P is a projective variety.

Explain the following results:

- Every parabolic contains a conjugate of any given Borel subgroup B.

  Solution: Consider B acting on G/P. We have exactly the preconditions to apply the Borel fixed point theorem. Complete by unraveling definition.

- Any 2 Borel subgroups are conjugate.

  Solution: Embed $G \to GL(V)$ and consider G acting on $\mathrm{Flags}\,(V)$, a projective variety. Choose an orbit $\mathcal{O}_F = G \cdot F$ of minimal dimension. Use minimal orbit lemma to conclude that this is a projective variety and furthermore $\mathcal{O}_F \cong G/H$ where $H = \mathrm{Stab}\,(F)$ so H is parabolic. By the previous lemma there exists some $B \subset H^0$. We know H is a subgroup of upper triangular matrices in the larger $GL(V)$ so is solveable. Therefore H^0 is connected solveable so by maximality of Borels we in fact have $B = H^0$. Starting with any Borel we see that it is in fact conjugate to $H^0 = (\mathrm{Stab}_G(F))^0$.

- Borel subgroups are parabolic.

  Solution: Use embedding theorem + Lie-Kolchin to show that $B = \mathrm{Stab}_G(L) = \mathrm{Stab}_G(F')$ where $\mathcal{O}_{F'}$ is not necessarily of minimal dimension.

  Then use the first result to show that B is conjugate to some $gBg^{-1} = H^0 = (\mathrm{Stab}_G(F))^0 \subset \mathrm{Stab}_G(F)$ where $\mathcal{O}_F$ does have minimal dimension as an orbit in $\mathrm{Flags}\,(V)$. Since connected components have finite index we have a map:

  $$G/gBg^{-1} = G/H^0 \to G/H$$

  With finite fibers, showing us that in fact $\mathcal{O}_{F'}$ has the same (minimal) dimension as $\mathcal{O}_F$. Therefore $G/B \cong G/gBg^{-1}$ is closed. Therefore B is parabolic.

- $P \subset G$ is parabolic iff it contains some Borel.

  Solution: $\implies$: P contains the conjugate of a Borel, which is a Borel.

  $\impliedby$: $P \supset B$. So we get a map:

  $$G/B \to G/P$$

  That is surjective. Furthermore Borel subgroups are parabolic and so the image of the map, namely G/P is closed. Therefore P is parabolic.

6.2 The Radical, Reductivity and Semisimplicity

The radical $R(G)$ of an algebraic group is defined to be:

$$R(G) := \left(\bigcup_{g \in G} gBg^{-1} \right)^{\circ}$$

Calculate the radical of the following algebraic groups. Are they reductive? semisimple?

- $GL_n(k)$:

  Solution: U_n, the upper triangular matrices are a Borel subgroup. So are the lower triangular mattrices L_n. Therefore:

  $$R(GL_n(k)) \subset U_n \cap L_n = D_n$$

  Now since the $R(GL_n(k))$ is a normal subgroup we use that fact that any non-scalar matrix has a conjugate that is not diagonal. Therefore $R(GL_n(k)) = \{\lambda I : \lambda \in k^{\times}\} \cong \mathbb{G}_m$ which is a torus, Therefore $GL_n(k)$ is reductive but not semisimple.

- $G = \mathbb{G}_m^n$

  Solution: G is solveable, connected, normal and maximal. Therefore $R(G) = G$ so G is always reductive, and is semisimple only when $m = 0$

- G: connected solveable.

  Solution: $R(G) = G$ so G is radical if G is a torus and semisimple if $G = \{1\}$.

6.3 The Union of Borel Subgroups

Prove that $G = \bigcup_{g \in G} gBg^{-1} =: X$.

Solution: We show that it is closed, and then in the $k = \mathbb{C}$ case that it is dense.

We can see closed by noting that it may as well be a union over G/B which is projective, which is close to compact which is close to finite.

We use Lie theory to show that X contains a euclidean neighbourhood of 1.

6.4 Semi-direct Product Subtlety

If we have two closed algebraic subgroups $G_1, G_2 \subset G$ such that $G = G_1 \rtimes G_2$ as abstract groups, it is not always the case that the map $G_1 \rtimes G_2 \to G$ is an isomorphism of algebraic groups.

6.5 Splitting Solveable Groups

Show that in any connected solveable group B there exists a torus T such that $T \to B \to B/U$ is an isomorphism. Where U is normal subgroup of unipotent elements (defined because B is solveable).

Solution:

Proof. We will take as given that for any torus T there exists a generator such that $\overline{\langle s \rangle} = T$. this is only true in characteristic 0 using the equivalence of categories and the neet fact that there are an infinite number of primes in a characteristic 0 field.

We know by Lie-Kolchin theorem that B/U is a torus. Let gU be a generator. That is $\overline{\langle gU \rangle} = B/U$. We have $b \in B$. Consider the jordan decomposition $g = su$. I claim that $T := \overline{\langle s \rangle}$ gives the necessary isomorphism.

Note that since uU is trivial $sU = suU = gU$. We can see that T is an algebraic subgroup. We need to check that the composition is an isomorphism.

To check injectivity. If $k \in \overline{\langle s \rangle} = T$ such that $kU = U$ then we know that $k \in U$ so k is unipotent. Furthermore since s is semisimple and T is commutative we conclude that $k \in T$ is also semisimple. Therefore $k = 1$. Therefore the composition if injective.

For surjectivity note that:

$$\begin{aligned}
\pi(T) &= \pi(\overline{\langle s \rangle}) && \text{by definition of } T \\
&= \overline{\langle sU \rangle} && * \\
&= \overline{\langle gU \rangle} \\
&= B/U
\end{aligned}$$

$*\subset$: Because π is continuous.

$*\supset$: Because the image of π is a closed subgroup containing sU.

We conclude that connected solveable groups have the structure of a semi-direct product. $B = T \rtimes U$. □

6.6 Unipotent Groups in Characteristic 0

- If $g \in GL_n(k)$ is unipotent then $G(g) := \overline{\langle g \rangle}$ is isomorphic to $\mathbb{G}_a = \{g^t : t \in k\}$.

  Solution: Using Lie theory and the fact that the infinte sums are finite in this case, so the exp map defines an isomorphism to the lie algebra.

- Any unipotent algebraic group is connected.

  Solution: It is the image of a vector space, which is connected.

- Any commutative unipotent algebraic group is isomorphic to k^n under addition.

  Solution: In this case $\exp(X)\exp(Y) = \exp(XY)$. Therefore exp is in fact an algebraic group morphism.

6.7 Theorem: Semisimple Elements of Connected Solveable Groups

Let $B \cong U \rtimes T$ be a connected solveable algebraic group over a field of characteristic 0. Let $s \in B$ be semisimple. Then s is conjugate to an element of T.

Solution:

Proof. We will induct on the dimension of U.

$\dim(U) = 0$: From the results on unipotent groups we know that U is contected and therefore U is trivial. Therefore $s = ut = t \in T$ as required.

$\dim(U) = 1$: Write $s = ut$ according to the semi-direct product decomposition. In the case where u and t commute this is the jordan decomposition of s, and so $u = 1$ and we are done.

Therefore we may assume that u and t do NOT commute. We claim that for all $h \in sU$ the B-conjugacy class $C(h)$ of h is sU.

Now I will show that this claim gives us the result. Suppose it is true. Then we have $t = u^{-1}s \in Us = sU = C(s)$. Therefore $t = wsw^{-1}$ which then shows us that s is conjugate to $t \in T$. As required.

Claim:

Proof. Now we will prove the claim. Let $h \in sU$. First note that $h \in C(h)$, so it is non-empty. Note that since U is of dimension 1 and solveable it is commutative ($[U, U] < U$). h does not commute with u. If it did then:

$$\begin{aligned}
hu &= uh && \\
su'u &= usu' && \text{Because } h \in sU \\
suu' &= usu' && [U,U] = 1 \\
su &= us && \\
utu &= uut && \text{Because} s = ut \\
tu &= ut &&
\end{aligned}$$

But we are working in the case when u and t do not commute. We conclude that h does not commute with u.

This tells us that $h \neq uhu^{-1} \in C(h) \implies |C(h)| \geq 2$.

We know by the lemma that since U has dimension 1 $U \cong \mathbb{G}_a$. $C(h)$ is a B-orbit and so we know that $C(h)$ is connected and open in its closure. Also note that $C(h) \subset sU = hU \cong k$ where the final isomorphism is as quasi-projective varieties, since multiplication by h is biregular. Also the first subset inclusion comes from the fact that B/U is a commutative, therefore $u_1 h u_1^{-1} \in hU$ for any $u_1 \in U$.

Therefore $C(h)$ is a connected subset of k that is open in its closure. The only locally closed sets are complements of finite sets or finite sets themselves. Connectedness with the fact that $|C(h)| \geq 2$ implies that $C(h) = sU - \Sigma$, where Σ is a finite set.

Now note that B acts on $sU - \Sigma$ and sU. To see that B acts on sU consider $bsub^{-1}$. Now consider the projection to the quotient B/U, which is commutative. We get:

$$bUsUuUb^{-1}U = bUb^{-1}UsUuU = sU$$

Therefore $bsub^{-1} \in sU$. Therefore B also preserves Σ under conjugation. Now suppose there exists $h' \in \Sigma$, then $C(h') \subset \Sigma$, but now $2 \leq |C(h)| \leq |\Sigma| < \infty$, contradicting the connectedness of $C(h')$. Thus we conclude that $\Sigma = \emptyset$.

Therefore $C(h) = sU$, as the claim required. $\square$

We have now completed the dimensions 0 and 1 case and all that remain is the induction step.

Suppose $\dim(U) \geq 2$. Let $s \in B = U \rtimes T$ be semisimple. Choose $V \subset U$ to be a normal, closed co-dimension 1 subgroup of U, as in the lemma.

Now define $B' := B/V$ and $U' := U/V$. Then B' is connected and solveable so it ought to be a semi-direct product. We know that $(B')_u = U'$ because taking the unipotent part of a group commutes with taking its image under the projection map $\pi : B \to B'$ Note that:

$$B'/(B')_u \cong (B/V)/(U/V) \cong B/U \cong T$$

Therefore we call the image of T under the projection map (an isomorphism onto its image) also T.

Therefore $B' = U' \rtimes T$. By construction $\dim(U') = 1$ therefore the theorem holds for $\pi(s)$ which is semisimple. Therefore there exists some conjugate $\pi(b)\pi(s)\pi(b)^{-1} \in \pi(T)" = "T$. Therefore $bsb^{-1} \in \pi^{-1}(T) = VT$. But now VT is a connected solveable group and is isomorphic to $V \rtimes T < U \rtimes T$.

$\dim(V) = \dim(U) - 1 < \dim(U)$ so by the inductive hypothesis $cbsb^{-1}c^{-1} \in T$. Therefore we are done. $\square$

6.8 Lemma: co-dimension 1 Subgroups of Unipotent Groups

Let $B = U \rtimes T$ be as above and the characteristic of k is 0. Suppose $U \neq 1$, then there exists $V \leq U$ such that:

- V is a normal algebraic subgroup of B.
- $\dim(U/V) = 1$.

Solution:

Proof. Keep in mind throughout the proof that this is commonly achieved by setting a super-diagaonal entry to 0 inside some faithful trigonalised representation.

Since unipotent groups are nilpotent, consider the following chain of subgroups:

$$U =: U_0 \supset U_1 \supset \cdots \supset U_n = 1$$

Where $U_j = (U_j, U)$. First I claim that U_j are subgroups, and furthermore that they are preserved by any automorphism of U. Since (U, U_j) are exactly the products of commutators $[a, b]$ with elements $a \in U$ and $b \in U_j$ or vice versa (to be closed under inverses). To prove these are characteristic I will begin be showing that $U_1 = (U, U)$ is characteristic. Given an automorphism $\phi : U \to U$ I need only show that $\phi(U_1) \subset U_1$ because ϕ is a bijection and ϕ^{-1} is a morphism. But clearly a product of commutators is sent to a product of commutators so the base case is done. For the induction since $\phi(U_j) \subset U_j$ we have that U_{j+1} is characteristic.

Therefore this chain of subgroups is preserved by the action of B (conjugation is an action by automorphisms). Furthermore B acts on $U/(U, U)$, which is a commutative unipotent group. Therefore $U/(U, U) \cong \mathbb{G}_a^n$, a vectorspace. Also note that since the quotient is abelian the action of U is trivial, so in fact we have B/U acting on $U/(U, U)$ by conjugation.

We know that the series on subgroups terminates, which allows us to conclude that $U/(U, U) \neq 1$, that is, this is an action on a non-trivial vectorspace.

We are now in the position where we have a representation of $B/U = T$ in $U/(U, U)$. Since T is a torus, in particular abelian we know that this representation is a direct sum of 1-dimensional representations. Therefore there exists $\overline{V}$, a co-dimension 1 T-invariant subspace of $U/(U, U)$. Define V to be the preimage of this subspace under $U \to (U, U)$.

We now need to check that this is a subgroup of the desired properties.

It is a subgroup because it's a preimage of a subgroup.

We can see that V is normal because consider $v \in V$, we have: To see if bvb^{-1} is in V we have to project to $U/(U, U)$, and by assumption $bvb^{-1}(U, U) \in \overline{V}$ because $\overline{V}$ is invariant. Therefore $bvb^{-1} \in V$

We can see that it is co-dimension 1 because:

$$\dim (U/V) = \dim ((U/(U, U))/V/(U, U)) = 1$$

□

6.9 Theorem: Semisimple Elements and Maximal Tori

A maximal torus is a torus not contained in any other torus.

Let G: a connected algebraic group and $T \leq G$ a maximal torus. Let $s \in G$ be semisimple. Then $C(s)$ meets T. That is, some conjugate of s is in T.

Solution:

Proof. Since T is connected and solvable we know there exists some Borel subgroup B such that $T \leq B \leq G$.

At the same time we know that $s \in B' = U \rtimes S$ for some other Borel and that $C(s)$ meets S.

Now we also know that B and B' are conjugate because they are both borel subgroups. Using characteristic 0 pick a generator $G(t) = T$. We know that t

and thus all of its conjugates are semisimple. There exists a conjugate of t in B'. It is semisimple therefore is also conjugate to an element in S.

Summing up we know that there is a conjugate t' of t such that $t' \in S$.

Now we know that $G(t) = T$ is conjugate to $T' := G(t') \subset S$ because S is an algebraic subgroup containing t'. But we know since T' is the conjugate of a maximal torus T that T' is maximal. Therefore $T' = S$.

Therefore s is conjugate to an element in $S = T'$ which is then conjugate to T. Since being conjugate is a transitive relation we have that s is conjugate to an element of T, as required. □

6.10 Results on Semisimple Elements and (Maximal) Tori

Breifly explain the proof of the following corollaries when G: a connected algebraic group and $\mathrm{char}(k) = 0$:

1. Any two maximal tori are conjugate.

   Solution: Fix a maximal torus T_0 and let $T = G(t)$ be some other maximal torus. Here we use characteristic zero find a generator. t is semisimple so by the previous result $gtg^{-1} \in T_0$. In particular $gTg^{-1} = gG(t)g^{-1} = G(gtg^{-1}) \subset T_0$. But the conjugate of a maximal torus is maximal so $gTg^{-1} = T_0$ as required.

2. Any torus S has a conjugate within a given maximal torus T.

   Solution: S is contained in some maximal torus T', which is itself conjugate to T.

3. If $s \in Z(G)_s$ then $s \in \bigcap_{T:\text{ max'l torus}} T$

   Solution: There is a conjugate of s in every maximal torus, but since it is central all of its conjugates are s iself.

6.11 Centralisers

If $H \subset G$ then we define the centraliser of H in G to be the set:

$$Z_G(H) := \{g \in G : ghg^{-1} = h \, \forall h \in H\}$$

Some facts about the centraliser of a group:

- $Z_G(H)$ is a subgroup of G.

  Solution:

  Proof. $Z_G(H)$ is clearly closed under inversion and multiplication, therefore it is a subgroup. □

- $Z_G(H)$ need not contain H.

  Solution:

Proof. Take a non-abelian subgroup H. □

- If $H \subset K$ then $Z_G(H) \supset Z_G(K)$

  Solution:

  Proof. Let $g \in Z_G(K)$ and let $h \in H \subset K$ then $gh = hg$ because $h \in K$. Therefore $g \in Z_G(H)$ □

- $Z_G(Z_G(H)) \supset H$.

  Solution:

  Proof. Let $h \in H$. Then for any $k \in Z_G(H)$ we have $hk = hk$ by the definition of $Z_G(H)$. Therefore $h \in Z_G(Z_G(H))$. □

- $Z_G(H)$ is the largest subgroup of G that contains H in its center.

  Solution:

  Proof. Let K be some other subgroup such that $H \subset Z_K(K)$. Then everything in H commutes with everything in K and so $K \subset Z_G(H)$. □

- If H is a characteristic subgroup then so is $Z_G(H)$

  Solution:

  Proof. Let $\phi : G \to G$ be an automorphism. Suppose $\phi(Z_G(H)) \subset Z_G(H)$ for any automorphism ϕ. Then in particular for ϕ^{-1} we have $\phi^{-1}(Z_G(H)) \subset Z_G(H)$ and taking the image under ϕ of both sides we get the reverse inclusion.

  Therefore suppose $k \in Z_G(H)$, then since H is characteristic every element of H is $\phi(h)$ for some H. Therefore $\phi(k)\phi(h) = \phi(h)\phi(k)$ because ϕ is a homomorphism. Therefore $\phi(k) \in Z_G(\phi(H)) = Z_G(H)$. As required. □

- If S, T are two subsets of G then $T \subset Z_G(S) \iff S \subset Z_G(T)$.

  Solution:

  Proof. This one is clear from the symmetry in the definition. □

- If $H \subset G$ is an algebraic subgroup then $Z_G(H)$ is an algebraic subgroup.

  Solution:

  Proof. $Z_G(H) = \{g \in G : [g, h] = 1\}$ for all $h \in H$. That is equivalent to the preimage of a closed set $\{1\}$ under the algebraic map:

  $$G \times H \to G, (g, h) \mapsto [g, h]$$

  □

6.12 Normalisers

If $H \subset G$ then we define the normalizer of H in G as:

$$N_H(G) = \{g \in G : ghg^{-1} \in H$$

Some facts about the normaliser of a group:

- $N_G(H)$ is a subgroup of G

  Solution:

  Proof. The tricky thing to test is inversion. Suppose $ghg^{-1} \in H$ for all $h \in H$. Then $(ghg^{-1})^{-1} = g^{-1}h^{-1}g$ is also in H because H is closed under inversion. Again because h is closed under inversion we see that g^{-1} satisfies the requirements of being in $N_G(H)$. □

- $N_G(H) \supset H$

  Solution:

  Proof. Let $h \in H$, then since H is a subgroup $hHh^{-1} = H$. □

- $N_G(H)$ is the largest subgroup of G in which H is normal.

  Solution:

  Proof. Let $K \supset H$ be a subgroup in which H is normal. Then for all $k \in K$ and $h \in H$ we have that $khk^{-1} \in H$. Therefore $k \in N_G(H)$. Since $k \in K$ was arbitrary we have that $K \subset N_G(H)$. □

- If H is a characteristic subgroup then so is $N_G(H)$

  Solution:

  Proof. Let $k \in N_G(H)$, $h \in H$.

  Then $\phi(k)\phi(h)\phi(k)^{-1} = \phi(khk^{-1}) \in \phi(H)$. Therefore $\phi(N_G(H)) \subset N_G(\phi(H)) = N_G(H)$ because H is characteristic. □

- If H is an algebraic subgroup of G then so is $N_H(G)$

  Solution:

  Proof. $N_G(H) = \bigcap_{h \in H} \phi_h^{-1}(H)$ where

  $$\phi_h : G \to G,\ g \mapsto ghg^{-1}$$

  The intersection of preimages of closed sets under regular map. □

6.13 Normalisers and Centralisers

The following are some results about normalisers and centralisers:

- If $a \in G$ then $Z_G(a) = N_G(a)$

  Solution:

  Proof. Clear $\square$

- $Z_G(H) \subset N_G(H)$ is a normal subgroup

  Solution:

  Proof. $g \in Z_G(H)$, then $ghg^{-1} = h \in H$ for all H by the defining property of $Z_G(H)$. $\square$

- $N_G(H)/Z_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}\,(H)$.

  Solution:

  Proof. Consider a map:

  $$\phi : N_G(H) \to \mathrm{Aut}\,(H), k \mapsto (\phi_k : H \to H,\, h \mapsto khk^{-1})$$

  The kernel of this map is exactly $Z_G(H)$. Therefore $N_G(H)/Z_G(H)$ is isomorphic to the image of ϕ, a subgroup of $\mathrm{Aut}\,(H)$. $\square$

6.14 Centralisers of Tori

- Let $S \leq G$ be a torus inside a connected algebraic group G. Let $g \in G$ be semisimple and commute with S. Then there exists a larger torus $T \supset S \cup \{g\}$.

  Solution:

  Proof. Since S is connected $S \subset Z_G(g)^0$, a connected algebraic group. Now note that g is semisimple and central in $Z_G(g)^0$. Therefore g is in every maximal torus of $Z_G(g)^0$. Therefore we are frr to pick a maximal torus containing S that will suit the condition we are after. $\square$

- If $S \leq G$ is a torus then $Z_G(S)$ is connected.

  Solution:

Proof. Let $g \in Z_G(S)$. $g = g_s g_u$ so it is enough to show the fact for when g is either semisimple of unipotent.

If g is semisimple then by the previous result we know that there exists a torus T such that $g \in T \subset Z_G(S)^0$ because tori are connected.

If g is unipotent the $G(g) \cong k$ or the trivial group, in particular it is connected. We know that $G(g) \subset Z_G(S)$ because the group generated by g is, and $Z_G(S)$ is a closed subgroup. Therefore we have that $G(g) \subset (Z_G(H))^0$, as required. □

6.15 Groups of Dimension ≤ 2 are Solveable

The key Lemma for this theorem is:

If G is a connected algebraic group with $B \leq G$ a Borel subgroup, if B is nilpotent, then $B = G$, which means of course that G is solveable.

Solution:

Proof. We induct on the dimension of B:

$\dim(B) = 0$: B is connected so $B = \{1\}$. We know that B is parabolic, therefore $G/B \cong G$ is a projective variety. Since G is affine we know G is defined by its regular ring $\mathcal{O}(G)$ consisting of maps $G \to k$, but since G is projective and connected any regular map is constant. Therefore $\mathcal{O}(G) = k$ which means that $G = \{1\} = B$.

$\dim(B) \geq 1$: We know B is nilpotent so write out the series:

$$B =: B_0 \supset B_1 \supset \cdots \supset B_n \supset \{1\} = B_{n+1}$$

Where $B_j = (B, B_{j-1})$. Now define $Z := Z_B(B)^0$. We see from the definitions that $Z \supset B_n$. Therefore Z is not trivial and is connected.

I claim now that $Z \subset Z(G)$:

Let $z \in Z$ and define the map:

$$\phi : G/B \to G, gB \to gzg^{-1}$$

Note that this is a well defined regular map on the quotient because $x \in Z_B(B)$. Note also that it is a map from a projective variety to an affine variety, therefore is constant. And since $\phi(1) = z$ we must have for all $g \in G$ that $gzg^{-1} = \phi(g) = z$. Therefore $z \in Z_G(G)$, as required.

This tells us that Z is normal in G and so we get an inclusion of quotient groups:

$$B/Z \hookrightarrow G/Z$$

Now we know that B/Z is connected and solveable, we need only prove it is Borel, i.e. maximal among connected solveable groups. To do this we now that it is in fact parabolic:

$$(G/Z)/(B/Z) \cong G/B$$

Therefore the quotient is projective. Therefore $B/Z \subset G/Z$ is parabolic and hence Borel.

Now since Z has dimension greater than 1 and B/Z is a nilpotent Borel in G/Z we conclude by induction that $B/Z = G/Z$ and by the correspondence theorem we have that $B = G$, as required. □

From here it will be easy to show that algebraic groups of dimension less than 2 are solveable:

Solution:

Proof. Let G be a connected algebraic group of dimension ≤ 2. Let $B \leq G$ be a Borel subgroup. If $B = G$ then G is solveable and we are done.

Therefore we may assume that $B < G$ and therefore $\dim(B) \leq 1$. We know that B decomposes as $B = U \rtimes T$.

If $U = \{1\}$ then B is a torus, and thus nilpotent, therefore by the lemma we are done.

If $U \neq 1$ then we know that U is connected in the characteristic 0 case. Therefore $\dim(U) = 1$ and thus $B = U$. Since U is solveable it is commutative and therefore nilpotent. Therefore B is nilpotent $G = B$ by the earlier lemma. □

6.16 Low-dimensional Groups

Solution:

6.16.1 Dimension 0

A connected groups of dimension 0 is the trivial group $\{1\}$.

6.16.2 Dimension 1

By the key theorem G is solveable. Therefore for the derived series to terminate we must have that $(G, G) = \{1\}$. Therefore G is commutative.

Since G is commutative we have $G \cong G_s \times G_u$.

If $G = G_s$, then G is diagonaliseable and connected, meaning $G = \mathbb{G}_m$.

If $G = G_u$ then G is commutative and unipotent of dimension 1, therefore (char=0) $G = \mathbb{G}_a$.

6.16.3 Dimension 2

$G = U \rtimes T$, and we have three choice for the dimensions of each:

$G = T$: G is dimension 2, diagonaliseable, connected $\implies G = \mathbb{G}_m^2$.

$G = U$: (G, G) is a commutative normal unipotent subgroup if dimension 1. Therefore:

$$0 \to (G, G) \to G \to G/(G, G) \to 0$$

Where both groups on the side are commutative unipotent algebraic groups of dimension 1, therefore each isomorphic to $\mathbb{G}_a$. Therefore $G \cong G_a \rtimes \mathbb{G}_a$.

$G = U \rtimes T$: We see that G is isomorphic to the matrix group:

$$\begin{pmatrix} \mathbb{G}_m & \mathbb{G}_a \\ 0 & 1 \end{pmatrix}$$

6.17 Nilpotent Groups Revisited

If G is a connected algebraic group then the following are equivalent:

1. G is nilpotent
2. Each maximal torus T of G satisfies $T \subset Z_G(G)$
3. G has a unique maximal torus.

Solution:

Proof. (1) $\implies$ (2): We know from Exercise 13 that $G \subset A_n$ and that $G \cong G_s \times G_u$.

Let T be a maximal torus and let $t \in T$. t is semisimple because it is in a torus. Due to the decomposition $G \cong G_s \times G_u$ we have that $t \in Z_G(G)$. Therefore $T \subset Z_G(G)$.

(2) $\implies$ (3): We know any two maximal tori are conjugate. However by assumption all tori are central, that is, fixed pointwise under conjugation. Therefore there exists a unique maximal torus $T \leq G$.

(3) $\implies$ (1): We will begin with the lemma that for a torus T we have:

$$T = \overline{\bigcup_{n \geq 0} T[n]}$$

Where $T[n] := \{t \in T : t^n = 1\}$. Note that we can use induction because for a product of tori the right hand side includes each factor by induction. Then since the right hand side is a group we get that it is true for the product.

Therefore we may assume $T = \mathbb{G}_m$. To prove this case let $f \in \mathcal{O}(\mathbb{G}_m) = k[x, 1/x]$, and suppose f vanishes on the union of the torsion subgroups. Multiplying out by x we may assume that f is a polynomial such that $f(\alpha) = 0$ for all $\alpha \in T[n]$. But the union of the torsion subgroups is infinite. Therefore $f = 0$ and thus vanishes on all of T. This gives us the desired result.

Now let G be a connected algebraic group with a unique maximal torus. We can see that $gTg^{-1} = T$ for all $g \in G$. Therefore T acts on T and $T[n]$ by conjugation.

However G is connected and each $T[n]$ is finite and so the action must be trivial on the $T[n]$. That is $T[n] \subset Z_G(G)$.

Therefore since $Z_G(G)$ is a closed subgroup of G we have that:

$$T = \overline{\bigcup_{n \geq 0} T[n]} \subset Z_G(G)$$

using the lemma for the first equality.

Now returning to the problem at hand, let G be a connected algebraic group with a unique maximal torus. Take a Borel subgroup, it decomposes as $B = U \rtimes T$, but from the preliminary results we have that $T \subset Z_G(G)$, in particular the action on T by conjugation is trivial. Therefore $B = U \times T$. B is the product of two nilpotent groups, therefore B is nilpotent. By the result in the nilpotent groups revisited section we have that $B = G$ because B was nilpotent. Therefore G is nilpotent, as required. □

6.18 The Weyl Group

For a connected algebraic group G with a maximal torus T we define the Weyl group:

$$W := W(G,T) := N_G(T)/Z_G(T)$$

6.18.1 Definition

Prove that this definition of $W(G,T)$ is well defined.

Solution:

Proof. Let gTg^{-1} be some other maximal torus. Then $N_G(gTg^{-1}) = gN_G(T)g^{-1}$ and similarly for the corresponding centralisers. Therefore conjugation by G gives an isomorphism:

$$W(G,T) \cong W(G, gTg^{-1})$$

□

6.18.2 Finititude

Prove that the Weyl group is finite.

Solution: I claim that $(N_G(T))^0 = (Z_G(T))^0$. One direction is easy. To show the other direction let $n \in (N_G(T))^0$ and consider the action by conjugation on S. By connectedness this must fix all torsion groups $S[n]$ pointwise, therefore n commutes with all $S[n]$ and thus $n \in Z_G(S)$.

Therefore there is a finite fiber map:

$$N_G(T)/(N_G(T))^0 = N_G(T)/(Z_G(T))^0 \to N_G(T)/Z_G(T)$$

Where the left hand side is finite.

6.18.3 Example $GL_n(k)$

Solution: Note that a maximal torus $T \leq GL_n(k)$ is D_n, the subgroup of diagonal matrices.

$N_G(T)$ are the permutation matrices with any non-zero entry. This is because we may it normalises a matrix will all diagonal entries different.

$Z_G(T)$ are the diagonal matrices, because it must truly fix a diagonal matrix with distinct eigenvalues.

Therefore $W(GL_n(k), T) \cong S_n$.

6.19 Borel Subgrous are Self Normalising

6.19.1 Lemma

Let $S \leq B = B_u \rtimes T \leq G$ be a torus contained in a Borel subgroup of G. Then $Z_B(S) \leq Z_G(S)$ is a borel subgroup.

Solution:

Proof. Note that since centralisers of tori are connected that both centralisers are connected.

First let $B_u := U$ I will begin with the claim that:

$$Z_G(S)B = \{g \in G : g^{-1}sg \in sU \,\forall s \in S\}$$

$\subset$: Let $g = zb \in Z_G(S)B$ and $s \in S$. Now:

$$\begin{aligned} g^{-1}sg &= b^{-1}z^{-1}szb \\ &= b^{-1}sb \end{aligned}$$

Since B/U is commutative $g^{-1}sg = b^{-1}sb \in sU$.

$\supset$: Let $g \in$ RHS. Then:

$$g^{-1}Sg \subset SU \subset B$$

We may assume that $S \subset T$ and so $SU = U \rtimes S$. Therefore S is a maximal torus in SU. Also, since S is closed in T we get that SU is closed in B.

Therefore by conjugacy of maximal tori there exists $b \in SU \subset B$ such that $b(g^{-1}Sg)b^{-1} = S$. Set $G \ni z := gb^{-1}$, therefore $g = zb$. We know that $b \in B$, so we need to show that $z \in Z_G(S)$.

By construction we have that $z^{-1}Sz = S$. Therefore for all $s \in S$ we have:

$$z^{-1}sz = b(g^{-1}sg)b^{-1} \in g^{-1}sgU = sU$$

Using the fact that $g^{-1}sg \in SU \subset B$, B/U is commutative and the fact that $g^{-1}sg \in sU$.

This now tells us that:

$$z^{-1}sz \in S \cap sU = \{s\}$$

Therefore $g = zb \in Z_G(S)B$, as required.

Now recall that $Z_B(S) \subset B$ is connected and solveable, hence it is enough to check that $Z_B(S) \leq Z_G(S)$ is parabolic. Now consider the set $Z_G(S)B$. Consider the map:

$$\phi_s : G \to G,\ g \mapsto g^{-1}sg$$

Then:

$$Z_G(S)B = \{g \in G : g^{-1}sg \in sU\ \forall s \in S\} = \bigcap_{s \in S} \phi_s^{-1}(sU)$$

So $Z_G(S)B$ is closed in G. Now consider the map:

$$G \to G/B$$

Consider the image of $Z_G(S)B$ under the projection map $Z_G(S)/B$. This is a closed set because its preimage under the projection map is closed. Furthermore the projection map $G \to G/B$ factors through $G/Z_B(S)$ and so taking the preimage we get that $Z_G(S)/Z_B(S)$ is closed. Therefore the subgroup is parabolic, therefore it is Borel in $Z_G(S)$. □

6.19.2 Theorem

$B \leq G :$ connected affine algebraic group, then $N_G(B) = B$.

Solution:

Proof. We will induct on the dimension of G.

$\dim(G) = 0$: Done because $G = \{1\}$.

$\dim(G) = 1$: G is commutative, so $B = G$.

$\dim(G) \geq 2$: Let $T \leq B \leq G$ be a maximal torus inside a Borel subgroup. Let $x \in N_G(B)$, we wish to show that $x \in B$. Since maximal tori in B are B conjugate we have that:

$$x^{-1}Tx = b^{-1}Tb,\ b \in B$$

And since $N_G(B) \supset B$ we have $x \in N_G(B) \iff b^{-1}x \in N_G(B)$. Therefore we may assume that:

$$x^{-1}Tx = T$$

Now consider the regular map:

$$\rho : T \to T, t \to xtx^{-1}t^{-1}$$

Note that since $x^{-1}Tx = T$, ρ_x is in fact an algebraic group morphism. ρ is either surjective or not.

ρ is not surjective: Then $\dim(\text{Image}(\rho)) < \dim(T)$ because the image is a closed subgroup of T. Hence $\dim(\ker(\rho)^0) > 0$, it is also connected and therefore $S := \ker(\rho)^0 \leq T$ is a nontrivial torus.

By construction x centralises S and normalises B.

Hence x normalises $Z_B(S) \ni z$ because $x^{-1}zx \in B$ and $Z_G(S)$. Therefore $x^{-1}zx \in B \cap Z_G(S) = Z_B(S)$.

As we proved previously $Z_B(S) \leq Z_G(S)$ is a Borel subgroup, therefore our inductive hypothesis tells us that $x \in N_G(Z_B(S)) = Z_B(S) \subset B$, as required. (note that if the dimension does not decrease then G was commutative)

ρ is surjective: Then:

$$T = \{xtx^{-1}t^{-1} : t \in T\} \subset (N_G(B), N_G(B))$$

Choose a linear representation $G \to GL(V)$ such that $N_G(B) = \mathrm{Stab}\,(L)$. The action of $N_G(B)$ on L is described by a homomorphism:

$$\gamma : N_G(B) \to GL(L) \cong \mathbb{G}_m$$

Now since the codomain is semisimple we have that:

$$\gamma|_{N_G(B)_u} = 1, \gamma|_{(N_G(B),N_G(B))} \implies \gamma|_T = 1 \implies \gamma|_B = 1$$

The first implication comes from the surjectivity of ρ and the second comes from the fact that $B = U \rtimes T$. Now pick a basis element v of L and define:

$$G/B \to V, gB \mapsto gv$$

This is now well defined. It is also a map from a projective variety to an affine variety, so is constant. However $N_G(B) = \{g \in G : gL = L\}$ therefore $G = N_G(B)$.

The key now is to notice that B is normal in G and since G is the union of all Borels we find in fact that $G = B$. $\square$

6.19.3 Corollary

There is now a bijection $G/B \to \mathcal{B}$ that maps $gB \to gBg^{-1}$. This gives the set of Borel subgroups the structure of a projective variety.

6.20 Borel Subgroups Containing a Given Maximal Torus

In this section we build up to the result that the Weyl group acts freely and transitively on $\mathcal{B}^T$, the set of Borel subgroups containing a maximal torus T.

6.20.1 Lemma

Let $T \leq G$ be a maximal torus. The action of $Z_G(T)$ on $\mathcal{B}^T$ is trivial. That is, for all $z \in Z_G(T)$ and $B \in \mathcal{B}^T$ we have that $zBz^{-1} = B$. Equivaliently $z \in N_G(B) = B$.

Solution:

Proof. Recall $Z_G(T)$ is connected because it is the centralizer of a torus. Note that T is also a maximal torus if $Z_G(T)$, and is clearly central in $Z_G(T)$. Therefore T is the unique maximal torus in $Z_G(T)$ and by the major theorem on nilpotent groups we have that $Z_G(T)$ is nilpotent. In particular it is solveable. Therefore there exists a borel $B_0 \leq G$ containing $Z_G(T)$ and clearly $B_0 \in \mathcal{B}^T$.

Now choose $g \in G$ such that $B = gB_0g^{-1}$. Since maximal tori in B are B-conjugate we may assume that $gTg^{-1} = b^{-1}Tb$. Now $gb \in Z_G(T)$ and still satisfy the defining property of G. Now by a group theory exercise $gb \in N_G(Z_G(T))$.

Therefore: $gbZ_G(T)(gb)^{-1} = Z_G(T) \subset B_0$ and so by conjugating:

$$Z_G(T) \subset (gb)^{-1}B_0gb = b^{-1}Bb = B$$

Therefore $Z_G(T) \subset B$, and in particular acts trivially on $\mathcal{B}^T$ by conjugation. □

6.20.2 Corollary

This result tells us the the Wely group $W(G,T) := N_G(T)/Z_G(T)$ acts on $\mathcal{B}^T$. And by our proof method we know that the action is transitive (particularly the part where we ensure $gb \in N_G(T)$)

6.20.3 Corollary

Since W is finite and acts transitively on $\mathcal{B}^T$ we have that $|\mathcal{B}^T| \leq |W(G,T)| < \infty$

6.20.4 Theorem

The action of $W(G,T)$ on $\mathcal{B}^T$ is free and transitive.

Recall that acting freely is that same as claiming that if the action has a fixed point then it is the action of the identity element. Being free and transitive is the same as the property that for all $x, y \in X$ there exists a unique $g \in G$ s.t. $gx = y$.

Solution:

Proof. We have transitivity from the corollary above. To prove freeness suppose that $b \in N_G(T)$ and $nBn^{-1} = B$ for some $B \in \mathcal{B}^T$. We want to show that b is trivial in the Weyl group, that is $b \in Z_G(T)$.

This boils down to showing that $N_G(T) \cap N_G(B) \subset Z_G(T)$. Keep in mind that $B = N_G(B)$.

Recall that $B = U \rtimes T$ and suppose that $b \in N_G(T) \cap N_G(B) = N_B(T)$. Therefore $b = ut$ and then $T = bTb^{-1} = uTu^{-1}$. Note that $b \in Z_G(T) \iff u \in Z_G(T)$.

Therefore it is enough to prove that $U \cap N_B(T) \subset Z_B(T)$.

Let $n \in U \cap N_B(T)$ and $t \in T$. Write $t' = ntn^{-1} \in T$. Now recall that:

$$T \to B \to B/U$$

is an isomorphism. Therefore $t = t'$. Therefore since this holds for any $t \in T$ we get that $n \in Z_B(T)$, as required. $\square$

6.20.5 Corollary

If $T \leq G$: a connected algebraic group then $|W(B,T)| = |\mathcal{B}^T|$ because the action is free and transitive.

6.20.6 Corollary

If $T \leq B$: a connected solveable algebraic group then $W(B,T) = 1$ because $|\mathcal{B}^T| = 1$.

6.20.7 Proposition

Let $T \leq B \leq G$ be a non-solveable algebraic group. Then $|W(G,T)| \geq 2$. Moreover $|W(G,T)| = 2 \iff \dim(G/B) = 1$.

Solution: For the proof sketch recall the bijection:

$$\mathcal{B} \leftrightarrow G/B$$

Note that under this bijection $\mathcal{B}^T \leftrightarrow gB : tgB = gB$ for all $t \in T$.

The proof idea from here is to show that the action of T on G/B is non-trivial using non-solveablility and that it has at least two fixed points.

7 Reductive Groups and Root Data

7.1 Groups of Semisimple Rank 1

7.1.1 Rank and Semisimple Rank Definition

If G is an algebraic group then $\text{Rank}\,(G) := \dim\,(T)$ for a maximal torus T.

The semisimple rank of G is defined to be $\text{Rank}\,(G/R(G))$.

Note that if G is connected and $\text{Rank}\,(G) = 0$ then G is unipotent, because any semisimple element is conjugate to the identity, and thus is the identity.

7.1.2 Table of Groups with Rank and Semisimple Rank

G	s.s.Rank (G)	Rank (G)
$SL_2(k)$	1	1
$PGL_2(k)$	1	1
GL_2	1	2
GL_n	$n-1$	n

7.1.3 Proposition

If G has semisimple rank 1 then there exists a surjective morphism:

$$\rho : G \to PGL_2(k)$$

such that $\ker(\rho)^0 = R(G)$. In particular if G is semisimple ($R(G) = 1$) $\ker(\rho)$ is finite.

Solution:

Proof. First note that by first projecting:

$$G \to G/R(G)$$

it is enough to prove it for G semisimple of rank 1.

This tells us the G is not solveable. Therefore by the last proposition on Weyl groups we have that $|W(G,T)| \geq 2$, but we know that:

$$W(G,T) = N_G(T)/Z_G(T) \hookrightarrow \text{Aut}\,(T) = \text{Aut}\,(\mathbb{G}_m) \cong \mathbb{Z}/2$$

The last isomorphism comes from knowledge of the character group of $\mathbb{G}_m$.

Therefore $W(G,T) = \mathbb{Z}/2$. Now again by the last result on Weyl groups this tells us that $\dim\,(G/B) = 1$.

Now we use the fact from algebraic curves that any smooth 1-dimensional projective variety X admitting a non-trivial action by a non-trivial connected algebraic group H statisfies $X \cong \mathbb{P}^1$.

Applying this fact without proof to the action of G on G/B we get that $G/B \cong \mathbb{P}^1$.

Also recall the fact that $\mathrm{Aut}\,(\mathbb{P}^1) = PGL_2(k)$, so we define the map:

$$\rho : G \to PGL_2(k),\, g \mapsto (hB \mapsto ghB)$$

Now clearly:

$$\ker(\rho) = \{g \in G : gxB = xB \forall x \in G\} = \bigcap_{x \in G} xBx^{-1} = \bigcap_{\text{Borels}B} B$$

Therefore $\ker(\rho)^0 = R(G) = 1$ because G is semisimple.

It remains to show surjectivity. Recall that by non-solveablility we have that $\dim(G) \geq 3$. Since the kernel, and hence all fibers are finite we have by dimensionality that $\dim(\mathrm{Image}\,(\rho)) \geq 3$ which means that since the image is cloased that ρ is surjective. ($PGL_2(k)$ is dimension 3 and connected). $\square$

7.1.4 Proposition 2

Let G be a reductive algebraic group ($R(G)$ is a torus) of semisimple rank 1. Let $\rho \to PGL_2(k)$ be surjective as in the previous theorem. Then $\ker(\rho)$ is diagonaliseable.

Solution:

Proof. Fix a maximal torus $T \leq G$. It is enough to show that $\ker(\rho) \subset T$. We know that $|\mathcal{B}^T| = |W(G,T)| = 2$. So there are exactly 2 borel subgroups B^+, B^- containing T.

We will prove that $B^+ \cap B^- = T$ because then:

$$\ker(\rho) = \bigcap_{g \in G} gBg^{-1} \subset B^+ \cap B^- = T$$

Furthermore we can reduce to proving that $B_u^+ \cap B_u^- = 1$ because $B^{\pm} = B_u^{\pm} \rtimes T$.

Recall that G is of semisimple rank 1, therefore $G \neq B^+$ because G is not solveable. Therefore due to the characterisation of nilpotent groups we know that B^+ is not nilpotent. This tells us that B_u^+ is non-trivial and connected (as the image of B^+), so $\dim(B_u^+) \geq 1$.

We also have that $\ker(\rho)^0 \cap B_u^{\pm} \subset R(G)_u = 1$. The last equality because G is reductive.

This tells us that $\ker(\rho|_{B_u^{\pm}})$ is finite, thus $\dim(\rho(B_u^+)) \geq 1$. However now we note that $\rho(B_u^{\pm})$ is a connected unipotent subgroup of $PGL_2(k)$. Now since we have $\rho(B_u^{\pm}) < \rho(B^{\pm}) < PGL_2(k)$. The first inclusion is strict because $B^{\pm}$ is note nilpotent, and the second is strict because $PGL_2(k)$ is not solveable. Since the dimension of $PGL_2(k)$ is 3 we get the other inequality, namely that $\dim(\rho(B_u^{\pm})) \leq 1$. Therefore since the fibers of $\rho|_{B_u^{\pm}}$ are finite we have that:

$$\dim(B_u^{\pm}) \leq 1$$

Therefore $\dim(B_u^{\pm}) = 1$ and since it is unipotent we have that $B_u^{\pm} \cong \mathbb{G}_a$ (only proved by us in characteristic 0).

Now recall that T acts on $B_u^{\pm}$ by conjugation, and that this action is non-trivial because $B^{\pm} = B_u^{\pm} \rtimes T$ is not nilpotent. This action gives us a map:

$$T \to GL(\mathbb{G}_a) \cong \mathbb{G}_m$$

If this map is not trivial it is surjective by the characterisation of characters.

Now suppose for the sake of contradiction that there exists $x \in B^+ \cap B^- - T$ then Tx is dense in both B_u^+ and B_u^- because it is $B_u^{\pm} - \{0\}$ ($x \neq 0$ because $0/1 \in T$).

Therefore $B_u^+ = \overline{Tx} = B_u^-$ and therefore $B^+ = B^-$, which is a contradiction.

Therefore no such x exists and so $B^+ \cap B^- = T$. □

7.1.5 Semisimple Rank 1 Classification

A connected reductive group of semisimple rank 1 is isomorphic to one of the following:

$$SL_2(k) \times T,\ PGL_2(k) \times T,\ GL_2(k) \times T$$

7.2 Isogenies and Simply-connectedness

Under the following definitions the previous subsection proves that if G is reductive of semisimple rank 1 then there is a multiplicative isogeny:

$$\rho : G \to PGL_2(k)$$

The following discussion mainly focuses on semisimple groups.

7.2.1 Definitions

A morphism of algebraic groups $\phi : G_1 \to G_2$ is an isogeny if it is surjective with finite kernel.

It is a multiplicative isogeny if the kernel is diagonaliseable.

A semisimple G is defined to be simply-connected if every multiplicative isogeny $H \to G$ is in isomorphism. (H is semisimple)

7.2.2 Theory

It can be shown that if G is a semisimple algebraic group then there exists G^{sc} simply connected, semisimple and an isogeny $\pi : G^{sc} \to G$ such that for any multiplicative isogeny $\rho : H \to G$ of semisimple groups we have a multiplicative isogeny $\lambda : G^{sc} \to H$ such that $\pi = \rho \circ \lambda$

It is a fact that $SL_2(k)$ is simply connected and $SL_2(k) \to PGL_2(k)$ is an isogeny.

Furthermore $SL_2(k) = (PGL_2(k))^{sc}$ and the above proves with some algebraic geometry that every semisimple algebraic group of rank 1 is $SL_2(k)$ or $PGL_2(k)$.

7.3 Reductive Groups Structural Results

For this section let G be a connected reductive algebraic group.

7.3.1 $R(G) = Z(G)^0$

Solution:

Proof. $\supset$: $Z(G)^0$ is commutative hence solveable, connected and normal in G, therefore $G(z) \subset gBg^{-1}$ for all $g \in G$. Therefore:

$$Z(G)^0 \subset R(G)$$

$\subset$: G is reductive, so $R(G)$ is a torus. Therefore $Z_G(R(G))^0 = N_G(R(G))^0$ by a result from the Weyl group section (the gist is take the action of conjugation on the torsion group and connectedness to show that $N_G(R(G))^0 \subset Z_G(R(G))$).

Therefore:

$$Z_G(R(G))^0 = N_G(R(G))^0 = G^0 = G$$

Therefore $Z_G(R(G)) = G$ and so $R(G)$ commutes with all of G, that is $R(G) \subset Z(G)^0$ because the radical is connected. □

7.3.2 $R(G) \cap [G, G]$ is Finite

Solution:

Proof. Consider an imbedding $G \hookrightarrow GL(V)$. Then $R(G) \leq G$ is a torus and so we can split V into:

$$V = \bigoplus_{\chi : R(G) \to \mathbb{G}_m} V_\chi$$

the eigenspaces of $R(G)$. Now $R(G)$ is normal, which implies that G acts on the individual V_χ. So we get:

$$\rho_\chi : G \to GL(V_\chi)$$

Now note that $\rho_\chi([G, G]) \subset SL(V_\chi)$ and $\rho_\chi(R(G)) \subset Z(GL(V_\chi)) = k\{I_{V_\chi}\}$.

Therefore $\rho_\chi([G, G] \cap R(G)) \subset \mu_{\dim(V_\chi)}$. Therefore the image $\rho(R(G) \cap R(G))$ are matrices in block form roots of unity. In particular finite. □

7.3.3 $[G, G]$ is Semisimple

Solution:

Proof. Let B be a Borel subgroup of $[G,G]$, gBg^{-1} is still a maximal connected solveable subgroup of $[G,G]$ because $[G,G]$ is normal. Therefore:

$$R([G,G]) = \left(\bigcap_{x\in[G,G]} xBx^{-1}\right)^0 = \left(\bigcap_{g\in G} gBg^{-1}\right)^0$$

Since $B \leq B'$ for some Borel subgroup of G we get:

$$R([G,G]) = \left(\bigcap_{g\in G} gBg^{-1}\right)^0 \cap [G,G] \subset \left(\bigcap_{g\in G} gB'g^{-1}\right)^0 \cap [G,G] = R(G) \cap [G,G]$$

But the right hand side is finite so by connectedness $R([G,G]) = \{1\}$ as required. $\square$

7.3.4 Proposition 4: Reductivity of Centralisers of Tori

Let $T \leq G$ be a torus, then $Z_G(T)$ is reductive. Furthermore if T is maximal then $Z_G(T) = T$.

Solution:

Proof. We recall from earlier that $Z_G(T)$ is connected. We know that $Z_B(T) = Z_G(T) \cap B$ is a Borel subgroup of $Z_G(T)$ for any B that is a Borel subgroup of G. Furthermore we know that any Borel subgroup of $Z_G(T)$ is contained in a borel subgroup of G. Therefore:

$$R(Z_G(T)) = \left(\bigcap_{B':\text{Borel in } Z_G(T)} B'\right)^0 = \left(\bigcap_{B:\text{Borel in } G} B \cap Z_G(T)\right)^0 \subset R(G)$$

Therefore since the right hand side is torus then $R(Z_G(T))$ is also a torus by connectedness $\square$

8 References

In the text I refer to the following using the following Abbreviations:

- OV: A.L. Onischik, E.B Vinberg. *Lie Groups and Algebraic Groups.*
- TS: Tamáz Szamuely. *Lectures on Linear Algebraic Groups*
- Course Synopsis: Paul Nelson. *Introduction To Algebraic Groups ETH Zrich Spring 2020 Synopsis of Lectures*
- Milne: James S. Milne *Algebraic Groups*