3-Manifolds Paper: Finite Type Invariants of Pure Braids

Lachlan Potter

June 4, 2021

1 Introduction

The following is a summary paper I am writing for my course in 3-manifolds (2021 Semester 1). The material is based on the paper "Free Groups and Finite Type Invariants of Pure Braids" written by Jacob Mostovoy and Simon Willerton [Mostovoy and Willerton, 2002]. I discussed the material extensively with Louis Carlin before writing.

2 Setting the Scene

To start talking about the interesting things in the paper we need to get some definitions out of the way first.

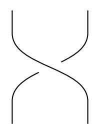
Definition 1. The braid group on k strands B_k can be defined formally as the fundamental group of the configuration space of k points in $\mathbb{R}^2$.

For the purposes of this paper it is likely sufficient to just think of the braid group on k strands as the set of braid diagrams with k strands, where composition is vertical concatenation. We may at times also use the generators and relations presentation:

$$\langle \sigma_1, ..., \sigma_{k-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$$

where σ_i is a twist of the i and $i+1$st strands introducing a negative crossing.

For example, $\sigma_1 \in B_2$ is drawn below:

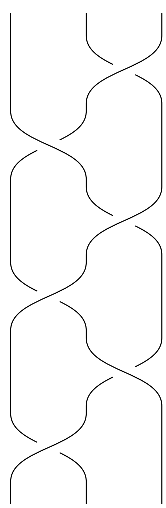


This paper talks almost exclusively about a certain subgroup of braids known as pure braids.

Definition 2. The group of pure braids on k strands $\mathcal{P}_k$ is the kernel of the homomorphism $B_k \to S_k$ that simply tracks how the k start points are permuted by the braid.

I find it best to think of these simply as braids such that each strand starts and ends at the same spot.

An example of an element $b \in \mathcal{P}_3$ is the following:



Now we will lay some of the algebraic foundations for the interplay between algebra and topology that happens in this paper.

For the rest of the paper let R be a commutative ring.

Definition 3. An (R-valued) invariant v of pure braids is a map of **sets** $v : \mathcal{P}_k \to R$

Since both $\mathcal{P}_k$ and R have algebraic structure it is easy to think that an invariant should behave nicely with respect to composition of braids, but this is not the definition. It is simply an assignment of an element of R to each braid type.

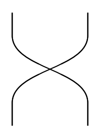
Now let $R\mathcal{P}_k$ be the group algebra of $\mathcal{P}_k$. It will be fine to think of $R = \mathbb{Z}$ if you prefer. We can now naturally consider "singular braids"

Definition 4. A singular braid is a braid with a finite number of "double points" where two strands intersect transversally.

We will not really interact with singular braids topologically, but rather we will think of them as elements of $R\mathcal{P}_k$ via the Vassiliev relations.

Definition 5. To realise a singular braid as an element of $R\mathcal{P}_k$ resolve the double points (locally) one at a time, with a sign equal to the type (positive or negative) of crossing introduced.

For example, if we consider the following singular braid:



This resolves as $\sigma_1^{-1} - \sigma_1 \in R\mathcal{P}_k$. Notice that for a braid with multiple singularities this resolution is well defined because the resolutions are all happening locally.

It is worth asking whether is it possible to easily "see" the singular braids algebraically. The answer, as it happens, is yes.

Definition 6. The augmentation ideal JG of any group algebra RG is the kernel of the augmentation map, which is the ring homomorphism defined as:

$$RG \to R, \sum_i r_i g_i \mapsto \sum_i r_i$$

Theorem 7. *$J\mathcal{P}_k$ is generated by the pure braids with one singularity.*

The argument is a quick one, courtesy of the paper.

Proof.

$$\begin{aligned} J\mathcal{P}_k &= \langle p - 1 | p \in \mathcal{P}_k \rangle \\ &= \langle (p - p_1) + (p_1 - p_2) + ... + (p_m - 1) | \text{paired terms differ by one crossing change} \rangle \\ &= \langle p - q | p \text{ and } q \text{ differ by one crossing change} \rangle \\ &= \langle \text{pure braids with one singularity} \rangle \end{aligned}$$

The first line is because for any element in the ideal, the coefficients must sum to 0. Hence, by adding and subtracting the same number of the identity braid we can pair every term with the identity braid of the opposite sign.

The second line comes from the fact that every pure braid can be turned into the trivial braid by a finite number of crossing changes. Simply make one strand lie over all other braids by swapping finitely many crossings, then proceed by induction.

The last line is because of the Vassiliev relations. □

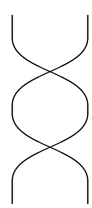
One result that we will need for later is a statement about powers of the Augmentation ideal.

Theorem 8. *The n-th power of the augmentation ideal ($J^n\mathcal{P}_k$) is generated by pure braids with n singularities*

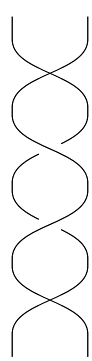
Proof. It is certainly clear that $J^n\mathcal{P}_k \subset \langle$n-singular braids$\rangle$ because $J^n\mathcal{P}_k$ is generated by products of n pure braids each with 1 singularity (by the previous theorem).

Now if we take a pure braid with n singularities this is certainly a product of n braids, each with exactly one singularity, but the issue is that these braids may not be pure. However this is easily fixed by possibly tangling the braid between each of the singularities to make it look like a product of pure 1-singular braids.

I think this is best seen with an example. The follwing braid is certainly 2-singular:



However if we simply think of it as the product of the two 1-singular braids, they are not pure. This can be easily fixed by isotoping this braid in the following way:



Now we can clearly see that this pure braid is a product of 2 1-singular pure braids.

Therefore $J^n \mathcal{P}_k = \langle$n-singular braids$\rangle$, as required. $\square$

Now we introduce the notion of finite type invariants. Note that any R-valued invariant of pure braids can be extended linearly to an R-module homomorphism $R\mathcal{P}_k \to R$. In light of this the following definition makes sense

Definition 9. A type n type invariant v is an invariant of pure braids such that v vanishes on all braids with more than n singularities. A finite type invariant is an invariant that is type n for some $n \in \mathbb{N}$.

To get a feel for what these finite type invariants are actually like, let's classify type 0 invariants.

Theorem 10. *All type-0 invariants of pure braids on k strands are constant.*

Proof. First note that the Vassiliev relations tell us that if v is a type-0 invariant then v must agree on any two braids that differ by a single crossing.

This, combined with the fact that any pure braid can be turned into the trivial braid after a finite number of crossing changes, tells us that for any $p \in \mathcal{P}_k$, $v(p) = v(1)$ where 1 is the trivial braid. I.e. v is fully determined by where it sends the trivial braid. Therefore v is constant. $\square$

The higher type invariants are a little more complicated to understand, but I think the best intuition is that an invariant being type-n imposes relations on the values of braids that differ by only n crossings.

Happily, we can transfer this type-n condition into algebra.

Theorem 11. *The type n invariants are in bijection with R-module homomorphisms:*

$$\mathrm{Hom}_R(R\mathcal{P}_k / J^{n+1}\mathcal{P}_k, R)$$

Proof. The proof follows from the previous theorems regarding higher powers of the augmentation ideal. $\square$

At this point it would be very surprising to find out that these types of invariants separate braids...

Read Louis' paper to see how this works.

3 Combing

Before reading about braid groups I always viewed their structure as particularly wild and hard to deal with, so the following part of the paper came as a shock to me. It turns out that the pure braid groups can be written as a semi-direct product of free groups in a delightfully natural way. This for me was the highlight of the paper.

First note that there is a group homomorphism $\mathcal{P}_k \to \mathcal{P}_{k-1}$ given simply by forgetting the k-th strand. We will call this the reduction map.

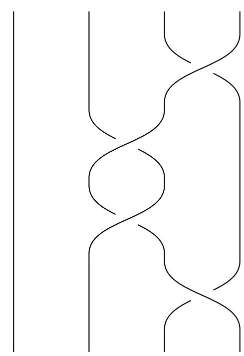
A natural question to ask is what the kernel of this map looks like. Intuitively it consists of all pure braids that can be drawn with the first $k-1$ strands not interacting, and the k-th strand allowed to weave between the rest.

This description actually allows us to understand this subgroup. The path the k-th strand takes describes a loop in $\mathbb{R}^2 - \{x_1, ..., x_{k-1}\}$. Therefore:

$$\ker(\mathcal{P}_k \to \mathcal{P}_{k-1}) \cong \pi_1(\mathbb{R}^2 - \{x_1, ..., x_{k-1}\}) \cong F_{k-1}$$

because $\mathbb{R}^2$ with $k-1$ points removed deformation retracts onto a wedge sum of $k-1$ circles. This identification actually gives us generators of the kernel. If we view F_{k-1} as the kernel of the map $\mathcal{P}_k \to \mathcal{P}_{k-1}$ we can take the generating set $\{x_{1,k}, ..., x_{k-1,k}\}$. Where $x_{i,k}$ loops the kth strand once around the ith strand, and crosses on top of all strands between.

As an example here is a picture of $x_{2,4} \in F_3 \triangleleft \mathcal{P}_4$:



This whole analysis gives us a short exact sequence:

$$0 \to F_{k-1} \to \mathcal{P}_k \to \mathcal{P}_{k-1} \to 0$$

It is even better than this. The map $\mathcal{P}_k \to \mathcal{P}_{k-1}$ has a section defined by simply adding in a kth strand that does not interact with anything else. So our exact sequence is in fact split:

$$\begin{tikzcd} 0 \arrow[r] & F_{k-1} \arrow[r] & \mathcal{P}_k \arrow[r, "\phi"'] & \mathcal{P}_{k-1} \arrow[l, bend right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

This tells us that $\mathcal{P}_k \cong F_{k-1} \rtimes \mathcal{P}_{k-1}$. This was all so far independent of k, so applying this iteratively we see that:

$$\mathcal{P}_k \cong F_{k-1} \rtimes F_{k-2} \rtimes \cdots \rtimes F_1$$

This is telling us that any pure braid $p \in \mathcal{P}_k$ can be written as a product $p_{k-1} \cdots p_1$ of pure braids, where each $p_i \in F_i$, where we implicitly identify F_i as the kernel of the map $\mathcal{P}_i \to \mathcal{P}_{i-1}$. We call such braids p_i i-free.

In picture terms this is saying that we can draw every pure braid as a product of i-free braids, as i descends from $k-1$ to 1.

I can't help but include a diagram I drew of this happy occurence, which the reader may or may not find useful:

$$
\begin{tikzcd}
 & & F_{k-2} \arrow[d] & & & \\
F_{k-1} \arrow[r] & \mathcal{P}_k \arrow[r, "\phi"'] & \mathcal{P}_{k-1} \arrow[l, bend right, "s"'] \arrow[d, "\phi"'] & F_{k-4} \arrow[d] & & \\
 & F_{k-3} \arrow[r] & P_{k-2} \arrow[r, "\phi"'] \arrow[u, bend right, "s"'] & P_{k-3} \arrow[l, bend right, "s"'] \arrow[d, "\phi"'] & F_{k-6} \arrow[d] & \\
 & & F_{k-5} \arrow[r] & P_{k-4} \arrow[r, "\phi"'] \arrow[u, bend right, "s"'] & P_{k-5} \arrow[l, bend right, "s"'] & \ddots \\
 & & & \ddots & \ddots & \ddots
\end{tikzcd}
$$

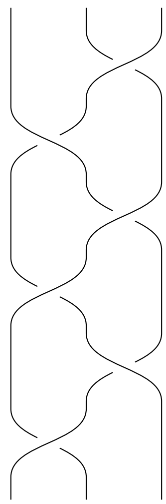
I would like to wrap the paper up with a low-dimensional topology style proof of this fact that does not mention algebra at all, inspiration courtesy of [Bar-Natan, 1995], and then a concrete example of a combing.

Theorem 12.

$$\mathcal{P}_k \cong F_{k-1} \rtimes \cdots \rtimes F_1$$

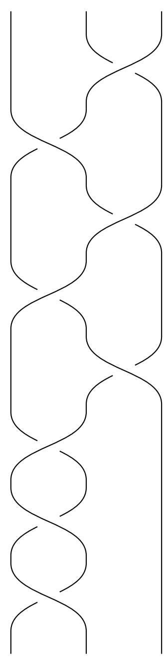
Proof. We will prove this by induction. Let $p \in \mathcal{P}_k$. Fix the first $k-1$ strands in place. Imagine they are made of metal. Then imagine that the k-th strand is made of soft spaghetti. Flip your metal model upside down so that the spaghetti strand falls to the top, where the first $k-1$ strands do not interact. We have just realised p as $p = p_{k-1}p'$, where $p_{k-1} \in F_{k-1}$ and p' is an element of $\mathcal{P}_k$ where the final strand does not interact. Therefore by induction we can comb p', and so we have shown that all $p \in \mathcal{P}_k$ can be written uniquely as $p_{k-1} \cdots p_1$. $\square$

Now with that delicious proof behind us we can move on to the specific example of combing. Let $b \in \mathcal{P}_3$ be the example of a pure braid I gave right at the start of the paper. I have drawn it below if you have forgotten:

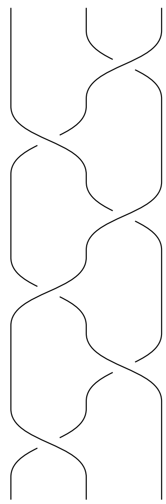


We know we have a split exact sequence $F_2 \to \mathcal{P}_3 \xrightarrow{\phi} \mathcal{P}_2$. How do we go from this to expressing b as an element of $F_2 \rtimes F_1$. Well, motivated by the algebra of the situation, we write $b = b \cdot s(\phi(b))^{-1} \cdot s(\phi(b))$.

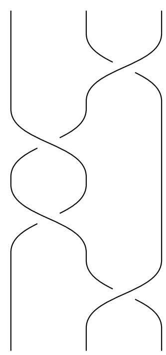
Now $b \cdot s(\phi(b))^{-1} \in \ker(\mathcal{P}_3 \to \mathcal{P}_2)$, so this should give us an idea that we are on the right track. $s(\phi(b)) = x_{2,1}^{-1}$, so $b \cdot s(\phi(b))^{-1}$ is drawn below



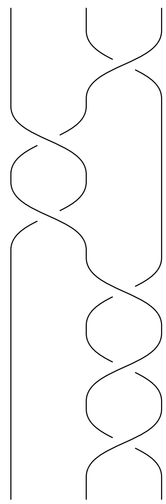
We know that this should be an element of F_2, and after some isotopies we can see this:



The above is a drawing of $b \cdot s(\phi(b))^{-1}$, which we can further simplify to the below drawing:



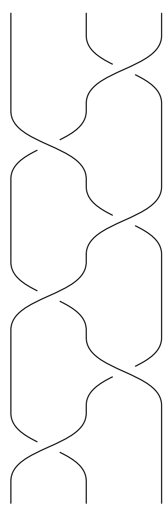
This we can see is an element of F_2. In fact in terms of the generators $b \cdot s(\phi(b))^{-1} = x_{3,1}^{-1}x_{3,2}$, though we need one more isotopy to totally see this:



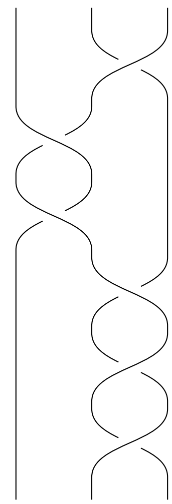
We have now successfully combed $b \cdot s(\phi(b))$. Since $s(\phi(b))$ is a pure braid on two strands it is already combed, so we have:

$$b = \left(b \cdot s(\phi(b))^{-1}\right) \cdot s(\phi(b)) = (x_{3,1}^{-1} x_{3,2}) \cdot x_{2,1}$$

Topologically we have performed the following isotopy of the following braid:



To the braid representated by $(x_{3,1}^{-1} x_{3,2}) \cdot x_{2,1}$.



References

[Bar-Natan, 1995] Bar-Natan, D. (1995). Vassiliev and quantum invariants of braids. In *Proceedings of Symposia in Applied Mathematics*, volume 51, pages 129–144. AMERICAN MATHEMATICAL SOCIETY.

[Mostovoy and Willerton, 2002] Mostovoy, J. and Willerton, S. (2002). Free groups and finite-type invariants of pure braids. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 132, pages 117–130. Cambridge University Press.