Impossibilities

Arrovian impossibilities:

Arrow:	[IIA_2	PO_2	TRat	non-dict_2	m >= 3]
	[IIA	PO		non-dict	m >= 3] (SWF)
Cond-May:	[Anon	Neutral	PR	m, n >= 3]	
Mas-Col-Sonn:	[IIA_2	PR_2	Rat	non-olig_2	n >= 4]
Strong Cond-May:	[Neutral_2	PR_2	α	Anon_2	(m,n?)]
Strong Mas-Col-Sonn:	[IIA	PR_2	α	non-weak-dict	(m,n?)]

Condorcet extensions and reinforcement: [Cond-Ext | Reinf | m >= 3]
resoluteness: [Anon | Neutral | Resolute for all n, m]
scoring rules and α^: [non-triv. monotonic scoring rule | α^]

Manipulability:
Gibb-Satt: [Resolute | non-Imp | Stratpr | non-dict | m >= 3]
[Resolute | Cond-Ext | StMon | m, n >= 3]
[Resolute | Cond-Ext | Particip | n >= 12, m >= 4]

Extension Manipulability:
[non-Imp | QuTRat | RK-Stratpr | non-weak-dict | m >= 3]
[PO | PR | RK-Stratpr | non-dict | m >= 3]
[PO | Maj | RF-Particip | m >= 5]
[PO | Anon | RF-Stratpr | m >= 5] (with weak pref.)

SDS:
[SDS puts prob. 1 on Cond winners | non-manip. | m, n >= 3]
[non-Imp | non-manip. | non-rand-dict | m >= 3]

Exercise theorems (1 - 32):

2a	Anti-Plurality is Monotonic, but not PO
2b	Baldwin's rule is PO, but not Monotonic
3	instant-runoff suffers from strategic abstention and manipulation
4b	knock-out elim. tree are monotonic
4c	m >= 5 => knock-out tree not PO
8	resolute => (rat ⇔ α) and (rat => strictly + trans. rat)
11	mon_2 + (∀x,y∃RN: ∀RN': RN | {x,y}=RN' | {x,y} => f(RN',{x,y})={x}) => PO_2
12	∃anon+neutr+PO+res strict SCF ⇔ n not divided by any q in {2,...,m}
14	narrow Borda: PO, IIA, not TrRat. Broad Borda: PO, TrRat, not IIA
26	Tournaments contain Ham. paths/Kemeny ranking is a Ham. path
32a	a selected by some elim. tree ⇔ a in TC

Characterizations

Borda's rule ←	Neutral, PO, Reinf., Cancellation SCF
Borda's rule ←	only scoring rule that never selects a Cond. loser
Kemeny's Rule ←	Neutral, Condorcet consistent, Reinf. SPF (Young I)
Kemeny's Rule ←	Anon., Neutral, PO, Reinf., LIIA SPF (Young II)
Top Cycle ←	finest Maj., Anon, Neutral, PR_2, β+ SCF
Top Cycle ←	only Maj., R^F-Stratpr., set-non-imp. SCF
Uncovered Set ←	finest Maj., γ SCF
Uncovered Set ←	largest Maj., PO SCF
Banks Set ←	finest Maj., ρ+ SCF
Tourn. EQ Set ←	(finest Maj., ρ SCF ?)
Copeland winners	
Bipartisan Set ←	most discriminating Maj, Stable SCF (non-unique)

⊆

SCF Properties

Relating to rationalizability:

α (contraction):	B ⊆ A => S(A)∩B ⊆ S(B)
γ (expansion):	S(A)∩S(B) ⊆ S(A u B)
β+ (str. expansion):	B⊆A, S(A)∩B!={} => S(B) ⊆ S(A)
ρ+ (str. retentiveness):	∀A, x in A, $\bar{D}$(x) != {}: S($\bar{D}$(x)) ⊆ S(A)
ρ (retentiveness):	∀A, x in S(A), $\bar{D}$(x) != {}: S($\bar{D}$(x)) ⊆ S(A)
S-retentive set B != {}:	∀x in B, $\bar{D}$(x) != {}: S($\bar{D}$(x)) ⊆ B

α and γ (Schwartz): For all x in A∩B:
α: x in S(A u B) => x in S(A) ∩ S(B)
γ: x in S(A) ∩ S(B) => x in S(A u B)

α and β+ (WARP): If B⊆A, S(A)∩B != {}:
α: S(A) ∩ B ⊆ S(B)
β+: S(B) ⊆ S(A) ∩ B

Relating to set rationalizability:

$\hat{\alpha}$ (set contraction):	X = S(A u B) ⊆ A ∩ B => S(A) = S(B) = X
$\hat{\gamma}$ (set expansion):	X = S(A) = S(B) => S(A u B) = X
set rat.:	∃R⊆F(U)²: X = S(A) ⇔ X in Max(R, F(A))
stable:	$\hat{\alpha}$ and $\hat{\gamma}$
quasi-trans. rat:	stable and α

trans. rat. rat.

$\hat{\alpha}$ (alternative characterization):
∀V, W, S(V) ⊆ W ⊆ V, S(V) = S(W)

Set base relation:
X R_s Y ⇔ X = S(X u Y)

SCFs, SWFs, SPFs and SDSs

Please use with care, I copied some of these from another sheet and didn't check all of them

	γ	α	rat	$\hat{\gamma}$	$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti	SP	CExt	Reinf	IIA	PR	Canc	C1/2/3	Maj	Stable	
Plurality	γ				$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti	SP	CExt			PR	Canc			Stable	Select alternative(s) ranked first by most voters
Seq Maj Comparisons					$\hat{\alpha}$	Anon	Neut	Mon	PO			CExt			PR				Stable	Make majority comparisons in fixed sequence
Plurality with runoff					$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti		CExt			PR				Stable	Two alternatives with highest plurality scores face off in majority comparison
Instant runoff					$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti	SP				PR				Stable	Succ. delete alternatives with lowest plurality score
Borda's rule					$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti	SP	CExt	Reinf		PR	Canc	C2		Stable	Score of alternative = number of alternatives below it, summed over all voters, highest score wins
Scoring rules																				Score of alternative summed up over all voters, highest score wins
Baldwin's rule						Anon	Neut	Mon	PO			CExt								Succ. delete alternatives with lowest Borda score
Black's rule					$\hat{\alpha}$	Anon	Neut					CExt			PR				Stable	Condorcet winner if it exists, else Borda winner
Kemeny's rule					$\hat{\alpha}$		Neut	Mon	PO	Parti		CExt	Reinf	IIA	PR				Stable	Maximize number of pairwise matches: $\arg\max_R \sum_i \lvert R \cap R_i \rvert$
Young's rule					$\hat{\alpha}$												C3		Stable	Alternatives that can be made Cond winners by removing a minimal number of voters
Copeland's rule CO					$\hat{\alpha}$				PO								C1		Stable	Alternatives with maximal number of pairwise victories
Top Cycle TC				$\hat{\gamma}$	α^	Anon	Neut	Mon	PO		RK-SP	CExt			PR			Maj	Stable	inclusion-minimal dominating set = undominated alternatives in the transitive closure
Uncovered Set UC	γ				$\hat{\alpha}$	Anon	Neut	Mon	PO		RK-SP	CExt						Maj	Stable	undominated alternatives in C ($x \, C \, y \Leftrightarrow D(y) \subseteq D(x) \Leftrightarrow \bar{D}(x) \subseteq \bar{D}(y)$) = alternatives that can reach everything in two steps
Banks Set BA					$\hat{\alpha}$	Anon	Neut	Mon	PO			CExt			PR			Maj	Stable	maximal el. of all inclusion-maximal transitive subsets = maximal el. of trans. subsets that cannot be extended from above
Tourn. Equil. Set TEQ						Anon	Neut	Mon			SP	CExt			PR			Maj		unique fixpoint of ° = union of inclusion-minimal TEQ-retentive sets = for each selected x, TEQ($\bar{D}(x)$) must be selected
Bipartisan Set BP				$\hat{\gamma}$	$\hat{\alpha}$	Anon	Neut	Mon			RK-SP	CExt			PR			Maj	Stable	alternatives that the unique optimal lottery assigns pos. prob. (optimal lottery $p \Leftrightarrow \forall x: p(\bar{D}(x)) \geq p(D(x)) \Leftrightarrow \forall x: u_p(x) = p(\bar{D}(x)) - p(D(x)) \geq 0$
Random Dictator	γ	α	rat	$\hat{\gamma}$	$\hat{\alpha}$	Anon	Neut	Mon	PO	Parti	SP	CExt	Reinf		PR				Stable	select a random voter's top choice
Maximal Lottery						Anon	Neut	Mon	PO	Parti		CExt	Reinf							unique lottery s.t. $p^T M \geq 0$ (component-wise), $M_{xy} = n_{xy} - n_{yx}$ = no other lottery is preferred by an exp. maj., i.e. $p^T M q \geq 0$
Probabilistic Copeland										Parti	SP									probability ~ to Copeland score
Probabilistic Borda										Parti	SP									probability ~ to Borda score

(but imposing!)

Algorithms

Single-Peakedness Algorithm:

```
A = alternatives still to be placed
while |A| >= 2:
  l, r = current left/right innermost alt.
  B = <the bottom-ranked alternatives>
  L = {x in B|∃i:r Pi x Pi l and ∃y in A: y Pi x}
  R = {x in B|∃i:l Pi x Pi r and ∃y in A: y Pi x}
  if |B|>2 or |L|>1 or |R|>1 or L,r overlap:
    return False
  <place x in L next to l, y in R next to r,
   z in B arbitrarily>
```

Computing TC:

```
B = CO(A,PM) # start with Copeland winners
while <B just got bigger>:
  B = B ∪ (∪{D̄(a)|a in B})
```

Computing UC:

- by matrix multiplication and two-step characterization
- I + M + M^2 for adj. matrix M, return rows without 0s

Computing some Banks Set element:

```
B = {}
C = A
while C != {}:
  pick a from C
  B = B ∪ {a}
  C = ∩{D̄(b)|b in B}
return a
```

Computing TEQ:

- for all a compute TEQ($\bar{D}$(a)) recursively
- for t in TEQ($\bar{D}$(a)), mark the edge (t -> a)
- use the TC algorithm (without CO init., i.e. $O(n^3)$)
  on the subgraph given by the marked edges

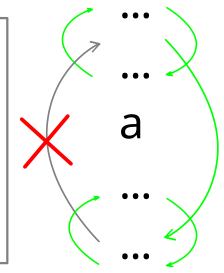
Theorems

Consistency condition relationships:

α and γ ⇔ rationalizability
α and β+ ⇔ transitive rationalizability
γ in maj. SCF => ρ+
$\hat{\alpha}$ + monotonicity + IIA => strong monotonicity

Strategyproofness:

preferences trans. in domain => majority winner SCF satisf. stratpr. and particip.
strongly monotonic => R^K-Stratpr.
maj. + R^K-stratpr. => R^K-particip.
BP, TC, UC are R^K-Stratpr.
resolute => (strong mon. ⇔ any voter can move around alternatives as long as
alternatives below the winner stay below, without changing the result)



Arrow's Proof:

Field expansion lemma:
g is SWF, sat. IIA + PO. If a $\tilde{D}$G b for some a, b, G, then x DG y for all x, y.

Group contraction lemma:
g is SWF, sat. IIA + PO. If G is decisive, |G| >= 2, there exists a decisive proper subset of G.

Exercise theorems (32 - 52):

32b	|A|>= 8 => no elim. tree always selects a CO winner
36a	(∃(A',PM'): UC-ind. subgr. is (A,PM)) ⇔ |A|=1 or (A,PM) has no Cond. winner
46	monotonic scoring rule, lex tiebreak. => participation
50a	UC is R^F-manipul
50b	TC is R^F-stratpr.
51	maj. + R^K-stratpr. + non-imp => Cond-cons.

More Theorems:

Cond. winners are never Borda losers, Cond. losers are never Borda winners
BP: p(x) > 0 <=> u_p(x) = 0; |BP| is odd; p(x) is 0 or quotient of odd numbers
Max. lottery: Condorcet winners picked with prob. 1

Definitions

SCF properties:

- IIA: ∀A,RN,RN': ∀i: Ri|A = Ri'|A => f(RN,A) = f(RN',A)
- monotonicity: a in f(RN, A) => a in f(RN', A) (RN ~> RN', i reinforces a)
- positive responsive: a in f(RN, A) and Ri|A != Ri'|A => f(RN',A) = {a}
- binary: ∀A,R_{N},R_{N'}: (∀x,y: f(RN,{x,y}) = f(R_N',{x,y})) => f(RN,A) = f(RN',A)
- majoritarian: Anon + Neutr + PR_2 + binary (only depends on majority rule base relation)
- reinforcement: ∀A,R_{N},R_{N'}: f(RN,A)∩f(RN',A) != {} => f(RN u RN', A) = f(RN,A)∩f(RN',A)
- cancellation: ∀A,RN: (∀x,y: n_xy = n_yx => f(RN,A) = A)
- strong monotonicity: RN = RN' except x Pi y, y Pi' x for some x notin f(RN) => f(RN') = f(RN)
  (Weakening of unchosen alternatives)

SPF properties:

- SPF anon: R_i = R'_π(i) => f(RN) = f(RN'), π: N -> N
- SPF neutr: RN = π(RN') => π(f(RN)) = f(RN'), π: U -> U
- SPF Cond-cons: ∀RN,R in f(RN): x, y adj. in R and x R y => x RM y
- SPF Reinf: f(RN)∩f(R_{N'}) != {} => f(RN) ∩ f(R_{N'}) = f(RN u R_{N'})
- SPF LIIA: R in f(RN), R' in f(RN'), x, y adj in R and R'
  and ∀i: Ri|{x,y} = Ri'|{x,y} => R|{x,y} = R'|{x,y}
- SPF PO: ∀RN,x,y,R in f(RN): (∀i: x Pi y) => x P y

Fishburn's Classification

- C1: depends on RM
- C2: depends on (n_xy)_xy, not C1
- C3: neither C1 nor C2

Preference domains:

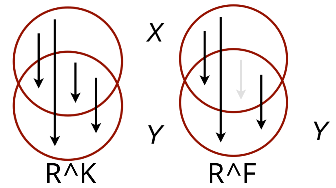
- dichotomous: D_DI(U) = {R in R(U) | ∀x,y,z: xPy => zIx or zIy}
- single-peaked: D^>_SP: ∀x,y,z,i: (x>y>z) or (z>y>x) => (x Pi y => y Pi z)
- single-caved: D^>_SC: ∀x,y,z,i: (x>y>z) or (z>y>x) => (y Pi x => z Pi y)
- value-restricted: ∀x,y,z ∃a in {x,y,z}: (a never worst) or (a never best) or (a never middle)

Dictatorship and variants:

- decisive group (a vs. b): ∀RN: (∀i in G: a Pi b) => a P b "a DG b"
- semidecisive group: (a vs. b) ∀RN: (∀i in G: a Pi b and ∀j notin G: b Pj a) => a P b "a D̃G b"
- dictator: x Pi y => x P y
- weak dictator: x Pi y => x R y
- oligarchy: decisive group of weak dictators
- collegium: intersection of all decisive groups

SCFs and variants:

- SCF: f: $R(U)^n$ x F(U) -> F(U)
- SWF: f: $R(U)^n$ -> R(U)
- SPF: f: $R(U)^n$ -> F(R(U))
- SDS: f: $R(U)^n$ -> $[0,1]^U$, Σ f(RN)(x) = 1



Kelly/Fishburn extensions:

- X R^K Y ⇔ ∀x in X, y in Y: x R y
- X R^F Y ⇔ (∀x in X\Y, y in Y: x R y) and (∀x in X, y in Y\X: x R y)

Transitivity notions:

- transitive: xRy and yRz => xRz
- quasi-transitive: xPy and yPz => xPz
- acyclic: $x_1Px_2P...Px_n$ => x_1Rx_n

More Definitions:

- more discriminating: fewer alt. on average, over all labelled tournaments of size m
- S°: S°(A) = {B⊆A | B incl-min. S-retentive}
- SDS manipul.: ∃RN,RN',u: U -> R: (u(x) >= u(y) ⇔ x Ri y) and E(f(RN')) > E(f(RN))
- dominant set: B in Dom(A,PM) ⇔ ∀x in B, y notin B: x PM y