Animating Newton's Cradle

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1 Aim

The purpose of this project is to animate Newton's Cradle from first principles. Newton's Cradle has been animated before however typically via a key-frame method or one reliant on a complicated physics engine. This project will demonstrate how strikingly real world behavior can be computed with a simple, first-principles approach mathematical framework. Our elementary model will predict how the system's behavior changes when varying numbers of pendulums are raised. Additionally, it will provide insight on how a real cradle works and offer an explanation for some of the more curious non-ideal behavior.

2 Method

"Newton's Cradle can be modelled as multiple pendulums confined to swing along a single axis which undergo elastic collisions with one another. The pendulums rest such that there is a small amount of horizontal displacement between their masses."

2.1 Fundamental Framework

This framework was implemented in python, and the following graphics generated with the matplotlib python package.

We will define the mathematics required for a generalised set of n independent pendulums, then compute their interactions in a cartesian coordinate system (x, y) which we will call the Cradle's Frame. For each $i \in \{1, 2, 3, ..., n\}$ let the ith pendulum have length ℓ, radius r_i and mass m_i. It will also need some displacement Δx_i along the x-axis, so that it 'almost touches' the others at rest. Each pendulum has a single degree of freedom along the axis (θ_i) in its own reference frame which obeys the differential equation of a pendulum;

$$\ddot{\theta}_i = -\frac{g}{\ell}\sin(\theta_i)$$

A transform to the Cradle's Frame can be computed by;

$$T(\theta_i, \Delta x_i) = (\ell \sin(\theta_i) + \Delta x_i, -\ell \cos(\theta_i))$$

We will say that a collision occurs if for any two pendulums i, j;

$$|T(\theta_i, \Delta x_i) - T(\theta_j, \Delta x_j)| = r_i + r_j$$

If we say that such a collision is elastic, from first principles we will arrive at;

$$\dot{\theta}_{i,\text{final}} = \frac{m_i - m_j}{m_i + m_j}\dot{\theta}_{i,\text{initial}} + \frac{2m_j}{m_i + m_j}\dot{\theta}_{j,\text{initial}} \quad \text{and} \quad \dot{\theta}_{j,\text{final}} = \frac{2m_i}{m_i + m_j}\dot{\theta}_{i,\text{initial}} + \frac{m_j - m_i}{m_i + m_j}\dot{\theta}_{j,\text{initial}}$$

We may also add the condition that each pendulum mass has the same density ρ such that m_i and r_i are related via $m_i = \frac{4\pi\rho}{3} r_i^3$. For the traditional Newton's Cradle with equal mass and radii pendulums the above collision formulae simplify. However, such a non traditional cradle will be explored.

3 Results

Click here to view the results discussed. (Same title names)

3.1 Traditional Cradle

The simulations for five pendulums reflect the same results seen with a real Newton's Cradle.

3.2 Middle Pendulum Motion

Watching these animations closely, we notice that the once stationary middle masses begin to move after some time. The same phenomena can be seen in the video of a real cradle at the link above. This is a remarkable result.

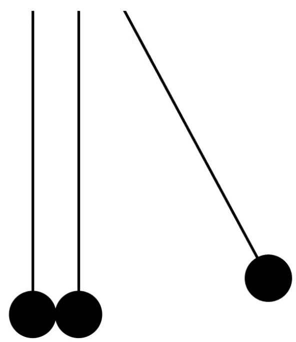

Figure 1: A 3-pendulum cradle

The explanation for this originates from what is meant by the separation between pendulums being small.

Our simulation will aid us in arriving at an explanation. We can place the pendulums closer together, observe how this increases the time before the motion of the middle masses becomes obvious.

Define the $\theta = 0$ position for a pendulum to have zero gravitational potential energy (GPE). Consider the 3-pendulum system where the masses have a non-zero seperation between them.

The right pendulum is released from a height, it swings just past its zero GPE point before transfering all of its momentum to the middle pendulum. The middle pendulum then swings past its zero GPE point before transfering all of its momentum to the left pendulum - the middle pendulum has 'stolen' some of the system's energy. The left pendulum swings up and down again, meanwhile the middle pendulum has also been accellerating due to its positive GPE (however just a tiny bit.) So long as the middle pendulum is not touching the right pendulum when the left pendulum hits it (which it is statsitically unlikely to be) then the middle pendlum will again 'steal' some of the system's energy. Thus with each subsequent collision the middle pendulum will almost certainly 'steal' some energy from the external pendulums. This thinking can be extended to a larger system with middle pendulum*s*.

It is then obvious why a real cradle inevitably reaches this state; there will always be some kind of seperation between the middle pendulums some of the time. Stationary middle pendulums is an inherently unstable system state.

This is my attempt at an explanation yet I don't find it completely satisfying. It is interesting how this seemingly simple system suddenly becomes complex when we look at it in detail. One of the great things about computational physics is we don't need to care, it is simply what our model predicts - an explanation for humans is nice but of secondary importance.

3.3 Non-equal Masses

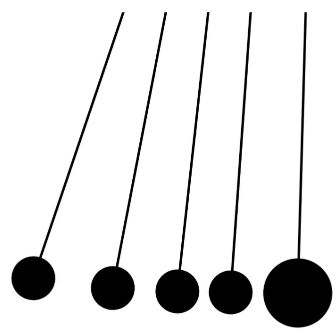

Figure 2: A novel varied mass cradle

While the prior simulations could have been computed with a simplified elastic collision formula, these could not have. It is interesting to note how the non-equal mass systems quickly exhibit chaotic behavior.

3.3 Computational Issues

Due to the discrete nature of the time steps used, collisions are not detected until the masses are slightly inside one another. This is problematic and sometimes leads to the masses getting stuck inside one another. To mitigate this problem, I have triggered an error to be raised if a collision is detected too late. I then added some error handling to attempt the simulation again with smaller time steps. The time steps required seem to be largely dependent on the system's complexity. It is favourable to choose a larger time step as this reduces the computation time. This error handling has largely fixed the issue but results in long computation times for more complicated systems. It is a work-around rather than an ideal solution. If I were to do this again I would want to reduce the collision detection algorithm's complexity.

4 Conclusions

We replicated the results of a real pendulum showing the expected behavior with one, two and three pendulums raised in a five pendulum system. The most interesting result of this project is we were able to explain the origins of middle pendulum motion in Newton's Cradle. Through simulation, we confirmed the hypothesis that it depended on their stationary separation and subsequently deduced a physical argument that supports this hypothesis. Consequently, we can infer that this is likely a contributing factor of middle pendulum motion in a real Newton's Cradle.