Computational Social Choice

Common Voting rules

Plurality

Whoever is ranked first by more voters than any other candidate wins.

Usage: most democratic elections

Borda's Rule

The i-th ranked alternative of each voter gets $m - i$ points (the bottom alternative gets zero points). The points from each voter are summed together, and the alternative with the greatest total score wins.

Usage: in Slovenia for elections, in academic institutions, in the Eurovision Song Contest.

Sequential majority comparisons (SMC)

There is a fixed sequence of comparisons of the alternatives, e.g. (((a vs. b) vs. c) vs. d). Majority comparisons are performed between the first two alternatives, then between the winner and the next alternatives, etc. The winner of a majority comparison between two alternatives is the alternative that is preferred by the majority of voters.

Usage: e.g. for the US congress *amendment procedure*.

Plurality with runoff

The two alternatives with the highest plurality scores (i.e. which are first-ranked by the most voters) face off in a final round with a majority comparison.

(Runoff = "Stichwahl")

Usage: In France, Brazil and Russia for elections.

Instant Runoff

Successively delete alternatives that are ranked first by the lowest number of voters (by deleting an alternative on top of some voter's profile, the next undeleted alternative below becomes the top alternative of this voter). Repeat until all alternatives that remain are first-ranked by the same number of voters: these are the winners.

Usage: Canada and UK (for elections?), Oscar nominations.

Example: "A Curious Preference Profile"

Consider the preference profile

5	4	3	2
a	e	d	b
c	b	c	d
b	c	b	e
d	d	e	c
e	a	a	a

Then

- by plurality, a wins
- by Borda's rule, b wins
- by SMC, c wins
- by instant-runoff, d wins
- by plurality with runoff, e wins

Desirable Properties (Axioms)

Here are informal definitions of some axioms that will be defined more formally.

- anonymity - the voting rule treats voters equally
- neutrality - the voting rule treats alternatives equally
- monotonicity - a chosen alternative will still be chosen if it is ranked higher in some individual rankings (and nothing else changes)
- Pareto-optimality - no alternative is chosen if there is another alternative which all voters prefer to it

	Anonymity	Neutrality	Monotonicity	Pareto
Plurality	✓	✓	✓	✓
Borda	✓	✓	✓	✓
SMC	✓	-	✓	-
Plurality w/ runoff	✓	✓	-	✓
Instant-runoff	✓	✓	-	✓

SMC fails neutrality (obviously), and Pareto-optimality (say, a Pareto-dominates c, but a loses against b and b loses against c).

Runoff rules fail monotonicity: Say, voters reinforce b, and because of that b faces off against c (against which it loses) instead of a (against which it would have won).

Strategic Manipulation / Strategic Abstention

Manipulation: mis-represent preferences to obtain a better outcome. By *Gibbard-Satterthwaite impossibility theorem*: every reasonable single-winner voting rule is prone to manipulation!

Abstention: don't participate in the election to obtain a better outcome. Plurality and Borda's rule are resistant to strategic abstention.

Week 2 - Choice Theory, Rationalizability, Consistency

Individual Choice

U is a universe of alternatives, $F(U) = 2^U \setminus \{\emptyset\}$ are the *feasible sets*.

Choice function: $S : F(U) \to F(U)$ s.t. $\forall A : S(A) \subseteq A$.

Define the set of maximal elements ("maximal set") wrt. a relation as $Max(R, A) = \{x \in Y \mid \nexists y \in A : yPx\}$.

Transitivity Notions

We use the following transitivity notions: A binary relation R is

1. *transitive* if $\forall x, y, z$: $xRy \wedge yRz \implies xRz$
2. *quasi-transitive* if $\forall x, y, z : xPy \wedge yPz \implies xPz$
3. *acyclic* if for all $x_1, \ldots, x_n : x_1Px_2P \ldots Px_n \implies x_1Rx_n$

Transitivity is stronger than quasi-transitivity, which in turn is stronger than acyclicity.

Lemma. If R is acyclic, $Max(R, A) \neq \emptyset$.

Rationalizability

A choice function S is *rationalizable* if there is a preference relation R s.t. $\forall A \in F(U) : S(A) = \max(R, A)$.

The *base relation* R_S is defined by $xR_Sy \Leftrightarrow x \in S(\{x, y\})$.

Lemma. S is rationalizable if and only if it is rationalized by its base relation.

Inconsistencies

Contraction consistency α

S satisfies contraction α if $\forall A, B \in F(U) : B \subseteq A \Rightarrow S(A) \cap B \subseteq S(B)$

Lemma 3: S satisfies α ⇒ base relation R_S is acyclic.

However, α is not sufficient for rationalizability. Example:

X	S(X)
ab	ab
bc	b
ac	a
abc	a

Expansion consistency γ

S satisfies expansion γ if $\forall A, B \in F(U) : S(A) \cap S(B) \subseteq S(A \cup B)$

Plurality fails to satisfy γ: e.g. it is possible that $S(\{a,b\}) = \{b\}$, $S(\{b,c\}) = \{b\}$, but $S(\{a,b,c\}) = \{a\}$.

α and γ are independent of each other.

Rationalizability and Consistency

> **Theorem** (Sen's Theorem, 1973).
> S is rationalizable if and only if it satisfies α and γ.

Alternative characterization of α, γ and rat. by Schwartz: (for all A, B **and** $x \in A \cap B$ it holds that ....)

- α: $S(A \cup B) \subseteq S(A) \cap S(B)$
- γ: $S(A \cup B) \supseteq S(A) \cap S(B)$
- rationalizability: $S(A \cup B) = S(A) \cap S(B)$

Transitive Rationalizability

Strong Expansion

Example: choice function S which is rationalizable, but not by a quasi-transitive relation.

X	S(X)
ab	a
bc	b
ac	ac
abc	a

Define *strong expansion* $\beta+$: For all $B \subseteq A$: $S(A) \cap B \neq \emptyset \Rightarrow S(B) \subseteq S(A)$.

$\beta+$ implies γ.

Transitive Rationalizability

> **Theorem** (Arrow, 1959).
> A choice function is rationalizable by a transitive relation iff it satisfies α and $\beta+$.

Conjunction of α and $\beta+$ is known as *weak axiom of revealed preference* (WARP).

Characterization: if $B \subseteq A$ and $S(A) \cap B \neq emptyset$, then

- WARP: $S(B) = S(A) \cap B$
- α: $S(B) \supseteq S(A) \cap B$
- $\beta+$: $S(B) \subseteq S(A) \cap B$

Week 3 - Formalizing Social Choice, May's Theorem, Condorcet Paradox

Fairness Conditions of SCFs

- *anonymous*: $f(R_N, A) = f(R'_N, A)$ if the voters in R' are a permutation of the ones in R
- *neutral* if $\pi(f(R_N, A)) = f(R'_N, B)$ if $\pi : A \to B$ is a bijection that satisfies $xR_iy \iff \pi(x)R'_i\pi(y)$ for all i

As defined here: Neutrality ⇒ SCFs independent of preferences over alternatives that are not in the feasible set (let $A = B$, $\pi(x) = x$)

Pareto Optimality

For given R_N and $x, y \in U$, then x *Pareto-dominates* y if xP_iy for all $i \in N$. A *Pareto-optimal* alternative is an alternative which is not Pareto dominated.

Warning: stronger notion of Pareto dominance than usual, as *everybody has to strictly prefer* here.

Pareto SCF

$f_{\text{Pareto}}(R_N, A)$: returns all Pareto-optimal alternatives in A.
- Pareto SCF is anon. and neutral

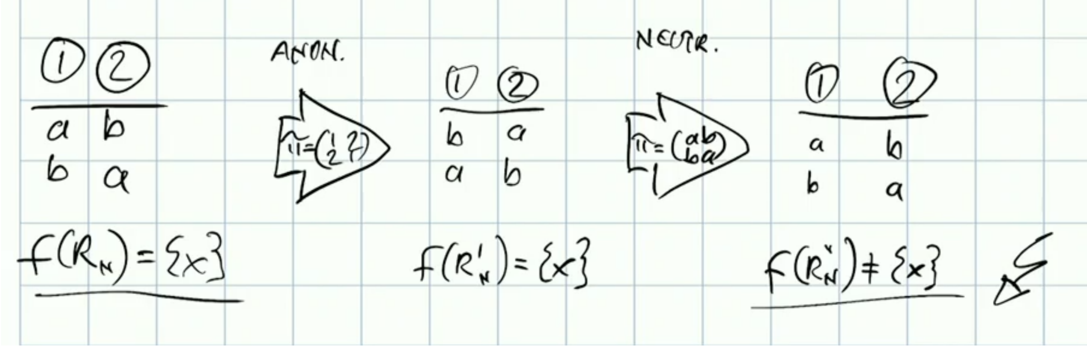
An arbitrary SCF f satisfies *Pareto-optimality* if $f(R_N, A) \subseteq f_{Pareto}(R_N, A)$ for all R_N, A
(if it never returns a Pareto-dominated alternative).

Resoluteness

- *resolute*: $|f(R_N, A)| = 1$

In general: anonymity and neutrality forbids resoluteness. (Intuition: sometimes we cannot break ties without violating anonymity or neutrality).

Formalized counterexample:



Bonus question: for which combinations of $m = |U|$ and $n = |N|$ are there such SCFs? Answer (Moulin 1983): There is a SCF that is anonymous, neutral, Pareto-optimal and resolute iff n cannot be divided by any q with $2 \leq q \leq m$.

Social Choice from Pairs

For now: restrict ourselves to two alternatives. Notation: N_{ab} is the set of voters who prefer a to b, $n_{ab} = N_{ab}$.

Still, several possible SCFS. Examples:

- f_{majority}
- f_{Pareto}: a if $n_{ba} = 0$, b if $n_{ab} = 0$, else $\{a, b\}$
- f_{silly}

⇒ symmetry conditions are not enough: silly SCF should be ruled out by further axioms

Strategic Manipulation

Manipulation is undesirable:

- difficult/impossible to detect, and also perfectly legal ("My scheme is intended only for honest men" - Borda)
- possibility to spend resources on manipulation (e.g. information gathering/compute power) not evenly distributed
- Predictions/theoretical statements become very difficult (e.g. silly rule)

Definition: Strategic Manipulation

For now: make some (strong) simplifying assumptions:

- we only know preferences over singletons - but now formally write $\{x\} R_i \{y\} \Leftrightarrow x R_i y$
- every i knows the *submitted preferences* of all other voters

Definition An SCF f is *manipulable* by voter i if there exist R_N, R'_N and A such that $R_j = R'_j$ for all $j \neq i$ and $f(R'_N, A) P_i f(R_N, A)$. It is *strategyproof* if it is *not manipulable by any voter*.

Monotonicity

Let R_N and R'_N s.t. for some voter i and some alternative a, for all $j \neq i$, $R_j = R'_j$ and for all $x, y \in U \setminus \{a\}$,

$$(x R_i y \Leftrightarrow x R'_i y),\ (a R_i y \Rightarrow a R'_i y),\ (a P_i y \Rightarrow a P'_i y)$$

(in other words: a rises in voter i's preferences. Only so complicated because we have non-strict preferences.)

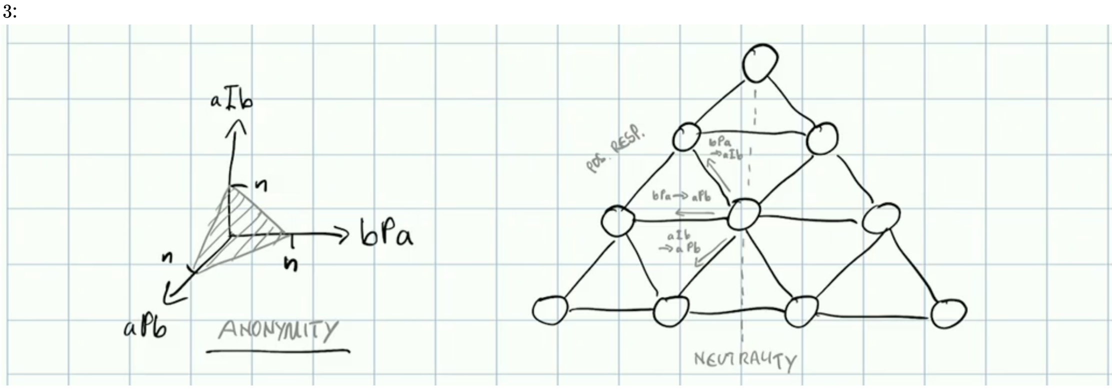
- An SCF f is *monotonic* if $a \in f(R_N, A)$ implies $a \in f(R'_N, A)$.
  - (in other words: if a is chosen, then it is also chosen if reinforced.)
- An SCF f is *positive responsive* if $a \in f(R_N A)$ and the restrictions of R_i, R'_i on A differ, then $\{a\} = f(R'_N, A)$
  - (in other words: if a is chosen in the first place, it is even chosen uniquely if reinforced.)

> **Theorem**: A *resolute* SCF on two alternatives is strategyproof iff it is monotonic.

May's Theorem: Characterization of Majority Rule

> **Theorem** (May, 1952).
> *Majority rule is the only SCF on two alternatives that satisfies anonymity, neutrality, and positive responsiveness.*

Proof sketch for $n = 3$:

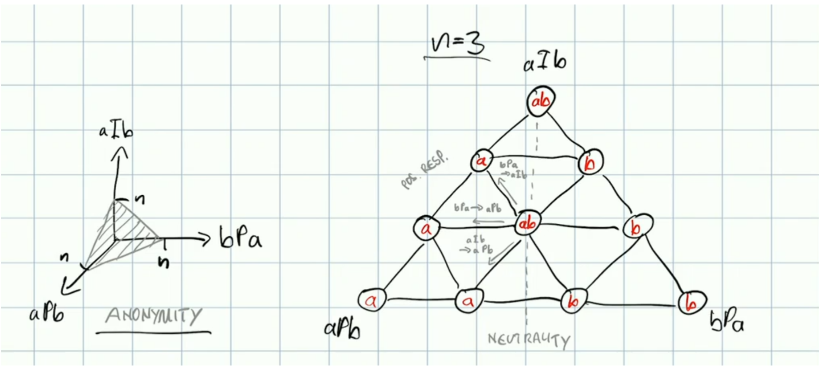


- Anonymity: exploited by the fact that we are only counting the number of voters per alternative

- Neutrality: diagram must be symmetric about the middle axis

Proof: start at middle axis (both ab). Then follow adjacency and exploit positive responsiveness to fill in the rest.

(note: the diagram also nicely shows that there are *three ways to reinforce an alternative*, as already mentioned above)



Another way to look at May's theorem: out of all anonymous, neutral and monotonic SCFs, majority rule is the *most decisive*.

Note: majority rule is uncontroversial *for two alternatives only*.

The Condorcet Paradox

- define *pairwise majority relation* R_M: $xR_My \Leftrightarrow n_{xy} \geq n_{yx}$.

> **Theorem** (Condorcet 1785; May 1952).
> There is no anonymous, neutral and positive responsive SCF that is *rationalizable* if $m \geq 3$ and $n \geq 3$.

Consider the preference profile:

1	1	1
a	b	c
b	c	a
c	a	b

Let f be an SCF with the desired properties, R its rationalizing relation. By Lemma 2, $R = R_f$. By May's theorem: $R_f = R_M$. R_M is cylic for this preference profile and therefore cannot rationalize f. Generalize the proof to $m, n > 3$ by adding indifferent voters and bottom-ranked alternatives.

Comment on the theorem: This is not strictly weaker than Arrow's theorem, actually not comparable, but most of the assumptions here are stronger, so the theorem "feels weaker".

Condorcet Winners

An alternative x is a *Condorcet winner* in A if xP_My for all $y \neq x$ - i.e. an alternative that wins against all alternatives in pairwise majority comparisons. They may not exist, but are of course unique if they exist.

How likely is it that no Condorcet winner exists? For $m = n = 3$ and uniformly distributed strict preferences: $p = 1/18$ (about 6%). In the limit for fixed m, $n \to \infty$:

m	1	2	3	4	5	10	15	20	40
p	0%	0%	9%	18%	25%	49%	61%	68%	81%

2-Versions of Axioms

We will use weakenings of axioms that only hold for agendas of size two:

- *positive responsiveness$_2$*
- *responsiveness$_2$*
- *monotonicity$_2$*
- *Pareto optimality$_2$*

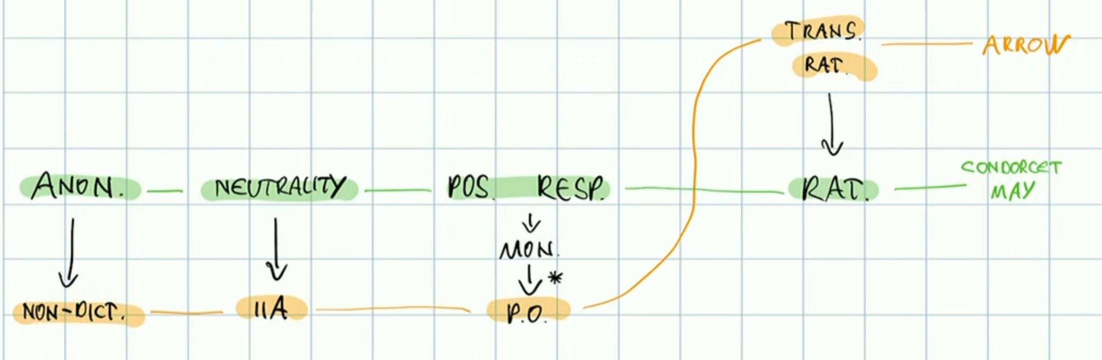
From Condorcet to Arrow

An SCF *f* satisfies

- *independence of infeasible alternatives* (IIA) if $\forall A, R_N, R'_N, i$, $R_i \mid_A = R'_i \mid A$ holds, then $f(R_N, A) = f(R'_N, A)$.
  - IIA is weaker than neutrality
- *Pareto optimality* if $\forall A, R_N, i$ and $x, y \in A$, xP_iy holds, then $y \notin f(R_N, A)$
  - Pareto optimality$_2$ weaker than monotonicity$_2$ under the additional assumption: $\forall x, y \in U$, there is R_N s.t $\forall R'_N$ with $R'_N \mid_{\{x,y\}} = R_N \mid_{\{x,y\}}$, $f(R'_N, \{x, y\}) = \{x\}$.
- *dictatorial* if there exists a voter i such that $\forall A, R_n$, and $x \in A$:

$$(\forall y \in A \setminus \{x\} xP_iy) \implies f(R_N, A) = \{x\}$$

  (namely, if voter i prefers x the most, x is uniquely chosen).



Arrow's Impossibility Theorem

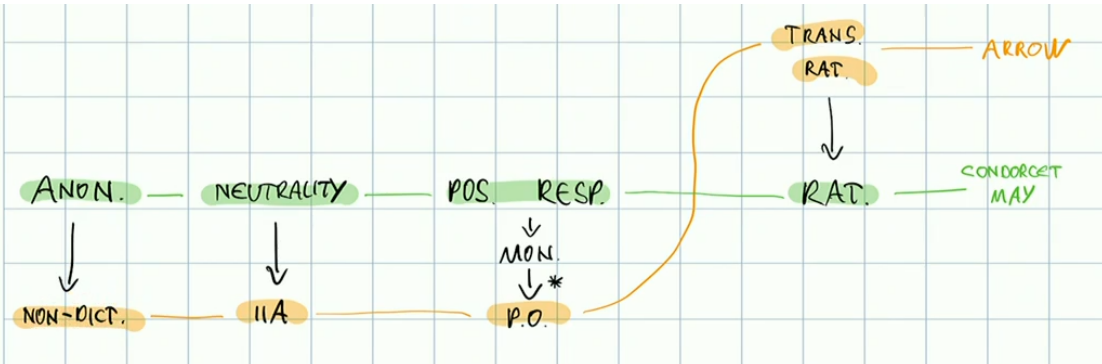
> **Theorem** (Kenneth Arrow, 1951; fixed in 1963).
> *There is no SCF that satisfies IIA, Pareto optimality, non-dictatorship, and transitive rationalizability if* $m \geq 3$.

Even a stronger version holds for IIA_2, $\text{Pareto-optimality}_2$ and $\text{non-dictatorship}_2$.

Week 4 - Arrow's Impossibility Theorem

Arrow's Impossibility

Recall:



While the stronger axioms like anonymity might not be so important and be violated in very special cases without making things too unreasonable, the much weaker non-dictatorship is definitely undesirable.

Arrow's Impossibility Theorem:

> **Theorem** (Kenneth Arrow, 1951; fixed in 1963).
> *There is no SCF that satisfies IIA, Pareto optimality, non-dictatorship, and transitive rationalizability if* $m \geq 3$.

Version for *Social Welfare Functions*

Social Welfare Function (SWF): $g : R(U)^n \to R(U)$ (returns a *collective preference relation* instead of a set).

SWFs and SCFs are connected: $xg(R_N)y \iff x \in f(R_n, \{x,y\})$ (since we assume the SCF is rationalizable).

QUESTION: are SWFs always transitive? ⟹ Yes! $R(U)$ is the set of *complete and transitive relations*.

Properties reformulated for SWFs

Denote the collective preference relation for a given R_N as $R = g(R_N)$.

g satisfies

- *IIA* if for all R_N, R'_N, x, y: if $\forall i \in N : R_i\mid_{\{x,y\}}$, then $R\mid_{\{x,y\}} = R'\mid_{\{x,y\}}$
- *Pareto-optimality* if for all R_N, x, y: if for all i, xP_iy, then xPy
- is *dictatorial* if for some $i \in N$ and all R_N, x, y: xP_iy implies xPy

> **Theorem** (Arrow, 1951; 1963).
> *Every SWF that satisfies IIA and Pareto-Optimality is dictatorial if* $m \geq 3$.

Illustration of Arrow's Theorem

...for $m = 3, n = 2$, strict preferences. See slides. Especially IIA has a lot of power to fill the table quickly.

Interlude: Computer-Aided Theorem Proving

Lin & Tang, 2008: induction-based proof (induction step for increasing alternatives and increasing voters).

Base case ($m = 3, n = 2$) proved by SAT solver (about 10^{28} SWFs for the base case).

Proof of Arrow's Impossibility Theorem

Let g be an SWF and $m \geq 3$.

- a group of voters $G \subseteq N$ is *decisive for a against b* (aD_Gb) if for all R_N, ($\forall i \in G: aP_ib$) implies aPb
- a group G is *decisive* if it is decisive for all pairs a, b.
- a group G is *semidecisive* for a against b, denoted $a\tilde{D}_gb$ if for all R_N: ($\forall i \in G: aP_ib \land \forall j \notin G: bP_ja$) implies aPb
  - *decisiveness* is stronger than *semidecisiveness*.

Lemma (Field Expansion Lemma): Let g be an IIA and Pareto-optimal SWF, let $G \subseteq N$. Then if for some $a \neq b$, $a\tilde{D}_Gb$, this implies that for all x, y: xD_Gy. (If a group is decisive for some pair of alternatives, it is decisive for all pairs of alternatives.)

Proof.
Let $x \neq a, b$.

(1) aD_Gx holds: Consider the partially specified preference profile (actually, a whole class of preference profiles):

G	N\G
a	b
b	
x	

Because of the semidecisiveness, aPb. Because of Pareto optimality, bPx. Because of transitivity, aPx. Because of IIA, aD_Gx. Fixing of b in above profile is valid because of IIA.

(2) bD_Gx holds:

G	N\G
b	
a	
x	a

By (1), aPx. By PO, bPa. By transitivity, bPx. By IIA, bD_Gx.

- Repeated application of (2) shows that D_G is a complete relation.
- D_G is symmetric: If xD_Gy, by (2), yD_Gz. Then by (1), yD_Gx.

Now let G be some decisive group with $|G| \geq 2$ (always exists $G = N$ because of PO). Partition G into two nonempty subgroups. Consider the preference profile:

G_1	G_2	$N \setminus G$
a	b	c
b	c	a
c	a	b

- Since bD_Gc, we get bPc.
- Case 1, aPc: by IIA, $a\tilde{D}_{G_1}c$. By the field expansion lemma, G_1 is decisive.
- Case 2, cRa: By transitivity, bPa. By IIA, $b\tilde{D}_{G_2}a$. By the field expansion lemma, G_2 is decisive.

Now, one of G_1, G_2 is decisive. Therefore repeatedly apply this, until there is a decisive group with *only one element*, i.e. a *dictator!*

Some additional comments about the proof

- The final part is called *Group Contraction Lemma*.
- The *Field Expansion Lemma* also holds for a quasi-transitive relation

Undesirable Groups of Voters

Dictators: extreme example of high concentration of power. But this notion can be generalized to other undesirable groups.

- Dictator: Decisive group with one element
  - $xP_iy \implies xPy$
- Weak dictator: Voter who can force an alternative into the choice set
  - $xP_iy \implies xRy$
  - you can think of a weak dictator as a *vetoer* who can veto yPx
- Oligarchy: Decisive group of weak dictators
  - seems a bit odd, but apparently quite natural because three papers considered it independently shortly after Arrow's theorem
- Collegium: Non-empty intersection of all decisive groups
  - if the collegium is non-empty

Implications: Dictator ⟹ Oligarchy ⟹ Weak Dictator ⟹ (under PO) Collegium

Overview over Arrovian Impossibility Results

Anonymity2	Neutrality	Pos. Resp.	Rat	$n \geq 3$	Condorcet/May (1785; 1952)
No Dictator2	IIA2	Par.Opt.2	Trans.Rat.		Arrow (1951)

Anonymity2	Neutrality	Pos. Resp.	Rat	$n \geq 3$	Condorcet/May (1785; 1952)
No Oligarchy2	IIA2	Par.Opt.2	Quasi-Trans.Rat.		Gibbard (1969)
No Oligarchy2	IIA2	Pos.Resp.2	Rat.	$n \geq 4$	Mas-Colell & Sonnenschein (1972)
No Weak Dictator2	Neutrality	Monotonicity2	Rat.	$m \geq n$	Blau & Deb (1977)
No Collegium	--	Par.Opt.2	Rat.	$m \geq n$	Brown/Banks (1975, 1995)
No Weak Dictator2	Neutrality	Par.Opt.2	Rat.	$m > n$	Austen-Smith & Banks (1999)

QUESTION: Aren't there also boundary conditions on m or n for Arrow's/Gibbard's results?

Ways to escape impossibility results

Four ways:

- consider restricted domain of preferences
  - e.g. approval voting, median voting
- replace consistency (rationalizability) with a variable-electorate condition
  - e.g. conditions similar to α, β, γ for sets of *voters* instead of alternatives
  - $\Longrightarrow$ basically scoring rules in some sense ([TODO... check this later])
- only require expansion consistency γ, not contraction consistency α
  - top cycle, uncovered set, Banks set, tournament equilibrium set
- weaken both consistency conditions
  - *bipartisan set*, related to mixed strategies in game theory

---

Week 5 - Escape Route 1, Domain Restrictions

Now we will explore "Escape Routes" from Impossibility.

Escape Route 1: Domain Restrictions

Goal: restrict the *set of possible preference relations*.

An SCF satisfies some property *in domain* D if it satisfies it for all $R_N \in D(U)^n$ where $D(U) \subseteq R(U)$.

- Example (that we saw before): the linear preferences $D_{LIN}(U) = \{R \in R(U) : \forall x, y \in U : xPy \vee yPx\}$.
  - Interesting note: Arrow's impossibility theorem still holds, but the two versions for strict and non-strict preferences are independent in the sense that none can be directly proven from the other. (Reason: someone might be a dictator in a smaller domain and not in a wider one).

Transitivity, Strategyproofness, Participation

Recall the majority relation from Week 3 - Formalizing Social Choice, May's Theorem, Condorcet Paradox > The Condorcet Paradox.

- an SCF f can be *manipulated by strategic abstention* if for some R_N, A and i:

$$f(R_{n-i}, A) P_i f(R_N, A)$$

QUESTION: how does (Set P_i Set) work? Probably works only for singleton sets.

- an SCF satisfies *participation* if it cannot be manipulated by strategic abstention.

The SCF $R_M \mapsto \max(R_M, A)$ is a well-defined SCF within domain D if R_M is acyclic in D (Lecture 2, Lemma 1).

> **Theorem**.
> If R_M is *transitive* in D, then $\max(R_M, A)$ satisfies Strategy-Proofness and Participation.

Proof: (intuition: show that you cannot manipulate from one Condorcet winner to another). Manipulating cannot improve the majority relation winner. Strategic abstention also doesn't work. [TODO... actual proof]

Dichotomous Preferences and Approval Voting

Dichotomous Preferences: only two indifference classes. There are *some alternatives that you like*, and *some alternatives that you dislike*. Formally:

$$D_{DI}(U) = \{R \in R(U) \mid \forall x, y, z \in U : xPy \Rightarrow zIx \vee zIy\}$$

> **Theorem** (Inada, 1964).
> R_M is transitive in domain D_{DI}.

Proof. For $x \in U$, define $n(x) = |\bigcup_{y \in U} N_{xy}|$ (the number of voters who (weakly) prefer x to any alternative; Recall the N_{ab} and n_{ab} notations from Week 3 - Formalizing Social Choice, May's Theorem, Condorcet Paradox > Social Choice from Pairs).

We get, where the last step uses dichotomousness:

$$xR_My \Leftrightarrow n_{xy} \geq n_{yx} \Leftrightarrow n(x) \geq n(y)$$

This equivalence directly implies transitivity by embedding x, y in the transitively ordered set $\mathbb{N}$.

In the domain D_{DI}, $\max(R_M, A)$ is called *approval voting*: The alternatives with the highest number of approvals win.

Important note: not only the submitted preferences should be dichotomous, but the intrinsic preferences of the voters.

QUESTION: do the strategyproofness guarantees break down if intrinsic preferences are non-dichotomous?

Simplifying Assumptions: "Declaration of Oddity"

For the remainder of the course, we assume:

- all R_i are strict (anti-symmetric), and thus linear orders
- the number of voters n is odd

Taken together, these imply that there are *no ties* in the majority relation.

That is: from now on, $R(U)$ is the set of *anti-symmetric, transitive and complete relations* (i.e. the linear orders).

Impact of these assumptions

- some results hold witout these restrictions (but might be harder to prove)
- some results hold in weaker forms
- some results have been generalized with additional axioms
- some results have *not* been generalized in a satisfactory way

Single-Peaked Preferences

Assumption: there is a natural linear order of the alternatives, and voters have one single peak as most-preferred alternative, and the preference decreases as we move away from this peak.

Definition. R_N is *single-peaked* wrt. some linear order $>$ over U if for all $x, y, z \in U$ and voters i,

$$(x > y > z) \vee (z > y > x) \Rightarrow (xP_iy \Rightarrow yP_iz)$$

QUESTION: I have doubts that a peak might be "skipped over" with this. Look at it again to be sure it is correct.

QUESTION: How many single-peaked preference profiles are there, for a peak in a given position?

The domain of single-peaked preferences wrt. $>$ is denoted $D_{SP}^{>}$.

Note: this concept can be generalized to multiple dimensions in principle, but all the nice results break down in this case.

Example applications

- left/right political spectrum
- location of desirable facilities on road
- temperature for joint thermostat
- grading system
- tax rate
- size of public park
- etc.

Transitive R_M on Single-Peaked Preferences

> **Theorem** (Black 1948, Arrow 1951).
> On single-peaked preferences, R_M is transitive

Proof.
Show: xP_My and yP_Mz implies xP_Mz (transitivity).

Case 1: x > y > z (wlog., or z > y > x).
Because of single-peakedness, $N_{xy} \subseteq N_{yz}$. By transitivity of the individual preferences: $N_{xy} \subseteq N_{xz}$. Therefore $|N_{xz}| \geq |N_{xy}| > n/2$.

Case 2: z > x > y (wlog., or y > x > z).
Because of SP, $N_{zx} \subseteq N_{xy}$. Because of transitivity, $N_{zx} \subseteq N_{zy}$. Therefore and because a majority prefers y to z, $n/2 > |N_{zy}| \geq |N_z x|$, and finally $|N_{xz}| > n/2$.

Case 3: y > z > x (wlog., or x > z > y). (*this case is different and therefore interesting*).
$N_{yz} \subseteq N_{zx}$ by SP, $N_{yz} \subseteq N_{yx}$ by transitivity. Therefore $n/2 < |N_{yz}| \leq |N_{yx}|$. Therefore $|N_{xy}| < n/2$, which is a contradiction! This last case can never occur.

Condorcet Winner on Single-Peaked Preferences

On single-peaked preferences, there is always a unique Condorcet winner. Namely, the Condorcet winner is the top choice of the median voter.

Proof: Let x, y be adjacent alternatives, denote by t_i the top choice of voter i. Then xP_My iff $|\{i : t_i \leq x\}| > |\{i : t_i \geq y\}$. Then by transitivity of P_M: x is a Condorcet winner if $|\{i : t_i \leq x\}| > \frac{n}{2}$ and $|\{i : t_i \geq x\}| > \frac{n}{2}$.

The SCF $Max(R_M, A)$ is known as *median voting* in the domain $D_SP^{>}$.

Some more observations:

- knowing the top choices is enough to compute the median voter, i.e. the Condorcet winner
- even with single-peaked preferences, the plurality winner might be different from the Condorcet winner!

Strategy-Proofness of Median rule

Median voting satisfies *strategy proofness* and *participation* in the domain of single peaked preferences.

Deciding Single-Peakedness of a Profile

Assume a preference profile is given and we want to decide whether there *exists a linear ordering of the alternatives* wrt. which it is single-peaked.

Observe that the last-ranked alternative of each voter must lie at the very left or the very right of the underlying order. For example, if there are more than two last-ranked alternatives accross all voters, the profile cannot be single-peaked.

This observation can be extended to an algorithm.

Single-Peaked Algorithm

Linear-time algorithm.

1. set leftmost alternative to z_l and rightmost one to z_r
2. let $A = U$ be the set of alternatives that still need to be placed
3. while $|A| \geq 2$:
    1. Let l and r be the current left-innermost and right-innermost alternative
    2. Define B, L, R:
        - $B = \{x \in A \mid \exists i : \forall y \in A : yR_ix\}$
        - $L = \{x \in B \mid \exists i : rP_ixP_il \wedge \exists y \in A : yP_ix\}$
        - $R = \{x \in B \mid \exists i : lP_ixP_ir \wedge \exists y \in A : yP_ix\}$
    3. If $|B| \leq 2$, $|L| \leq 1$ and $|R| \leq 1$ as well as $L \cap R = \emptyset$:
        1. place the alternative in L (if any) next to l
        2. place the alternative in R (if any) next to r
        3. place the alternatives in $B \setminus (L \cup R)$ (if any) arbitrarily in empty slots next to l and r
    4. Else: R_N is not single-peaked, terminate.
    5. $A = A \setminus B$
4. If $|A| = 1$, put $x \in A$ into the last remaining slot.

Characterizing Domains: Value Restriction

Majority rule is transitive within

- the dichotomous preferences D_{DI},
- all domains of single-peaked preferences $D_{SP}^{>}$, and
- all domains of single-caved preferences D_{SC}.

There are however other preference profiles on which R_M is transitive. For example:

1	1	1
a	c	d
b	a	a
c	b	b
d	d	c

We want a complete characterization of such domains (under the declaration of oddity, which rules out dichotomous preferences for $m > 2$).

> **Definition** (value-restricted).
> A domain D is *value-restricted* if for each $x, y, z \in U$, there is some alternative (say x) s.t.
>
> - x is never the worst alternative ($\forall R \in D : xPy \vee xPz$)
> - OR x is never the best alternative ($\forall R \in D : yPx \vee zPx$)
> - OR x is never the middle alternative ($\forall R \in D : (xPy \wedge \ldots$)

Some intuition: A domain is value-restricted if it does not contain the Condorcet cycle as a forbidden substructure.

x	y	z
y	z	x
z	x	y

> **Theorem** (Sen & Pattanaik, 1969).
> R_M is transitive in domain D if and only if D is value-restricted. *(Proof: T exercise)*

Whether a domain is value-restricted can be checked in polynomial time (using the naive triple-checking algorithm).

Week 6 - Escape Route 2, Variable-Electorate Condition

Escape Route 2: Replace Consistency with Variable-Electorate Condition

A Brief History Lesson: Borda vs. Condorcet

- Jean-Charles Chevalier de **Borda** (1733 - 1799)
  - scientist; adventurer
  - involved in standard-meter definition (1/1e7-th distance north pole - equator)
- Marie Jean Antoine Nicolas Caritat, Marquis de **Condorcet** (1743 - 1794)
  - early advocate for equal rights

Family of Scoring Rules

Feasible set A fixed.

- a *score vector* is a vector $s \in \mathbb{R}^{|A|}$
  - voter ranks alternative in position $i \longrightarrow$ it recieves s_i votes

Set $s^1, \ldots, s^m$ be a collection of score vectors, s^i of dimension i. Define the corresponding *scoring rule*:

$$f(R_N, A) = \arg\max s(x, A) \quad \text{where} \quad s(x, A) = \sum_i s^{|A|}_{|\{y \in A : yR_i x\}|}$$

Examples:

- Borda's rule: $s^{|A|} = (|A| - 1, |A| - 2, \ldots, 0)$
- Plurality rule: $s^{|A|} = (1, 0, \ldots, 0)$
- Anti-plurality rule: $s^{|A|} = (1, 0, \ldots, 0)$

But since we said that A is fixed, we don't really need a collection of score vectors.

Properties of Scoring Rules

- Scoring rules are *invariant* under componentwise *positive affine transformations* $x \mapsto ax + b,\ a > 0$.
  - In particular, for any score vector (s_1, s_2, s_3) we can find an equivalent score vector $(1, s, 0)$.
- Scoring rules can be efficiently computed
- A scoring rule satisfies monotonicity if $s_1 \geq \cdots \geq s_{|A|}$: see next lemma
- A monotonic scoring rule is one that satisfies $s_1 > s_{|A|}$

> **Lemma (Monotonicity Characterization).** Assume there are enough voters. A scoring rule is monotonic if and only if $s_1 \geq s_2 \geq \cdots \geq s_m$. (the only if direction of the proof only works with $m = n$ voters)

Proof (one direction). Assume that for some k, $s_{k+1} > s_k$. Consider a Condorcet cycle style profile on m alternatives: every alternative obtains every position exactly once. Therefore all have the same score $s(x_i)$ and therefore the chosen set is $f(R_N) = A$. Now switch the positions of a_k, a_{k+1} in the first preference profile. Then $f(R_N') = \{a_k\}$: i.e. a_{k+1} dropped out even |}$:though it was reinforced.

Family of Condorcet Extensions

The scoring rules we saw don't offer that much variety. The *Condorcet extensions* family on the other hand is much more varied.

- an SCF f is a *Condorcet extension* if it uniquely picks Condorcet winners whenever they exist: $f(R_N, A) = \{x\}$ if x is a Condorcet winner in A according to R_N.

Classification of SCFs based on their Level of Abstraction

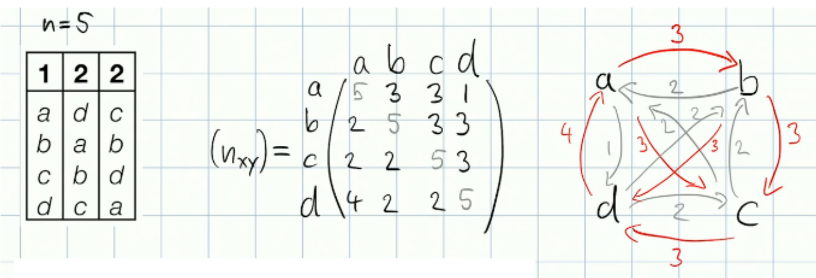
(...Fishburn 1977)

The way that this classification works: Higher number $\Longrightarrow$ the SCF is allowed to depend on more information.

- **C1**: f only depends on the majority relation R_M
  - degree counting in the majority graph: *Copeland's rule* (maximize degree)
- **C2**: f only depends on $(n_{xy})_{x,y \in U}$ and f is not C1
  - maximin: $f_{maximin}(R_N, A) = \arg\max_{x \in A} \min_{y \in A \setminus \{x\}} n_{xy}$
  - Borda's rule: $f_{Borda}(R_N, A) = \arg\max_{x \in A} \sum_{y \in A \setminus \{x\}} n_{xy}$
    - apparently Borda's rule is the only scoring rule that is C2
- **C3**: f is neither C1 nor C2, i.e. it depends on the full preference profile.
  - Young's rule: picks *all alternatives* that can be made Condorcet winner *by removing as few voters as possible* (well-defined: removing all voters but one obviously yields a Condorcet winner)
    - Computing Young's rule is hard: namely, Θ_2^P-complete (P with logarithmically many NP oracle queries)

The classification is usually applied to Condorcet extensions, but the "C" does not stand for Condorcet! It can also contain non Condorcet extensions

Examples: Copeland, Maximin, Young



(red edges are majority edges)

- Copeland: a, b (those both have 2 outgoing edges each)

- Maximin: b, c, d
  - the smallest outgoing edge labels are: (1, 2, 2, 2)
- Borda: b, d (obvious)
- Young: d, c (not so obvious)
  - removing one voter is not enough: by declaration of oddity, removing one voter could only lead to a tie, not a strict Condorcet winner
  - remove one from first, one from second → c wins
  - remove one from first, one from third → d wins

Scoring Rules ⟷ Condorcet Extensions

Borda's Rule is no Condorcet Extensions

3	2
a	b
b	c
c	a

Here a is the Condorcet winner, but b wins.

Scoring Rules are no Condorcet Extensions

Scoring rules are no Condorcet extensions in general if $m \geq 3$:

For monotonic rules:

	4	3	2	2
s_1	c	b	a	b
s_2	b	a	c	c
s_3	a	c	b	a

Wlog. assume $s_1 = 1, s_2 = s, s_3 = 0$ due to the positive affine transfomations property.

Then $s(a) = 2 + 3s$, $s(b) = 5 + 4s$, $s(c) = 4 + 4s$. $\implies b$ wins. The Condorcet winner however is c.

For non-monotonic rules, make a case distinction. Assume $s_2 > s_1$:

	1	1	1
s_1	b	b	c
s_2	a	c	a
s_3	c	a	b

(Condorcet winner b, but either a or c wins in the scoring rule)

Assume $s_3 > s_2$:

	1	1	1
s_1	a	b	c
s_2	c	c	a
s_3	b	a	b

(Condorcet winner c, but b wins in the scoring rule)

Properties of Borda's Rule

- Borda's rule picks alternative with *highest average rank*

Condorcet winner are never Borda losers:

> **Theorem (Smith, 1973)**. A Condorcet winner is *never the alternative with the lowest Borda score*. Borda's rule is the unique scoring rule for which this is the case.
> Also, a Condorcet loser never has the highest Borda score.

Borda's rule maximizes the probability of Condorcet winners across all scoring rules:

> **Theorem (Gehrlein et al., 1978)**. When all preference profiles are equally likely, Borda's rule maximizes the probability over all scoring rules that a Condorcet winer is chosen if it exists.

Unifying Borda & Condorcet

Can we unify the advantages of Borda's and Condorcet's Rules?

Black's Rule

Return the Condorcet winner if one exists, the Borda winner otherwise

- obviously a Condorcet extension

- but somewhat ad-hoc

Baldwin's Rule

(Recall first exercise sheet)

Runoff method based on Borda's rule

- delete alternatives with minimal Borda scores
- also a Condorcet extension

Another rule: delete alternatives with below-average Borda scores

Variable Electorates

Now: define properties similar to the consistency condtions (α, γ etc.), but with respect to sets of voters ("electorate"), not sets of alternatives.

Reinforcement: the alternatives chosen simultaneously by two disjoint electorates are precisely the alternatives chosen by the union of the electorates.

> **Definition (Reinforcement)**. An SCF f satisfies *reinforcement* if for all A, disjoint N, N' and all $R_N \in R(U)^N$, $R_{N'} \in R(U)^{N'}$ that satisfy $f(R_N, A) \cap f(R_{N'}, A) \neq \emptyset$ the following holds:
>
> $$f(R_N, A) \cap f(R_{N'}, A) = f(R_N \cap R_{N'}, A)$$
>
> - When dealing with reinforcement, we do *not* assume an odd number of voters

This is the equivalent of $\alpha \wedge \gamma$ for variable electorates!

- $\alpha \wedge \gamma:$ $\quad x \in f(R_N, A) \cap f(R_N, A') \Leftrightarrow x \in f(R_N, A \cup A')$
- Reinforcment: $x \in f(R_N, A) \cap f(R_{N'}, A) \Leftrightarrow x \in f(R_N \cup R_{N'}, A)$

Characterization of Scoring Rules

We can compose scoring rules to break ties: An SCF f is a *composed scoring rule* if there are scoring rules $f_1, \ldots, f_k$ s.t. $f(R_N, A) = f_1(R_N, f_2(R_N, \ldots (f_k(R_N, A))))$

> **Theorem (Smith 1973, Young 1975)**. A neutral and anonymous SCF is a composed scoring rule if and only if it satisfies reinforcement.

In other words: If you want reinforcement in your SCF, you need to use a scoring rule.

Proof: $\Rightarrow$ is easy, but $\Leftarrow$ isn't and is left out here.

Furthermore:

- every non-trivial anon. + neutral SCF refines a non-trivial scoring rule
- the non-composed scoring rules can also be characterized, by adding a continuity axiom

Reinforcement is the defining property of scoring rules!

Characterization of Borda's Rule

An SCF satisfies *cancellation* if for all A and R_N: $\forall x, a \in A : n_{xy} = n_{yx} \implies f(R_N, A) = A.$

(Cancellation is a rather technical axiom only needed for the characterization).

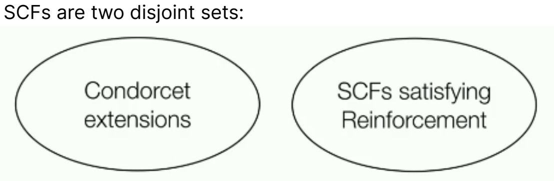
> **Theorem (Young, 1974)**. Borda's rule is *the only SCF* satisfying neutrality, Pareto-optimality, reinforcement, and cancellation.

Proof: Exercise. Pareto-optimality is only required in 1-voter profiles.

Young-Levenglick Theorem: Incompatibility of Borda/Condorcet principles

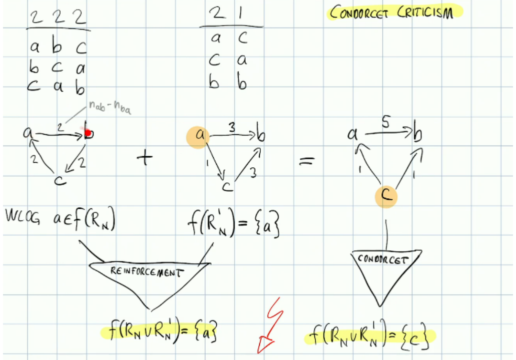
> **Theorem (Young and Levenglick, 1978)**. No Condorcet extension satisfies reinforcement when $m \geq 3$.

Condorcet extensions and reinforcement-satisfying SCFs are two disjoint sets:



This shows that **the rationales between Condorcet's and Borda's ideas are incompatible**.

Proof.



(note: the arrows in the graphs are labelled with $n_{ab} - n_{ba}$).
(another note: this proof was pointed out by a student of this course, the original one is more complicated)

A Condorcet Criticism

Above proof is seen by some as a criticism of Condorcet's paradigm: adding one "symmetric" profile to another one changes the Condorcet winner. Rebuttal: The added profile is not completely symmetric. It cycles clockwise, there is also another anticlockwise cycle. Also it only shows that Condorcet extensions and reinforcements are incompatible, so one could equally critizice the reinforcement property.

A Borda Criticism

In the following profile:

	99	1
100	a	b
99	b	c
...	...	...
0		a

$\Longrightarrow$ b wins just because of one voter who feels very strongly about placing a at the bottom.

Kemeny's Rule

Kemeny's rule

The two disjoint set above do overlap with exactly one neutral function when aggregating preference relations to sets of preference relations: Kemeny's rule.

Week 7 - Maximum Likelihood SCFs and Kemeny's Rule

Voting as Maximum Likelihood Estimation

> "Democracy is the recurrent suspicion that more than half the people are right more than half the time" (E. B. White, 1943)

Probabilistic model (due to Condorcet):

- assume there exists a "true" ("god-given") preference ranking
- Each voter selects a "true" pairwise comparison with probability "p \in (0.5, 1)": voters are imperfect, but right more often than wrong.

An SCF is a *maximum likelihood SCF* for a given p if it yields all alternatives that are most likely to be top-ranked in the true ranking. In other words, it yields the ranking(s) that maximize(s) the likelihood of observing the given votes, assuming the probabilistic model described above.

> **Theorem (Condorcet Jury Theorem, Condorcet 1785)**. For *two alternatives* and any p, majority rule is the maximum likelihood SCF.

Proof. see presentation. Not too complicated. $p_{ab}(R_N) > p_{ba}(R_N) \Leftrightarrow p^{n_{ab}-n_{ba}}(1-p)^{n_{ba}-n_{ab}} > 1$
$\Longleftrightarrow \left(\frac{p}{1-p}\right)^{n_{ab}-n_{ba}} > 1$ which means $n_{ab} - n_{ba} > 0$.

"The more members you have in a jury, the more likely you are to get the right outcome."

> **Theorem (Young, 1988)**. If p is sufficiently close to 0.5, the maximum likelihood SCF is *Borda's rule*.

Proof left out. Idea: Borda winners are alternatives that receive most "pairwise votes".

QUESTION: WHY "sufficiently close to 0.5"? Shouldn't p be close to 1 for this to work well?

Kemeny's Rule

A *social preference function* (SPF) - which is a set-valued version of a social welfare function - is a function

$$f : R(U)^N \to F(R(U))$$

Definition:

$$f_{Kemeny}(R_N) = \arg\max_{R \in R(U)} \sum_i |R \cap R_i|$$

i.e. all the R such that the sum of all pairwise preference relations that it shares with any voter i is maximized. Here $R(U)$ is the set of *strict* rankings (declaration of oddity)!

Related (shortly mentioned): Kendall-Tau Distance.

> **Theorem ((Condorcet 1785), Young 1988)**. Kemeny's rule is the maximum-likelihood SPF for any p.

Kemeny's rule can be interpreted as a "scoring rule on rankings".

Example (Condorcet, 1785)

23	17	10	8	2
a	b	c	c	b
b	c	a	b	a
c	a	b	a	c

ab bc ac

23+10 + 23+17+2 23+2 = 100

abc	100
acb	76
bac	94
bca	104
cab	86
cba	80

Kemeny's Rule and Majority Graphs

A Kemeny ranking is an *acyclic subgraph with maximum weight* in the majority graph: If cycles were allowed, R_M would have maximal Kemeny score; with cycles not allowed, every Kemeny edge (y, x) that does not coincide with the corresponding majority edge (x, y) leads to a penalty of $n_{xy} - n_{yx}$.

The *majority graph* has weights $n_{xy} - n_{yx}$. If we find a set of edges with minimal accumulated weight whose removal breaks all cycles, the inversion of these edges yields a Kemeny ranking.

> **Lemma**. Let $G = (V, E)$ be a directed graph, $E' \subseteq E$. G can be made acyclic by inverting a subset of edges in E' if and only if $(V, E \setminus E')$ is acyclic.

(in other words: removing and inverting can both be used to break cycles)

Social choice axioms for SPFs

Adapt definitions of anonymity/neutrality to SPFs:

- SPF f is anonymous if $f(R_N) = f(R'_N)$ for all R_N, R'_N s.t. a permutation $\pi : N \to N$ exists with $R_i = R'_{\pi(i)}$.
- SPF f is neutral if $\pi(f(R_N)) = f(R'_N)$ for all R_N, R'_N such that a permutation $\pi : U \to U$ exists with $R_N = \pi(R'_N)$

Note: the SPF version of neutrality does *not* imply IIA.

Furthermore, define *reinforcement* and *Condorcet consistency* for SPFs:

- SPF f satisfies reinforcement if for all disjoint N, N' and all R_N, $R_{N'}$ with $f(R_N) \cap f(R_{N'}) = \emptyset$, $f(R_N) \cap f(R_{N'}) = f(R_N \cup R_{N'})$
  - "seems easier to satisfy for SPFs than for SCFs".
- SPF f satisfies *Condorcet consistency* if for all R_N and $R \in f(R_N)$ and all x, y that are adjacent in R, xRy implies xR_My.
  - (x, y are adjacent in R if there is no z with $xPzPy$ or $yPzPx$)

Also define *Local Independence of Irrelevant Alternatives* (LIIA) and *Pareto optimality* for SPFs:

- SPF f satisfies *LIIA* if for all R_N, R'_N, $R \in f(R_N)$, $R' \in f(R'_N)$ and x, y adjacent *in both* R *and* R',

$$(\forall i : R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}) \implies R|_{\{x,y\}} = R'|_{\{x,y\}}$$

  - LIIA is weaker than Condorcet consistency
- SPF f satisfies Pareto optimality if for all R_N, x, y and $R \in f(R_N)$, $(\forall i : xP_iy) \implies xPy$

Young's Characterizations of Kemeny's Rule

Young's first characterization, using reinforcement and Condorcet consistency:

> **Theorem** (Young & Levenglick 1978).
> Kemeny's rule is the *only SPF* that satisfies neutrality, reinforcement and Condorcet consistency.

Young's second characterization, using LIIA and Pareto optimality (among others):

> **Theorem** (Young 1988).
> Kemeny's rule is the *only SPF* that satisfies anonymity, neutrality, Pareto optimality, reinforcement and LIIA.

In other words: weakening Condorcet consistency to LIIA requires adding anonymity and Pareto optimality to the mix.

Week 8 - Computing Kemeny, Escape Route 3, Top Cycle

Computing Kemeny Rankings

Intuition beforehand: the whole problem smells like NP hardness.

> **Theorem (McGarvey, 1953)**. For every majority graph G with weight 1 on every edge, there exists a preference profile with an odd number of voters that induces G.

Proof. Idea: for each edge, add two voters s.t. only this edge is induced, and all other edges remain unchanged. Do this for an edge (a, b) by adding the following voters:

1	1
a	z
b	...
c	c
...	**a**
z	**b**

Do this for every edge of the majority graph to finish.

QUESTION: Where does "weight 1 on every edge" come into play? How do we ensure oddness of the number of voters? (right now they are even).

Answer to the oddness question: just add one voter in the beginning, then flip all wrong edges.

Note: McGarvey's construction requires $O(m^2)$ voters, but due to Erdos and Moser: you only require $\Theta(m/logm)$ voters.

NP-Completeness of Computing Kemeny Rankings

Feedback Arc Set (FAS) problem: "Is it possible to make a given directed graph *acyclic* by removing k edges?".

> **Theorem (Karp, 1972)**. FAS is NP complete.

> **Theorem (Alon 2006; Charbit et al. 2007; Conitzer 2006)**. FAS is NP-complete even when restricted to *tournaments*, i.e. complete oriented graphs.

We introduce a Kemeny score decision problem: "Does there exists a ranking with Kemeny score at least s?"

> **Theorem (Bartholdi et al. 1989)**. The Kemeny ranking decision problem is NP-complete.

Proof. Reduce tournament-restricted FAS to the Kemeny problem. Assume we have a tournament-restricted FAS instance: a complete oriented graph G and an integer k. Construct a preference profile with an odd number of voters corresponding to G using McGarvey's theorem.

The lemma from last week implies that G can be made acyclic by removing k edges $\iff$ R_N admits a ranking with Kemeny score at least

$$s = \frac{m(m-1)}{2}\frac{n+1}{2} - k$$

In addition, we also show that finding a Kemeny ranking is also hard: Just consider the Turing reduction where we have a Kemeny-ranking-computing oracle. This trivially lets us solve the decision problem.

More advanced results:

> **Thereom (Dwork et al. 2001; Bachmeier et al. 2016)**. Finding a Kemeny ranking is NP-hard for even $n \geq 4$ and odd $n \geq 7$.

The question for $n = 3$ and $n = 5$ is open.

> **Theorem (Hemaspaandra et al. 2005; Fitzsimmons et al. 2021)**. Deciding whether a given ranking is a Kemeny ranking is coNP-complete and deciding whether a given alternative is a Kemeny winner is Θ_2^P-complete (even worse than NP-completeness).

> **Thereom (Kann 1992)**. FAS is APX-hard, i.e. it cannot be approximated efficiently.

One positive result:

> **Theorem (Kenyon-Mathieu et al. 2007)**. There exists a *polynomial-time approximation scheme* (PTAS) for *weighted tournament* FAS.

Escape Route 3 - Only Require Expansion Consistency

Recall the basics of choice theory: α, γ, β^+, furthermore rationalizability = $\alpha \wedge \gamma$ and trans. rat. = $\alpha \wedge \beta^+$.

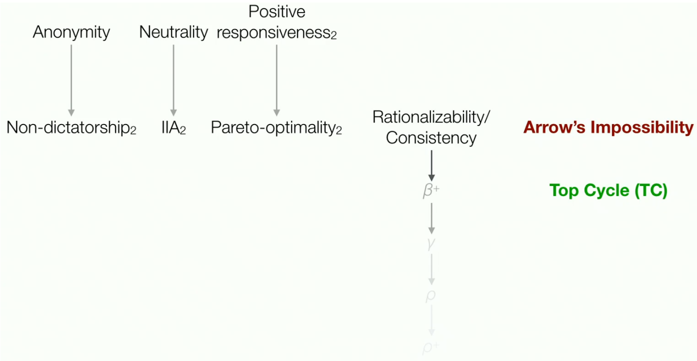
Also recall the Arrovian impossibility results.

- Sen (1977): all these impossibility proofs are statements about the base relation; every impossibility involving rationalizability can be turned into one involving only α.
- *Strong Condorcet-May Impossibility*: No SCF satisfies anonymity$_2$, neutrality$_2$, positive responsiveness$_2$ and α
- *Strong Mas-Colell/Sonnenschein impossibility*: Every SCF with IIA, positive responsiveness$_2$ and α admits a weak dictator.

Dropping α

Takeaway: α is the culprit, without α we can get better results! β^+ has no implications on the acyclicity of the base relation (example of cyclic CF satisfying β^+ see slides).

Contraction consistency: **devastating**, even in its weakest form $\longleftrightarrow$ Expansion consistency: **much less harmful**, even in its strongest form.



The Top Cycle

Due, among others, to John I. Good.

- a *dominant set* is a non-empty subset of alternatives B such that for all $x \in B$ and $y \in A \setminus B$, xP_My.

Denote the set of dominant sets by $Dom(A, P_M)$.

The set $Dom(A, P_M)$ is ordered by $\subseteq$: If $X, Y \in Dom(A, P_M)$, $x \in X \setminus Y$ and $y \in Y \setminus X$, then xP_My and yP_Mx (contradiction).

The *minimal dominant set* is called the *Top Cycle* (TC).

$$TC(A, P_M) = \bigcap Dom(A, P_M)$$

The same construction is also nown as *GETCHA* or the *Smith Set*.

Axioms fulfilled by TC

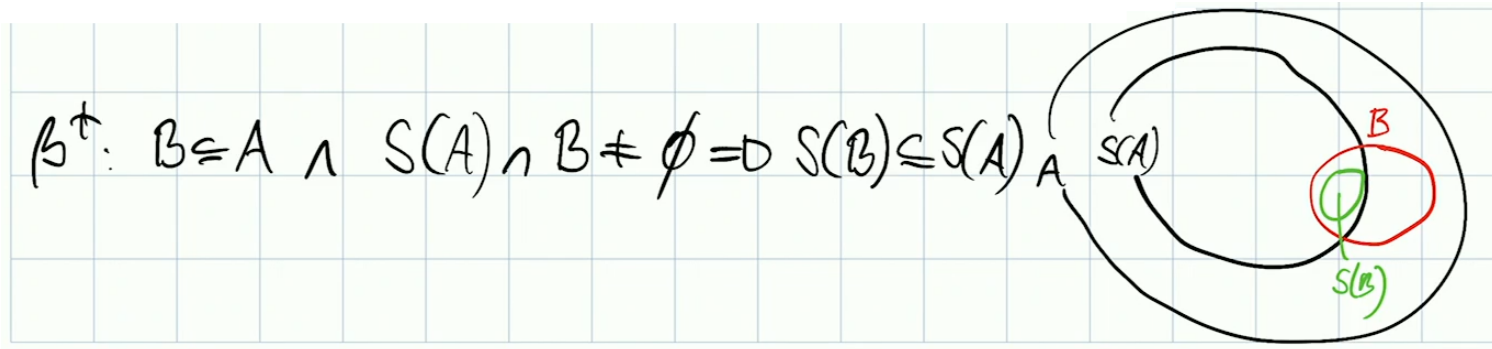
TC is a Condorcet extension.

> **Theorem (Bordes, 1976)**. The top cycle is the *finest* SCF that satisfies anonymity, neutrality, positive responsiveness$_2$, and β^+.

An SCF f is *finer* than f' if $f(X) \subset f'(X)$ for all X. Some trivial SCFs also satisfy above axioms, but are coarser than the top cycle.

Proof. In two steps: every SCF that satisfies the axioms contains the top cycle; the top cycle itself satisfies the axioms.

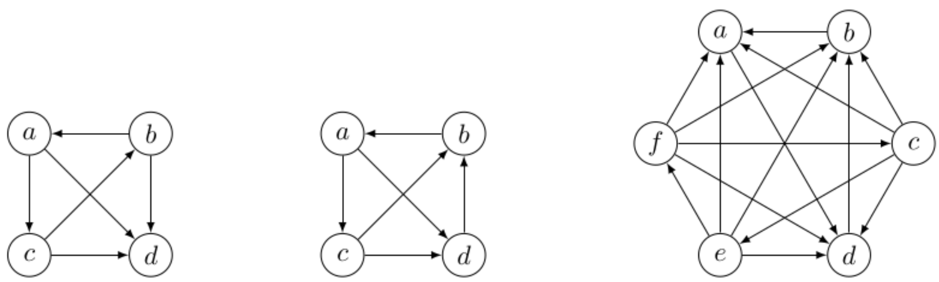
Recall β^+:



Step 1, let f satisfy the axioms. Let $B = \{x, y\}$ with $x \in f(A)$ and $y \in A \setminus f(A)$. By β^+, $S(B) = \{x\}$ (y cannot be contained because it is not in $S(A)$). By May's Theorem, xP_My. Therefore $f(A)$ is a dominant set, and the top cycle is containted in it.

Step 2, show that TC satisfies the axioms. [TODO... I think we only showed β^+]

Top Cycle examples



$TC(A,P_M)=\{a,b,c\}$ $TC(A,P_M)=\{a,b,c,d\}$ $TC(A,P_M)=\{c,e,f\}$

Majoritarian SCFs

The top cycle only depends on majority rule (i.e. the base relation).

- an SCF is *binary* if for all R_N, R'_N: If f agrees on pairs for R_N, R'_N, it agrees on all sets.

A *majoritarian SCF* is an SCF that satisfies anonymity, neutrality, positive responsiveness$_2$ and *binarity*.

Properties of majoritarian SCFs:

- majoritarian SCFs are a subset of the C1 functions: additionally, have to be neutral + make pairwise choices according to majority rule
- three of Arrow's conditions are satisfied by maj. SCFs: IIA, PO$_2$, and non-dictatorship.

The strict part of majority rule P_M for a given preference profile defines a *tournament* (= complete directed graph) on A, written (A, P_M). We denote majoritarian SCFs as function of tournaments: e.g. $CO(A, P_M)$ and $TC(A, P_M)$.

> **Lemma**. Let S be majoritarian and $aP_MbP_McP_Ma$. Then $S(\{a,b\}, P_M) = \{a\}$ and $S(\{a,b,c\}, P_M) = \{a,b,c\}$.

For odd numbers of alternatives, you can find quite symmetric tournaments (called *cyclones*) - has to do with the complete automorphism group.

Some more new notation:

- Dominion of x $D(x) = \{y \in A \mid xP_My\}$
- Dominator $\overline{D}(x) = \{y \in A \mid yPMx\}$

(In graph theory language: outgoing/incoming neighbors).

Furthermore, define inductively $D^k(x)$ and $\overline{D}^k(x)$ as the vertices that can be reached in k steps, or can reach x in k steps (including x itself). Finally, define $D^*(x) = D^{|N|}(x)$ and $\overline{D}^*(x) = \overline{D}^{|N|}(x)$ as the alternatives that can be reached from x/that can reach x.

Week 9 - Computing Top Cycle, Uncovered Set

Top Cycle Linear-Time Algorithm

Linear in the number of edges (i.e. quadratic in the number of alternatives).

- computing the *minimal dominant set containing a given* x:
  - start with $B = \{x\}$
  - iteratively: add all alternatives that dominate some alternative in B, until no more such alternatives exist

This gives a different characterization of dominant sets: they are all of the form $\overline{D}^*(x)$ for some $x \in A$ ($\overline{D}^*(x)$ are the alternatives that can reach x on some path).

$$Dom(A, P_M) = \{\overline{D}^*(x) \mid x \in A\}$$

So: $Dom(A, P_M)$ can be computed in $O(m^3)$ time.

To improve runtime: we want to find an alternative in linear time which we *know* is contained in the TC. Claim: this is the case for Copeland winners.

$$CO \subseteq TC$$

...and CO is computed in linear time (linear in the number of edges).

Proof that $CO \subseteq TC$: Assume some Copeland winner a outside the TC dominates a set of alternatives $D(a)$, also outside the TC. But then the TC nodes all point to a as well as to $D(a)$, i.e. have higher degree than a.

Interlude: alternative way to draw tournaments

Draw all nodes on a vertical line, only draw the edges from bottom to top.

Transitive Closure

Idea: the problem we are always dealing with are cycles in the base relation. How about we take the reflexive transitive closure of P_M?

> **Theorem (Deb, 1977)** The maximal elements of the transitive closure P_M^* of P_M are equal to the top cycle:
>
> $$TC(A, P_M) = Max(P_M^*, A)$$

Proof. see notes; intuitively quite clear why this holds:

- P_M^* adds edges in both directions along the cycles
- the top cycle elements then have no strict ingoing edges in P_M^*

This leads to another, alternative algorithm:

- find strongly connected components
- find source in the DAG of strongly connected components

Top Cycle and Pareto Optimality

Now we come to the drawbacks. General problem: the TC can be very large.

Therefore, it fails to be Pareto-optimal on more than 2 alternatives!

The Uncovered Set

Covering Relation

In a tournament, x *covers* y if $D(y) \subseteq D(x)$ (Notation: xCy).

C is a *transitive subrelation* of P_M.

- transitivity: obvious since $\subseteq$ is transitive
- subrelation of P_M: assume instead yP_Mx. Then $x \in D(y)$, i.e. $x \in D(x)$, contradiction.

Equivalent definition using dominators:

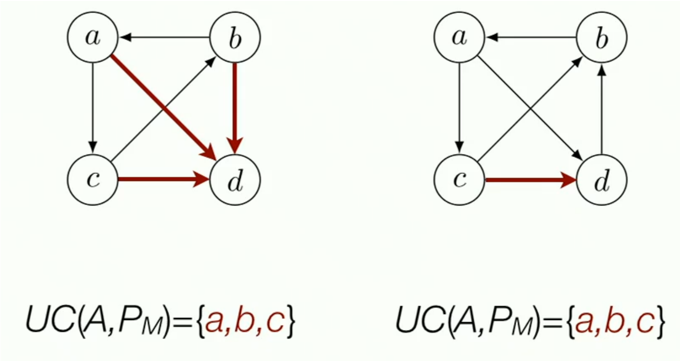
$$xCy \iff \overline{D}(x) \subseteq \overline{D}(y)$$

Uncovered Set

> **Definition (Uncovered Set)**. The Uncovered Set consists of all *uncovered alternatives*:
>
> $$UC(A, P_M) = Max(C, A).$$

- is a Condorcet extension
- alternatively: define as Condorcet winners of all inclusion-maximal subtournaments that have one

Example



In the second example: we are doing better than the top cycle! Recall that TC selects all edges in this case, which makes it non-Pareto-optimal.

Characterization of UC

> **Theorem (Characterization of UC; Moulin, 1986).** The uncovered set is the *finest majoritarian SCF* satisfyfing [expansion consistency γ](#).

Proof.

1. show that for any maj. and γ SCF S, $UC \subseteq S$.
   - let $a \in UC(A)$, show $a \in S(A)$
   - we have $a \in UC(A)$ iff $\forall x \in \overline{D}(a) \exists y \in D(a) : yP_Mx$
   - (full proof see notes)
2. show that UC satisfies maj. and γ
   - by contradiction: let $x \in UC(A) \cap UC(B)$, $x \notin UC(A \cup B)$. Then for some y: yCx. $\implies y \in A \cup B$.
   - (full proof see notes)

Comment: This is a prime example of where the declaration of oddity helps immensely. Without it we would need some very technical extra conditions.

Also, since β^+ implies γ, $UC \subseteq TC$.

UC and Pareto Optimality

Also: UC satisfies PO.

> **Theorem**. UC satisfies PO.

Proof. Let aP_ib for all i. Then if c is covered by b, then bP_Mc and therefore aP_Mc. Therefore aCc.

> **Theorem (Brandt & Geist, 2014)**. UC is the largest majoritarian SCF satisfying Pareto-optimality.

Therefore: UC the *only* majoritarian SCF that satisfies PO and γ.

Uncovered Set Algorithm

- Naively: in $O(m^3)$
- We can do better. However if a linear $O(m^2)$ algorithm exists is open

Two-Step Principle for UC

First, an equivalent characterization of UC:

> **Theorem (Shepsle & Weingast, 1984)**: UC consists precisely of all alternatives that reach every other alternative in *at most two steps*, i.e.
>
> $$UC(A, P_M) = \{x \in A \mid D^2(x) = A\}$$
>
> In graph theory terms, vertices that reach any other vertex on a path of at most 2 are called *kings*.

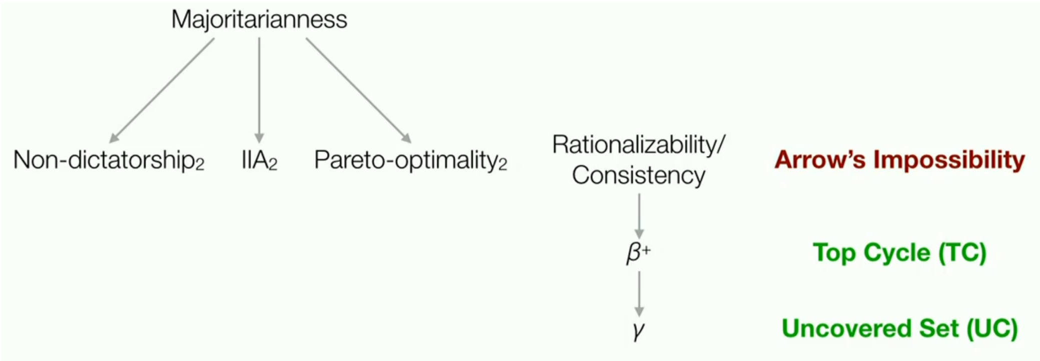
Proof. Claim: this was already shown via the proof we saw above. [TODO think about this]

Matrix Multiplication algorithm for UC

```
def UC(A, PM):
        M = matrix(lambda i, j: 1 if PM(i,j) else 0)
        U = M^2 + M + I
        B = [i for i in A if all(U[i,j] != 0 for j in A)]
        return B
```

Matrix multiplication, asymptotically, can be done in $O(m^{2.37286})$ (Alman et al. 2021). Strongly based on the Coppersmith and Winograd algorithm (1990).

A conjecture is: matrix multiplication is believed to be possible in $O(m^2)$ time.



Week 10 - Banks Set, Tournament Equilibrium Set

Week 10

Banks Set

A transitive subset of a tournament is a set of alternatives within which P_M is transitive.

Let $Trans(A, P_M) = \{B \subseteq A : B \text{ is transitive}\}$.

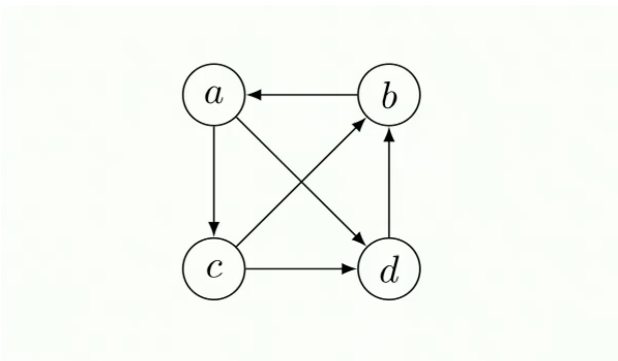
The *Banks Set* BA consists of the maximal elements of all inclusion-maximal transitive subsets:

$$BA(A, P_M) = \{Max(P_M, B) \mid B \in Max(\supseteq, Trans(A, P_M))\}$$

Characterization: $x \in BA(A, P_M)$ if and only if there exists some $B \in Trans(A, P_M)$ such that $x \in Max(P_M, B)$ and $(\nexists a \in A : \forall b \in B : aP_Mb)$. In other words: x is the maximum of some transitive subset of the tournament, and this transitive subset cannot be extended from above.

Examples

The "only interesting 4-tournament up to isomorphism": $a \to c \to d \to b$, $a \to d$, $c \to b$.



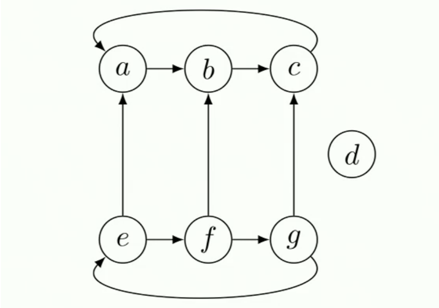
Then

- $TC(A, P_M) = \{a, b, c, d\}$
- $UC(A, P_M) = \{a, b, c\}$
- $BA(A, P_M) = \{a, b, c\}$

- a from $\{a, b, d\}$
- b from $\{a, b\}$
- c from $\{b, c, d\}$
- d could only be from $\{d, b\}$, but this can be extended from above by c

An example where $BA \neq UC$:

(All missing edges are pointing downwards.)



Then

- $TC(A, P_M) = \{a, b, c, d, e, f, g\}$
- $UC(A, P_M) = \{a, b, c, d\}$
- $BA(A, P_M) = \{a, b, c\}$
  - a from $\{A, d, f, g\}$, b and c by symmetry
  - not d (d + two alternatives from the bottom can be extended by an alternative from the top)
  - not e, f and g

We will soon see: $BA(A, P_M) \subseteq UC(A, P_M)$.

Strong Retentiveness ρ^+

A choice function S satisfies *strong retentiveness* ρ^+ if for all $A \in F(U)$ and $x \in A$ with $\overline{D}(x) \neq \emptyset$, $S(\overline{D}(x)) \subseteq S(A)$.

(In other words: the best elements from all dominator sets have to be chosen.)

> **Theorem** (majoritarian + $\gamma \Longrightarrow \rho^+$)
> A majoritarian SCF that satisfies γ also satisfies ρ^+.

Proof. Let S be such an SCF, $a \in A$. Show: $S(\overline{D}(a)) \subseteq S(A)$.

Idea: find enough sub-tournaments where x is chosen, use γ to show that it is chosen from their union = the whole set.

Let $x \in S(\overline{D}(a))$. For all $y \in D(a) : x \in S(\{a, x, y\})$.

- either $x \to a, x \to y, a \to y$, then x is selected from $\{x, a\}$ and from $\{x, b\}$, hence also from their union
  - QUESTION: does this mean that γ implies Condorcet extension?
- or $x \to a \to y \to x$: then from this 3-cycle, by neutrality (from majoritarianness), everything must be selected.

ρ^+ Characterization of the Banks set

> **Theorem** (Brandt, 2011): The Banks set is the finest majoritarian SCF satisfying ρ^+.

Proof.

Lemma 1: If S is majoritarian and satisfies ρ^+, then $BA \subseteq S$.
Let $x \in BA(A)$. To show: $x \in S(A)$.
There is some inclusion-max. trans. $B = \{x = x_0, x_1, \dots, x_k\}$, $x_i P_M x_j \forall i < j$.

Define $C = \bigcap_{i=1}^{k} \overline{D}(x_i)$. Claim: $C = \{x\}$. (if any other z were in C, then B would not be inclusion-maximal).

Then

$$S(A) \overset{\rho^+}{\supseteq} S\left(\overline{D}(x_k)\right) \overset{\rho^+}{\supseteq} S\left(\overline{D}(x_k) \cap \overline{D}(x_{k-1})\right) \overset{\rho^+}{\supseteq} \cdots \overset{\rho^+}{\supseteq} S\left(\overline{D}(x_k) \cap \cdots \cap \overline{D}(x_1)\right) = S(C) = \{x\}$$

Lemma 2: show that BA satisfies major. and $\rho+$.
Let $a \in A$ and $x \in BA(\overline{D}(a))$. Show: $x \in BA(A)$.
Have some inclusion-max. $B \subseteq \overline{D}(a)$, with max. $\{x\}$.
[TODO see notes for the complete proof]

Computing the Banks set

- some alternative in the Banks set can be found efficiently:
  - in linear time
- finding if *some specific alternative* is a Banks winner is *NP-complete*: computing the Banks set is NP hard.

Computing one alternative from the Banks set

```
def banks_set_element(A, PM):
        B = []
        a = A[0]
        while True:
                B = B + [a]
                C = (interesection of dominators)
                if C = []:
                        return a
                a = C[0]
```

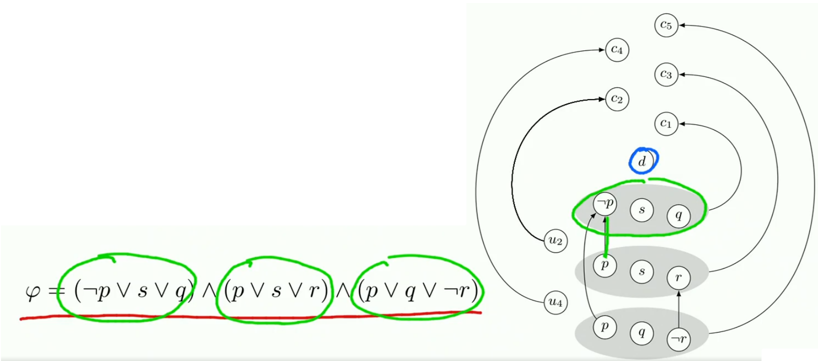
Apparently by choosing a from A or from C in some other way, we could in principle obtain any element from the Banks set; maybe a route of attack for a randomized algorithm?

Computing the Banks set is NP-complete

Computing the Banks set is NP-complete (Woeginger, 2003); It remains NP complete even for 5 voters (even though you can only get a subset of majority tournaments with 5 voters) (Bachmeier et al. 2013).

Proof sketch:

- reduction from CNF-SAT (conjuctive normal form SAT)
- a maximal transitive set with maximal element d has to contain an element of every level below d
- A maximal transitive set with maximal element d cannot contain upward edges
- for some formula ϕ, construct a graph with some node d where d is in the Banks set iff ϕ is satisfiable



Idea:

- for each clause, one grey area
- upward edges for literals that are negations of each other
- from each grey area, at least one element must be taken: otherwise one of the elements on top kick in and dominate d
- the u_2, u_4 elements make sure that no upward edges between the grey areas can be included, and hence no contradicting nodes can be chosen

The Bachmeier et al. paper shows that these kinds of graphs can always be realized with 5 voters.

More on the Banks set

- the Banks set is a singleton $\{x\}$ iff x is a Condorcet winner
  - $\Longrightarrow$ every trans. set without x is eventually extended by x
  - $\Longleftarrow$ Condorcet winner extends all trans. sets
- The same property holds for UC and TC
  - $\Longrightarrow$ because $BA \subseteq UC \subseteq TC$
  - $\Longleftarrow$ was shown earlier ([TODO... where?])

Retentiveness ρ

Strong retentiveness ρ^+ can be further weakened to *retentiveness* ρ:

A choice function S satisfies *retentiveness* ρ if for all $A \in F(U)$ and $x \in S(A)$ with $\overline{D}(x) \neq \emptyset$, we have $S(\overline{D}(x)) \subseteq S(A)$.

Tournament Equilibrium Set

Let S be an arbitrary choice function.

A non-empty set of alternatives B is S-retentive if $S(\overline{D}(x)) \subseteq B$ for all $x \in B$ with $\overline{D}(x) \neq \emptyset$.

Compare with the top cycle:

- in the TC, *no incoming edges allowed*
- with S-retentiveness, this notion is relaxed: incoming edges to x are allowed as long as among the dominators of x, the "best" (i.e. chosen by S) options are in B
  - "no alternative should be properly dominated"

$\mathring{S}$ is a new choice function that yields the *union of all inclusion-minimal S-retentive sets*.

$\mathring{S}$ satisfies retentiveness ρ.

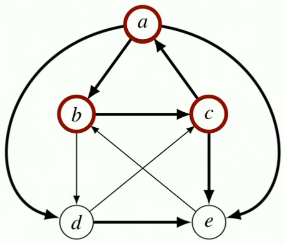
Example: $T\mathring{RIV} = TC$.

Define the *tournament equilibrium set*:

> The *Tournament Equilibrium Set* (TEQ) of a tournament is defined as solution to $TEQ = T\mathring{E}Q$, i.e. the unique fixpoint of $(\mathring{\ })$.

> **Theorem** (Schwartz, 1990):
> $TEQ \subseteq BA$

TEQ Example



A thick edge from y to x
denotes that $y \in TEQ(\overline{D}(x))$.

- $TEQ(\overline{D}(a)) = TEQ(\{c\}) = \{c\}$
- $TEQ(\overline{D}(b)) = TEQ(\{a, e\}) = \{a\}$
- $TEQ(\overline{D}(c)) = TEQ(\{b, d\}) = \{b\}$
- $TEQ(\overline{D}(d)) = TEQ(\{a, b\}) = \{a\}$
- $TEQ(\overline{D}(e)) = TEQ(\{a, c, d\}) = \{a, c, d\}$

Pick *the inclusion-minimal set which has no thick ingoing edge*.

Properties of TEQ

Computing TEQ is NP-hard (Brandt et al. 2010) and remains so even for 7 voters (Bachmeier et al. 2015)
- best known upper bound: PSPACE

> **Theorem**
> The following are equivalent:
>
> - Every tournament contains a unique minimal TEQ-retentive set (Schwartz' conjecture, 1990)
> - TEQ is the unique finest maj. SCF satisfying ρ
> - TEQ satisfies monotonicity
> - TEQ satifies independence of unchosen alternatives
> - TEQ satisfies $\hat{\alpha}$ and $\hat{\gamma}$
> - TEQ is R^K-strategyproof

Schwartz' conjecture is false!
⇒ All the cool properties *do not hold*!

Disproving Schwartz' conjecture

The first proof was *highly non-constructive!*.

Proof: using the probabilistic method.

Smallest counterexample *of the type from the non-constructive proof* requires 10^136 alternatives.

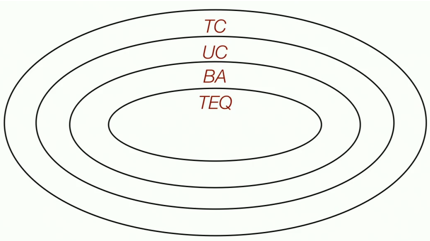
The conjecture holds for up to 14 alternatives.

Counterexample with 24 alternatives constructed (Brandt et al. 2016)

"In principle, TEQ severely flawed; but not clear if there are any practical consequences"

Does this even cast some doubt on the axiomatic method?

Summary: Expansion-Consistent SCFs



Top Cycle (1971)	*TC*	β^+	$O(m^2)$
Uncovered Set (1977)	*UC*	γ	$O(m^{2.38})$
Banks Set (1985)	*BA*	ρ^+	$2^{O(m)}$
Tournament Equilibrium Set (1990)	*TEQ*	(ρ)	$2^{O(m)}$

Week 11 - Escape Route 4, Weaken Both Consistency Conditions; Bipartisan Set

Escape Route 4 - Weaken Both Consistency Conditions

Recall [Week 2 - Choice Theory, Rationalizability, Consistency > Contraction consistency alpha] and [Week 2 - Choice Theory, Rationalizability, Consistency > Expansion consistency gamma].

Redefine α and γ with sets: SCFs yield *sets of alternatives*, but rationality and consistency conditions defined in terms of *alternatives*.

α and γ: An alternative at the intersection of two feasible sets is chosen in both sets iff it is also chosen in the union of both sets.

Set Rationalizability

S is *set-rationalizable* if there is a relation $R \subseteq F(U) \times F(U)$ such that for all $A, X \in F(U)$:

$$X = S(A) \iff X \in Max(R, F(A))$$

The base relation can be extended to sets:

$$X \, R_S \, Y \iff X = S(X \cup Y)$$

Set Consistency

Consistency conditions (as we've seen before):
Let $x \in A \cap B$.

- α: if $x \in S(A \cup B)$, then $x \in S(A)$ and $x \in S(B)$
- γ: if $x \in S(A)$ and $x \in S(B)$, then $x \in S(A \cup B)$

New *Set Consistency Conditions*:

> **Definition**: Set Consistency Conditions $\hat{\alpha}$ and $\hat{\gamma}$.
> **Let** $X \subseteq A \cap B$.
>
> - $\hat{\alpha}$: if $X = S(A \cup B)$ then $X = S(A)$ and $X = S(B)$
> - $\hat{\gamma}$: $X = S(A)$ and $X = S(B)$ then $X = S(A \cup B)$

You can see: syntactically quite similar. However, the notions are actually quite different.

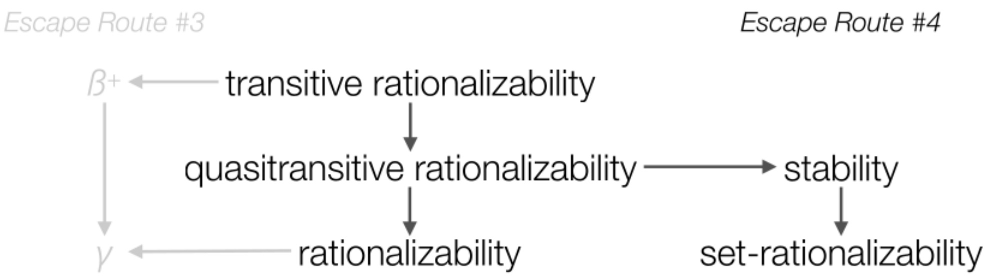
> **Lemma**.
> An SCF S satifies $\hat{\alpha}$ iff for all V, W with $S(V) \subseteq W \subseteq V$, $S(V) = S(W)$.

Stability

> **Theorem** (Brandt & Harrenstein, 2011).
> S is set-rationalizable iff it satisfies $\hat{\alpha}$.

S is *stable* if it satisfies $\hat{\alpha}$ and $\hat{\gamma}$.

S is *quasi-transitively rationalizable* iff it satisfies α, $\hat{\alpha}$, $\hat{\gamma}$.



Set-Rationalizable SCFs

All non-trivial monotonic scoring rules violate $\hat{\alpha}$: e.g.

3	2	1
a	b	c
c	a	b
b	c	a

- $f(R, \{a, b, c\}) = \{a\}$
- $f(R, \{a, b\}) = \{a, b\}$

In fact, *almost every SCF* studied during this course violate $\hat{\alpha}$.

- instant-runoff
- plurality with runoff
- Baldwin's rule
- Black's rule
- Kemeny's rule
- maximin
- Young's rule
- Copeland's rule
- uncovered set
- Banks set

(If a losing alternative is removed, the winning set should not change).

However a handful of Condorcet extensions are set-rationalizable, and even stable. For instance, *the top cycle satisfies $\hat{\alpha}$ and $\hat{\gamma}$.*

The Bipartisan Set

Let (A, P_M) be a tournament and $p : A \to [0,1]$ be a probability distribution over the alternatives. p is *optimal* if

$$u_p(x) := \sum_{y \in \overline{D}(x)} p(y) - \sum_{y \in D(x)} p(y) \geq 0 \text{ for all } x \in A$$

In other words: For every x, the alternative selected by p is at least as likely to dominate x as to be dominated by x. u has that name referring to *utilities*.

> **Theorem** (Laffond et al. 1993; Fisher & Ryan 1995)
> Every tournament admits a *unique* optimal probability distribution.

The theorem is a consequence of the Minimax theorem (1928).

This can be related to game theory (roughly like this?):

- use the skew adjacency matrix of the majority relation (which has $+1$ and -1 for outgoing/incoming edges and 0s on the diagonal)
- use the matrix game defined by this matrix
- apply the min-max theorem to it (which is a special case of the Nash Equilibrium existence theorem)

Proof.
Two observations:

1. For every p, $\sum_{x \in A} p(x) u_p(x) = 0$:

$$\sum_{x \in A} p(x) u_p(x) = \sum_{x \in A} p(x) \left(\sum_{y \in \overline{D}(x)} p(y) - \sum_{y \in D(x)} p(y) \right) = \sum_{x,y \in A} p(x)p(y) sign(n_{xy} - n_{yx}) = 0$$

2. If p is *optimal*, then $\forall x \in A : p(x) > 0 \implies u_p(x) = 0$.

Existence proof: Base case $|A| = 1$ clear.
Consider $|A| > 1$. Let $p \in \arg\max_{pin\Delta(A)} \min_{x \in A} u_p(x)$ and assume for contradiction that $\min_{x \in A} u_p(x) =: v < 0$.

Pick some $z \in A$ s.t. $u_p(z) \geq 0$ (otherwise $u_p(x) = v < 0$ for all x but $\sum_{x \in A} p(x) u_p(x) = 0$, contradiction).

By the IH find some $q \in [0,1]^A$ s.t. $q(z) = 0$ and $\forall x \neq z : u_q(x) \geq 0$.

Let $v = (1-\varepsilon)p + \varepsilon q$ for $\varepsilon in (0,1]$, i.e. some convex combination of p and q. Then for all $x \neq z$ (since $v < 0$),

$$u_v(x) \geq (1-\varepsilon)v + \varepsilon \cdot 0 > v.$$

Also,

$$u_v(z) = (1-\varepsilon)u_p(z) + \varepsilon u_q(z) > v, \text{ given } \varepsilon \text{ is small enough.}$$

So we have found a lottery which performs better than p did:

$$\min_{x \in A} u_r(x) > v$$

which contradicts (*).

Uniqueness proof: Assume $p \neq q$ are both optimal. Assume wlog. that p, q have the same support. [TODO... otherwise?]

Let $v(x) = p(x) - q(x)$ for all x. Then

1. for all $x \in B$:

$$u_r(x) = \sum_{y \in \overline{D}(x)} (p(y) - q(y)) - \sum_{y \in D(x)} (p(y) - q(y)) = u_p(x) - u_q(x) = 0.$$

2. $\sum_{x \in B} r(x) = 0$

(1) and (2) define a *homogeneous system of linear equations with integer coefficients*, which has a non-zero solution r. It also has a non-trivial solution r^* in *integer* numbers.
(probably because if it has non-zero solutions, it is underdetermined, in which case we can multiply through any rational solution).

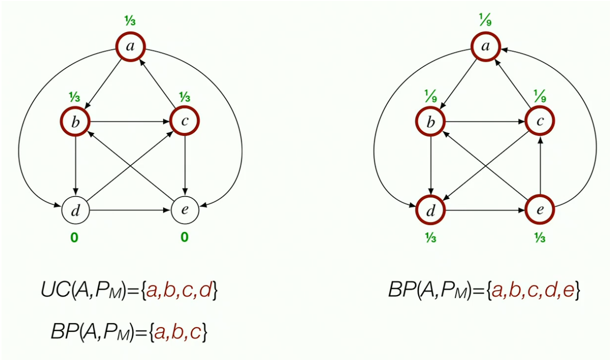
Assume wlog. $r^*(b)$ is odd for some $b \in B$ (otherwise divide all by some 2^k).

Let $x \in B$:

$$0 = \sum_{x \in B} r^*(x) = r^*(x) + \sum_{y \in \bar{D}(x)} r^*(y) + \sum_{y \in D(x)} r^*(y) = r^*(x) + 2 \sum_{y \in \bar{D}(x)} r^*(y).$$

Therefore for all $x \in B$, $r^*(x)$ is even. Contradiction!

Bipartisan Set Example



Properties of the Bipartisan Set

Some "Odd" properties

1. $p(x) > 0$ if and only if $u_p(x) = 0$
2. $|BP(A, P_M)|$ is odd
3. $p(x)$ is the *quotient of odd numbers* for all $x \in BP(A, P_M)$

Proof:
Let p be optimal, i.e. $u_p(x) \geq 0$ for all $x \in A$, and $p(x) \in \mathbb{Q}$. Let l be the lowest commond denominator of all $p(x) \neq 0$ and $p^*(x) = l \cdot p(x) \in \mathbb{N}_0$ for all $x \in A$. Then there is some $b \in As.t. p^*(b)$ is odd.

$$\sum_{y \in A} p^*(y) = p^*(b) + \sum_{y \in \bar{D}(b)} p^*(y) + \sum_{y \in D(b)} p^*(y) = p^*(b) + 2 \sum_{y \in \bar{D}(b)} p^*(y)$$

In the last term, we add an odd and an even number, which is odd. Therefore $\sum_{y \in A} p^*(y)$ is odd (this is the denominator of the fraction).

Let $x \in A$ s.t. $u_p(x) = 0$. Then

$$\sum_{y \in A} p^*(y) = p^*(x) + 2 \sum_{y \in \bar{D}} p^*(y):$$

i.e. an odd number is the sum of $p^*(x)$ and an even number. Hence $p^*(x)$ is odd (this is the numerator of the fraction), hence (1). This also shows that $p^*(x) > 0$, i.e. (2). Since $\sum_{y \in A} p^*(y)$ is odd, there must be an odd number of non-zero summands, which shows (3).

Relationship to other SCFs

1. $BP \subseteq UC$
   - therefore BP is a Condorcet extension
2. Open question: can BP and BA return disjoint choice sets?
3. BP is stable, i.e. it satisfies $\hat{\alpha}$ and $\hat{\gamma}$.
4. BP satisfies *strong monotonicity*: this means it is invariant under the weakening of unchosen alternatives. (Also called "independence of unchosen alternatives").

Characterization of the Bipartisan Set

Introduce a new notion of one SCF being *more discriminating* than another one.

> **Definition** (more discriminating).
> For two majoritarian SCFs S and S', S is *more discriminating* than S' if for some m, S selects *fewer alternatives* than S' *on average* (averaged over all labelled tournaments of size m).

> **Theorem** (Brandt et al. 2017)
> There is no more discriminating stable majoritarian SCF than BP.

In particular, no majoritarian refinement of BP is stable; However BP is not the *unique* stable majoritarian function, as there is no such unique function.

BP can be characterized as the *unique most discriminating majoritarian SCF satisfying stability, monotonicity*, and two further axioms.

Computing the Bipartisan Set

BP can be computed in polynomial time by solving a linear feasibility program.

In game theoretic terms, we are just solving a symmetric zero-sum game.

The Bipartisan Set is P-complete: *it is among the hardest problems in P*. [TODO look up: how P hardness works exactly?]

Week 12 - Strategyproofness

Goal: Resistance against

- misrepresentation of preferences (*strategyproofness*)
- strategic abstention (*participation*)

Plurality and Borda, actually any monotonic scoring rule, satisfy participation.

Resolute SCFs on two alternatives: strategyproofness $\longleftrightarrow$ monotonicty

In any domain with R_M acyclic, $Max(R_M, A)$ satisfies strategyproofness and participation. In particular, approval voting and median voting satisfy these properties in their respective domain.

Our notion of strategy-proofness is most reasonable for resolute SCFs. We usually study non-resolute SCFs (in order to allow for anonymity and neutrality).

Strong Monotonicity

> **Definition** (*Strong Monotonicity*)
> An SCF f is *strongly monotonic* if for all R_N, R'_N and $i \in N$, with $R_N = R'_N$ except xP_iy and yP_ix for some x, y with $x \notin f(R_N)$, implies $f(R'_N) = f(R_N)$.

In other words: f is strongly monotonic if it is *invariant under the weakening of unchosen alternatives*.

Strong monotonicity implies monotonicity (compare with Sheet 1, Exercise 43)

A *resolute* SCF is strongly monotonic if and only if for all R_N, R'_N and $i \in N$ such that $R_j = R'_j$ for all $j \neq i$, it holds that

$$f(R_N) = \{x\} \wedge \left(\forall z : xP_iz \implies xP'_iz\right) \implies f(R'_N) = \{x\}$$

In other words: For a resolute strongly monotonic SCF, we can move any alternative below the winning alternative around however we want, without changing the outcome.

Comment: $\{z \mid xP_iz\}$ is also called *lower contour set* (in economics).

> **Theorem** (Muller & Satterthwait; 1977).
> A resolute SCF is strategyproof if and only if it is strongly monotonic.

Proof. Have R_N, R'_N that only differ for a single voter i, and f resolute.

Definition of strategyproofness we use now: "For all R_i, R'_i, $\neg(f(R'_N)\ P_i\ f(R_N))$". Slightly reformulated:

$$f(R_N) = \{x\} \neq \{y\} = f(R'_N) \implies yP_ix \wedge xP'_iy \qquad [SP^*]$$

Strong monotonicity formulation:

$$\forall R_i, R'_i : f(R_N) = \{x\} \wedge (\forall z : xP_iz \implies xP'_iz) \implies f(R'_N) = \{x\} \qquad [SMON]$$

Direction $SP^* \implies SMON$: Assume for contradiction that SMON, but $f(R'_N) = \{y\} \neq \{x\}$. By SP^*, $xP_iy \wedge yP'_ix$.

Direction $SMON \implies SP^*$: Assume for contradiction that $f(R_N) = \{x\} \neq \{y\} = f(R'_N)$, but yP_ix. Define R''_i where y is moved to the top. Since nothing below x was moved, by $SMON$, x is still selected: $f(R''_N) = \{x\}$. On the other hand, we can start with R'_i and go to R''_i via SMON-preserving operations. Therefore $f(R'_N) = \{y\}$, by SMON, $f(R''_N) = \{y\}$. Contradiction!

Strong Monotonicity: Number of Alternatives

For two alternatives, monotonicity $\iff$ strong monotonicity.

However, for more than two alternatives and more than two voters, things get bad again.

> **Theorem**.
> No resolute Condorcet extension satisfies strong monotonicity when $m, n \geq 3$.

Proof.
Consider the preference profile R_N:

1	1	1
a	b	c
b	c	a
c	a	b

Without loss of generality, assume $f(R_N) = \{a\}$. Create R'_N:

1	1	1
a	b	c
c	c	b

1	1	1
b	a	b

Then $f(R'_N) = \{a\}$ by strong monotonicity. But then f is not a Condorcet extension, which would require $f(R'_N) = \{c\}$.

For $m > 3$, add in bottom-ranked alternatives. For $n > 3$, n odd, we can add pairs of voters which cancel each other out; for n even, a somewhat other construction is needed (skipped here).

Gibbard-Satterthwaite Impossibility

> **Definition** (*non-imposing*).
> An SCF is *non-imposing* if it is surjective, i.e. any alternative x is returned for some R_N.

This gives the following well-known impossibility result, independently discovered by A. Gibbard and M. A. Sattherwaite.

> **Theorem** (*Gibbard & Satterthwaite; 1973; 1975*)
> Every non-imposing, strategyproof, resolute SCF on at least 3 alternatives is dictatorial.

All Gibbard-Satterthwaite conditions are necessary

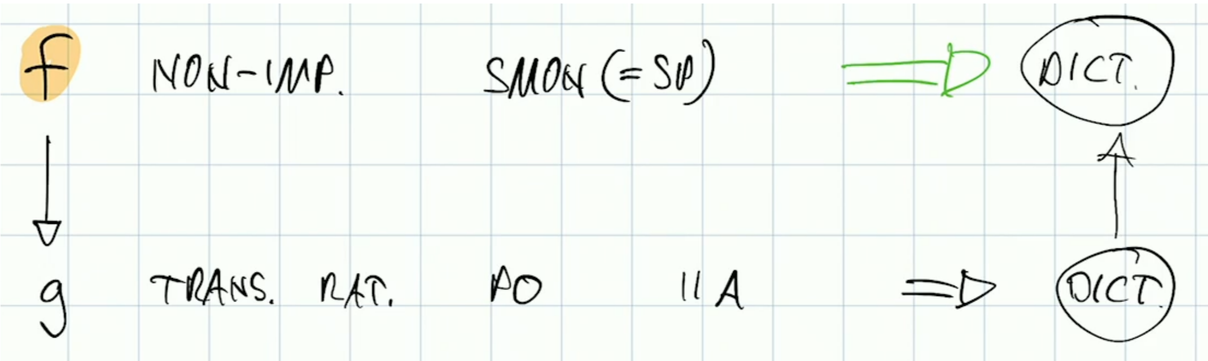
- omit resoluteness: SCF where 1) if all voters agree on x: return x; 2) else return everything
- etc.

One can check that all the conditions are necessary.

Proof of Gibbard-Sattherthwaite Impossibility

Proof here: by reduction to Arrow's impossibility theorem.

- given a non-imposing, strategyproof, resolute SCF f, construct an SCF g that satisfies all Arrovian conditions
- show that dictatorship of f implies dictatorship of g



For $S \subseteq U$, $R_i \in R(U)$, let R_i^S be the ranking obtained by moving all alternatives in S to the top in R_i and not changing their internal ordering. For profiles write $f(R_N^S) = f((R_1^S, \dots, R_n^S))$.

Lemma. f satisfies Pareto-optimality.
Proof: assume xP_iy for all i, but $f(R_N) = \{y\}$. By strong mon. $f(R_N^{\{x\}} = \{y\}$. There exists R'_N such that $f(R'_N) = \{x\}$ because of non-imposition. By strong mon., $f(R_N'^{\{x\}}) = \{x\} = f(R_N^{\{x\}})$, a contradiction.

Because of PO, $f(R_N^S) \subseteq S$ for all R_N and $S \neq \emptyset$.

Define $g(R_N, \{x,y\}) = f(R_N^{\{x,y\}})$. g is well-defined (by PO of f) and the base relation R_g is asymmetric (by resoluteness of f). We now check the Arrovian conditions:

Transitivity of R_g:

- Without loss of generality, $f\left(R_N^{\{x,y,z\}}\right) = \{x\}$ and $f\left(R^{\{}_N y,z\}\right) = \{y\}$.
- By strong monotonicity, $f\left(R_N^{\{x,y\}}\right) = \{x\}$ and $f\left(R_N^{\{x,z\}}\right) = \{x\}$.
- therefore xP_gy, yP_gz, and xP_gz.

Pareto-Optimality PO$_2$

- if xP_iy for all i, then $g(R_N, \{x,y\}) = f(R_N^{\{x,y\}}) = \{x\}$

Independence of infeasible alternatives IIA$_2$:

- If $R_{N|\{x,y\}} = R'_{N|\{x,y\}}$, by SMON,

$$g(R_N, \{x,y\}) = f(R_N^{\{x,y\}}) = f(R_N'^{\{x,y\}}) = g(R'_N, \{x,y\})$$

Since R_g is transitive, we can define $g(R, A) = Max(R_g, A)$. g is tr. rat. and satisfies PO$_2$ and IIA$_2$. *By Arrow's Theorem, g is dictatorial.*

Let i be the dictator fo g. Assume for contradiction that i is *not* a dictator for f: Then for some R_N, $Max(R_i, A) = \{x\} \neq \{y\} = f(R_N)$. By strong monotonicity, $g(R_N, \{x,y\}) = f(R_N^{\{x,y\}}) = \{y\}$. Therefore i is not a dictator for g: contradiction!

The No-Show Paradox: Participation

Condorcet extensions are ruled out if one wants to satisfy participation, with enough numbers of alternatives and voters - Hervé Moulin showed in 1988:

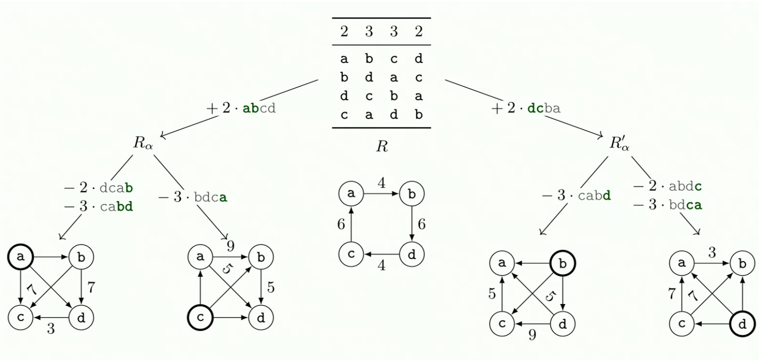
Theorem (*Moulin, 1988*).
No resolute Condorcet extension satisfies participation when $n \geq 25$, $m \geq 4$.

The numbers $n \geq 25$, $m \geq 4$ are *not* tight (but just an artifact of the proof). Examples:

- For $m \leq 3$, Kemeny's rule with appropiate tie-breaking satisfies participation.
- the bound of 25 was improved to 12. For $n \leq 11$, an SCF that satisfies the assumptions, found by a SAT solver, exists.

Theorem (Brandt et al., 2016).
No resolute Condorcet extension satisfies participation when $n \geq 12$, $m \geq 4$.

Proof.



Outlook: Escape Routes from Strategyproofness Impossibilities

- restrict domain of preferences
- computational hardness of manipulation
  - use computational complexity as a barrier to manipulation
- irresolute SCFs
  - requires limited information about tie-breaking
- probabilistic SCFs
  - requires preferences over lotteries

Week 13 - Hardness of Manipulation, Kelly and Fishburn Extensions, Randomized Social Choice

Hardness of Manipulation

Idea: use NP-hardness as a shield against manipulation.

Finding a beneficial manipluation for the following SCFs is NP-hard:

- second-order copeland
  - due to Bartholdi et al., 1989
- instant runoff
  - due to Bartholdi and Orlin, 1991
- Baldwin's rule (Borda score with iterated deletions)
  - Narodytska et al., 2011

Those are the most important results. Many more similar result that use some kinds of weighted voting, or coalitional manipulation.

Some limitations to these approaches:

- the hardness is *in the number of candidates*, and usually the number of candidates is not so large
- often the NP hardness comes from a few hard instances, but practical instances are sometimes much easier to manipulate - then: no barrier against manipulation
  - random manipulations work well
  - there is a large class of distributions/restricted domains that are easy to manipulate

Kelly's Extension

Idea to deal with irresoluteness: we need to have a tie breaking mechanism.

Strong assumption for Kelly's Extension: *a single alternative is eventually chosen, but the voters do not know anything about how the tie-breaking mechanism works.*

Under this assumption, the *preferences over choice sets* are given by *Kelly's preference extension* $R^K \subseteq F(U) \times F(U)$:

$$XR^KY \Leftrightarrow \forall x \in X, y \in Y : xRy$$

(many sets are of course not comparable by this preference extension; R^K is incomplete).

For the strict part, it holds that

$$XP^KY \Leftrightarrow \forall x \in X, y \in Y : (xRY) \wedge \exists x \in X, y \in Y : (xPy)$$

Watch out: even if $aPbPc$, $\{a, b\}$ and $\{a, b, c\}$ are incomparable!

Warning: By this definition, R^K is not reflexive. It does not matter for our purposes because we are only concerned with the strict part.

Kelly Strategyproofness

An SCF is R^K-strategyproof if there are no R_N, R'_N, i s.t. $R_j = R'_J$ for all $j \neq i$ and $f(R'_N)P_i^K f(R_N)$.

Most *$SCFs$ violate* R^K-strategyproofness: e.g. plurality, Borda's rule, Copeland's rule, Nanson's rule.

> **Theorem** (Brandt, 2015). Every strongly monotonic SCF is R^K-strategyproof.

(Compare the Muller-Satterthwait Theorem for the resolute case; However observe that this result only goes in one direction!)

> **Thereom**. If an SCF satisfies monotonicity, $\hat{\alpha}$ and IIA, it satisfies strong monotonicity.

Proof. See notes.

> **Corollary** (Kelly strategyproofness of BC and UC)
>
> - The Bipartisan Set BP is R^K-strategyproof
> - The Top Cycle TC is R^K-strategyproof

Comment: The Uncovered Set UC also satisfies R^K-strategyproofness, even though it does not satisfy $\hat{\alpha}$.
- (because $BP \subseteq UC$ and $BP = UC$ when BP returns a singleton)

Open question: *Is BA R^K-strategyproof?*

Kelly Participation

An SCF satisfies R^K*-participation* if for no $R_N, i \in N$: $f(R_{N-i})P_i^K f(R_N)$.

> **Theorem**. Every majoritarian R^K-strategyproof SCF also satisfies R^K-participation.

Proof.
Proof by contrapositive: simulate abstention by having a voter cancel out his own preferences. Show: No R^K-participation ⇒ not R^K-strategyproof. "Double" a preference profile where all voters are there twice, add preferences R_i and $\overline{R_i}$ (= R_i inverted) to it: by majoritarianness, it holds that

$$f(2R_{N_i} + R_i + \overline{R_i}) = f(R_{N-i})P_i^K f(R_N) = f(2R_N) = f(2R_N + R_i + R_i)$$

(i.e. only one voter changed, and could manipulate by it).

Comment: this argument works for any extension, not just Kelly's extension.

By this theorem, TC, UC, BP all also satisfy R^K-participation.

Kelly Strategyproofness: Negative Results

There are some impossibility results concerning R^K-strategyproofness. These are not very strong, since they impose strong conditions.

> **Theorem** (Barberà 1977; Kelly 1977).
> Every non-imposing R^K-strategyproof, quasi-transitively rationalizable SCF is weakly dictatorial when $m \geq 3$

Comment: the conditions for the theorem are quite restrictive; every Pareto-optimal, quasi-transitively rationalizable SCF that satisfies IIA is weakly dictatorial (see Exercise 16).

> **Theorem** (Barberà 1977).
> Every Pareto-optimal, R^K-strategyproof, positive responsive SCF is dictatorial when $m \geq 4$.

Comment: positive responsiveness requires a high degree of decisiveness; of the SCFs we studied, only Borda's rule and Black's rule satisfy it.

Fishburn's Extension

Alternative assumption: The tiebreaking is performed with respect to some unknown, but consistent ordering. (Whereas in Kelly's Extension, the tiebreaking did not have to satisfy any amount of consistency).

The preferences over choice sets given by *Fishburn's preference extension* $R^F \subseteq F(U) \times F(U)$:

$$XR^FY \Leftrightarrow (\forall x \in X \setminus Y, y \in Y : xRy) \wedge (\forall x \in X, y \in Y \setminus X : xRy)$$

- Fishburn's extension is incomplete as well: e.g. $\{a, c\}$, $\{b\}$ are incomparable
- if $aPbPc$, then $\{a, b\}P^F\{a, b, c\}P^F\{b, c\}$ (unlike in Kelly's extension): more sets are comparable under P^F than under P^K.

Fishburn Strategyproofness and Participation

Fishburn Strategyproofness and Participation are defined equivalently as Kelly Strategyproofness.

Fishburn Strategyproofness is *stronger* than Kelly Strategyproofness.

An SCF satisfies set non-imposition if for every $X \subseteq A$, there is some R_N such that $f(R_N) = X$.

> **Theorem** (Brandt & Lederer, 2021).
> TC is the *only majoritarian SCF* satisfying R^F-strategyproofness and set non-imposition.
>
> Furthermore, TC is the finest majoritarian R^F-strategyproof SCF.

There are a few other R^F-strategyproof SCFs, e.g. f_{Pareto}.

Fishburn Strategyproofness: Negative Results

The following impossibilities were shown using SAT solvers:

> **Theorem** (Brandl et al. 2015).
> There is no Pareto-optimal majoritarian SCF that satisfies R^F-participation if $m \geq 5$.

> **Theorem** (Brandt et al. 2018)
> There is no Pareto-optimal and R^F-strategyproof anonymous SCF if $m \geq 5$ *and preferences are weak*.

Comment how one can use SAT solvers for something like this: The SAT solver gives the induction base case, and there is a very easy induction step (e.g. add more indifferent voters) which allows to go all $m \geq 5$.

Probabilistic SCFs

A *social decision scheme* (SDS) f maps from preference profiles to lotteries over alternatives. Formally:

> An SDS is a function $f : R(U)^N \to [0,1]^U$ such that for all R_N and $p = f(R_N)$, $\sum_{x \in U} p(x) = 1$.

Idea: voters have an underlying utility function, but they only submit preferences; we argue about the possible utility representations that a voter might have.

An SDS is *manipulable* if a voter can increase their *expected utility* by misrepresenting their preferences. Formally:

> An SDS f is *manipulable* if for some $R_N, R'_N, i \in N$ and $u : U \to \mathbb{R}$ such that $R_j = R'_j$ for all $j \neq i$,
>
> $$\forall x, y \in U : (u(x) \geq u(y) \Leftrightarrow x R_i y),$$
>
> and
>
> $$f(R_N) = p, \quad f(R'_N) = p', \quad \sum_{x \in U} p'(x) u(x) > \sum_{x \in U} p(x) u(x).$$

> **Theorem**. Every SDS that puts probability 1 on a Condorcet winner can be manipulated when $m, n \geq 3$.

Proof. Similar to a proof from last week.

Consider the preference profile R_N:

1	1	1
a	b	c
b	c	a
c	a	b

Let $f(R_N) = p$, Without loss of generality, assume $p(a) > 0$. Create R'_N:

1	1	1
a	b	c
c	c	b
b	a	b

Then $f(R'_N) = p'$ and $p'(c) = 1$ (Condorcet winner!). If we choose the utility for a large enough (sufficiently much larger than c), the first voter has an incentive to mis-represent her true preferences (R'_N) and instead give R_N.

Random Dictatorship

Random dictatorships are SDSs of the following form:

- pick a voter at random (the *dictator*)
- choose the favorite alternative of the dictator.

In a uniform random dictatorship, the voters are picked uniformly at random. Uniform random dictatorships are anonymous!

An SDS is *non-imposing* if its image contains all (degenerate) lotteries that yield any given alternative with probability 1.

> **Theorem** (Gibbard, 1977).
> Every non-imposing, non-manipulable SDS is a random dictatorship when $m \geq 3$.

(This is kind of a negative result - it's from the 1970's, after all - but is it really a negative result? Other than dictatorships, random dictarships can be useful in some settings!)

Non-imposition (in the probabilistic context) seems a rather strong property: think of SDS that never put 1 on any alternative, but at most $1-\epsilon$. There is another result in this context

> **Theorem** (Barberà 1979).
> There are probabilistic variant of Borda's rule and Copeland's rule that are non-manipulable (but violate non-imposition).
>
> - in the probabilistic Borda's rule, every alternative gets probability proportional to its Borda score

Maximal Lotteries

Proposed independently by Germain Kreweras and Peter C. Fishburn.

Let $(M_{x,y})_{x,y\in A}$ be the *majority margin matrix*: $M_{x,y} = n_{xy} - n_{yx}$. This is a skew-symmetric matrix.

A lottery p is *maximal* if $p^\top M \geq 0$.

$$\begin{array}{ccc} 1 & 1 & 1 \\ \hline a & b & c \\ b & c & a \\ c & a & b \end{array} \qquad \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}^T \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \geq 0$$

- A maximal lottery essentially picks a "randomized Condorcet winner" (Kreweras 1965; Fishburn 1984); if there is a Condorcet winner, it is assigned probability 1.
- It can be interpreted as a C2 version of optimal distributions (as used for the bipartisan set BP).
- No other lottery q is preferred by an expected majority: $p^\top Mq \geq 0$

A *unique maximal lottery* always exists, and can be efficiently computed via linear programming.

	D. Felsenthal (2018): On Paradoxes Afflicting Voting Procedures	Plurality	Borda	Plurality w/ Runoff	Instant Runoff	SMC	Maximal Lotteries
single profile	Condorcet winner paradox	⚠	⚠	⚠	⚠	–	–
	Absolute majority paradox	–	⚠	–	–	–	–
	Condorcet loser paradox	⚠	–	–	–	–	–
	Absolute loser paradox	⚠	–	–	–	–	–
	Pareto paradox	–	–	–	–	–	–
multi profile	Monotonicity paradox	–	–	⚠	⚠	–	(–)
	Reinforcement paradox	–	–	⚠	⚠	⚠	–
	No-Show paradox	–	–	⚠	⚠	⚠	–
	Twin paradox	–	–	⚠	⚠	⚠	–
	Subset choice paradox	⚠	⚠	⚠	⚠	⚠	–
	Preference inversion paradox	⚠	–	⚠	⚠	–	–

(for lotteries, these properties are extended more or less naturally; for some, there is more than one possible reasonable extension).

Maximal Lotteries vs. Impossibility Theorems

Maximal lotteries

- can be characterized using reinforcement, Condorcet consistency and Cloning consistency (Brandl et al. 2016)
  - resolving the Dilemma of social choice!
- Can be characterized using participation and Condorcet consistency (Brandl et al. 2018)
- can be characterized using IIA and Pareto optimality (Brandl et al. 2020)
  - resolving Arrow's impossibility!

Takeaway: with randomization, several social choice impossibilities can be resolved!