Sorbonne Université

Master Thesis

An Introduction to the Geometric Satake Equivalence

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For the friends I made along the way

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Chapter 1

Introduction

Let G be a complex reductive group and $\Bbbk$ a field of characteristic zero. Denote by $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$ the category of L^+G-equivariant perverse sheaves with coefficients in $\Bbbk$ on the affine Grassmannian Gr_G and $\mathbf{Rep}_{G^\vee_\Bbbk}$ the category of $\Bbbk$-representations of the Langlands dual group $G^\vee_\Bbbk$. The celebrated result of Mirković and Vilonen [MV07] states that

$$P_{L^+G}(\mathrm{Gr}_G, \Bbbk) \simeq \mathbf{Rep}_{G^\vee_\Bbbk} \text{ as tensor categories,}$$

and is known as the geometric Satake equivalence.

This thesis follows part of the proof of this result. We follow often the exposition of Baumann and Riche [BR17], which outlines the arguments of [MV07] along with simplifications that become possible when $\Bbbk$ is a characteristic zero field. A notable exception is the proof given of Theorem 3.40, that is taken from more recent sources and allows one to avoid discussion of specific projective embeddings of Gr_G ([BR17, Section 3.3]).

We introduce the affine Grassmannian, its various decompositions, the category of perverse sheaves and finish at the decomposition $H^\bullet(\mathrm{Gr}_G, -)$ into a direct sum of local cohomology functors indexed by the cocharacters of $T \subset G$. In particular we do not mention anything about the fusion product on $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$ or Tannakian formalism.

Chapter 2 goes over background material on ind-schemes and algebraic groups. Section 2.1 discusses ind-schemes. Motivated by the definition of the affine Grassmannian we define ind-schemes in generality, after which we discuss properties of schemes and morphisms between them which can be sensibly applied to ind-schemes. We finish the section with a discussion of ind-schemes as topological spaces.

Section 2.2 discusses algebraic groups over an algebraically closed field. We give the definition of a reductive group G, its root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ and the Langlands dual group $G^\vee_k$. We continue to discuss Borel subgroups and the partial orders they induce on the cocharacter lattice $X_*(T)$, along with the bijection between parabolic subgroups containing a Borel and subsets of the corresponding simple roots. We tie all the results together in Subsection 2.2.5 with GL_n as an extended example.

Chapter 3 studies the affine Grassmanian for a reductive group G and makes up the bulk of the thesis. We begin in Section 3.1 studying the affine Grassmannian for GL_n and proving that it is ind-projective. Section 3.2 proves the same result for Gr_G with G reductive. Interestingly the proof actually relies on the fact being already established for GL_n. Section 3.3 introduces the affine Grassmannian as an étale quotient of LG by L^+G. This realisation gives rise to a natural action of LG on Gr_G, which we leverage in Section 3.4 to give the Cartan and Iwasawa decompositions of Gr_G. The final Section 3.5 is dedicated to proving Theorem 3.40, which analyses the interaction between Cartan and Iwasawa decompositions.

Chapter 4 covers perverse sheaves and results about their cohomology. Section 4.1 goes over some generalities for perverse sheaves on a stratified topological space X and Section 4.2 specialises to perverse sheaves on Gr_G. It is in this section where $\Bbbk$ being a characteristic zero field used to prove semisimplicity of $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$. Sections 4.3 and 4.4 use the results at the end of Chapter 3 to prove Theorem 4.24, which decomposes the total cohomology functor $H^\bullet(\mathrm{Gr}_G, -)$ from $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$ to $\mathbf{Vect}_\Bbbk$ as the sum of local cohomology functors of the strata in the Iwasawa decomposition. Since the strata of the Iwasawa decomposition are indexed by the cocharacters $X_*(T)$ and the representations of a reductive algebraic group are closely related to its characters this gives a first indication of the appearance of the dual group $G^\vee_\Bbbk$.

Chapter 2

Background

2.1 Ind-schemes

The aim of this section is to build from a basic understanding of schemes to the notion of ind-schemes. Our principle example will be that of the affine Grassmannian, which will be developed in Chapter 3.

2.1.1 Functor of Points of a Scheme

Our starting point is the functor of points perspective on schemes. We have from the Yoneda embedding a fully faithful functor

$$\mathbf{Sch} \to \mathbf{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}), X \mapsto h_X,$$

Where $h_X(Y) := \mathrm{Hom}(Y, X)$ and $f : Y \to Y'$ gives a map

$$h_X(f) = \mathrm{Hom}(f, X) = (-) \circ f : h_X(Y') \to h_X(Y)$$

Similarly $f : X \to X'$ gives a natural transformation defined by

$$h_f(Y) = \mathrm{Hom}(Y, f) = f \circ (-) : h_X(Y) \to h_{X'}(Y)$$

Remark 2.1. The Yoneda embedding holds for any locally small category in place of **Sch**.

Already this allows one to consider schemes as certain functors from $\mathbf{Sch}^{\mathrm{op}}$ to **Set**. It is natural to ask whether **Sch** can embed into the category of functors from a more wieldy subcategory of $\mathbf{Sch}^{\mathrm{op}}$ into **Set**. Affine schemes being the local model for schemes suggests and gives the proof of the following theorem.

Theorem 2.2 ([EH06, Proposition VI-2])**.**

$$h : \mathbf{Sch} \to \mathbf{Fun}(\mathbf{Rings}, \mathbf{Set})$$

is fully faithful. In other words, the category of schemes embeds as a full subcategory of $\mathbf{Fun}(\mathbf{Rings}, \mathbf{Set})$

We give a proof in the style of the reference, but with more detail. Throughout we identify the opposite category of affine schemes with **Rings**.

Proof. We need to prove that the map $\mathbf{Sch}(X, X') \to \mathrm{Hom}(h_X, h_{X'})$ is a bijection. Suppose that $\eta : h_X \to h_{X'}$ is a natural transformation.

Choose an open covering $\{U_a\}$ of X by affines. Let $j_a : U_a \to X$ be the open immersion. Then $\eta_{U_a}(j_a) : U_a \to X'$. We want to glue these to a function $X \to X'$.

Now applying functoriality to

$$\begin{tikzcd}
& & U_a \arrow[dr] & \\
V \arrow[r] & U_a \cap U_b \arrow[ur] \arrow[dr] & & X \\
& & U_b \arrow[ur] &
\end{tikzcd}$$

where $V \subset U_a \cap U_b$ is an affine open we see that $\eta_{U_a}(j_a)|_V = \eta_V(V \to X) = \eta_{U_b}(j_b)|_V$ for all affine opens inside $U_a \cap U_b$. In particular the $\{\eta_{U_a}(j_a)\}$ glue to a function $f : X \to X'$. Post-composition by f is η because $(f \circ g)|_{g^{-1}(U_a)} = \eta(g)|_{g^{-1}(U_a)}$ for all a and for all $g \in h_X(Y)$. To check this one needs to use naturality twice and cover $g^{-1}(U_a)$ by affines. This shows surjectivity.

For injectivity suppose that $f, f' : X \to X'$ are not equal. Then there exists some affine open U_a on which $f \neq f' : U_a \to X'$. It is important that these are opens because f and f' might only differ on stalks, which are captured when restricting to open subschemes. However this shows that $h_f(U_a)(j_a) = f \circ j_a \neq f' \circ j_a = h_{f'}(U_a)(j_a)$, hence $h_f \neq h_{f'}$. □

Remark 2.3. Everything we have done can equivalently be done for R-schemes and the category of R-algebras.

This theorem can dramatically change how we view schemes. Instead of building a topological space out of affines, we need only (in principle) show a functor from **Rings** to **Set** and argue that it is representable.

We now have the category $\mathbf{Fun}(\mathbf{Rings}, \mathbf{Set})$ in which schemes can be viewed as representable functors. The ind-schemes we wish to introduce live in this category, but are close enough to schemes that we can make sense of many of the features typically attributed to schemes.

2.1.2 Ind-schemes as Objects

Here we make note of a particular class of objects known as ind-schemes. From the functor of points perspective one can view ind-schemes as functors from **Rings** to **Set** which are in some sense too large to be representable. Primarily we are motivated by the example of the affine Grassmannian, which will be discussed more in Chapter 3.

Informally these can be viewed as "colimits of schemes" or "increasing unions of schemes". Formally they are defined as follows.

Definition 2.4. An ind-scheme is a functor $X : \mathbf{Rings} \to \mathbf{Set}$ such that $X = \varinjlim_{i \in I} X_i$ is a filtered colimit of schemes and the transition maps $X_i \to X_j$ are closed embeddings for all $i \leq j$.

Remark 2.5. The definition of ind-scheme we are using is sometimes referred to as a strict ind-scheme, where the term ind-scheme is reserved for the same notion without the condition of the transition maps being closed embeddings.

In this definition we are leveraging the functor of points perspective to conflate the functors X_i with their representing schemes. The colimit of functors is just taken object-wise with the induced maps between the colimits. This is well defined whenever the target category contains all filtered colimits (for example in **Set**).

The following lemma will show is some sense that ind-schemes are reasonable geometric objects to consider.

Lemma 2.6. Ind-schemes are sheaves for the fpqc topology.

Proof. We check conditions (1) and (2) as in [Zhu16, 0.3.1]. Let $\mathcal{F} = \varinjlim_{i\in I} \mathcal{F}_i$ be an ind-scheme. Proving (1) means proving that the map

$$\varinjlim_{i\in I} \mathcal{F}_i(\prod_j R_j) \to \prod_j \varinjlim_{i\in I} \mathcal{F}_i(R_j)$$

is an isomorphism. Since schemes are fpqc sheaves [Sta23, Tag 03O3] the $\mathcal{F}_i$ commute with finite products, so we are left to prove that

$$\varinjlim_{i\in I} \prod_j \mathcal{F}_i(R_j) \to \prod_j \varinjlim_{i\in I} \mathcal{F}_i(R_j)$$

is an iso, which follows immediately from the fact that filtered colimits commute with finite products.

To prove (2) let $R \to R'$ be an fpqc cover, then again since schemes are fpqc sheaves we have that

$$\mathcal{F}_i(R) \to \mathcal{F}_i(R') \rightrightarrows \mathcal{F}_i(R' \otimes_R R') \tag{2.1}$$

are equaliser diagrams for all $i \in I$.

Now consider the diagram

$$\begin{tikzcd} S \arrow[d, dashed] \arrow[dr, "\phi"] & & \\ \mathcal{F}(R) \arrow[r, "\iota"] & \mathcal{F}(R') \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \mathcal{F}(R' \otimes_R R'), \end{tikzcd}$$

Where $p_1 \circ \phi = p_2 \circ \phi$. We construct the vertical arrow. Let $s \in S$ we have that $p_1(\phi(s)) = p_2(\phi(s))$ in $\mathcal{F}(R' \otimes_R R')$ by assumption, which is represented by some element in $\mathcal{F}_i(R' \otimes_R R')$. Since 2.1 is an equaliser, there is a unique element x of $\mathcal{F}_i(R)$ such that $\iota_i(x) \in \mathcal{F}_i(R')$ represents $\phi(s)$. Now we can define the dashed arrow to send s to the image of x in $\mathcal{F}(R)$. We leave the verification that this map is well defined to the reader. Uniqueness and commutativity of the diagram are by construction. □

2.1.3 Properties of Ind-schemes

Here we describe properties of ind-schemes and maps of ind-schemes that appear in the thesis. For more details see [Ric19, Chapter 1] or [Zhu16, 0.2.3]. The reader is encouraged to skip this subsection on a first reading and come back when necessary.

For many properties of schemes we can define analogous properties of ind-schemes.

Definition 2.7. We call an ind-scheme X (i). ind-projective, (ii). reduced (iii). finite type over k if there exists a presentation $X = \varinjlim_{i\in I} X_i$ where the X_i are (i). projective, (ii). reduced, (iii). finite type over k.

We define properties of maps of ind schemes similarly, but need to be a bit careful about which adjectives remain well defined.

First, note that any morphism of ind-schemes $f : X \to Y$ can be written as a system of morphisms of schemes. In particular if $X = \varinjlim_{i\in I} X_i$ and $Y = \varinjlim_{j\in J} Y_j$ we get a new presentation of X as $\varinjlim_{(i,j)\in I\times J} X_i \times_X f^{-1}(Y_j)$ where $I \times J$ is given the product ordering. Therefore after changing the presentation for X, f can be written as a system $f_{i,j} : X_i \to Y_j$ of morphisms of schemes over a suitable index set.

Definition 2.8. For **P** be a property of morphisms of schemes which is stable under base change and Zariski local on the target. A morphism $f : X \to Y$ of ind-schemes is ind-**P** if there exist a presentation $f_{i,j} : X_i \to Y_j$ where each morphism is **P**.

Definition 2.9. A map of ind-schemes $f : X \to Y$ is called schematic if for all affine schemes T the fibre product functor $X \times_Y T$ is a scheme.

Definition 2.10. A map of ind-schemes has property **P** if it is ind-**P** and schematic.

In this thesis we will typically be interested in maps of (quasi-compact) schemes into ind-schemes, for which the schematic condition is automatic. See [Ric19, Exercise 1.26].

2.1.4 Topology of Ind-schemes

In this subsection we make the geometry of ind-schemes more apparent by defining their underlying topological spaces. We build up enough theory to define a stratification of an ind-scheme without ambiguity and discuss the analytic topology for ind-schemes.

Definition 2.11. A sub-ind-scheme of an ind-scheme X is a subfunctor $Z \subset X$ that is representable by an immersion. If Z is a scheme then we say Z is a subscheme of X.

Remark 2.12. We define open and closed sub(-ind-)schemes by inserting the corresponding adjective in the above definition.

Definition 2.13. If X is an ind-scheme we define $|X| := \varinjlim_k X(k)$ where the colimit is taken over the category of fields. We endow this set with the colimit topology coming from $\varinjlim_i |X_i|$, where $|X_i|$ are the underlying topological spaces of the schemes X_i.

Remark 2.14. There is a more canonical definition of the topology based on subfunctors of X which are representable by open immersions. These two topologies are equivalent by [Ric19, Lemma 1.12], so our definition is in fact independent of our choice of system X_i. There is also no ambiguity here when X itself is a scheme because of [Sta23, Tag 01J9].

With this definition we can now talk about closure and density of locally closed subsets of ind-schemes without ambiguity. Which allows us to make sense of stratifications of ind-schemes.

Definition 2.15. For X a topological space a stratification of X is a decomposition

$$X = \bigsqcup_{i \in I} X_i,$$

where each X_i is locally closed and $\overline{X_i}$ is a union of the other components.

Recall that if X is a finite type scheme over $\mathbb{C}$ then it can be endowed with an analytic topology that is locally compact and Hausdorff [Nee07, 4.1]. We can in fact do the same thing for any ind-finite type ind-scheme over $\mathbb{C}$.

Definition 2.16. For $X = \varinjlim_{i \in I} X_i$ an ind-finite type ind-scheme over $\mathbb{C}$ we define the analytic topology on X as the colimit topology when the X_i are given the analytic topology.

2.2 Algebraic Groups

In this section we give some background on the theory of reductive algebraic groups. Fix k a field.

2.2.1 Basic Definitions

Definition 2.17. A group scheme over k (or k-group scheme) is a functor $G : k\text{-}\mathbf{Alg} \to \mathbf{Grp}$ represented by a k-scheme.

Often k is dropped from the name to be inferred from context, especially with standard groups $\mathbb{G}_a$ and $\mathbb{G}_m$.

Examples 2.18. Some basic examples of group schemes are the following:

- $\mathbb{G}_a : R \mapsto (R, +)$,
- $\mathbb{G}_m : R \mapsto (R^\times, \cdot)$,
- $\mathrm{SL}_n : R \mapsto \mathrm{SL}_n(R)$,
- $\mathbb{G}_a^\infty : R \mapsto (\prod_{i=1}^\infty R, +)$
- $R \mapsto (\{x \in R : x^2 = 0\}, \cdot)$.

Definition 2.19. An algebraic group (over k) is a group scheme (over k) that is of finite type.

Definition 2.20. Affine algebraic groups are often referred to as linear algebraic groups.

All of the above are examples of affine group schemes, but the fourth is not an affine algebraic group. SL_n is represented by $\operatorname{Spec} k[x_{ij}]/(\det(x_{ij}) - 1)$, which is reduced and finite type. The group scheme $\mathbb{G}_a^\infty$ is reduced but not of finite type as it is represented by $\operatorname{Spec}(k[x_1, x_2, ...])$, while the last group scheme on the list is finite type and not reduced as it is represented by $\operatorname{Spec}(k[x]/x^2)$.

2.2.2 Reductive Groups

Definition 2.21. A torus T is a linear algebraic group such that $T_{\bar{k}} \cong \mathbb{G}_m^n$ over $\bar{k}$ for some n. A torus is (k-)split if $T \cong \mathbb{G}_m^n$ over k.

Example 2.22. If $k = \mathbb{R}$ then SO_2 is an example of a non-split torus.

Definition 2.23. For a split torus T we define its character and cocharacter groups as $X^*(T) = \mathrm{Hom}_k(T, \mathbb{G}_m)$ and $X_*(T) = \mathrm{Hom}_k(\mathbb{G}_m, T)$

One can see via the Hopf algebras that $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$. Therefore for split tori the character and cocharacter groups are free finitely generated abelian groups. As is suggested by the notation there is a duality between these two groups.

Proposition 2.24. *For T a split torus, there is a perfect pairing*

$$\langle \bullet, \bullet \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}, \langle \chi, \lambda \rangle := \chi \circ \lambda$$

In this thesis we care mostly about a certain subclass of linear algebraic groups called reductive groups, whose classification is rather rigid. In some of the following definitions we implicitly use the fact that linear algebraic groups over a field of characteristic 0 are smooth (see [Mil17, Theorem 3.23]).

Definition 2.25 ([Mil17, 6.44])**.** For G a connected linear algebraic group, the maximal connected solvable normal algebraic subgroup is called the radical of G, which we denote $R(G)$.

An important class of algebraic groups are unipotent groups. Unipotent algebraic groups can be thought of as closed subgroups of $\mathbb{U}_n$, the subgroup of upper triangular matrices with unique eigenvalue 1. ([Mil17, Corollary 14.6])

Definition 2.26 ([Mil17, 6.45])**.** An algebraic group G is said to be unipotent if every nonzero representation of G has a nonzero fixed vector.

Definition 2.27 ([Mil17, 6.46])**.** The unipotent radical $R_u(G)$ is the maximal connected normal unipotent algebraic subgroup of G.

Definition 2.28. A connected linear algebraic group G over an algebraically closed field is called:

- reductive if $R_u(G) = \{e\}$, equivalently $R(G)$ is a torus
- semisimple if $R(G) = \{e\}$

A connected linear algebraic group G over k is reductive or semisimple if $G_{\bar{k}}$ is.

Examples 2.29. SL_n and SO_n are semisimple, while GL_n is reductive but not semisimple.

Definition 2.30. A split reductive k-group is a reductive k-group G containing a maximal torus which is k-split.

2.2.3 Root Data

Now to a split reductive group we can associate an object known as a root datum. This is done by taking a split maximal torus $T \subset G$ and considering the adjoint action of T on $\mathfrak{g}$, the Lie algebra of G. This action is diagonalisable, hence we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha.$$

Where $\mathfrak{g}_\alpha$ is the eigenspace on which T acts by α. We define $\Phi = \{\alpha : \mathfrak{g}_\alpha \neq 0\}$ and call these the roots of (G, T). From here there is a particular way to associate to each root a coroot $\alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$, details of this process can be found in [Mil17, Theorem 21.11]. We define $\Phi^\vee$ to be the set of $\alpha^\vee$ where $\alpha \in \Phi$. We define $Q^\vee = \mathbb{Z}\Phi^\vee \subset X_*(T)$ and call it the coroot lattice.

Informally, one can think of a root datum as a pair of dual free finite rank $\mathbb{Z}$-modules (X and $X^\vee$) along with finite subsets of each (Φ and $\Phi^\vee$) such that $\langle \alpha, \alpha^\vee \rangle = 2$, which allow one to construct reflections s_α for $\alpha \in \Phi$ on X. We then require these reflections to preserve Φ and the subgroup of $\mathrm{Aut}(X)$ they generate to be finite. (This finite group is the Weyl group.) For a formal definition see [Mil17, C.28].

Proposition 2.31 ([Mil17, Corollary 21.12])**.** *For G a split reductive group and T a split maximal torus, the quadruple $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ forms a root datum.*

The surprising fact is how much information is captured by the root datum of a split reductive group.

Theorem 2.32 ([Mil17, 21.18, Theorem 23.25 and Theorem 23.55])**.** *The isomorphism class of a root datum associated to a split reductive group G is independent of the choice of split maximal torus. Furthermore, a root datum determines a unique split reductive group.*

As is suggestive in the notation but not immediate from the informal definition, there is a duality inherent to root data. If $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum, then so is $(X^\vee, \Phi^\vee, X, \Phi)$. Therefore to a reductive group G we can define its Langlands dual group $G_k^\vee$ as the unique k-split reductive group whose root data is dual to that of G. Note that we can do this for any field k. Further generality is possible, but won't be used in this thesis.

2.2.4 Borel and Parabolic Subgroups

Before we give an example that ties all this together, we wish to talk about certain important types of subgroups. We will assume for this section that k is algebraically closed and that G is reductive. All algebraic groups will be over k.

Definition 2.33. A Borel subgroup of a linear algebraic group G is a maximal connected solvable algebraic subgroup.

Definition 2.34. Given a choice of maximal torus T and a Borel subgroup B containing it, we define the opposite Borel subgroup B^- as the unique Borel subgroup such that $B \cap B^- = T$. B and B^- are conjugate under the longest element in the Weyl group.

Definition 2.35. A parabolic subgroup of a linear algebraic group is a subgroup P such that G/P is a complete variety.

These two types of subgroups are closely connected by the following theorem

Theorem 2.36 ([Mil17, Theorem 17.16])**.** *A subgroup P of G is parabolic if and only if it contains a Borel.*

As one might suspect, the Borel and parabolic subgroups have interpretations in the root datum of (G, T).

Given a Borel subgroup B such that $T \subset B \subset G$ it is natural to consider the decomposition of the Lie algebra of B under the action of T. The roots that appear in the decomposition we call positive roots, denoted $\Phi^+ \subset \Phi$. For all $\alpha \in \Phi$ exactly one of α or $-\alpha$ is in Φ^+. The corresponding simple roots $\Phi_s \subset \Phi^+$ are the positive roots that can not be expressed as a positive sum of other positive roots. Dually we have the notion of positive and simple coroots as the image of Φ^+ under $^\vee$.

A choice of Borel subgroup also determines an ordering on the cocharacter lattice $X_*(T)$ where we define

$$\lambda \geq 0 \iff \lambda = \sum_{\alpha \in \Phi^+} n_\alpha \alpha^\vee, \quad n_\alpha \in \mathbb{N} \tag{2.2}$$

and $\lambda \geq \mu$ if $\lambda - \mu \geq 0$. We define the dominant cocharacters $X_*(T)^+$ as the cocharacters such that $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Phi_s$. There is also a dual ordering on $X^*(T)$ which we will not use.

Remark 2.37. If we define $2\rho := \sum_{\alpha \in \Phi^+} \alpha$ and have $\lambda \geq \mu$ then certainly $\langle 2\rho, \lambda - \mu \rangle$ is an even positive integer.

Given a triple (G, B, T), there is also a process for attaching a parabolic subgroup containing B to any dominant cocharacter $\lambda \in X_*(T)^+$.

Theorem 2.38 ([Mil17, Theorem 21.91])**.** *For each subset $I \subset \Phi_s$ there is a unique parabolic subgroup P_I and in fact all parabolic subgroups are of this form.*

Definition 2.39. For $\lambda \in X_*(T)^+$ define $P(\lambda) := P_I$ where

$$I = \{\alpha \in \Phi_s : \langle \alpha, \lambda \rangle = 0\}$$

Instead of giving details of the construction in Theorem 2.38, which uses the Weyl group, we will put the theory in context with an extended example.

2.2.5 A Worked Example

Let $T \subset B \subset G$ where $G = \mathrm{GL}_n$ over $\mathbb{C}$, T is the maximal torus of diagonal matrices and B is the Borel subgroup of upper triangular matrices.

The unipotent radical N of B is exactly the upper triangular matrices with 1 on the diagonal. B^- is the subgroup of lower triangular matrices, B^- will induce positive roots that are the complement of Φ^+ in Φ. Thus B^- induces the opposite order on cocharacters.

We have that $X^*(T) = \bigoplus_{i=1}^n \mathbb{Z}\chi_i \cong \mathbb{Z}^n$ with $\chi_i : \mathrm{diag}(t_1, ..., t_n) \to t_i$ and $X_*(T) = \bigoplus_{i=1}^n \mathbb{Z}\lambda_i \cong \mathbb{Z}^n$ with $\lambda_i : t \mapsto \mathrm{diag}(1, ..., 1, \underset{i}{t}, 1, ..., 1)$. The perfect pairing is given by $\langle \chi_i, \lambda_j \rangle = \delta_{ij}$.

Now to find the roots we need to analyse the action of the torus on the Lie algebra $\mathfrak{gl}_n = M_n(\mathbb{C})$. If $A = (a_{ij})$ we have that

$$\left(\mathrm{diag}(t_1, ..., t_n) A \,\mathrm{diag}(t_1, ..., t_n)^{-1}\right)_{ij} = \frac{t_i}{t_j} a_{ij}$$

Therefore if E_{ij} are the elementary matrices and $\alpha_{ij} := \chi_i - \chi_j$ we have that T acts on $\mathbb{C}E_{ij}$ as α_{ij}. Hence

$$\mathfrak{gl}_n = \bigoplus_{i=1}^n \mathbb{C}E_{ii} \oplus \bigoplus_{1 \le i \ne j \le n} \mathbb{C}E_{ij} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha_{ij}} \mathfrak{g}_{\alpha_{ij}}.$$

Therefore our set of roots are $\Phi = \{\alpha_{ij} : 1 \le i \ne j \le n\}$. It turns out that the coroot $\alpha_{ij}^\vee$ is $\lambda_{ij} := \lambda_i - \lambda_j$. Indeed we note that

$$\langle \alpha_{ij}, \lambda_{ij} \rangle = \langle \alpha_i, \lambda_i \rangle + \langle -\alpha_j, -\lambda_j \rangle = 2.$$

Hence our set of coroots is $\Phi^\vee = \{\lambda_{ij} : 1 \le i \ne j \le n\}$. See [Mil17, Example 21.16] for details. Our root datum is $(X^*(T), \Phi, X_*(T), \Phi^\vee)$. We can see under the identifications that this is a self-dual root datum.

Since B is the subgroup of invertible upper triangular matrices the Lie algebra $\mathfrak{b}$ is simply upper triangular matrices. Therefore

$$\mathfrak{b} = \bigoplus_{i=1}^n \mathbb{C}E_{ii} \oplus \bigoplus_{1 \le i < j \le n} \mathbb{C}E_{ij} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha_{ij} : i<j} \mathfrak{g}_{\alpha_{ij}}$$

Therefore our positive roots are $\Phi^+ = \{\alpha_{ij} : 1 \le i < j \le n\}$. Since we can always write $\alpha_{ij} = \alpha_{i,j-1} + \alpha_{j-1,j}$ whenever $j - i \ge 2$ we see that the corresponding simple system is $\Phi_s = \{\alpha_{i,i+1} : 1 \leq i < n\}$; we sometimes identify Φ_s with $\{1, ..., n-1\}$ in the obvious way.

Now Φ_s defines an order on $X_*(T)$ where

$$\eta \leq \lambda \iff \langle \alpha, \lambda - \eta \rangle \geq 0 \quad \forall \alpha \in \Phi_s$$

In particular if $\eta = (b_1, ..., b_n)$ and $\lambda = (a_1, ..., a_n)$ under the identification with $\mathbb{Z}^n$ we have $\eta \leq \lambda$ if and only if $a_1 - b_1 \geq a_2 - b_2 \geq \cdots \geq a_n - b_n$. The positive coroots $X_*(T)^+$ are the $(a_1, .., a_n)$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$.

We have defined a lot of things, so let's put them in a table so they are easier to digest:

(GL_n, B, T)	characters	cocharacters
(co)character lattice	$X^*(T) = \bigoplus_{i=1}^n \mathbb{Z}\chi_i \cong \mathbb{Z}^n$	$X_*(T) = \bigoplus_{i=1}^n \mathbb{Z}\lambda_i \cong \mathbb{Z}^n$
(co)roots	$\Phi = \{\alpha_{ij} : 1 \leq i \neq j \leq n\}$	$\Phi^\vee = \{\lambda_{ij} : 1 \leq i \neq j \leq n\}$
positive (co)roots	$\Phi^+ = \{\alpha_{ij} : 1 \leq i < j \leq n\}$	$\{\lambda_{ij} : 1 \leq i < j \leq n\}$
simple (co)roots	$\Phi_s = \{\alpha_{i,i+1} : 1 \leq i < n\}$	$\{\lambda_{i,i+1} : 1 \leq i < n\}$
dominant (co)roots	$\{\sum_{i=1}^n a_i\alpha_i : a_1 \geq \cdots \geq a_n\}$	$X_*(T)^+ = \{\sum_{i=1}^n a_i\lambda_i : a_1 \geq \cdots \geq a_n\}$

In this case the GL_n root datum is self dual, so $(\mathrm{GL}_n)_k^\vee = \mathrm{GL}_{n,k}$.

Finally we give an example of a parabolic subgroup containing B attached to a positive cocharacter. Let $n = 6$ and $\lambda = \lambda_1 + \lambda_2 - \lambda_6 = (1, 1, 0, 0, 0, -1)$. In other words $\lambda : \mathrm{diag}(t_1, ..., t_6) \mapsto t_1 t_2 t_6^{-1}$. Indeed $\lambda \in X_*(T)^+$ and

$$I = \{\alpha \in \Phi_s : \langle \alpha, \lambda \rangle = 0\} = \{\alpha_{1,2}, \alpha_{3,4}, \alpha_{4,5}\} = \{1, 3, 4\}.$$

Following the more detailed statement from [Mil17, Theorem 21.91] we have that $P(\lambda)$ is the parabolic subgroup:

$$\begin{bmatrix} \mathrm{GL}_2 & M_{2,3} & M_{2,1} \\ 0 & \mathrm{GL}_3 & M_{3,1} \\ 0 & 0 & \mathrm{GL}_1 \end{bmatrix}$$

which certainly contains B. As a heuristic, each $\alpha \in \Phi_s$ defines a hyperplane in the cocharacter lattice given by the kernel of $\langle \alpha, \cdot \rangle$, and the more of these hyperplanes that λ lies on the larger the corresponding parabolic subgroup will be. In particular $P(0) = G$ and $P(\lambda_{1,2} + \cdots + \lambda_{n-1,n}) = B$.

Chapter 3

The Affine Grassmannian

3.1 The Affine Grassmannian for the General Linear Group

We begin our study of the affine Grassmannian with the case of GL_n, which is important for the general case of G a reductive group. We define the affine Grassmannian by its functor of points. Unfortunately this functor is not representable and so the affine Grassmannian is not a scheme. The main result of the section is that $\mathrm{Gr}_{\mathrm{GL}_n}$ is an ind-projective scheme (Theorem 3.4).

Throughout this section we take k to be a field and write Gr for $\mathrm{Gr}_{\mathrm{GL}_n}$.

Definition 3.1. For R a k-algebra an R-family of lattices in $k((t))^n$ is a finitely generated projective $R[[t]]$-submodule Λ of $R((t))^n$ such that $R((t))\otimes_{R[[t]]}\Lambda \cong R((t))^n$.

Note that tensor product here is really just the process of localising t. We define the affine Grassmannian as the functor parameterising R-families of lattices in $k((t))^n$ for various R.

Definition 3.2. The affine Grassmannian Gr for GL_n is the functor k-**Alg** $\to$ **Set**

$$R \mapsto \{R\text{-families of lattices in } k((t))^n\}.$$

Before doing anything else it is important to verify a claim embedded in the previous definition.

Lemma 3.3. The assignment of objects in Definition 3.2 in fact defines a functor k-**Alg** $\to$ **Set**.

Proof. For $\phi : S \to R$ we take $\Lambda \mapsto S[[t]] \otimes_{R[[t]]} \Lambda =: \Lambda'$. Since finite generation and projectiveness are preserved under base change we know that Λ' is a finitely generated projective module. The fact that tensor product commutes with localisation implies that Λ' is indeed an S-family of lattices in $k((t))^n$. For a more detailed proof see Lemma A.1 in the appendix. □

We can now state the main theorem of this section.

Theorem 3.4. *The affine Grassmannian* Gr *for* GL_n *is represented by an ind-projective scheme.*

The major insight for the proof is that due to finite generation we can pick a finite spanning set for any $\Lambda \in \mathrm{Gr}(R)$. Since inverting t reverts Λ to $R((t))^n$ we have that there exists some $N \in \mathbb{N}$ such that $t^N R[[t]]^n \subset \Lambda \subset t^{-N}R[[t]]^n$. This gives us a natural family of subfunctors of Gr, which we define below.

Definition 3.5. For any $N \in \mathbb{N}$ we define the following subfunctors of Gr

$$\mathrm{Gr}^{(N)}(R) := \{\Lambda \in \mathrm{Gr}(R) : t^N R[[t]]^n \subset \Lambda \subset t^{-N}R[[t]]^n\}.$$

Certainly $\mathrm{Gr} = \varinjlim_{N\in\mathbb{N}} \mathrm{Gr}^{(N)}$ and the inclusions $\mathrm{Gr}^{(N)} \to \mathrm{Gr}^{(N+1)}$ are closed embeddings. The proof strategy for Theorem 3.4 will be to show that these functors are representable by projective schemes.

Definition 3.6.

$$\mathrm{Gr}^{(N),f}(R) := \left\{ R[[t]]\text{-quotient modules of } \frac{t^{-N}R[[t]]^n}{t^N R[[t]]^n} \text{ that are } R\text{-projective} \right\}.$$

Proposition 3.7. *For $\mathrm{Gr}^{(N),f}$ defined above we have that:*

1. $\mathrm{Gr}^{(N)} \cong \mathrm{Gr}^{(N),f}$.
2. $\mathrm{Gr}^{(N),f}$ *is represented by a closed subscheme of* $\mathrm{Gr}(2nN)$,

Where $Gr(2nN)$ is the usual grassmannian classifying vector subspaces of k^{2nN}.

Proof idea. 1. We have an equivalence $\mathrm{Gr}^{(N)} \cong \mathrm{Gr}^{(N),f}$ given by the map

$$\Lambda \mapsto t^{-N}k[[t]]^n/\Lambda.$$

See [Zhu16, Lemma 1.1.5] for details.

2. The idea is easy to see on k-points since

$$t^{-N}k[[t]]^n/t^N k[[t]]^n \cong k^{2nN}.$$

Multiplication by t is a nilpotent operator on this space and $\mathrm{Gr}^{(N),f}$ takes the subspaces that are stable under this operator, which should be a closed condition. □

The proof of Theorem 3.4 is now immediate.

Proof of Theorem 3.4. Certainly

$$\mathrm{Gr} = \varinjlim_{N\in\mathbb{N}} \mathrm{Gr}^{(N)} \overset{3.7}{\cong} \varinjlim_{N\in\mathbb{N}} \mathrm{Gr}^{(N),f}.$$

Since $\mathrm{Gr}^{(N)} \to \mathrm{Gr}^{(N+1)}$ are closed immersions so are $\mathrm{Gr}^{(N),f} \to \mathrm{Gr}^{(N+1),f}$. The $\mathrm{Gr}^{(N),f}$ are projective as closed subschemes of the projective scheme $\mathrm{Gr}(2nN)$. Therefore Gr is ind-projective. □

3.2 The Affine Grassmannian for Reductive Groups

For this section fix G a connected reductive group over a field k.

In this section we will define the affine Grassmannian Gr_G for a reductive group G via its functor of points and prove that it is ind-projective. For this we will need the concept of a torsor.

Definition 3.8. An (étale) G-torsor over a base X is a G-equivariant map of schemes $\mathcal{E} : E \to X$ where X is given the trivial G action and such that there exists an étale cover $X' \to X$ such that $E \times_X X' \cong G \times X'$.

A map of torsors over X is a G-equivariant map of X-schemes.

Notation 3.9. If $U \subset X$ is an open subset of the base for a torsor $\mathcal{E}$ over X we denote $\mathcal{E}|_{\mathcal{E}^{-1}(U)}$ by $\mathcal{E}|_U$. Equivalently one can think of $\mathcal{E}|_U$ as the base change of $\mathcal{E}$ by the open immersion $U \hookrightarrow X$.

Example 3.10. The trivial G-torsor over X, which we denote $\mathcal{E}^0$, is $G \times X \to X$. Where the action is simply defined on the G factor.

Remark 3.11. The base change of the trivial torsor remains trivial, so we allow the base of $\mathcal{E}^0$ to be inferred from context.

Before defining the affine Grassmannian we need some notation that will appear again in the following sections.

Notation 3.12. Define $\mathbb{D}_R := \operatorname{Spec} R[[t]]$ and $\mathbb{D}_R^* := \operatorname{Spec} R((t))$. These are often referred to as the formal disk and the punctured formal disk respectively. Clearly $\mathbb{D}_R^*$ sits inside $\mathbb{D}_R$ as a principal open affine.

Definition 3.13. The affine Grassmannian, denoted Gr_G is the functor

$$\mathrm{Gr}_G(R) = \left\{(\mathcal{E}, \beta) | \mathcal{E} \text{ is a } G\text{-torsor over } \mathbb{D}_R,\ \beta : \mathcal{E}|_{\mathbb{D}_R^*} \cong \mathcal{E}^0\right\} / \mathrm{iso}.$$

Remark 3.14. To resolve the two definitions of $\mathrm{Gr}_{\mathrm{GL}_n}$ consider an element of $\mathrm{Gr}_{\mathrm{GL}_n}(R)$. Note that [GR02, Proposition 5.1, Page 232] implies that GL_n-torsors are all in fact locally trivial for the Zariski topology. The bottom of page 6 of [Zhu16] takes us from a Zariski GL_n-torsor to a rank n vector bundle over $\mathbb{D}_R$. The standard result [GW10, Proposition 11.7] brings us to a finitely generated Zariski locally free $\mathcal{O}_{\mathbb{D}_R}$-module, which is of course equivalent to a finitely generated locally free (i.e. projective) $R[[t]]$-module Λ. Following the trivialisation $\mathcal{E}|_{\mathbb{D}_R^*} \cong \mathcal{E}^0$ through the same steps you get the isomorphism $R((t)) \otimes_{R[[t]]} \Lambda \cong R((t))^n$ from Definition 3.1.

It is non-trivial theorem that Gr_G is ind-projective for G reductive, and it in fact relies on the case of GL_n, as we will see.

Theorem 3.15. *For G a reductive group Gr_G is ind-projective.*

The proof of the above theorem has two major steps, encapsulated in the propositions below. Together these propositions establish a closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$. Hence the ind-projectiveness of Gr_G relies on the ind-projectiveness of $\mathrm{Gr}_{\mathrm{GL}_n}$ being already established.

Proposition 3.16. *If G is reductive there exists a linear representation $\rho : G \to \mathrm{GL}_n$ such that GL_n/G is affine.*

Proof. See [Alp14, Corollary 9.7.7.] □

Proposition 3.17. *If $\rho : G \to \mathrm{GL}_n$ is a linear representation with GL_n/G affine, then $f_\rho : \mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{GL}_n}$ is a closed embedding.*

Proof Sketch. Let $\operatorname{Spec} R \to \mathrm{Gr}_{\mathrm{GL}_n}$ be represented by a torsor-trivialisation pair $(\mathcal{E}, \beta)$. We need to show that the base change map

$$\mathcal{F} := \operatorname{Spec} R \times_{\mathrm{Gr}_{\mathrm{GL}_n}} \mathrm{Gr}_G \to \operatorname{Spec} R$$

is a closed immersion. We denote by $\pi : \mathcal{E} \to \mathbb{D}_R$ and regard β as a section of π over $\mathbb{D}_R^*$. Furthermore denote $\overline{\pi} : [\mathcal{E}/G] \to \mathbb{D}_R$ be the induced map on the étale quotient. We represent all this information and more in the following diagram.

$$\begin{tikzcd}
& \mathcal{E} \arrow[d] \arrow[dd, bend left, "\pi"] \\
& {[\mathcal{E}/G]} \arrow[d, "\overline{\pi}"] \\
\mathbb{D}_R^* \arrow[uur, bend left, "\beta"] \arrow[ur, "\overline{\beta}"] \arrow[r, hook] & \mathbb{D}_R
\end{tikzcd}$$

Since GL_n/G is affine, by [Sta23, Tag 0244] we have that $[\mathcal{E}/G]$ is represented by an affine scheme W of finite presentation over $\mathbb{D}_R$.

The section β of $\mathcal{E}$ induces a section $\overline{\beta}$ of $[\mathcal{E}/G]$. Now, giving a reduction of $\mathcal{E}$ to a G-torsor is equivalent to choosing a section of $\overline{\pi}$. Therefore $\mathcal{F}$ is the presheaf over $\operatorname{Spec} R$ that assigns to any $R \to R'$ the set of sections β' of $\overline{\pi}$ over $\mathbb{D}_{R'}$ such that $\beta'|_{\mathbb{D}^*_{R'}} = \overline{\beta}|_{\mathbb{D}^*_{R'}}$. We can write this in notation as

$$\mathcal{F} : (R \to R') \mapsto \{\beta' : \mathbb{D}_{R'} \to [\mathcal{E}/G] \text{ sections of } \overline{\pi} \text{ such that } \beta'|_{\mathbb{D}^*_{R'}} = \overline{\beta}|_{\mathbb{D}^*_{R'}}\}.$$

By $\overline{\beta}|_{\mathbb{D}_{R'}}$ we of course mean $\overline{\beta}$ pre-composed with the morphism $\mathbb{D}^*_{R'} \to \mathbb{D}^*_R$.

Applying Lemma 3.19 identifies $\mathcal{F}$ as a closed subscheme of $\operatorname{Spec} R$, as required. $\square$

Remark 3.18. For a description of the induced map on affine Grassmannians see [Ric19, Remark 3.2.(4)].

It remains to prove the following lemma.

Lemma 3.19. Let $p : W \to \mathbb{D}_R$ be an affine scheme of finite presentation, and s be a section of p over $\mathbb{D}^*_R$. Then the presheaf over $\operatorname{Spec} R$ defined by

$$\mathcal{F} : (R \to R') \mapsto \{s' : \mathbb{D}_{R'} \to [\mathcal{E}/G] \text{ sections of } p \text{ such that } s'|_{\mathbb{D}^*_{R'}} = s|_{\mathbb{D}^*_{R'}}\},$$

is represented by a closed subscheme of $\operatorname{Spec} R$.

Proof. We can embed $W \subset \mathbb{A}^N_{\mathbb{D}_R}$ as a closed subset of some large affine space. Then the section s can we written in coordinates as

$$s = (s_1(t), ..., s_N(t)), \quad s_i(t) = \sum s_{ij} t^j \in R((t))$$

The s_i of course must satisfy certain equations defining V, but they will not be relevant. Now, let $(R \to R')$ be given. Our aim is to find an ideal of R that determines whether $(R \to R')$ is in the image of $\mathcal{F}$. We put the data we have in a diagram. Note that the section $s' : \mathbb{D}_{R'} \to W$ can be described by $(u_1(t), ..., u_N(t))$ such that $u_i(t) \in R'[[t]]$.

$$\begin{tikzcd}
& & W \arrow[dr, "\pi"] & \\
\mathbb{D}_{R'} \arrow[urr, "s'"] & \mathbb{D}^*_{R'} \arrow[l, hook'] \arrow[r] & \mathbb{D}^*_R \arrow[u, "s"] \arrow[r, hook] & \mathbb{D}_R
\end{tikzcd}$$

Since everything is affine this becomes a diagram of maps of rings.

$$\begin{tikzcd}
& & R[[t]](x_1, ..., x_N)/I \arrow[dll, "s'"'] \arrow[d, "s"] & \\
R'[[t]] \arrow[r, hook] & R'((t)) & R((t)) \arrow[l, "\phi"] & R[[t]] \arrow[l, hook'] \arrow[ul, "\pi"']
\end{tikzcd}$$

The condition $s'|_{\mathbb{D}^*_{R'}} = s|_{\mathbb{D}^*_{R'}}$ corresponds to both maps being equal at $R'((t))$. Therefore

$$\begin{aligned}
&\{s' : \mathbb{D}_{R'} \to [\mathcal{E}/G] \text{ sections of } p \text{ such that } s'|_{\mathbb{D}^*_{R'}} = s|_{\mathbb{D}^*_{R'}}\} \\
&= \{(u_1(t), ..., u_N(t)) \in R'[[t]] \mid (u_1(t), ..., u_N(t)) = (\phi(s_1(t)), ..., \phi(s_N(t)))\} \\
&= \begin{cases} \{*\} & \text{if } (s_{ij})_{j<0, i=1,...,N} \subset \ker(R \to R') \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}$$

In short, the restriction on the section is that after mapping to R' there should be no poles, which is closed because it is equivalent to setting the negative coefficients of $s_i(t)$ to 0. □

3.3 The Affine Grassmannian as a Quotient

It turns out there is a different way to view the affine Grassmannian as a quotient. This alternate perspective will equip Gr_G with various group actions that will be critical in what follows.

For this section let G be a reductive group over k.

Definition 3.20. For X a scheme over k we define the loop and positive loop spaces of X as the presheaves

$$LX(R) = X(R((t))) \quad \text{and} \quad L^+X(R) = X(R[[t]]).$$

We note here a useful theorem regarding the representability of these objects.

Theorem 3.21 ([PR08, 1.a] or [BL94, Proposition 1.2])**.** *If X is affine of finite type then L^+X is represented by an affine scheme and LX is represented by an ind-affine scheme.*

Proof idea. The idea of the proof is to use the conditions on X to embed it as a closed subset of some $\mathbb{A}^n_k$. Then $L^+\mathbb{A}_k = \prod_{i\in\mathbb{N}} \mathbb{A}_k$ is affine, where each copy of affine space corresponds to a particular coefficient in the power series, since $L^+\mathbb{A}_k(R) = \mathbb{A}_k(R[[t]]) = R[[t]]$. Then any equation $f \in \Gamma(\mathbb{A}^n_k, \mathcal{O}_{\mathbb{A}^n_k})$ defining $X \subset \mathbb{A}^n_k$ translates into a polynomial condition on the coefficients of the various power series for $L^+X \subset L^+\mathbb{A}^n_k$.

For LX one takes a colimit bounding the size of the poles occurring in the Laurent series, along with a similar analysis to that for L^+X. □

Importantly, since G is affine and finite type we have that L^+G is representable by an affine group scheme and that LG is representable by an ind-affine scheme.

The proof of Theorem 3.21 is well illustrated by an example.

Example 3.22. Let $G = \mathrm{SO}_n$. If R is a ring then we identify $L^+\mathrm{SO}_n(R)$ with the set of matrices $(A(t), B(t)) = \left(\sum_{r\geq 0} A_r \cdot t^r, \sum_{r\geq 0} B_r \cdot t^r\right)$ subject to the conditions $A(t)B(t) = 1$, $A_r^T = B_r$ and $\det A(t) = 1$. Since the set of $A(t)$ with no conditions is clearly representable by $\mathrm{Spec}\,(k[a^r_{ij}]_{i,j\in\{1,..,n\}, r\geq 0})$ we need only show that the conditions we add are polynomial in the entries of the A_r and B_r. Indeed, there are polynomials P^r_{ij} with coefficients in $\mathbb{Z}$ such that

$$A(t)B(t)_{ij} = \sum_{r\geq 0} P^r_{ij}((A_s)_{s\leq r}, (B_s)_{s\leq r}) \cdot t^r$$

and likewise Q^r such that

$$\det A(t) = \sum_{r \leq 0} Q^r((A_s)_{s \leq r}) \cdot t^r$$

For the case of $L\mathrm{SO}_n$ we define for each $N \in \mathbb{N}$ the subfunctor $\mathrm{SO}_n^{(N)}(R)$ as the set of matrices $(A(t), B(t)) = \left(\sum_{r \geq -N} A_r \cdot t^r, \sum_{r \geq -N} B_r \cdot t^r\right)$ subject to the three same conditions. Again these are governed by polynomials with integer entries, so we conclude that $L\mathrm{SO}_n$ is ind-representable by noting that

$$L\mathrm{SO}_n(R) = \bigcup_{m=0}^{\infty} \mathrm{SO}_n^{(N)}(R).$$

The transition maps correspond simply to setting the entries of certain A_i and B_i to zero, which give closed immersions.

Now we have the major result for this section.

Theorem 3.23. *The affine Grassmannian* Gr_G *is isomorphic to the étale quotient* $[LG/L^+G]_{et}$.

The method of proof here will be to use properties of torsors to construct an alternative realisation of LG, which will have an obvious map to Gr_G, then to do some work to prove that this will identify Gr_G with the étale quotient.

We start by identifying $G(S)$ with the automorphisms of the trivial G-torsor over the base S.

Lemma 3.24. If $\mathcal{E}^0$ is the trivial G-torsor on a base S, then $\mathrm{Aut}(\mathcal{E}^0) = G(S)$.

Proof. We construct inverse maps.

Given $\phi \in \mathrm{Aut}(\mathcal{E}^0)$ consider the map:

$$S \overset{e_S}{\to} G \times S \overset{\phi}{\to} G \times S \overset{\mathrm{pr}_1}{\to} G$$

The map e here is the base change by S of the antipode $\mathrm{Spec}\,(k) \to G$. The composition is in $\mathrm{Hom}(S, G) = G(S)$.

Given $A \in G(S) = \mathrm{Hom}(S, G)$ we construct a $\phi \in \mathrm{Aut}(\mathcal{E}^0)$ via the maps

$$G \times S \overset{(id,A)}{\to} G \times G \overset{m}{\to} G$$
$$G \times S \overset{\mathrm{pr}_2}{\to} S$$

where m here is the multiplication map. The constructed map is indeed an automorphism whose inverse is given by the same construction with A^{-1}.

Perhaps more intuitively but less literally we define ϕ by $(g, s) \mapsto (gA(s), s)$. $\square$

As a result of this we can come up with an alternate realisation of the loop group via torsors. This result serves as our first indication that LG might have anything to do with torsors at all.

Corollary 3.25. *There is an isomorphism of functors*

$$LG(R) \cong \left\{ (\mathcal{E}, \beta, \varepsilon) \;\middle|\; \begin{array}{l} \mathcal{E} \textit{ is a torsor over } \mathbb{D}_R \\ \beta : \mathcal{E}|_{\mathbb{D}_R^*} \cong \mathcal{E}^0|_{\mathbb{D}_R^*} \\ \varepsilon : \mathcal{E}^0 \cong \mathcal{E} \end{array} \right\} / iso.$$

Proof. The maps between the two are $A \mapsto (\mathcal{E}^0, A, \mathrm{id})$ and $(\mathcal{E}, \beta, \varepsilon) \mapsto \beta \circ \varepsilon|_{\mathbb{D}_R^*}$, where we implicitly use Lemma 3.24.

The first composition is the identity because $A \circ \mathrm{id}|_{\mathbb{D}_R^*} = A$.

The second composition is the identity because $(\mathcal{E}, \beta, \varepsilon) \simeq (\mathcal{E}^0, \beta \circ \varepsilon|_{\mathbb{D}_R^*}, \mathrm{id})$ via ε^{-1}. □

We will also use this isomorphism to view L^+G as a subfunctor of the right hand side.

Lemma 3.26. Every G-torsor on $\mathbb{D}_R$ can be trivialised over $\mathbb{D}_{R'}$ for some étale covering $R \to R'$.

Proof. First we note $\mathcal{E}$ is smooth and affine because G is smooth and affine ([Sta23, Tag 02VL] and [Sta23, Tag 02L5]). Therefore the base change $\mathcal{E}_R \to \operatorname{Spec} R$ is also affine smooth. Hence for some étale cover $R \to R'$ we have a section [Gro67, Corollary 17.16.3 (ii)] $\operatorname{Spec} R' \to \mathcal{E}_{R'}$.

Again $\operatorname{Spec} R' \to \mathcal{E}_{R'}$ is smooth, so we can use infinitesimal lifting property to lift this to compatible sections $\operatorname{Spec} R'(t)/(t^i) \to \mathcal{E}_{R'}$. Since $\mathcal{E}_{R'}$ is affine these then glue to give us a map $\mathbb{D}_{R'} \to \mathcal{E}_{R'}$, which becomes a section of $\mathcal{E}_{\mathbb{D}_{R'}}$.

Therefore $\mathcal{E}$ can be trivialised over $\mathbb{D}_{R'}$, as required. □

Lemma 3.27. If $\mathcal{F} \to \mathcal{G}$ is a map from an étale presheaf to an étale sheaf such that:

- $\mathcal{F} \to \mathcal{G}$ is injective.
- For all $g \in \mathcal{G}(R)$ there exists an etale cover $R \to R'$ and an $f' \in \mathcal{F}(R')$ such that f' is mapped to the image of g in $\mathcal{G}(R')$.

then this map identifies $\mathcal{G}$ as the étale sheafification of $\mathcal{F}$.

Proof. We begin with a map $\mathcal{F} \to \mathscr{H}$ into a sheaf $\mathscr{H}$, and we aim to use the properties we have to factor it uniquely through our map $\mathcal{F} \to \mathcal{G}$. Let R be a ring, for a section $g \in \mathcal{G}(R)$. We want to decide where to send g in $\mathscr{H}(R)$. Choose an étale cover $R \to R'$ and an $f' \in \mathcal{F}(R')$ such that f' has the same image as g in $\mathcal{G}(R')$.

$$
\begin{tikzcd}
\mathcal{F}(R) \arrow[r] \arrow[d] \arrow[dd, bend right] & \mathcal{F}(R') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[dd, bend right] & \mathcal{F}(R' \otimes_R R') \arrow[d] \arrow[dd, bend right] \\
\mathcal{G}(R) \arrow[r] \arrow[d, dotted] \arrow[dr, dashed] & \mathcal{G}(R') \arrow[r, shift left] \arrow[r, shift right] & \mathcal{G}(R' \otimes_R R') \\
\mathscr{H}(R) \arrow[r] & \mathscr{H}(R') \arrow[r, shift left] \arrow[r, shift right] & \mathscr{H}(R' \otimes_R R')
\end{tikzcd}
$$

A diagram chase on the above diagram gives a unique $h \in \mathscr{H}(R)$ to send g to (in short it is the unique element of $\mathscr{H}(R)$ given by the image of f' in $\mathscr{H}(R')$). This process is independent of choice (another diagram chase). Our dashed arrow is not quite literal here because our diagram in fact depends on our choice of g, but it illustrates the construction of the map $\mathcal{G}(R) \to \mathscr{H}(R)$.

So by universal property we have that $\mathcal{G}$ is the étale sheafification of $\mathcal{F}$. □

We are finally in a position to prove theorem 3.23.

Proof of Theorem 3.23. Since $(\mathcal{E}, \beta, \varepsilon) \simeq (\mathcal{E}^0, \beta \circ \varepsilon|_{\mathbb{D}_R^*}, \mathrm{id})$ we may assume that any element of $LG(R)$ is of the form $(\mathcal{E}^0, A, \mathrm{id})$, where $A \in G(R((t)))$.

There is an obvious map $LG \to \mathrm{Gr}_G$ given by $(\mathcal{E}, \beta, \varepsilon) \mapsto (\mathcal{E}, \beta)$. In fact this factors through the quotient presheaf to a map $LG/L^+G \to \mathrm{Gr}_G$ because if $A \in G(R[[t]]) \subset G(R((t)))$ then the trivialisation A over $\mathbb{D}_R^*$ in fact extends to a trivialisation over $\mathbb{D}_R$, so $(\mathcal{E}^0, A, \mathrm{id}) \simeq (\mathcal{E}^0, \mathrm{id}, A^{-1})$ via A.

We now prove that this map is injective. Suppose that $(\mathcal{E}^0, A) \simeq (\mathcal{E}^0, A')$. This implies the existence of an isomorphism ϕ such that $\phi|_{\mathbb{D}_R^*} = A'^{-1} \circ A$. Translating the fact that ϕ is defined over $\mathbb{D}_R$ tells us that A and A' indeed came from the same coset of $G(R[[t]])$. Therefore the map is injective.

Note that Lemma 3.26 implies that the second dot point in Lemma 3.27 is satisfied. Applying this lemma we have that the map $LG/L^+G \to \mathrm{Gr}_G$ identifies $[LG/L^+G]$ with Gr_G.

Note that we use Theorem 3.15 with 2.6 for the fact that Gr_G is an étale sheaf. $\square$

3.4 Geometry of the Affine Grassmannian

Fix for the rest of the thesis a triple (G, B, T) of a reductive group over $\mathbb{C}$ and a Borel subgroup containing a maximal torus T.

Given our description of the affine Grassmannian as a quotient in hand, we will now use it to describe even more the geometry of this object.

If T is a choice of maximal torus then we have special points of Gr_G that can be associated to a any cocharacter $\lambda \in X_*(T)$.

Definition 3.28. If $\lambda \in X_*(T)$ then we have maps $R((t))^\times \to LG(R)$ functorial in R. Denote the image of t by t_R^λ. The collection of t_R^λ form a compatible choice of elements in $LG(R)$. In other words a functor $\operatorname{Spec} \mathbb{C} \to LG$. Therefore by composing with the quotient map we get a point in Gr_G, which we denote L_λ

Remark 3.29. Another way to define L_λ is to note that we have a composition $L_\lambda : \operatorname{Spec} \mathbb{C}((t)) \to \mathbb{G}_m \xrightarrow{\lambda} G$ given by the obvious ring map $\mathbb{C}[t, t^{-1}] \to \mathbb{C}((t))$. This composition is exactly the data of a point $G(\mathbb{C}((t))) := LG(\mathbb{C})$.

The following decomposition arises from considering the orbits of these L_λ under the left action of L^+G on Gr_G.

3.4.1 Cartan Decomposition

Proposition 3.30 (Cartan Decomposition)**.** *There is a stratification*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in X_*(T)^+} \mathrm{Gr}_G^\lambda, \qquad \textit{where} \quad \mathrm{Gr}_G^\lambda = L^+G \cdot L_\lambda$$

Gr_G^λ is an affine bundle over G/P_λ, where P_λ is the parabolic corresponding to the subset of simple roots $\{\alpha \in \Delta_s(G, B, T) : \langle \alpha, \lambda \rangle = 0\}$. In fact Gr_G^λ is a quasi-projective variety and if $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ then

$$\dim(\mathrm{Gr}_G^\lambda) = \langle 2\rho, \lambda \rangle.$$

Furthermore we have a nice description of the closures of each of the orbits as

$$\overline{\mathrm{Gr}_G^\lambda} = \bigsqcup_{\substack{\eta \in X_*(T)^+ \\ \eta \leq \lambda}} \mathrm{Gr}_G^\eta,$$

where $\leq$ is the order introduced in Equation 2.2 defined by $\lambda \geq 0$ if λ is a positive sum of positive coroots.

Remark 3.31. The closure relation implies that Gr_G^λ is locally closed since

$$\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda = \bigcup_{\alpha \in \Phi^+} \overline{\mathrm{Gr}_G^{\lambda-\alpha}}.$$

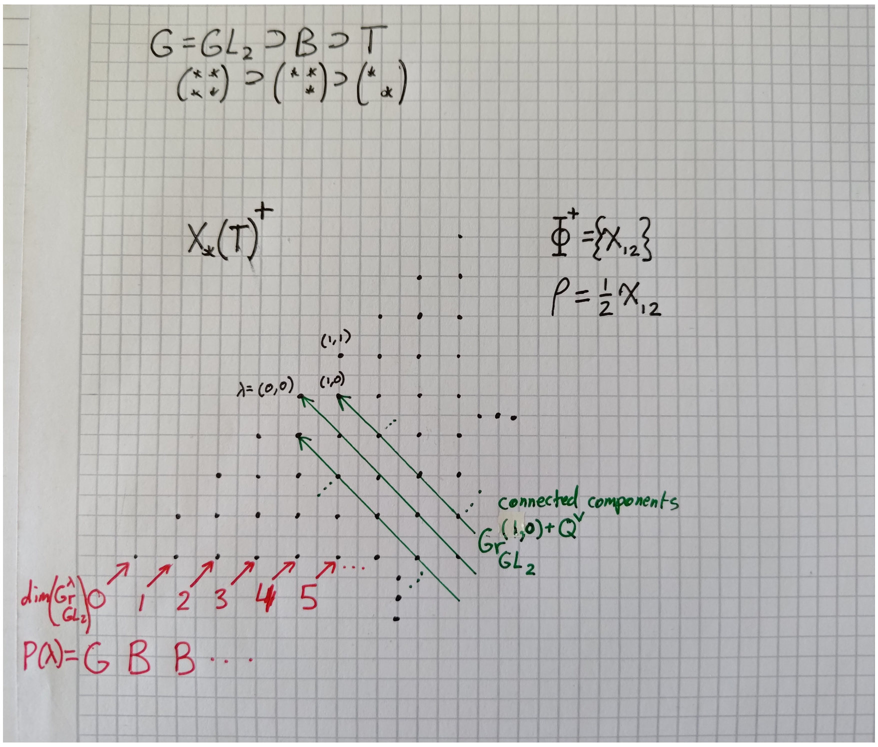
Finiteness of root data is crucial here. Furthermore, one can interpret the closure relations either in the Zariski or analytic topology by [GR02, Exposé XII, Proposition 2.2], which will be important in Chapter 4.

Remark 3.32. We also get a descriptions of the connected components of Gr_G, which are indexed by $c \in X_*(T)/Q^\vee$, where $Q^\vee$ is the coroot lattice.

$$\mathrm{Gr}_G^c := \bigsqcup_{\substack{\lambda \in X_*(T)^+ \\ \lambda + Q^\vee = c}} \mathrm{Gr}_G^\lambda$$

In general $\langle \rho, \lambda \rangle \in \mathbb{Z}$ when $\lambda \in Q^\vee$, which means that the parity of $\dim(\mathrm{Gr}_G^\lambda)$ is constant on connected components. We call a connected component even or odd according to the dimensions of its strata.

Example 3.33. This decomposition can become less abstract with an example. Let $G = GL_2$ over $\mathbb{C}$ and take the standard torus and Borel . We draw $\mathrm{Gr}_{\mathrm{GL}_2}$ as follows.



The dominant cocharacters $X_*(T)^+$ can be viewed as a half-lattice, where each point corresponds to $\mathrm{Gr}_{\mathrm{GL}_2}^\lambda$. The dimensions are exactly the first coordinate minus the second, so are constant on the anti-diagonals and increasing to the right.

We use Subsection 2.2.5 to find the parabolic subgroups corresponding the various λ. In fact, by ([Zhu16], Lemma 2.1.13) we note that $\mathrm{Gr}_{\mathrm{GL}_2}^{(i,i)} = \{*\}$ and $\mathrm{Gr}_{\mathrm{GL}_2}^{(i+1,i)} = G/B = P_{\mathbb{C}}^1$.

The connected components are given by the disjoint union of the diagonals in green. We see that every connected component consists entirely of either even or odd dimensional strata, in line with Remark 3.32. In fact each connected component has a unique stratum of every odd/even dimension.

To take the closure of a stratum one simply takes the disjoint union of all strata in the direction of the green arrow it lies on. For example

$$\overline{\mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)}} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \sqcup \mathrm{Gr}_{\mathrm{GL}_2}^{(0,0)} = \mathrm{Gr}_{\mathrm{GL}_2}^{(1,-1)} \sqcup \{*\}.$$

3.4.2 Iwasawa Decomposition

We now let N be the unipotent radical of B. The Iwasawa decomposition describes the orbits of the action of LN on $[LG/L^+G] = \mathrm{Gr}_G$.

Proposition 3.34.

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} S_\mu \qquad \textit{where} \quad S_\mu := LN \cdot L_\mu$$

Where the closure of an orbit is given by

$$\overline{S_\mu} = \bigsqcup_{\nu \leq \mu} S_\nu.$$

Remark 3.35. Analogously to Remark 3.31 one can see that then S_μ are locally closed. This gives them an ind-scheme structure.

In contrast to the Cartan Decomposition the orbits S_μ are not contained in any finite type subscheme of Gr_G.

We can also consider the analogous decomposition if we take the opposite Borel subgroup B^- and its unipotent radical N^-.

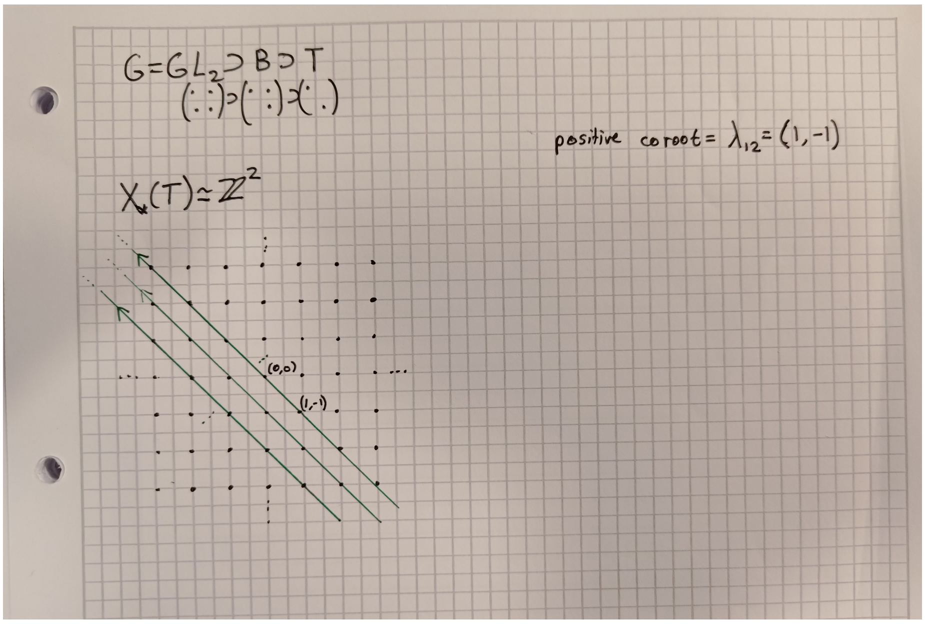
Corollary 3.36.

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)} T_\mu \qquad \textit{where} \quad T_\mu := LN^- \cdot L_\mu$$

In this case the closures are given by a dual condition

$$\overline{T_\mu} = \bigsqcup_{\nu \geq \mu} T_\nu.$$

Example 3.37. Again we can draw a picture in the case of $G = \mathrm{GL}_2$



A lattice point here represents S_μ. While the picture for T_μ would be identical except for the directions of the arrows being reversed, the lattice points themselves would correspond to different orbits.

The orbits S_μ and T_μ are in some sense maximally different in the same way that $LN \cap LN^- = I$. We encapsulate this idea in a lemma that will be used later.

Lemma 3.38. Let $\mu, \nu \in X_*(T)$. Then $\overline{S_\mu} \cap \overline{T_\nu} = \emptyset$ unless $\nu \leq \mu$ and $\overline{S_\mu} \cap \overline{T_\mu} = \{L_\mu\}$

This disjointness of orbits is not completely captured in the pictures we have given, but can be seen in Serre tree pictures of the affine Grassmannian. See [BR17, Section 3.2] for details.

3.5 Dimension Estimates

Here we prove an important theorem about the interaction between the Cartan and Iwasawa decompositions. This result will be the fountain from which the results of the final chapter flow. Recall that G is complex reductive with maximal torus T and that $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$.

Definition 3.39. For any coroot $\mu \in X_*(T)$ we define μ^+ (resp. μ^-) to be the unique dominant (resp. anti-dominant) element in the Weyl orbit of μ.

Theorem 3.40. *For $\lambda, \mu \in X_*(T)$ with λ dominant, we have the following.*

1.
$$\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu \neq \emptyset \iff L_\mu \in \overline{\mathrm{Gr}_G^\lambda} \iff \mu^+ \leq \lambda,$$

2. *If μ satisfies the condition of 1, then the intersection $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ has pure dimension $\langle \rho, \lambda + \mu \rangle$.*

3. *If μ satisfies the condition of 1, then $\mathrm{Gr}_G^\lambda \cap S_\mu$ is open dense in $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$; in particular, the irreducible components of $\mathrm{Gr}_G^\lambda \cap S_\mu$ and $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ are in canonical bijection.*

Proof. We prove each statement separately.

1. The backwards direction of the first equivalence is clear because $L_\mu \in S_\mu$ by definition.

For the forwards direction we need to realise S_μ as the attractive variety associated to the point L_μ. For $\eta \in X_*(T)^+$ strictly dominant we get an action of $\mathbb{G}_m$ on Gr_G given by conjugation whose fixed points are exactly the L_μ for $\mu \in X_*(T)$. Then we have

$$S_\mu \subset \{x \in \mathrm{Gr}_G : \eta(a)x \to L_\mu \text{ as } a \to 0\}.$$

Then the Iwasawa decomposition (Proposition 3.34) implies that is is an equality. Then the fact that $\overline{\mathrm{Gr}_G^\lambda}$ is stable by the action of T gives the forwards implication.

To formally makes sense of this attractor variety one uses the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ see [AR, Subsections 1.1.2 and 1.1.3] for details. We can however show this informally for the case of $G = \mathrm{GL}_2$ (the GL_n case is similar but the notation becomes cumbersome).

Take $\eta \in X_*(T)^+$, which determines a map

$$\mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{C}((t)))/\mathrm{GL}_2(\mathbb{C}[[t]]),\ a \mapsto \begin{pmatrix} a^{n_1} & 0 \\ 0 & a^{n_2} \end{pmatrix}$$

with $n_1 > n_2$.

Now for all elements of LN the action of $\eta(a)$ is given by:

$$\begin{pmatrix} a^{n_1} & 0 \\ 0 & a^{n_2} \end{pmatrix} \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-n_1} & 0 \\ 0 & a^{-n_2} \end{pmatrix} = \begin{pmatrix} 1 & a^{n_1-n_2} f(t) \\ 0 & 1 \end{pmatrix}$$

Which tends to I as $a \to 0$ (informally). Therefore $LN \to I$ as $a \to 0$ and so $S_\mu = LN \cdot L_\mu \to L_\mu$.

For the second equivalence we need the fact that

$$W\lambda \subset \{\mu \in X_*(T) : L_\mu \in \overline{\mathrm{Gr}_G^\lambda}\}.$$

See [BD, 4.5.8.] for details. By the Cartan decomposition (Proposition 3.30) this is in fact an equality because each Weyl orbit has a unique dominant representative.

Therefore we have that $L_\mu \in \overline{\mathrm{Gr}_G^\lambda}$ if and only if the dominant weyl conjugate μ^+ of μ satisfies $\mu^+ \le \lambda$.

2. Here we show a different, faster proof than the one given in [BR17, 5.2]. It begins by noting from 1 that

$$\overline{\mathrm{Gr}_G^\lambda} = \bigcup_{\nu\,:\, S_\nu \cap \overline{\mathrm{Gr}_G^\lambda} \neq \emptyset} S_\nu \cap \overline{\mathrm{Gr}_G^\lambda}$$

is a finite union, as it is the union over ν such that $\nu^+ \le \lambda$.

From the closure relations in the Iwasawa decomposition (Proposition 3.34) we have for all $d \in \mathbb{Z}$ that

$$H(d) := \bigcup_{\nu\,:\, \langle 2\rho, \nu\rangle \le d} S_\nu \cap \overline{\mathrm{Gr}_G^\lambda} \tag{3.1}$$

is closed in $\overline{\mathrm{Gr}_G^\lambda}$. Furthermore $H(d) \subset H(e)$ is closed for all $d \le e$. The proof strategy is to analyse $\dim(H(d))$ for various d in order to conclude.

If $d = -\langle 2\rho, \lambda\rangle$ we have

$$H(-\langle 2\rho, \lambda\rangle) = S_{\lambda^-} \cap \overline{\mathrm{Gr}_G^\lambda},$$

where λ^- is the unique anti-dominant element in $W\lambda$. By 1 we have that S_ν only meets $\overline{\mathrm{Gr}_G^\lambda}$ if $\nu \leq \lambda$, so $\overline{\mathrm{Gr}_G^\lambda} \subset \overline{S_\lambda}$. If we take w_0 to be a lift of the longest element in the Weyl group, such that $w_0\lambda = \lambda^-$, we can conjugate this inclusion by w_0 to get $\overline{\mathrm{Gr}_G^\lambda} \subset \overline{T_{\lambda^-}}$. Using these inclusions along with Lemma 3.38 we get

$$S_{\lambda^-} \cap \overline{\mathrm{Gr}_G^\lambda} \subset \overline{S_{\lambda^-}} \cap \overline{\mathrm{Gr}_G^\lambda} \subset \overline{S_{\lambda^-}} \cap \overline{T_{\lambda^-}} \overset{3.38}{=} \{L_{\lambda^-}\}.$$

By 1, $H(\langle 2\rho, \lambda\rangle) = \overline{\mathrm{Gr}_G^\lambda}$. Therefore

$$\dim(H(-\langle 2\rho, \lambda\rangle)) = 0 \text{ and } \dim(H(\langle 2\rho, \lambda\rangle)) = \dim(\overline{\mathrm{Gr}_G^\lambda}) = \langle 2\rho, \lambda\rangle.$$

We have a descending chain of closed subsets

$$H(\langle 2\rho, \lambda\rangle) \supset H(\langle 2\rho, \lambda\rangle - 2) \supset \cdots \supset H(-\langle 2\rho, \lambda\rangle)$$

By [AR, Remark 1.2.9] the open complements

$$H(d) \setminus H(d-2) = \bigsqcup_{\nu : \langle 2\rho, \nu\rangle = d} S_\nu \cap \overline{\mathrm{Gr}_G^\lambda}$$

are affine schemes of finite type over $\mathbb{C}$. It is important here that $S_\nu \cap \overline{\mathrm{Gr}_G^\lambda}$ is only non-empty for finitely many ν. Now, applying [Sta23, Tag 0BCV] and [GW10, Proposition 5.30] we see that the open complements being affine inside a locally Noetherian scheme (because $H(d)$ is finite type affine over $\mathbb{C}$) means the dimension can drop by at most one at each step. However we know from the cases we calculated that it drops by $\langle 2\rho, \lambda\rangle$ in $\langle 2\rho, \lambda\rangle$ steps. Therefore the dimension drops by exactly one at each step. In particular

$$\dim H(\langle 2\rho, \lambda\rangle - 2d) = \langle 2\rho, \lambda\rangle - d, \quad \text{for } d \in \{0, ..., \langle \rho, \lambda\rangle\},$$

and even stronger $H(\langle 2\rho, \lambda\rangle - 2d) \setminus H(\langle 2\rho, \lambda\rangle - 2d - 2)$ is equidimensional of the same dimension.

Therefore since $S_\mu \cap \overline{\mathrm{Gr}_G^\lambda} \subset H(\langle 2\rho, \mu\rangle) \setminus H(\langle 2\rho, \mu\rangle - 2)$ we have that $\dim S_\mu \cap \overline{\mathrm{Gr}_G^\lambda}$ is equidimensional of dimension $\langle \rho, \lambda + \mu\rangle$. (Set $d = \langle \rho, \lambda - \mu\rangle$).

3. Openness is a consequence of Remark 3.31.

To prove openness it suffices to prove that $\mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}_G^\lambda}$ is open, and indeed

$$\overline{\mathrm{Gr}_G^\lambda} \setminus \mathrm{Gr}_G^\lambda = \bigcup_{\alpha \in \Phi^+} \overline{\mathrm{Gr}_G^{\lambda - \alpha}}.$$

To prove density let $Z \subset \overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ be an irreducible component. It suffices to prove that Z meets $\mathrm{Gr}_G^\lambda \cap S_\mu$. Suppose not, then $Z \subset H(\langle 2\rho, \mu\rangle - 2)$, so

$$\langle \rho, \lambda + \mu\rangle = \dim Z \leq \langle \rho, \lambda + \mu\rangle - 1,$$

which is a contradiction. □

There is a corresponding dual result for the T_μ which mutatis mutandis has the same proof as above. Recall $\overline{T_\mu}$ is the union of T_ν for ν greater than μ, not less, as in the case of S_μ. Therefore the descending chain of closed subsets in the proof of Theorem 3.40 runs in the opposite direction. This is the cause of $\overline{\mathrm{Gr}_G^\lambda} \cap T_\mu$ having a different dimension to $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$.

Corollary 3.41. *Let $\lambda, \mu \in X_*(T)$ with λ dominant, we have the following.*

1. $$\overline{\mathrm{Gr}_G^\lambda} \cap T_\mu \neq \emptyset \iff L_\mu \in \overline{\mathrm{Gr}_G^\lambda} \iff \mu^+ \leq \lambda,$$

2. *If μ satisfies the condition of 1, then the intersection $\overline{\mathrm{Gr}_G^\lambda} \cap T_\mu$ has pure dimension $\langle \rho, \lambda - \mu \rangle$.*

3. *If μ satisfies the condition of 1, then $\mathrm{Gr}_G^\lambda \cap T_\mu$ is open dense in $\overline{\mathrm{Gr}_G^\lambda} \cap T_\mu$; in particular, the irreducible components of $\mathrm{Gr}_G^\lambda \cap T_\mu$ and $\overline{\mathrm{Gr}_G^\lambda} \cap T_\mu$ are in canonical bijection.*

Chapter 4

Total Cohomology

Throughout this chapter we fix $\Bbbk$ a field of characteristic 0. A $\Bbbk$-sheaf is a sheaf of $\Bbbk$-modules, hence $\Bbbk$-vector spaces.

4.1 Perverse Sheaves

The aim of this section is to introduce the category of perverse sheaves over $\Bbbk$ and some of its basic properties.

In this section fix X a topological space with stratification $\mathscr{S}$ satisfying certain technical conditions. The conditions ensure that for a locally closed subset $h : Y \subset X$, the functors $h_*, h^*, h_!$ and $h^!$ preserve constructibility (see [Ach21, Lemma 2.3.22]).

Remark 4.1. For the case of the affine Grassmannian with the Cartan decomposition these technical conditions are not a problem because we consider the category of perverse sheaves on Gr_G as the union of categories of of perverse sheaves on the complex projective varieties $\overline{\mathrm{Gr}_G^\lambda}$.

We start by defining $D^b_{\mathscr{S}}(X, \Bbbk)$, the bounded derived category of $\Bbbk$-sheaves constructible with respect to $\mathscr{S}$, which arises as a modification of $D^b(X, \Bbbk)$, the bounded derived category of $\Bbbk$-sheaves on X. The category of perverse sheave will then come from a t-structure on this derived category.

Definition 4.2. A local system on a topological space is a locally free sheaf of finite rank.

Definition 4.3. A bounded complex of $\Bbbk$-sheaves $\mathcal{F} \in D^b(X, \Bbbk)$ is constructible with respect to $\mathscr{S}$ if for all strata $i_S : S \hookrightarrow X$ and for all $n \in \mathbb{Z}$ the restriction $i_S^* \mathscr{H}^n \mathcal{F}$ is a local system.

Definition 4.4. We denote by $D^b_{\mathscr{S}}(X, \Bbbk)$ the bounded derived category of sheaves of $\Bbbk$-vector spaces which are constructible with respect to $\mathscr{S}$.

In short we wish to consider complexes of $\Bbbk$-sheaves such that the restriction to any stratum becomes locally free.

The category $D^b_{\mathscr{S}}(X, \Bbbk)$ has the standard t-structure $(D^{\leq 0}_{\mathscr{S}}(X, \Bbbk), D^{\geq 0}_{\mathscr{S}}(X, \Bbbk))$ defined by the grades in which the cohomology sheaves are zero ([Ach21, Example A.7.2.]). We denote by $\tau_{\leq i}$ the projection functors $D^b_{\mathscr{S}}(X, \Bbbk) \to D^{\leq i}_{\mathscr{S}}(X, \Bbbk)$. The category we are looking to define however needs a different t-structure for its definition.

Definition 4.5 ([BBD82, Corollary 2.1.4.])**.** The perverse t-structure on $D^b_{\mathscr{S}}(X, \Bbbk)$ is

$$ {}^pD^{\leq 0}_{\mathscr{S}}(X, \Bbbk) := \{\mathcal{F} \in D^b_{\mathscr{S}}(X, \Bbbk) : \forall S \in \mathscr{S}, \forall n > -\dim S, \mathscr{H}^n(i_S^* \mathcal{F}) = 0\} $$
$$ {}^pD^{\geq 0}_{\mathscr{S}}(X, \Bbbk) := \{\mathcal{F} \in D^b_{\mathscr{S}}(X, \Bbbk) : \forall S \in \mathscr{S}, \forall n < -\dim S, \mathscr{H}^n(i_S^! \mathcal{F}) = 0\}. $$

We define $P_{\mathscr{S}}(X, \Bbbk)$, the category of perverse sheaves on X (constructible with respect to $\mathscr{S}$) as the heart of this t-structure. Therefore $P_{\mathscr{S}}(X, \Bbbk)$ is an abelian category.

The rest of this section will be spent describing what we know about the structure of this abelian category.

By [BBD82, Theorem 4.3.1] $P_{\mathscr{S}}(X, \Bbbk)$ is Artinian and Noetherian [1]. Equivalently ([Sta23, Tag 0FCJ]) every object $\mathcal{F} \in P_{\mathscr{S}}(X, \Bbbk)$ has a filtration

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n = \mathcal{F},$$

such that all $\mathcal{F}_{i+1}/\mathcal{F}_i$ are simple.

Therefore it is natural to try to understand the simple objects of $P_{\mathscr{S}}(X, \Bbbk)$ and their extensions. In fact the simple objects are classified in the following way.

Definition 4.6. Given a stratum $S \in \mathscr{S}$ and a simple local system $\mathcal{L}$ on S we define the intersection cohomology complex $\mathrm{IC}(S, \mathcal{L})$ as the unique object $\mathcal{F} \in P_{\mathscr{S}}(X, \Bbbk)$ satisfying the following:

- $\mathcal{F}|_{X \setminus \overline{S}} = 0$,
- $\mathcal{F}|_S = \mathcal{L}[\dim S]$,
- $i^*\mathcal{F} \in {}^pD^{\leq -1}(\overline{S} \setminus S, \Bbbk)$,
- $i^!\mathcal{F} \in {}^pD^{\geq 1}(\overline{S} \setminus S, \Bbbk)$.

Where $\overline{S}$ is the closure of S and $i : \overline{S} \setminus S \hookrightarrow X$.

Proposition 4.7 ([Ach21, Theorem 3.4.5.])**.** *When $\Bbbk$ is a field the simple objects in $P_{\mathscr{S}}(X, \Bbbk)$ are exactly the intersection cohomology complexes $\mathrm{IC}(S, \mathcal{L})$.*

In general the extensions between objects are difficult to understand, but in the case when X is the affine Grassmannian we can say more.

One of our principal tools when working with perverse sheaves will be applying cohomological functors to distinguished triangles. The following proposition accounts for a very important class of distinguished triangle, that will be used many times in the chapter.

Theorem 4.8 ([Ach21, Theorem 1.3.10] Recollement Triangles)**.** *Let $i : Z \hookrightarrow X$ be a closed embedding and $j : U \hookrightarrow X$ the complementary open embedding. Then for any $\mathcal{F} \in D^b(X, \Bbbk)$, we have two distinguished triangles.*

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_!i^*\mathcal{F} \to$$
$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^!\mathcal{F} \to$$

known as the recollement triangles associated to Z and U.

4.2 Perverse Sheave on the Affine Grassmannian

For the rest of the chapter fix $\mathscr{S}$ to be the stratification from the Cartan Decomposition. In this section we outline properties of the category $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$.

Remark 4.9. Whenever we consider sheaves on Gr_G we implicitly use the analytic topology, but when we consider properties such as dimension or irreducible components we use the Zariski topology. It should be clear from context which topology is being considered.

[1] An object in an abelian category is Artinian (resp. Noetherian) if every decreasing (resp. increasing) chain of sub-objects terminates. An abelian category is Artinian or Noetherian if every object is.

We know from Proposition 3.30 that each individual stratum $\mathrm{Gr}_G^\lambda \in \mathscr{S}$ is an affine bundle over G/P_λ. We start this chapter by analysing the topology of these strata.

Proposition 4.10. *For any* $\mathrm{Gr}_G^\lambda \in \mathscr{S}$ *we have* $\pi_1(\mathrm{Gr}_G^\lambda) = 0$ *for the usual fundamental group and the analytic topology on* Gr_G*.*

Proof. Since Gr_G^λ is an affine bundle we have a fibration $\mathrm{Gr}_G^\lambda \to G/P_\lambda$ with simply connected fibres (homeomorphic to $\mathbb{C}^n$ for some n). Therefore the long exact sequence on homotopy groups yields

$$0 = \pi_1(\mathbb{C}^n) \to \pi_1(\mathrm{Gr}_G^\lambda) \to \pi_1(G/P_\lambda) \to \pi_0(\mathbb{C}^n) = 0.$$

Therefore it suffices to prove that $\pi_1(G/P_\lambda) = 0$. Now consider the fibration given by $G \to G/P_\lambda$. This gives us

$$\pi_1(P_\lambda) \to \pi_1(G) \to \pi_1(G/P_\lambda) \to \pi_0(P_\lambda) = 0$$

Therefore $\pi_1(G)$ surjects onto $\pi_1(G/P_\lambda)$. The fundamental group of a topological group is abelian because for any two loops α, β we have that $\alpha * \beta \sim \beta * \alpha$ via the homotopy $F(s,t) := \alpha(s)\beta(t)$. Therefore $\pi_1(G/P_\lambda)$ is abelian, so

$$\pi_1(G/P_\lambda) \cong H_1(G/P_\lambda; \mathbb{Z}).$$

Finally, Bruhat decomposition [Mil17, Theorem 22.67] tells us that G/P_λ is paved by copies of $\mathbb{A}^1_{\mathbb{C}}$. Therefore $H_1(G/P_\lambda; \mathbb{Z}) = 0$, as required. □

Given the equivalence between local systems and representations of the fundamental group [Ach21, Theorem 1.7.9.] we see that the only local systems on Gr_G^λ are the constant sheaves $\underline{\Bbbk}^n_{\mathrm{Gr}_G^\lambda}$, of which only $\underline{\Bbbk}_{\mathrm{Gr}_G^\lambda}$ is simple. In light of this and Proposition 4.7 we see that the simple objects of $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ are indexed by $\lambda \in X_*(T)$

Notation 4.11. As a shorthand notation we define $\mathrm{IC}_\lambda := \mathrm{IC}(\mathrm{Gr}_G^\lambda, \Bbbk)$.

Now that we understand well the simple objects of $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ we turn our attention to extensions of these simple objects.

Theorem 4.12. *The category* $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ *is semisimple. That is, all objects are a finite direct sum of intersection cohomology complexes.*

Proof Sketch. It is necessary to prove that there is a unique extension between any two $\mathrm{IC}_\lambda, \mathrm{IC}_\mu$ for $\lambda, \mu \in X_*(T)^+$, which is equivalent by [Ach21, Proposition A.7.11] to proving that

$$\mathrm{Hom}_{D^b_{\mathscr{S}}}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu[1]) = 0.$$

This can be broken into three cases:

$$\lambda = \mu, \quad \mathrm{Gr}_G^\lambda \cap \overline{\mathrm{Gr}_G^\mu} = \mathrm{Gr}_G^\mu \cap \overline{\mathrm{Gr}_G^\lambda} = \emptyset \quad \text{and} \quad \mathrm{Gr}_G^\lambda \subset \overline{\mathrm{Gr}_G^\mu} \text{ or } \mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G^\lambda}.$$

The third case uses the fact [BR17, Lemma 4.5] that the cohomology sheaves $\mathscr{H}^n(\mathrm{IC}_\lambda)$ vanish unless n is of the same parity as Gr_G^λ.

It is clear that this holds if Gr_G^λ and Gr_G^μ have different parities, as IC_λ and IC_μ are supported on distinct connected components.

One can also use the work of [JMW14, Section 2] which requires $\Bbbk$ to be characteristic zero. □

Remark 4.13. $\operatorname{supp} \mathrm{IC}_\lambda = \overline{\mathrm{Gr}_G^\lambda}$ is compact (since $\overline{\mathrm{Gr}_G^\lambda}$ is projective). Therefore in light of Theorem 4.12 $\operatorname{supp} \mathscr{A}$ is compact for any $\mathscr{A} \in P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ because it is the finite union of compacts.

The semisimplicity of $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ will typically be used via the following corollary.

Corollary 4.14. *Any additive functor $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk) \to \mathcal{C}$ is automatically exact.*

Proof. Since $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ is semisimple all exact sequences split, and so additivity is equivalent to exactness. $\square$

There are various ways to define the category of L^+G-equivariant perverse sheaves. One way is to take the heart of the perverse t-structure on the category of constructible equivariant sheaves, as in [BL06]. Another is the more direct approach we take here, whose equivalence is proven in [BR17, A.1.]

Definition 4.15. Given X a complex variety and H a complex algebraic group acting on X let $p, a : G \times X \to X$ be the projection and action map respectively. The category $P_H(X, \Bbbk)$ of H-equivariant perverse sheaves on X is the full subcategory consisting of perverse sheaves $\mathcal{F}$ such that there is an isomorphism $p^*\mathcal{F} \cong a^*\mathcal{F}$.

Remark 4.16. Again we are using the fact the the category of perverse sheaves on the affine Grassmannian is really the union of the categories of perverse sheaves on the $\overline{\mathrm{Gr}_G^\lambda}$.

Corollary 4.17. *The forgetful functor $P_{L^+G}(\mathrm{Gr}_G, \Bbbk) \to P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ is an equivalence of abelian categories.*

Proof. The functor is fully faithful by construction, exact by Corollary 4.14 and the simple objects IC_λ of $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$ are all in the essential image. Therefore by semisimplicity the functor is essentially surjective. $\square$

We will use this corollary to justify treating objects of $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$ as if they came from $P_{\mathscr{S}}(\mathrm{Gr}_G, \Bbbk)$.

4.3 Local Cohomology

We start this section with a definition.

Definition 4.18. For $i : Y \hookrightarrow X$ the inclusion of a locally closed subspace of a topological space and $\mathcal{F} \in D^b(X, \Bbbk)$ we define the local cohomology groups $H^k_Y(X, \mathcal{F}) := H^k(Y, i^!\mathcal{F})$.

The aim of this section is to analyse the local cohomology groups, and cohomology with compact support of the locally closed subsets given in the Iwasawa Decomposition. Our work will culminate in the following proposition.

Proposition 4.19. *For $\mathscr{A} \in P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$, $\mu \in X_*(T)$ and $k \in \mathbb{Z}$, there exists a canonical isomorphism*

$$H^k_{T_\mu}(\mathrm{Gr}_G, \mathscr{A}) \xrightarrow{\sim} H^k_c(S_\mu, \mathscr{A}),$$

and both terms vanish if $k \neq \langle 2\rho, \mu \rangle$.

The strategy is to prove that $H^k_c(S_\mu, \mathscr{A}) = 0$ for $k > \langle 2\rho, \mu \rangle$ and $H^k_{T_\mu}(\mathrm{Gr}_G, \mathscr{A}) = 0$ for $k < \langle 2\rho, \mu \rangle$. We will begin with a critical vanishing lemma that uses the interactions between the Cartan and Iwasawa decompositions. The desired vanishing of the proposition will come after filtering S_μ by the Gr_G^λ. The second assertion can be proven analogously.

Lemma 4.20. For $\mathscr{A} \in P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$, $\mu \in X_*(T)$ and $i \in \mathbb{Z}$

$$H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{A}) = 0$$

for all $i > \langle 2\rho, \mu\rangle$.

Proof of Lemma 4.20. Let $\lambda \in X_*(T)^+$, since $\mathscr{A} \in {}^pD^{\leq 0}(\mathrm{Gr}_G, \Bbbk)$ we have that $\mathscr{H}^i(\mathscr{A}|_{\mathrm{Gr}^\lambda_G}) = 0$ for all $i > -\langle 2\rho, \lambda\rangle$. In particular $\mathscr{A}|_{\mathrm{Gr}^\lambda_G} \in D^{\leq -\langle 2\rho,\lambda\rangle}(\mathrm{Gr}^\lambda_G, \Bbbk)$, which will be used in Equation 4.1.

Consider $H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu; \Bbbk)$, we know this is 0 for all $i > \langle 2\rho, \lambda + \mu\rangle$ because of the dimension estimates of Theorem 3.40 and [Ach21, Theorem 2.7.4.]. Fix now any $i > \langle 2\rho, \mu\rangle$.

For $j > \langle 2\rho, \lambda\rangle$ we have the distinguished triangle

$$\tau_{\leq -j-1}\mathscr{A} \to \tau_{\leq -j}\mathscr{A} \to \mathscr{H}^{-j}(\mathscr{A})[j] \xrightarrow{\Sigma} .$$

Applying the cohomological functor $H^\bullet_c(\mathrm{Gr}^\lambda_G \cap S_\mu, -)$ we get the following exact sequence

$$\cdots H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \tau_{\leq -j-1}\mathscr{A}) \to H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \tau_{\leq -j}\mathscr{A}) \to H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{H}^{-j}(\mathscr{A})[j]) \cdots .$$

Since $\mathscr{A}$ is $\mathscr{S}$-constructible we have that

$$H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{H}^{-j}(\mathscr{A})[j]) \cong H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \underline{\Bbbk}^{c_0}[j]) \cong H^{i+j}_c(\mathrm{Gr}^\lambda_G \cap S_\mu; \underline{\Bbbk}^{c_0}) \cong 0,$$

where the last isomorphism comes from the fact that $i + j > \langle 2\rho, \lambda + \mu\rangle$. The result of this for various $j > \langle 2\rho, \lambda\rangle$ gives a sequence of surjections

$$H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \tau_{\leq -j}\mathscr{A}) \twoheadrightarrow \cdots \twoheadrightarrow H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \tau_{\leq -\langle 2\rho,\lambda\rangle}\mathscr{A}) = H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{A}) \quad (4.1)$$

By choosing j large enough that $\tau_{\leq -j}\mathscr{A} = 0$ we see that 0 surjects onto the right hand side, and so $H^i_c(\mathrm{Gr}^\lambda_G \cap S_\mu, \mathscr{A}) = 0$ for all $i > \langle 2\rho, \mu\rangle$. □

Remark 4.21. So far (and indeed in what follows) it is only necessary that $\dim(S_\mu \cap \overline{\mathrm{Gr}^\lambda_G}) = \langle \rho, \lambda + \mu\rangle$, with no need of purity. The purity of the dimension is important in the dual argument involving the T_μ. One uses the purity of dimension in Corollary 3.41 to conclude that $\overline{\mathrm{Gr}^\lambda_G} \cap T_\mu \subset \overline{\mathrm{Gr}^\lambda_G}$ is of codimension at least $\langle 2\rho, \lambda\rangle - \langle \rho, \lambda - \mu\rangle = \langle \rho, \lambda + \mu\rangle$. Combined with [Ive86, Theorem X.2.1.] it yields the fact that

$$H^i_{T_\mu \cap \overline{\mathrm{Gr}^\lambda_G}}(\overline{\mathrm{Gr}^\lambda_G}; \Bbbk) = 0 \quad \text{for all } i < \langle 2\rho, \lambda + \mu\rangle.$$

Instead of the induction generating a sequence of surjections starting at the zero object one gets a string of injections terminating at the zero object.

With this lemma in hand we are ready to prove Proposition 4.19.

Proof of Proposition 4.19. The isomorphism is given by Braden's Hyperbolic localisation theorem [Bra03, Theorem 1 & Equation 1]. In fact, a slight generalisation is needed [DG14] because the original paper assumes a group acting on a normal variety. Details for the case of the affine Grassmannian can be found in [Xue17, §1.8.1 (iii). Page 48]. We need only prove the vanishing.

To conclude we begin by proving that $H^k_c(\overline{\mathrm{Gr}^\lambda_G} \cap S_\mu, \mathscr{A}) = 0$ for all $k > \langle 2\rho, \mu\rangle$. Consider an order on the dominant cocharacters such that $\lambda_i \leq \lambda_j \implies i \leq j$, and let i_1 be the smallest integer such that and $\mathrm{Gr}_G^{\lambda_{i_1}} \cap S_\mu \neq \emptyset$. Then this intersection is a closed subset of $\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu$ and we have the following inclusion of a closed set and its complement

$$\mathrm{Gr}_G^{\lambda_{i_1}} \cap S_\mu \xhookrightarrow{i} \overline{\mathrm{Gr}_G^\lambda} \cap S_\mu \xhookleftarrow{j} (\mathrm{Gr}_G^{\lambda_{i_1}})^c \cap \overline{\mathrm{Gr}_G^\lambda} \cap S_\mu,$$

from which we form the distinguished triangle (see [Ach21, Theorem 1.3.10])

$$j_! j^* \mathscr{A} \to \mathscr{A} \to i_! i^* \mathscr{A} \xrightarrow{\Sigma} .$$

Taking cohomology with compact support we get the long exact sequence

$$\cdots \to H_c^k((\mathrm{Gr}_G^{\lambda_{i_1}})^c \cap \overline{\mathrm{Gr}_G^\lambda} \cap S_\mu, \mathscr{A}) \to H_c^k(\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu, \mathscr{A}) \to H_c^k(\mathrm{Gr}_G^{\lambda_i} \cap S_\mu, \mathscr{A}) \to \cdots .$$

By Lemma 4.20 we have that $H_c^k(\mathrm{Gr}_G^{\lambda_i} \cap S_\mu, \mathscr{A}) = 0$ and thus a surjection between the first two terms. Since there are finitely many λ_i such that $\mathrm{Gr}_G^{\lambda_i} \cap S_\mu \neq \emptyset$ we can continue by induction to see that $H_c^k(\overline{\mathrm{Gr}_G^\lambda} \cap S_\mu, \mathscr{A}) = 0$.

Finally, using Theorem 3.40 we can choose finitely many $\lambda \in X_*(T)^+$ such that

$$S_\mu = \bigsqcup_\lambda \overline{\mathrm{Gr}_G^\lambda} \cap S_\mu.$$

Hence $H_c^k(S_\mu, \mathscr{A}) = 0$ for all $k > \langle 2\rho, \mu \rangle$, as required. □

4.4 Total Cohomology

In this section we prove that the total cohomology functor decomposes as the sum of the local cohomology functors associated to the orbits in the Iwasawa decomposition, which are parameterised by the cocharacters of T.

Inspired by the proposition of the previous section we define the following functors F_μ for any $\mu \in X_*(T)$.

Definition 4.22. Define $F_\mu : P_{L^+G}(\mathrm{Gr}_G, \Bbbk) \to \mathbf{Vect}_\Bbbk$ by

$$F_\mu(\mathscr{A}) := H_{T_\mu}^{\langle 2\rho, \mu \rangle}(\mathrm{Gr}_G, \mathscr{A}) \cong H_c^{\langle 2\rho, \mu \rangle}(S_\mu, \mathscr{A}).$$

Remark 4.23. By Proposition 4.19 this is the same as taking total cohomology of the corestricted sheaf (for T_μ) or total cohomology with compact support of the restricted sheaf (for S_μ).

Theorem 4.24. *There exists a canonical isomorphism of functors $P_{L^+G}(\mathrm{Gr}_G, \Bbbk) \to$ $\mathbf{Vect}_\Bbbk$*

$$H^\bullet(\mathrm{Gr}_G, \mathscr{A}) \cong \bigoplus_{\mu \in X_*(T)} F_\mu(\mathscr{A}).$$

Furthermore, this functor is faithful and exact.

Proof. Let $\mathscr{A} \in P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$, our aim is to construct a canonical isomorphism

$$H^\bullet(\mathrm{Gr}_G, \mathscr{A}) \cong \bigoplus_{\mu \in X_*(T)} F_\mu(\mathscr{A}).$$

We will establish this grade by grade. In particular we will prove for all $k \in \mathbb{Z}$ that

$$H^k(\mathrm{Gr}_G, \mathscr{A}) \cong \bigoplus_{\mu \in X_*(T)\langle 2\rho, \mu\rangle = k} F_\mu(\mathscr{A}).$$

Since the functors are additive we can suppose that $\mathscr{A}$ is indecomposable. In particular that $\operatorname{supp} \mathscr{A}$ is connected. (A disconnection of the support of $\mathscr{A}$ gives rise to a decomposition $\mathscr{A} = j_{U!}\mathscr{A}|_U \oplus j_{V!}\mathscr{A}|_V$.)

We now define for any $n \in \frac{1}{2}\mathbb{Z}$ the locally closed subset

$$Z_n := \bigsqcup_{\substack{\mu \in X_*(T) \\ \langle \rho, \mu \rangle = n}} T_\mu.$$

The unions

$$\bigcup_{n \in \mathbb{Z}} Z_n \qquad \text{and} \qquad \bigcup_{n \in \frac{1}{2} + \mathbb{Z}} Z_n$$

consist of connected components of Gr_G (Remark 3.32). Now since we have assumed that $\operatorname{supp} \mathscr{A}$ is connected the support lies in one of these sets. We suppose that it is contained in the first, the reasoning for the other case is similar.

When Z_n is given the subspace topology from Gr_G it becomes a topological disjoint union of the relevant T_μ. Therefore $H^k_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) = \bigoplus_{\langle \rho, \mu \rangle = n} H^k_{T_\mu}(\mathrm{Gr}_G, \mathscr{A})$. As a result of this we get

$$H^k_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) = \begin{cases} 0 & \text{if } k \neq 2n \\ \bigoplus\limits_{\langle \rho, \mu \rangle = n} F_\mu(\mathscr{A}) & \text{if } k = 2n \end{cases} \tag{4.2}$$

Now recalling the closure relations of the T_μ in Corollary 3.36 we note that

$$\overline{Z_n} = Z_n \sqcup Z_{n+1} \sqcup Z_{n+2} \sqcup \cdots = Z_n \sqcup \overline{Z_{n+1}},$$

which gives a diagram of complementary open and closed immersions

$$\overline{Z_{n+1}} \overset{i}{\hookrightarrow} \overline{Z_n} \overset{j}{\hookleftarrow} Z_n.$$

Applying the functor $H(\overline{Z_n}, -)$ to the recollement triangle

$$i_* i^! \mathscr{A}_n \to \mathscr{A}_n \to j_* j^! \mathscr{A}_n \xrightarrow{\Sigma}$$

where $\mathscr{A}_n$ is the shriek pull-back of $\mathscr{A}$ to $\overline{Z}_n$ we get a long exact sequence whose terms look like

$$\cdots \to H^k_{\overline{Z_{n+1}}}(\mathrm{Gr}_G, \mathscr{A}) \to H^k_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) \to H^k_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) \to H^{k+1}_{\overline{Z_{n+1}}}(\mathrm{Gr}_G, \mathscr{A}) \to \cdots. \tag{4.3}$$

Now, $\mathscr{A}$ is a finite direct sum of IC_λ for various lambda. The support of each of these is $\overline{\mathrm{Gr}^\lambda_G}$, which is compact because it is projective [Zhu16, Proposition 2.1.5(2)]. Therefore $\operatorname{supp} \mathscr{A}$ is compact, and so there exists some sufficiently large $N \in \mathbb{Z}$ such that $\operatorname{supp} \mathscr{A} \cap Z_N = \emptyset$. We will now prove the following claims, using a descending induction on n for the first two.

1. $H^k_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) = 0$ for k odd or $n > \frac{k}{2}$;
2. $H^k_{\overline{Z_{k/2}}}(\mathrm{Gr}_G, \mathscr{A}) \cong H^k_{Z_{k/2}}(\mathrm{Gr}_G, \mathscr{A})$ for k even and $n \leq \frac{k}{2}$;
3. $H^k_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) \cong H^k_{\overline{Z_{k/2}}}(\mathrm{Gr}_G, \mathscr{A})$ for k even and $n \leq \frac{k}{2}$.

For 1. suppose k is odd or $n > \frac{k}{2}$ the first term of 4.3 vanishes by induction and the third term vanishes by 4.2 giving the result.

For 2. let $n = k/2$ with k even in 4.3. The first term of 4.3 vanishes by 1. because $k/2 + 1 > k/2$. The last term vanishes by 1. because $k + 1$ is odd. This gives the desired isomorphism.

For 3. we claim that $H^k_{\overline{Z_{n+1}}}(\mathrm{Gr}_G, \mathscr{A}) \to H^k_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A})$ in 4.3 is an isomorphism for k even and $n + 1 \leq k/2$. We know that $H^{k-1}_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) = 0$ because $k - 1$ is odd. Meanwhile $H^k_{Z_n}(\mathrm{Gr}_G, \mathscr{A}) = 0$ by 1. because $n < k/2$, giving us the isomorphism by exactness.

Now using compactness of $\operatorname{supp} \mathscr{A}$ again we may choose some n small enough that $\operatorname{supp} \mathscr{A} \subset \overline{Z_n}$. For this n we have a composition of isomorphisms

$$H^k(\mathrm{Gr}_G, \mathscr{A}) \to H^k_{\overline{Z_n}}(\mathrm{Gr}_G, \mathscr{A}) \overset{2.+3.}{\to} H^k_{k/2}(\mathrm{Gr}_G, \mathscr{A}) \overset{4.2}{=} \bigoplus_{\langle \rho, \mu \rangle = k/2} F_\mu(\mathscr{A}).$$

The first isomorphism comes from a recollement triangle for $\overline{Z_n} \subset \mathrm{Gr}_G$ and its complement, noting that the support condition implies that the sheaf corresponding to the open embedding is zero. Therefore we have proven the isomorphism.

It remains to prove exactness and faithfulness. Exactness is immediate from Corollary 4.14.

By exactness for any $\phi \in \mathrm{Hom}_{P_{L^+G}(\mathrm{Gr}_G, \Bbbk)}(\mathscr{A}, \mathscr{B})$ we can identify $\operatorname{im} H^\bullet(\mathrm{Gr}_G, \phi)$ with $H^\bullet(\mathrm{Gr}_G, \operatorname{im} \phi)$. Therefore it is enough to show that $H^\bullet(\mathrm{Gr}_G, -)$ does not kill any non-zero object. Since $P_{L^+G}(\mathrm{Gr}_G, \Bbbk)$ is semisimple and the functor is additive, it is enough to prove that $H^\bullet(\mathrm{Gr}_G, \mathrm{IC}_\lambda)$ for all $\lambda \in X_*(T)^+$.

We claim that $F_\lambda(\mathrm{IC}_\lambda) \neq 0$. First note by definition that $\mathrm{IC}_\lambda|_{\mathrm{Gr}^\lambda_G} \cong \underline{\Bbbk}[\langle 2\rho, \lambda \rangle]$ and that $\mathrm{Gr}^\lambda_G \subset \operatorname{supp} \mathrm{IC}_\lambda \subset \overline{\mathrm{Gr}^\lambda_G}$ by one of the characterising properties of IC_λ. Therefore by [Ach21, Lemma 9.6.7] we have that $F_\lambda(\mathrm{IC}_\lambda) \neq 0$, as required. $\square$

Appendix A

Appendix

Lemma A.1. The assignment of objects in Definition 3.2 in fact defines a functor k-**Alg** $\to$ **Set**.

Longer proof of Lemma 3.3. For $\phi : S \to R$ we take $\Lambda \mapsto S[[t]] \otimes_{R[[t]]} \Lambda =: \Lambda'$. We need to prove that Λ' is an S-family of lattices in $k((t))^n$.

The finitely many generators for Λ become generators of Λ', so it is finitely generated.

We can check that Λ' is projective using the Tensor-Hom adjunction.

$$\mathrm{Hom}_{S[[t]]}(S[[t]] \otimes_{R[[t]]} \Lambda, -) \cong \mathrm{Hom}_{R[[t]]}(\Lambda, \mathrm{Hom}_{S[[t]]}(S[[t]], -))$$

Since Λ and the free module $S[[t]]$ are projective this is a composition of exact functors, so $\Lambda' = S[[t]] \otimes_{R[[t]]} \Lambda$ is projective.

Finally, notice that taking the tensor product with $S((t))$ is simply inverting t. Therefore we have:

$$S((t)) \otimes_{S[[t]]} S[[t]] \otimes_{R[[t]]} \Lambda \cong S((t)) \otimes_{R((t))} R((t))^n \cong S((t))^n$$

$\square$

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