Modelling Pendulums

A. Student

1 Aim *(Question posed and why its interesting)*

The purpose of this project is to model single and double pendulums using various numerical methods. The Euler method and the Runge-Kutta method for solving ODE's will be compared for the simple case of a single pendulum, and the superior method then applied to the more chaotic double pendulum, which is harder to solve analytically.

2 Method *(Basic approach)*

A pendulum with length l that has an angle to the horizontal given by $\theta(t)$ will have equations of motion given by:

$$\begin{aligned} \theta' &= \omega, \\ \theta'' &\simeq \frac{-g\theta}{l}. \end{aligned} \tag{1}$$

This system can be sovled for using the well established Euler and or Runge-Kutta methods. Assuming for a simplistic case of $\theta(0) = \pi/4$ and $\omega(0) = 0$, we can compare the results we get to the true solution

$$\theta(t) = \frac{\pi}{4} \cos\left(t\sqrt{\frac{g}{l}}\right). \tag{2}$$

For the double pendulum, the more complicated equations of motion are:

$$\begin{aligned} 0 &= (m_1 + m_2)l_1\theta_1'' + m_2 l_2 \theta_2'' \cos(\theta_1 - \theta_2) + m_2 l_2 (\theta_2')^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 \\ 0 &= m_2 l_2 \theta_2'' + m_2 l_1 \theta_1'' \cos(\theta_1 - \theta_2) - m_2 l_1 (\theta_1')^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2. \end{aligned}$$

Though these do not have a simple solution to test against, we can mimic the single pendulum case by requiring $\theta_1 = \theta_2$. If this retains the cosine behaviour, then we consider implementation successful.

3 Results *(Notable results - the best or final output of code)*

3.1 Single Pendulum

The plots of the angular displacement of the pendulum are shown in Fig. 1, with the errors in Fig. 2. The error in the Euler method tends to increase over larger time periods, overestimating the angular displacement by around 200% after 10 seconds in the $h = 0.1$ s case. Increasing the number of steps (and so decreasing the value of h) helps to reduce this error, but it continues to trend towards larger values for longer time periods. The error of the Runge-Kutta method scales exactly with the value of h, and remains consistent across the total time period. Thus it is clear that Runge-Kutta is a much better method for computing the angular displacement and velocity of our single pendulum.

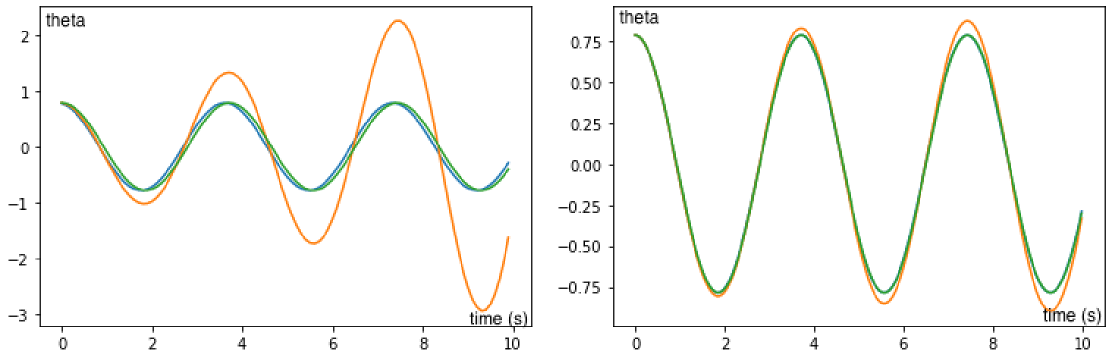

Figure 1: Position of pendulum as a function of time, with 100 steps, $h = 0.1$ s (left) and 1000 steps, $h = 0.01$ s (right). Euler method is shown in orange, RK4 in blue, and the true distribution in green.

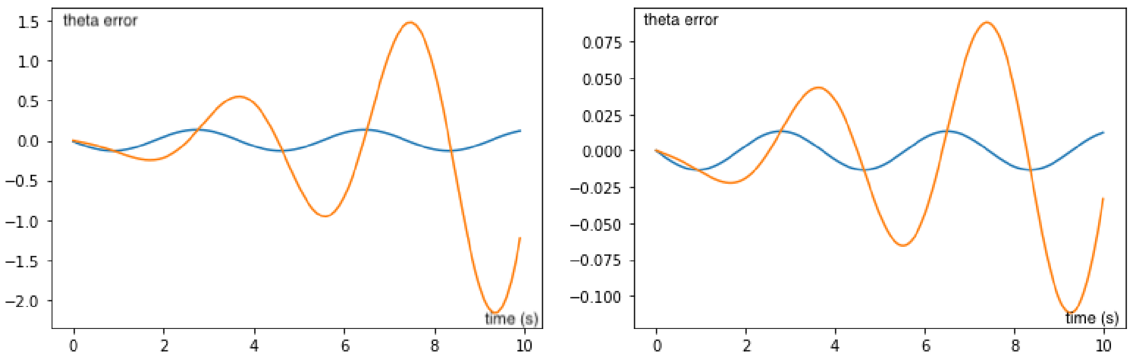

Figure 2: Error in pendulum position for Euler (orange) and Runge-Kutta (blue) methods as a function of time, with 100 steps, $h = 0.1$ s (left), and 1000 steps, $h = 0.01$ s (right)

3.2 Double Pendulum

An example of the movement of a double pendulum is given in Fig. 3, where the displacement of m_1 is shown in blue and m_2 in orange, assuming equal masses and length, and initial conditions $\theta_1(0) = \pi/2$, $\theta_2(0) = \pi/3$, $\theta_1'(0) = 0$, and $\theta_2'(0) = 0$. The departure from the regular cosine, and adoption of chaotic behaviour can readily be seen here. Although m_1 manages to keep a shaky cosine-like movement, m_2 drops to very high, negative values, indicating clockwise rotation around m_1, before climbing back to zero, indicating its rotation has reversed to anti-clockwise about the other mass.

4 Conclusions *(Answer to your question/Summary)*

While both the Euler and Runge-Kutta method are able, to an extent, to model the movement of a single pendulum, the Runge-Kutta tends to give more accurate results. Applying this to the more complex double pendulum (which is difficult to solve analytically), we can clearly see the more chaotic nature of the more complex system. This is demonstrated for a range of scenarios that can be observed in the accompanying code.

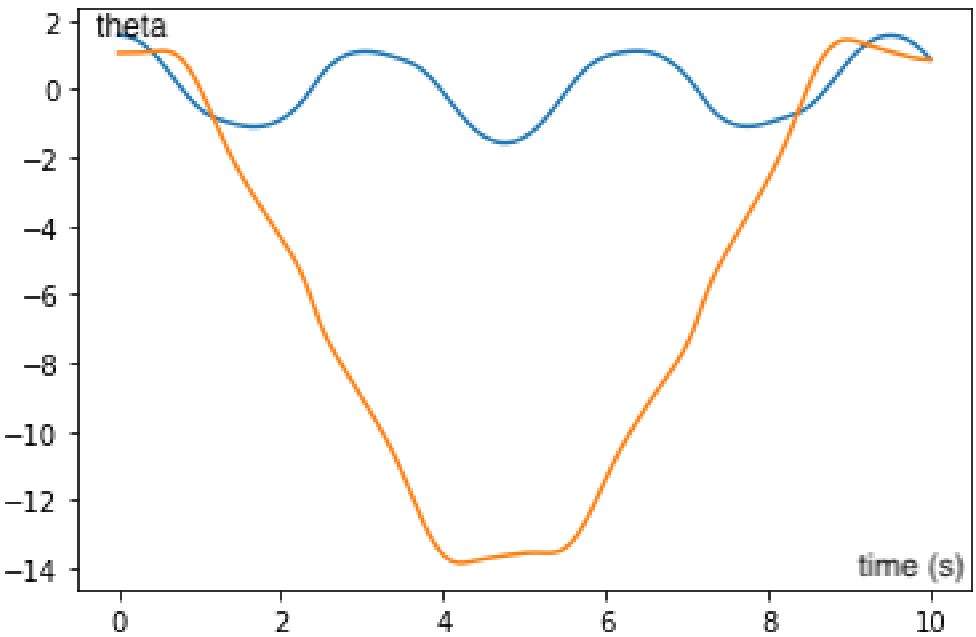

Figure 3: Angular displacement for a double pendulum. The path of m_1 is shown in blue, while m_2 is shown in orange.