Technische Universität München

Summary of the lecture MA4800

Foundations in Data Analysis

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Contents

1 Linear Algebra Review

- We work on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- $A^H = \overline{(A^T)}$.
- A Hermitian matrix A satisfies $A = A^H$.
- $A^{(i)}$ are rows and $A_{(j)}$ are the columns.
- $A^{(i)} = (a_{ij})_{j\in J}$ and $A_{(j)} = (a_{ij})_{i\in I} = \left(A^T\right)^{(j)}$
- The matrix-vector product between $A \in \mathbb{K}^{I\times J}$ and $x \in \mathbb{K}^I$ results in the vector in $Ax \in K'$ with entries

1.1 Matrices

$$(Ax)_i = \sum_{j\in J} a_{ij} x_j.$$

1.2 Matrix Multiplication

The matrix-matrix product between $A \in \mathbb{K}^{I\times J}$ and $B \in \mathbb{K}^{J\times L}$ yields the matrix in $\mathbb{K}^{I\times L}$ with entries

$$(AB)_{i\ell} = \sum_{j\in J} A_{ij} B_{j\ell}.$$

2 The Singular Value Decomposition

2.1 The Power Method

Lemma 2.1 *Let $x \in \mathbb{R}^d$ be a unit d-dimensional vector of components $x = (x_1, \ldots, x_d)$ with respect to the canonical basis and picked uniformly at random from the sphere $\{x : \|x\|_2 = 1\}$. The probability that $|x_1| \geq \alpha > 0$ is at least $1 - C\alpha\sqrt{d}$ for some absolute constant.*

Proof

We want the probability of y picked uniformly at random from

$$B^d(1) = \left\{y \in \mathbb{R}^d, ||y||_2 \leq 1\right\}$$

satisfies $|y_1| > \alpha$. In other words, we are looking for the fraction of $B^d(1)$ that satisfies $|y_1| > \alpha$. This corresponds to

$$V_\alpha := \mathrm{Vol}(B^d(1) \cap \{y : |y_1| \leq \alpha\})$$

$$= \int_{y\in B^d(1)\cap\{y:|y_1|\leq\alpha\}} 1 dy$$

$$= \int_{-\alpha}^{\alpha} \left(\int_{\mathbb{R}^{d-1}} 1_{y_2^2+\ldots+y_d^2 \leq 1-y_1^2} \, dy_2 \ldots dy_d \right) dy_1$$

$$= \int_{-\alpha}^{\alpha} \mathrm{Vol}\left(B^{d-1}\left(\sqrt{1-y_1^2}\right)\right) dy_1$$

Replacing $\text{Vol}\left(B^{d-1}\left(\sqrt{1-y_1^2}\right)\right)$ with $(\sqrt{1-y_1^2})^{d-1}\text{Vol}\left(B^{d-1}(1)\right)$ since the volume the unit ball with a factor proportional to radius in the power of $d-1$.

$$= \int_{-\alpha}^{\alpha} (\sqrt{1-y_1^2})^{d-1}\text{Vol}\left(B^{d-1}(1)\right) dy_1$$

$$= \text{Vol}\left(B^{d-1}(1)\right) \int_{-\alpha}^{\alpha} (1-y_1^2)^{(d-1)/2} dy_1$$

In the integral part, $\int_{-\alpha}^{\alpha}(1-y_1^2)^{(d-1)/2}dy_1$, notice that $(1-y_1^2)^{(d-1)/2} < 1$ in the whole integration domain. Thus we can write

$$= \text{Vol}\left(B^{d-1}(1)\right) \int_{-\alpha}^{\alpha} (1-y_1^2)^{(d-1)/2} dy_1$$

$$\leq \text{Vol}\left(B^{d-1}(1)\right) \int_{-\alpha}^{\alpha} 1 dy_1$$

$$= 2\alpha \text{Vol}\left(B^{d-1}(1)\right)$$

Recall that volume of unit ball in d dimensions is asymptotically

$$V_1 = \frac{1}{\sqrt{d\pi}} \left(\frac{2\pi e}{d}\right)^{d/2}$$

Hence the probability p we are interested in satisfies asymptotically

$$p = \frac{V_\alpha}{V_1} \propto \frac{2\alpha \frac{1}{\sqrt{(d-1)\pi}} \left(\frac{2\pi e}{d-1}\right)^{(d-1)/2}}{\frac{1}{\sqrt{d\pi}} \left(\frac{2\pi e}{d}\right)^{d/2}} = \frac{2\alpha \frac{1}{\sqrt{(d-1)\pi}} \left(\frac{2\pi e}{d-1}\right)^{(d-1)/2}}{\frac{1}{\sqrt{d\pi}} \left(\frac{2\pi e}{d}\right)^{(d-1)/2} \left(\frac{2\pi e}{d}\right)^{1/2}}$$

We simplify the last term

$$= 2\alpha * \left(\frac{d}{d-1}\right)^{1/2} * \left(\frac{d}{d-1}\right)^{(d-1)/2} * \left(\frac{d}{2\pi e}\right)^{1/2}$$

$$= 2\alpha * \left(\frac{d}{\sqrt{2\pi e(d-1)}}\right) * \left(\frac{d}{d-1}\right)^{(d-1)/2}$$

Since $\frac{d}{d-1} = 1 + \frac{1}{d-1}$

$$= 2\alpha * \left(\frac{d}{\sqrt{2\pi e(d-1)}}\right) * \left(1 + \frac{1}{d-1}\right)^{(d-1)/2}$$

We modify the power of the same term, to show it as

$$= 2\alpha * \left(\frac{d}{\sqrt{2\pi e(d-1)}}\right) * \left(\left(1 + \frac{1}{d-1}\right)^{(d-1)}\right)^{1/2}$$

Recall that

$$e = \lim_{n\to\infty} (1 + 1/n)^n$$

Thus this term is bounded with $\sqrt{e}$

$$\leq 2\alpha * \left(\frac{d}{\sqrt{2\pi e(d-1)}}\right) * \sqrt{e}$$

We reformulate as

$$= \alpha\sqrt{d}\sqrt{\frac{2d}{\pi(d-1)}}$$

Since $\sqrt{\frac{d}{d-1}} \leq 2$ for $d \geq 2$

$$\leq \frac{2\sqrt{2}}{\pi}\alpha\sqrt{d}$$

Given that all of this only holds asymptotically; we might need another multiplicative constant to make it hold in general. Hence the constant C in the theorem.

$$p \leq C\alpha\sqrt{d}$$