

Then $\#\{\theta_k(z) = 0\} = \sum_{k=1}^u |\theta_k|_0 = \text{rank}(z)$, and hence

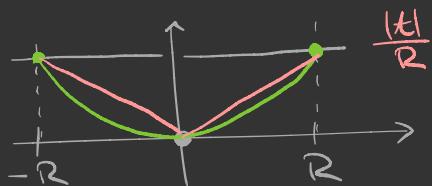
$$X = \underset{\{z_{ij} | (i,j) \in \Omega\} = Y}{\operatorname{argmin}} \sum_{k=1}^u |\theta_k(z)|_0.$$

Transform in easy problem:

Substitute with largest possible function, s.t. it is convex.
We call such a function the lower convex envelope.

We can show that the convex relaxation / lower convex envelope has the same absolute global minimum.

How do we get such function? In our case:



→ Assume that all singular values are contained in $[-R, R]$

→ convex function, but not the largest one

$$\frac{1}{R} |t| \leq |t|_0 \quad \forall t \in [-R, R]$$

is the largest convex function

$$\underbrace{\frac{1}{R} \sum_{k=1}^u |\theta_k|}_\text{convex and} \leq \sum_{k=1}^u |\theta_k|_0 = r$$

$\|z\|_* = \|z\|_1$
nuclear norm

Remember Schatten-norm:
 $\|z\|_p = \left(\sum_{k=1}^u |\theta_k|^p \right)^{1/p}$

Note: Nuclear norm gives lower bound to the rank function.

⇒ We can now write the convex relaxation

$$\boxed{X = \underset{\{z_{ij} | (i,j) \in \Omega\} = Y}{\operatorname{argmin}} \|z\|_*}$$

Then: If Ω is chosen at random and $\#\Omega$ is sufficiently large (but $\ll n \times d$), then

$$X = A^*$$

(again, a combination of LA, optim. and probability)

Lecture 9

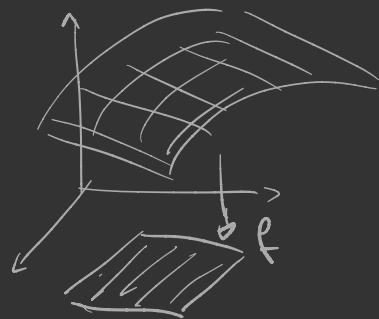
2.5.4. Pseudo-inverse matrices and least squares problems

Remark: Linear Algebra is all about constructing a solution to a linear system of the form

$$\begin{array}{|c|c|c|} \hline & A & x \\ \hline & = & y \\ \hline \end{array} \quad \begin{aligned} Ax &= y \\ \rightsquigarrow x &= A^{-1}y \end{aligned}$$

In physics, we can model things often using differential equations; e.g. what shape does a surface take on after being bent by a force?

- $\Delta u = f$ Poisson Equation
- $\Delta u = \frac{\partial^2}{\partial x_1^2} u + \dots + \frac{\partial^2}{\partial x_d^2} u$ Laplacian



{ we can solve this
by

Finite Element Method (FEM)
using linear systems

In Data Science: we usually don't have a square A , but see two different cases:

1)

Want to solve a matching problem, e.g. looking for a person having all the given characteristics.

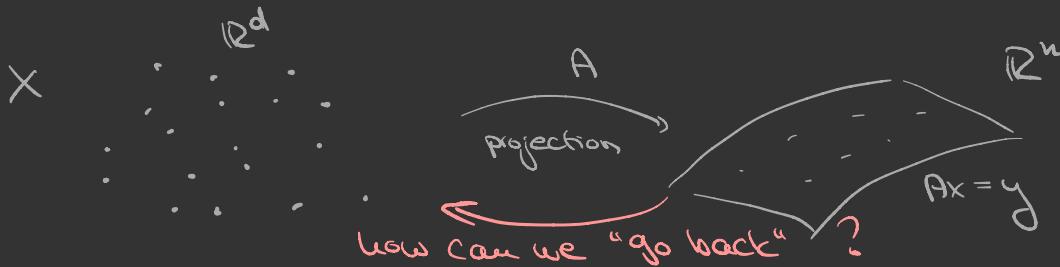
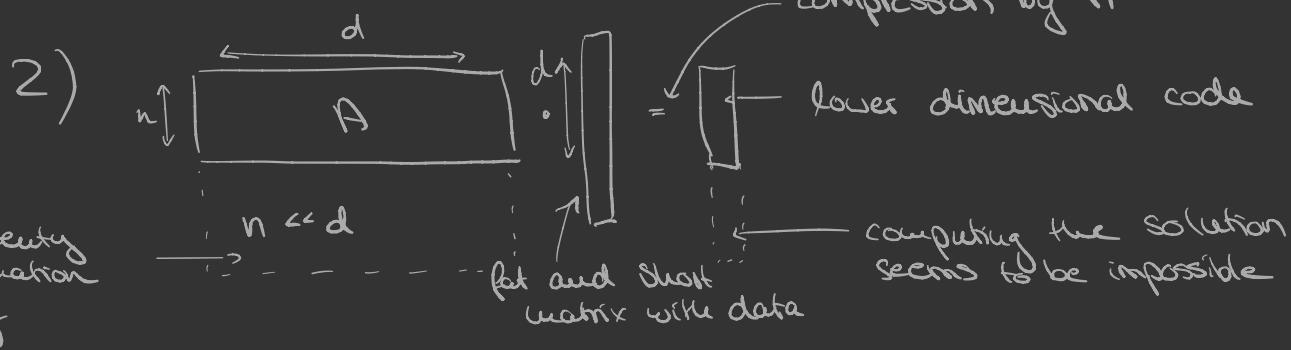
$$\begin{array}{|c|c|c|} \hline \xleftarrow{d} & \text{Characteristics} & \\ \hline & \text{Character 1} & \\ & \text{Character 2} & \\ \hline & \vdots & \\ \hline \xleftarrow{n} & u & \\ \hline \end{array} = \begin{array}{|c|} \hline \vdots \\ \hline \end{array}$$

↑
search for solution

$d \ll n$

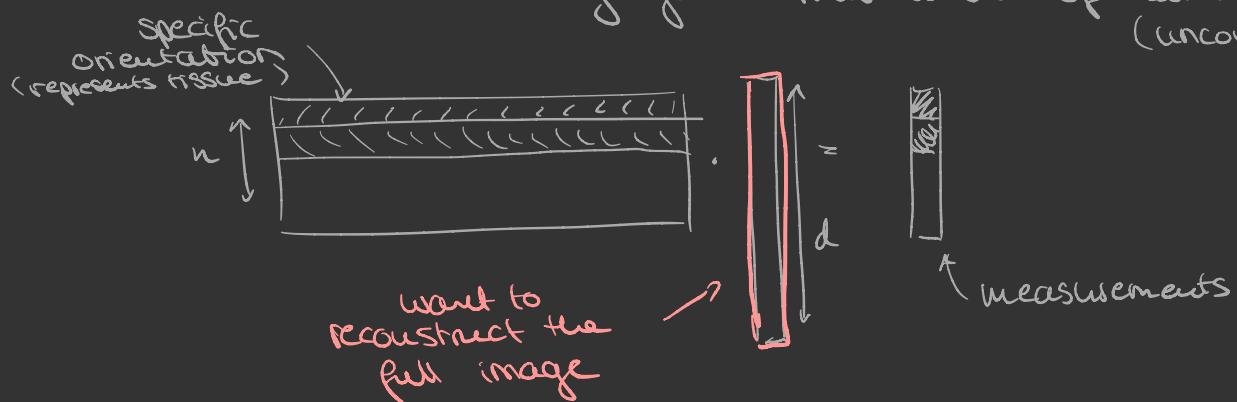
full, slim A

But there might not exist any solution.



Example: Magnetoresonance imaging (MRI)

- Depending on the tissue, H_2O molecules get reoriented.
- Reorientation gives signals which are measured.
- Can only get a small amount of data (unconvenient & expensive)



\Rightarrow Both problems are about reversing A , but since A is not square, there does not exist a true inverse.

Definition: Given $A \in \mathbb{K}^{I \times J}$ with $\text{rank}(A)=r$. Then its SVD is given by

$$A = U \Sigma V^H = \sum_{k=1}^r \sigma_k u_k v_k^H$$

We define the Moore-Penrose pseudo-inverse $A^+ \in \mathbb{K}^{J \times I}$ as

$$A^+ = \sum_{k=1}^r \sigma_k^{-1} v_k u_k^H$$

or equivalently

$$A^+ = V \Sigma^{-1} U^H$$

where we only invert the singular values.

Case 1: Let $d = \# \mathcal{J} \leq \# \mathcal{I} = n$. Then

$A^H A$ is invertible, if $r = \# \mathcal{J}$, i.e. A is full rank.

$$\Rightarrow A^+ = (A^H A)^{-1} A^H$$

$$\begin{aligned}\text{Proof: } (A^H A)^{-1} A^H &= (\underbrace{V \Sigma U^T U \Sigma V}_{})^{-1} \underbrace{V \Sigma U^T}_{} \\ &= (\underbrace{V \Sigma^2 V^H}_{})^{-1} \underbrace{V \Sigma U^H}_{} \\ &= \underbrace{V \Sigma^{-2} V^H}_{} \underbrace{V \Sigma U^H}_{} \\ &= V \Sigma^{-1} U^H \stackrel{\text{Def}}{=} A^+\end{aligned}$$

$$\boxed{n = \# \mathcal{I} \longleftrightarrow \# \mathcal{J} = d}$$

$$\boxed{\begin{bmatrix} A^H \\ A \end{bmatrix} = \begin{bmatrix} A^H \\ A \end{bmatrix}}$$

U, V are unitary.
 $U^H U = U U^H = E$

Properties: (i) $A^+ A = ((A^H A)^{-1} (A^H) A) = I$, i.e. For $d < n$, the pseudo-inverse matrix is a left-inverse for A .

$$\begin{aligned}\text{(ii)} \quad AA^+ &= A (A^H A)^{-1} A^H \\ &= (\underbrace{U \Sigma V^T}_{}) (\underbrace{V \Sigma^{-1} U^T}_{}) \\ &= U U^T\end{aligned}$$

For any $x \in \mathbb{R}^d$, we have

$$U U^T x = \sum_{k=1}^n \langle x, u_k \rangle u_k$$

$$\stackrel{\text{FP}}{=} P_{\text{span}\{u_k\}} x \quad \leftarrow \text{orthogonal projection}$$

Compare
Ex. 2.3

Given $\{u_k\}$ is an ONS for $\text{range}(A)$, this gives an orthogonal projection onto the range of the matrix A .

Case 2: Let $\# \mathcal{I} = n \leq d = \# \mathcal{J}$. Then

$A A^H$ is invertible if $\text{rank}(A) = n$.

(i.e. A is full rank)

$$\boxed{n \uparrow \begin{bmatrix} \xrightarrow{d} \\ A \end{bmatrix} \begin{bmatrix} \uparrow \\ A^H \end{bmatrix} = \begin{bmatrix} A A^H \end{bmatrix}}$$

Then

$$A^+ = A^H (A A^H)^{-1}.$$

Prop: (i) A^+ is a right inverse, i.e. $A A^+ = A (A^H (A A^H)^{-1}) = I$

Summary: Define $A^+ = V \Sigma^{-1} U^H$ | pseudo-inverse matrix.

Case 1 $n \begin{array}{|c|} \hline d \\ \hline A \\ \hline \end{array}$ $d = \text{rank}(A) \leq n \Rightarrow A^+ = (A^H A)^{-1} A^H$
and $A^H A = I$
 $AA^+ = P_{\text{range}(A)}$

Case 2 $n \begin{array}{|c|} \hline d \\ \hline \quad \\ \hline \end{array}$ $n = r \leq d \Rightarrow A^+ = A^H (A A^H)^{-1}$
and $AA^+ = I$
 $A^H A = P_{\text{range}(A^H)}$.

least Squares Problem $Ax = y$.

Case 1: $Ax = y$ is called overdetermined, i.e. we have more equations than variables.

\Rightarrow If we ask "too much" we might don't have a solution.

\Rightarrow We need the "best match" i.e. we look for a solution that minimizes the discrepancy

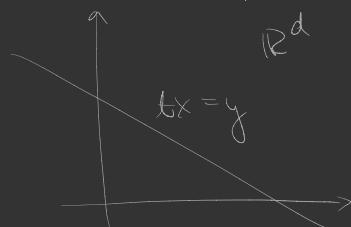
$$\|Ax - y\|_2^2.$$

$$\left\| \begin{array}{|c|} \hline n \\ \hline \quad \\ \hline \end{array} \begin{array}{|c|} \hline d \\ \hline A \\ \hline \end{array} \begin{array}{|c|} \hline \quad \\ \hline x \\ \hline \end{array} - \begin{array}{|c|} \hline y \\ \hline \end{array} \right\|_2^2 \rightarrow \text{minimize over } x$$

Case 2: $Ax = y$ is called underdetermined

$$\begin{array}{|c|} \hline \quad \\ \hline \quad \\ \hline \quad \\ \hline \quad \\ \hline \end{array} = \begin{array}{|c|} \hline \quad \\ \hline \quad \\ \hline \quad \\ \hline \quad \\ \hline \end{array}$$

geometrically, solutions are an affine subspace



\Rightarrow We have an infinite amount of solutions

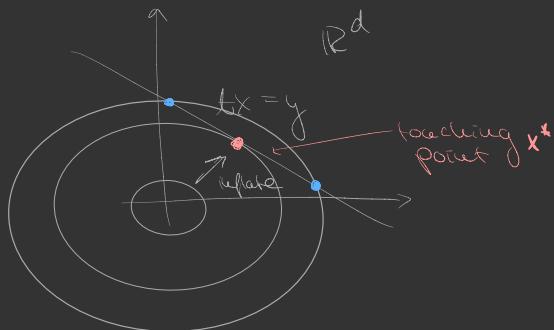
But: How do we decide which solution is the best one?

Remember the Netflix problem, where we optimized over the rank. So we know, that norms can help:

$$\triangleright \|x\|_2 = \left(\sum_{e=1}^d |x_e|^2 \right)^{1/2} \quad l_2\text{-norm / Euclidean norm}$$

Denote $\{\|x\|_2 = R\}$ the ball of radius R .

geometrically, solutions are
an affine subspace



$$\Rightarrow x^* = \underset{Ax=y}{\operatorname{argmin}} \|z\|_2^2$$

i.e. we choose the
solution which additionally
minimizes the $\| \cdot \|_2^2$.

\Rightarrow In both cases we minimize using the $\|\cdot\|_2$, i.e. minimizing the Squares.

Proposition 2.12: Both least squares solutions are computed with the same formula

$$x^* = A^+ y.$$

Remark: Remember the p -norms

$$\|x\|_p = \left(\sum_{e=1}^d |x_e|^p \right)^{1/p} \quad \text{for } 1 \leq p \leq \infty.$$

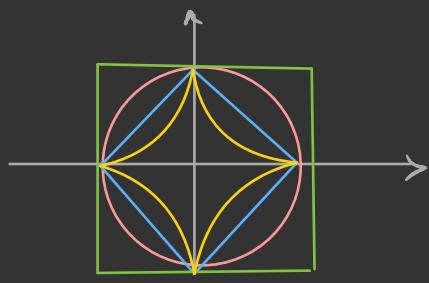
Then for $p \geq 1$, this is actually a norm and hence fulfills the triangle inequality

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p.$$

If $0 < p < 1$, then $\|\cdot\|_p$ is not a norm, but a quasinorm, since it fails the triangle inequality which is replaced by

$$\|x+y\|_p \leq c_p (\|x\|_p + \|y\|_p) \quad \text{for } c_p > 1.$$

We can also see this geometrically by looking at the corresponding R-balls $\|x\|_p = R$.



- $\|x\|_2 = 1 \Leftarrow$ linear, isotropic
i.e. direction is not of significant importance, gives easy solution
- $\|x\|_1 = 1 \quad \|x\|_\infty = 1 \quad \|x\|_{1/2} = 1 \quad \Downarrow$ non-linear, non-isotropic
pointy, non-convex shape

But: real data is not isotropic, because they are not identically distributed all over the space

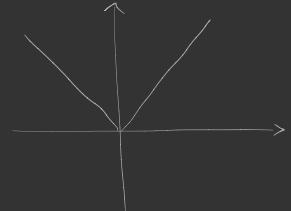
- \Rightarrow algorithms using isotropic norms are not biased
- \Rightarrow algorithms considering directions (i.e. that are not isotropic) are more intelligent

Example: Neural networks: intertwining non-linear activation functions with linear operations.

non-linear,
binary decision
"switch"

$$f_\alpha(t) = \begin{cases} 0, & t \leq \alpha \\ t - \alpha, & t > \alpha \end{cases}$$

"shifted ReLU"



Ex.
 $f_\alpha(t) = \underset{s \geq 0}{\operatorname{argmin}} |s - t|^2 + 2\alpha|s|$

absolute value is
 $\|x\|_1 = \sum_{l=1}^L |x_l|$ in the scalar case.

Each algorithm consists out of memory assignments, loops and decisions. We software them "writes themselves" according to the task, using these fundamental concepts only.

Neural networks do this because

get identified by a certain combination of inputs + outputs

- layers enable loops with feedback architecture
- switches using non-linear activation functions $f_\alpha(t)$.

Without the given structure, we can consider it as a mathematical problem and we can use optimization techniques.

Lecture 10

20.5.2020

Proof: of Proposition 2.12.

Let $A \in \mathbb{K}^{I \times J}$ and $y \in \mathbb{K}^I$. Define $\mathcal{M} \subset \mathbb{K}^d$ to be the set of minimizers of the map $x \mapsto \|Ax - y\|_2$. The convex optimization problem

$$\underset{x \in \mathcal{M}}{\operatorname{argmin}} \|x\|_2$$

has the unique solution $x^* = A^+y$.

1st Optimization
depending on the case, this can be $\#\mathcal{M} = \infty$, or $\#\mathcal{M} = 1$.

2nd Optimization

Proof: We define $\mathcal{M} = \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \|Ax - y\|_2$

Set which might contains many points.

using the SVD as $A = \sum_{k=1}^r \theta_k u_k v_k^T$, $r \leq \min\{n, d\}$

with $\{u_1, \dots, u_r, u_{r+1}, \dots, u_n\} \subseteq \mathbb{R}^n$

complete to be a basis of \mathbb{R}^n or \mathbb{R}^d

and $\{v_1, \dots, v_r, v_{r+1}, \dots, v_d\} \subseteq \mathbb{R}^d$.

Define

$$U = (u_1 \dots u_r) \in \mathbb{R}^{n \times r}, \tilde{U} = \begin{pmatrix} u_1 & u_2 & \dots & u_r & u_{r+1} & \dots & u_n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

and

$$V = (v_1 \dots v_r) \in \mathbb{R}^{d \times r}, \tilde{V} = \begin{pmatrix} v_1 & v_2 & \dots & v_r & v_{r+1} & \dots & v_d \end{pmatrix} \in \mathbb{R}^{d \times d}$$

Then

$$A = U \Sigma V = n \uparrow \left(\begin{array}{c|c} \xrightarrow{r} & \xleftarrow{n-r} \\ \diagdown & \diagup \\ \xrightarrow{r} & \xleftarrow{n-r} \end{array} \right) n \uparrow \left(\begin{array}{c|c} \xrightarrow{r} & \\ \diagdown & \diagup \\ \theta_1 & 0 \\ \vdots & \vdots \\ \theta_r & 0 \\ 0 & \ddots \\ 0 & \theta_{r+1} \\ \vdots & \vdots \\ 0 & \theta_d \end{array} \right) d \uparrow \left(\begin{array}{c|c} \xrightarrow{r} & \\ \diagdown & \diagup \\ \xrightarrow{r} & \xleftarrow{d-r} \end{array} \right) d \uparrow \left(\begin{array}{c|c} \xrightarrow{r} & \\ \diagdown & \diagup \\ \tilde{U} & \Sigma \tilde{V}^T \end{array} \right)$$

$$\theta_{r+1} = \dots = \theta_d = 0$$

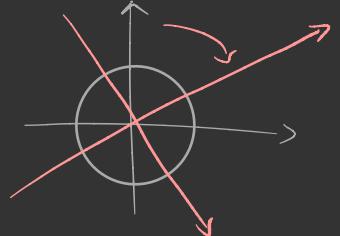
let $x \in \mathbb{R}^d$. Then we define

$$z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tilde{V}^T x, z_1 \in \mathbb{R}^r, z_2 \in \mathbb{R}^{d-r}$$

$$b := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \tilde{U}^T y, b_1 \in \mathbb{R}^r, b_2 \in \mathbb{R}^{n-r}$$

Recall that the Euclidean norm is invariant under rotations or unitary transformations, i.e. the direction doesn't matter.

Hence, for the first optimization, we have



$$\begin{aligned}
 \|Ax - y\|_2^2 &= \|\tilde{U}^T(Ax - y)\|_2^2 \\
 &= \|\tilde{U}^T(\tilde{U}\tilde{\Sigma}\tilde{V}^T x - y)\|_2^2 \\
 &= \|\tilde{\Sigma}\tilde{V}^T x - \tilde{U}^T y\|_2^2 \\
 \tilde{\Sigma} = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & 0 \end{pmatrix} &\quad = \left\| \begin{pmatrix} \Sigma z_1 \\ 0 \cdot z_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|_2^2 \\
 &= \left\| \begin{pmatrix} \Sigma z_1 - b_1 \\ -b_2 \end{pmatrix} \right\|_2^2 \\
 &\quad \text{↑ indep. of } x \\
 &= \underbrace{\|\Sigma z_1 - b_1\|_2^2}_{\text{↑ need to minimize this by choice of } x} + \underbrace{\|b_2\|_2^2}_{\text{fixed}}
 \end{aligned}$$

We can see that

↑ need to minimize this by choice of x

$$M = \left\{ x \in \mathbb{R}^d \mid \Sigma z_1 = b_1, z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tilde{V}^T x \right\}$$

we can write equality because all can explicitly be calculated using x, A and SVD components

$$\hat{M} = \left\{ x \in \mathbb{R}^d \mid z_1 = \Sigma^{-1} b_1 \right\}.$$

For the 2nd optimization, we rewrite

$$\begin{aligned}
 \|x\|_2^2 &= \|\tilde{V}^T x\|_2^2 = \|z_1\|_2^2 + \|z_2\|_2^2 \\
 &\quad \uparrow \text{invariant for unitary transfo} \quad \uparrow \text{fixed with } z_1 = \Sigma^{-1} b_1.
 \end{aligned}$$

and can optimize x such that $\|z_2\|_2^2 = 0$, for $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tilde{V}^T x$.

$$\begin{aligned}
 \Rightarrow x^* : z_1^* &= \Sigma^{-1} b_1, \quad z_2^* = 0. \\
 (\tilde{V}^T)^{-1} = \tilde{V} &\Rightarrow x^* = \tilde{V} \begin{pmatrix} z_1 \\ 0 \end{pmatrix} = \tilde{V} \underbrace{\begin{pmatrix} \Sigma^{-1} 0 \\ 0 0 \end{pmatrix}}_{\substack{z_1 = \Sigma^{-1} b_1 \\ \text{for } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \tilde{U}^T y}} \tilde{U}^H y = A^+ y.
 \end{aligned}$$

□

Remark: Prop. 2.12 holds no matter if $n \leq d$ or $d \leq n$,
i.e. it is "universal".

Case 1:

overdetermined

$d \gg n$

$$M = \{x^*\} \quad \text{so that} \quad \underset{x \in M}{\operatorname{argmin}} \|x\|_2 = x^*$$

minimizes $\|Ax - y\|_2$, doesn't have to be $Ax = y$.

$$\Rightarrow A^T y = \underset{x \in \mathbb{R}^d}{\operatorname{argmin}} \|Ax - y\|_2$$

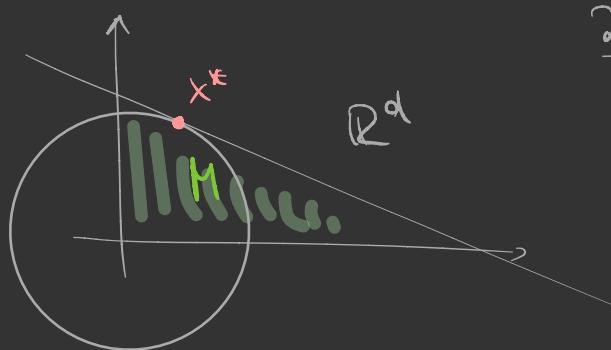
Case 2:

underdetermined

$d \ll n$

1st optimization

$M = \{Ax = y\}$ gives an infinite set



2nd optimization

$x^* = \underset{x \in M}{\operatorname{argmin}} \|x\|_2$
gives unique solution

$$\Rightarrow A^T y = \underset{Ax=y}{\operatorname{argmin}} \|x\|_2$$

here we have
equality, but too
many solutions

Least Squares problem: We have the sphere and $\|\cdot\|_2$
 that solutions are linear
 \Rightarrow all about PF-theorem, isotropy and linearity.

Example:

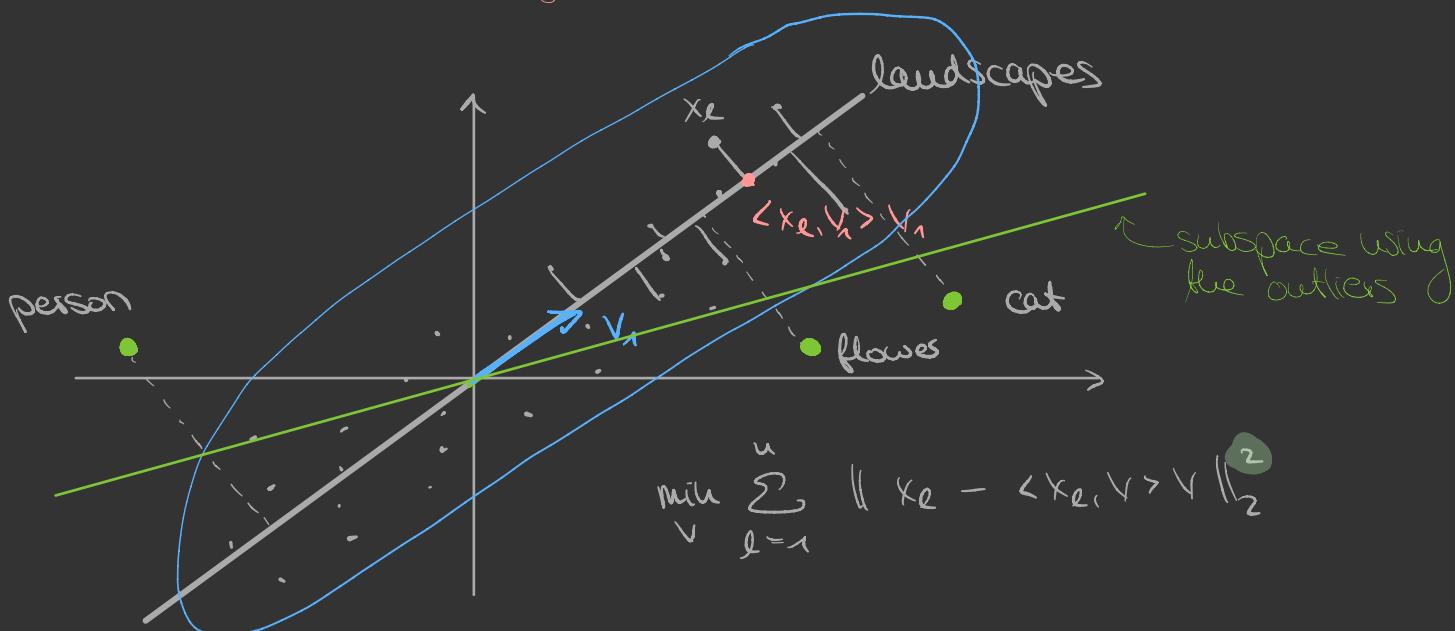
Remember $v_x = \underset{\|v\|_2=1}{\operatorname{argmin}} \sum_{l=1}^n \|x_l - \langle x_l, v \rangle v\|_2^2$
 Projection on 1-dim space $\text{span}\{x\}$

x rows in A \Rightarrow $= \underset{\|v\|_2=1}{\operatorname{argmin}} \sum_{l=1}^n \|A^{(l)} - \langle A^{(l)}, v \rangle v\|_2^2$

PF $= \underset{\|v\|_2=1}{\operatorname{argmax}} \|Av\|_2^2$ isotropy comes at a cost

But this procedure is not robust w.r.t. outliers. | Indeed:

since we treat everyone equally, we loose the possibility to distinguish "good" from "bad" data points

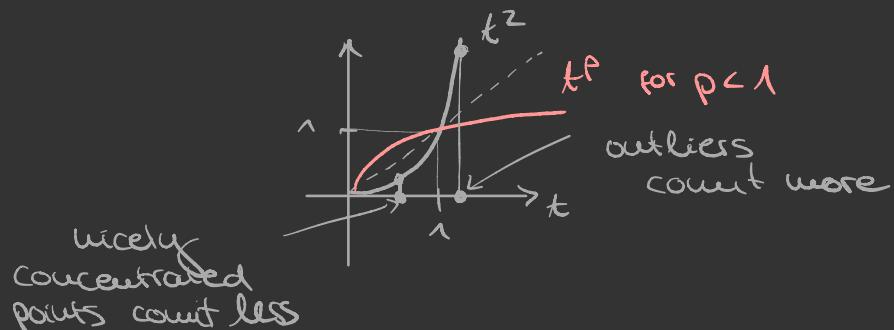


all perfect for linearly aligned data (only pictures of landscapes)

but now we add one picture of a cat, a person, a flower which are outliers for landscapes

Intelligent algorithms need to be able to "differentiate" nicely concentrated subsets from outliers.

Due to the square of $\| \cdot \|_2$ we even emphasise the outliers with a larger distance:



What can we do instead?

Robust PCA | Robust Subspace detection can be formulated as the following optimisation problem:

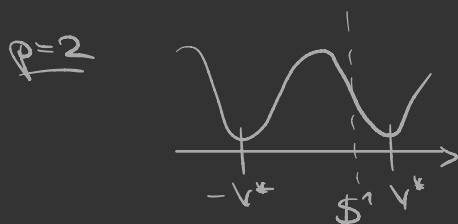
$$\hat{v}_* = \underset{\|v\|_2=1}{\operatorname{argmin}} \sum_{l=1}^n \|x_l - \langle x_l, v \rangle v\|_2^p \quad \text{with } p < 1.$$

cost function

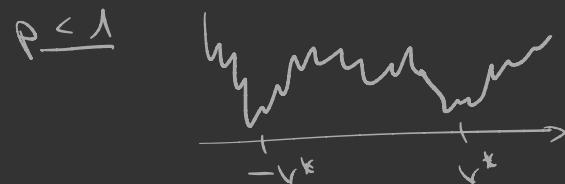
With this, outliers don't repeat the solution that much, i.e. more intelligent in distinguishing the data by counting them less.

Problem: can't use PF-theorem

the smaller the p , the more complicated the cost function



two local minima
which are both allowed



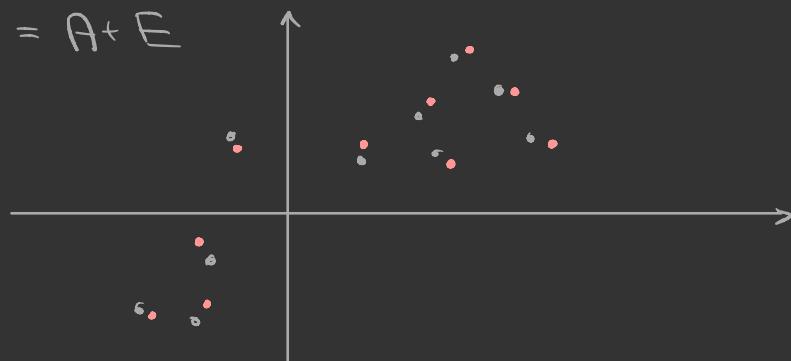
lots of local minima

~ So we have to get more creative to solve these optimisation problems.

2.6. Stability of the SVD

Data might contain some error, e.g. due to errors in measuring.

$$\rightarrow \tilde{A} = A + E$$



exact data A
we can't access

available/real data \tilde{A}
with error E

How does the SVD on available data compare to the one for the undisclosed data, i.e. how good is our approximation.

$$\tilde{A} = \tilde{U} \tilde{\Sigma} \tilde{V}^H \quad \text{real data SVD}$$

$$A = U \Sigma V^H \quad \text{undisclosed data SVD}$$

Wish, as long as $\|E\|$ is not too large: $\tilde{U} \approx U, \tilde{V} \approx V, \tilde{\Sigma} \approx \Sigma$.

But, this is not true in general, i.e. a very small perturbation could completely screw up our SVD.

"SVD is hackable", meaning in certain cases of perturbation, it can break.

Question: To what extend can we rely on the SVD of \tilde{A} ?
How wrong can we be?

Definition: For $A \in \mathbb{K}^{I \times I}$, we call a non-zero vector $v \in \mathbb{R}^I$ an eigenvector of A if for some $\lambda \in \mathbb{K}$: $Av = \lambda v$.
The scalar λ is called the corresponding eigenvalue.

Spectral Theorem: $A \in \mathbb{K}^{I \times I}$ hermitian, i.e. $A = A^H$. Then there exists an ONB $\{v_1, \dots, v_n\}$ consisting of eigenvectors of A with real corresponding eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ such that

$$A = \sum_{k=1}^n \lambda_k v_k v_k^H.$$

called the spectral decomposition.

Remark: Writing $u_k = \text{sign}(\lambda_k)$ and $\sigma_k = |\lambda_k|$, we get

$$A = \sum_{k=1}^n \sigma_k u_k v_k^T$$

↑ not necessarily positive

the SVD of A .

Weyl's bounds

Property: Assume $E \in \mathbb{R}^{I \times I}$ to be hermitian. Then for λ 's numerated in descending order, we have

$$\begin{aligned} \lambda_1(\tilde{A}) &= \max_{\|v\|_2=1} v^T (A+E)v && \leftarrow \text{Rayleigh-Quotient} \\ &\leq \max_{\|v\|_2=1} v^T Av + \max_{\|v\|_2=1} v^T Ev \\ &\stackrel{\text{Def.}}{=} \lambda_1(A) + \lambda_1(E). && \Rightarrow \text{upper bound} \end{aligned}$$

Now, let v_1 be the eigenvector associated with the largest eigenvalue λ_1 of A . Then

$$\lambda_1(\tilde{A}) = \max_{\|v\|=1} v^T (A+E)v$$

Smaller since we take a specific vector

https://de.wikipedia.org/wiki/Satz_von_Courant-Fischer#Verwendung

$$\begin{aligned} &\geq v_1^T (A+E)v_1 \\ &= \lambda_1(A) + v_1^T Ev_1 \\ &\geq \lambda_1(A) + \lambda_n(E) \end{aligned}$$

Courant-Fischer: $\lambda_{\min} \leq R_A(x) \leq \lambda_{\max}$
for $R_A(x) = \frac{\langle x, Ax \rangle}{\|x\|^2}$

$$\Rightarrow \lambda_1(A) + \lambda_n(E) \leq \lambda_1(A+E) \leq \lambda_1(A) + \lambda_1(E)$$

{ generalizing for all eigenvalues

Theorem 2.16: Weyl's bounds.

If $A, E \in \mathbb{K}^{I \times I}$ are two hermitian matrices, then for all $k=1, \dots, n$

$$\lambda_k(A) + \lambda_n(E) \leq \lambda_k(A+E) \leq \lambda_k(A) + \lambda_k(E).$$

↑ minimum eigenvalue of E

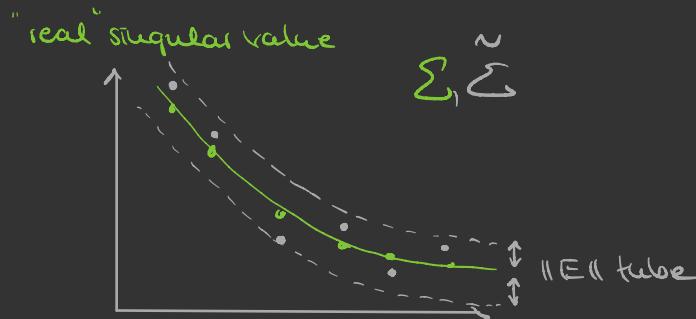
Corollary 2.17: If $A, E \in \mathbb{K}^{I \times I}$ two arbitrary square matrices, then for all $k=1, \dots, n$

$$|\theta_k(A+E) - \theta_k(A)| \leq \|E\|.$$

↑ spectral norm, i.e. $\rho = \infty$

Vorlesung 11

We found out that the singular values of the perturbed matrix are close to the "real" ones.



The result of 2.17 extends to all p -Schatten-norms, i.e. for any $1 \leq p \leq \infty$:

$$\left(\sum_{k=1}^n (\sigma_k(A+E) - \sigma_k(A))^p \right)^{1/p} \leq \|E\|_p.$$

So actually, there is no problem with the singular values, when there are perturbations to the true values.

{ What else can go wrong? The singular vectors!

Example / Exercise 5.2

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$ for $\varepsilon > 0$. Try to prove that the right singular vectors are composed $\tilde{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

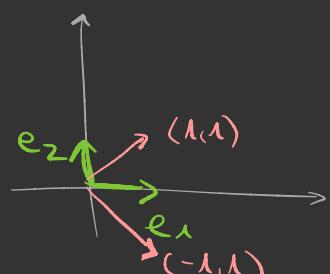
For $\tilde{A} = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$, we have

$$\underbrace{\|\tilde{A} - A\|_F}_{= E} = \sqrt{\varepsilon^2 + \varepsilon^2 + \varepsilon^2} = \sqrt{3} \cdot \varepsilon.$$

$$E = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & \varepsilon \end{pmatrix}$$

and

$\tilde{V} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which is in particular independent



of the magnitude of the error. And the vectors \tilde{V} is the "furthest possible" other ONB of \mathbb{R}^2 compared to $\{e_1, e_2\}$.

In other words, the following inequality

$$\text{dist}(V, \tilde{V}) \leq c \cdot \|E\|_F = c \cdot \sqrt{3} \cdot \varepsilon^{\frac{2-\alpha}{2}} \rightarrow 0$$

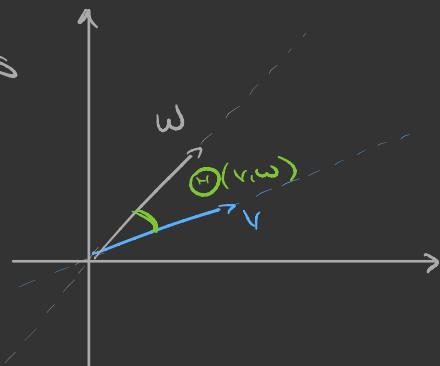
can not hold in general.

How can we measure the distance between two subspaces.

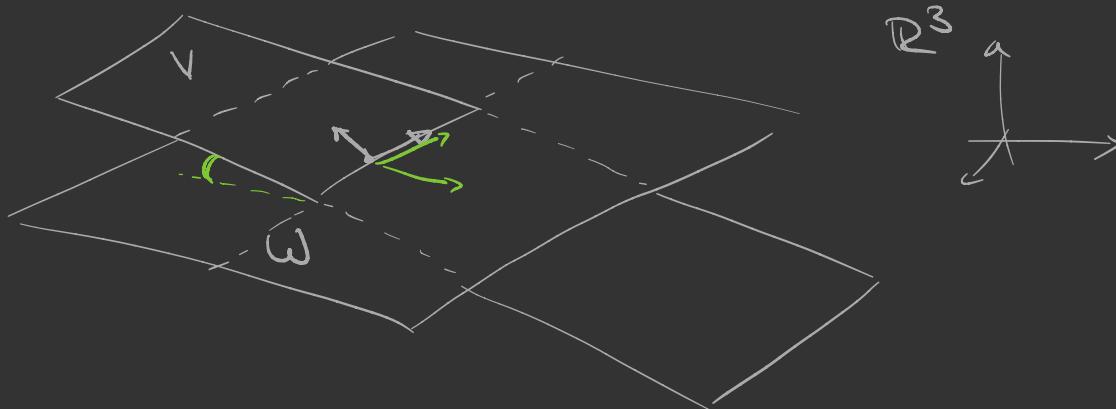
Let v, w be two vectors.

1) Use the angle between the subspaces

$$\cos \Theta(v, w) = \frac{\langle v, w \rangle}{\|v\|_2 \|w\|_2}$$



2) generalize this to higher dimensions:



→ Principle Angles

Def. 2.19: Let $V, W \in \mathbb{K}^{I \times J}$ orthogonal (columns are orthogonal), with

$$V = \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline v_1 & \dots & \dots & v_n \\ \hline 1 & & & \end{array} \right) \xrightarrow{d} \quad \begin{array}{l} \langle v_i, v_j \rangle = \delta_{ij} \\ V = \text{span}(v_1, \dots, v_n) \end{array}$$

$$W = \left(\begin{array}{c|c|c|c} 1 & & & \\ \hline w_1 & \dots & \dots & w_n \\ \hline 1 & & & \end{array} \right) \xrightarrow{d} \quad \begin{array}{l} \langle w_i, w_j \rangle = \delta_{ij} \\ W = \text{span}(w_1, \dots, w_n) \end{array}$$

We define the principle angles $\alpha_1, \dots, \alpha_n$ between V and W as

$$\cos \Theta(V, W) := (\cos(\alpha_1), \dots, \cos(\alpha_n))$$

$[-1, 1]$ is sufficient,
because V, W are
in particular normalized

$$= \sum_i (W^T V) = (\xi_1, \dots, \xi_n) \text{ with } 0 \leq \xi_i \leq 1.$$

↑ listing the singular values

and $A = W^T V$

$$\left. \begin{aligned} A^T A &= I \Rightarrow A^T A A^{-1} = A^{-1} \\ &\Rightarrow A A^T = A A^{-1} \\ &\Rightarrow A A^T = I \end{aligned} \right\} \text{orthogonal, since } A^T A = \underbrace{V W^T W V}_I = V V^T = I$$

composition
of two
rotations is
a rotation

$$\text{and } A A^T = W^T V V^T W = I.$$

$$\text{and } \|A\| = \|W^T V\| \leq \|W\| \cdot \|V\| = 1. \text{ Hence the largest singular value } \sigma_1(A) \leq 1.$$

↑
Submultiplicative norm

<https://www.quora.com/Is-the-product-of-two-unitary-matrices-always-unitary>

We know from prev. lectures that for the orthogonal projection onto V and W , we have for a vector x :

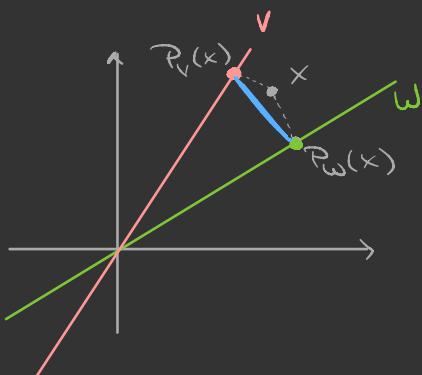
$$P_V(x) = \sum_{e=1}^n \langle x, v_e \rangle v_e = V V^T x$$

and

$$P_W(x) = \sum_{e=1}^n \langle x, w_e \rangle w_e = W W^T x$$

Other option (besides the angles)

Compute the projection of an arbitrary point and measure the distance:



→ Consider the worst case as distance:

$$\sup_{\|x\|_2=1} \|P_W(x) - P_V(x)\|$$

↑ normalized by x

$$= \sup_{\|x\|_2=1} \|(W W^T - V V^T)x\|_2$$

$$= \|W W^T - V V^T\|$$

largest singular value \sim

↑ spectral norm

Remark: we already saw that $|\tilde{\sigma}_k - \sigma_k| \leq \|\mathbb{E}\| \rightarrow 0$,
i.e. the same singular values can't be far apart.

For Wedin's bounds we look at successive singular values.

Interpretation:



If we know $\|\mathbb{E}\| = \delta$, i.e. we know the magnitude of the noise,
we also get that

$$\tilde{\sigma}_k - \sigma_{k+1} \geq \tilde{\sigma}_k - \tilde{\sigma}_{k+1} - \delta > 0$$

↑ ↓
always, if we have a gap
due to ordering!

↑ ↑
max. perturbation
from Cor. 2.17

Vorlesung 12

27.05.20

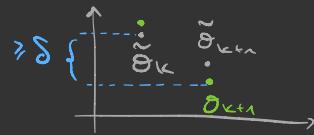
THEOREM 2.22. (Wedin's bound [Wed72]) Under the notations so far used in this section, assume that there exist an $\alpha \geq 0$ and $\delta > 0$ such that

$$(2.14) \quad \sigma_k(\tilde{A}) \geq \alpha + \delta \text{ and } \sigma_{k+1}(A) \leq \alpha.$$

Then for every unitarily invariant norm $\|\cdot\|_*$ (in particular for the spectral and Frobenius norm)

$$(2.15) \quad \max\{\|\sin \Theta(\tilde{V}_1, V_1)\|_*, \|\sin \Theta(\tilde{U}_1, U_1)\|_*\} \leq \frac{\max\{\|R_{11}\|_*, \|R_{21}\|_*\}}{\delta} \leq \frac{\|E\|_*}{\delta}.$$

$$\Rightarrow |\tilde{\theta}_k - \theta_{k+1}| \geq \delta$$



Recovery:

$$\begin{aligned} \text{Consider } \tilde{A} &= A + E = A_1 + A_0 + E \\ &= \tilde{A}_1 + \tilde{A}_0 = \tilde{A} \end{aligned}$$

$$\text{with } \tilde{A}_1 = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^T, \quad \tilde{A}_0 = \tilde{U}_0 \tilde{\Sigma}_0 \tilde{V}_0^T$$

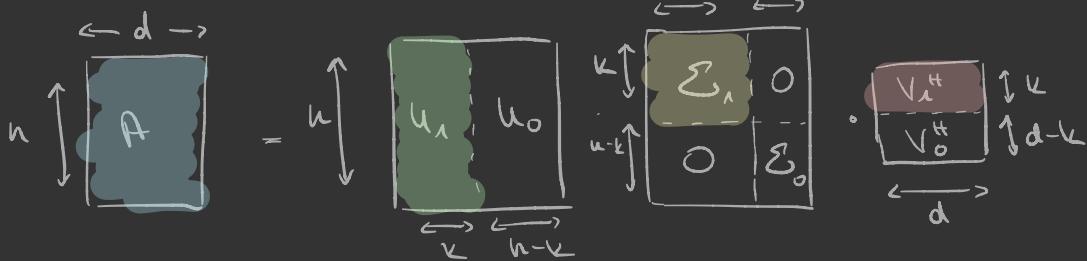
unitarily invariant
 $\|\cdot\|$ is unitarily invariant
 if $\|A\| = \|UAV\|$
 for unitary U, V in \mathbb{C} .

\tilde{U}_1 is ONS for range(A_1), \tilde{V}_1 is ONS for range(A_1^T) and
 $\tilde{U}_1 \perp \tilde{U}_0$ and $\tilde{V}_1 \perp \tilde{V}_0$.

Introduce residual (error) matrices:

$$\bullet R_{11} = \underbrace{AV_1}_{(n \times d) \times (d \times k)} - \underbrace{\tilde{U}_1 \tilde{\Sigma}_1}_{(n \times k) \times (k \times k)} \in \mathbb{K}^{n \times k}$$

$$\bullet R_{21} = \underbrace{A^T \tilde{U}_1}_{(d \times n) \times (n \times k)} - \underbrace{\tilde{V}_1 \tilde{\Sigma}_1^T}_{(d \times k) \times (k \times k)} \in \mathbb{K}^{d \times k}$$



$$\text{with } A = U_1 \Sigma_1 V_1^T + U_0 \Sigma_0 V_0^T.$$

\Rightarrow we don't necessarily have that $\Sigma_0 = 0$, because k is chosen arbitrarily!

We want to clarify the relationship between R_{11} , R_{21} and E .

$$\begin{aligned} R_{11} &= A \tilde{V}_1 - \tilde{U}_1 \tilde{\Sigma}_1 \\ &= (\tilde{A} - E) \tilde{V}_1 - \tilde{U}_1 \tilde{\Sigma}_1 \\ &\stackrel{\textcircled{1}}{=} -E \tilde{V}_1 \in \mathbb{K}^{n \times k} \end{aligned} \quad (1)$$

$$\begin{aligned} \tilde{A} \tilde{V}_1 &= \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^T \tilde{V}_1 \xrightarrow{\tilde{V}_1^T \tilde{V}_1 = I} \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^T \tilde{V}_1 \xrightarrow{\tilde{V}_1^T \tilde{V}_1 = 0} 0 \\ &= \tilde{U}_1 \tilde{\Sigma}_1 = \tilde{U}_1 \tilde{\Sigma}_1 \quad \tilde{\Sigma}_0 \text{ is not taken by } \tilde{U}_1 \end{aligned}$$

Analogously, we get that

$$R_{21} = -E^T \tilde{U}_x \in \mathbb{K}^{d \times k}. \quad (2)$$

Proof: only prove it for Frobenius norm.

In abuse of notation, we define for $j=0,1$

$$\begin{aligned} U_j &= \text{range}(A_j) \\ V_j &= \text{range}(A_j^\top) \end{aligned}$$

matrix Subspace
matrix Subspace

Because of $\|P_{U_0} P_V\|_F = \|\sin \Theta(V, U)\|_2$, we get that

$$\|\sin \Theta(V_0, \tilde{V}_x)\|_2 = \|P_{V_0} P_{\tilde{V}_x}\|_F = \|P_{V_0} P_{V_x}\|_F$$

and

$$\|\sin \Theta(U_0, \tilde{U}_x)\|_2 = \|P_{U_0} P_{\tilde{U}_x}\|_F.$$

By using the cyclicity property of traces (or the unitary invariance of the Frobenius norm), we can observe that

$$\begin{aligned} \|R_M\|_F &\stackrel{(1)}{=} \|E^T \tilde{V}_x\|_F \stackrel{\text{unitary invariance}}{=} \left\| \underbrace{E^T}_{P_{V_x}} \tilde{V}_x V_x^\top \right\|_F \\ &= \left\| \underbrace{E^T}_{P_{V_x}} \right\|_F. \end{aligned}$$

Similarly

$$\|R_{21}\|_F = \|E^T \tilde{U}_x\|_F = \left\| \underbrace{P_{\tilde{U}_x} E}_{P_{U_x}} \right\|_F.$$

Remember for $M = U \Sigma V^\top$, we defined the pseudo inverse as

$$M^+ = V \Sigma^{-1} U^\top$$

and

$$P_{\text{range}(M)} = MM^+ = (M^+)^T M^\top.$$

$$P_{\text{range}(M^\top)} = M^\top M = M^\top (M^+)^T.$$

The rest of the proof is based on the following decompositions:

Putting all these information together, we get that

$$\left\{ \begin{array}{l} t_1 = \frac{\mu + \alpha t_2}{\alpha + \delta} \quad t^* = \max\{t_1, t_2\} \\ t_2 = \frac{\mu + \alpha t_1}{\alpha + \delta} \end{array} \right. \implies t^* \leq \frac{\mu + \alpha t^*}{\alpha + \delta}$$

$$\implies t^* \leq \frac{\mu}{\delta}$$

Since $\mu = \max\{\|R_{11}\|_F, \|R_{21}\|_F\}$ with $\|R_{11}\|_F = \|\mathbb{E} \tilde{V}_1\|_F \leq \|\mathbb{E}\|_F$.

Schematic overview of the proof

Want to estimate: $\|\sin \Theta(U_1, \tilde{V}_1)\|_2 = \|\underbrace{P_{U_0} P_{\tilde{V}_1}}_{\text{in}}$

$$\|\sin \Theta(U_1, \tilde{U}_1)\|_2 = \|\underbrace{P_{U_0} P_{\tilde{U}_1}}_{\text{in}}\|_F.$$

These admit very special decompositions:

$$\underbrace{P_{U_0} P_{\tilde{U}_1}}_{\text{in}} = \left[P_{U_0} \mathbb{E} P_{\tilde{V}_1} + A_0 (\underbrace{P_{U_0} P_{\tilde{V}_1}}_{\text{in}}) \right] \tilde{A}_1^+$$

$$\underbrace{P_{U_0} P_{\tilde{V}_1}}_{\text{in}} = \tilde{A}_1^+ [P_{\tilde{U}_1} \mathbb{E} P_{V_0} + \underbrace{P_{\tilde{U}_1} P_{U_0}}_{\text{in}} A_0].$$

Translate these decomposition into scalars:

$$\begin{aligned} \mu &= \max\{\|P_{U_0} \mathbb{E} P_{\tilde{V}_1}\|_F, \|\underbrace{P_{U_0} P_{\tilde{V}_1}}_{\text{in}}\|_F\} \\ &\leq \max\{\|R_{11}\|_F, \|R_{21}\|_F\}. \end{aligned}$$

$$\begin{aligned} t_1 &:= \|\sin \Theta(\tilde{U}_1, U_1)\|_2 = \|\underbrace{P_{U_0} P_{\tilde{U}_1}}_{\text{in}}\|_F \\ t_2 &:= \|\sin \Theta(\tilde{V}_1, V_1)\|_2 = \|\underbrace{P_{V_0} P_{\tilde{V}_1}}_{\text{in}}\|_F \end{aligned}$$

Adding our assumptions, we get

$$\Theta_{\text{Lip}}(A) = \Theta_{\max}(A_0) = \|A_0\| \leq \infty$$

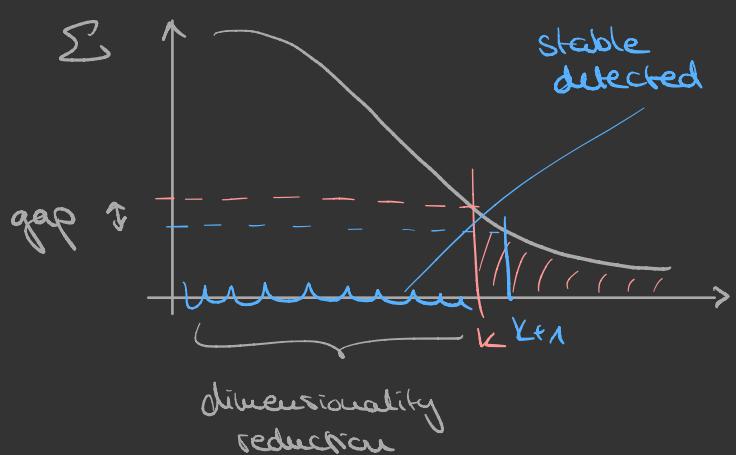
$$\frac{1}{\|\tilde{A}_1^+\|} = \Theta_{\min}(\tilde{A}_1) \geq \alpha + \delta.$$

Plugging everything together, we get:

$$\left\{ \begin{array}{l} t_1 = \frac{\mu + \alpha t_2}{\alpha + \delta} \quad t^* = \max\{t_1, t_2\} \\ t_2 = \frac{\mu + \alpha t_1}{\alpha + \delta} \end{array} \right. \implies t^* \leq \frac{\mu + \alpha t^*}{\alpha + \delta}$$

$$\implies t^* \leq \frac{\mu}{\delta}$$

Visualization



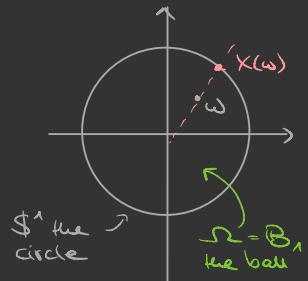
If we have any spectrum like that (with no significant gap) the data does not admit a "reduceable" structure



3. Basic Tools from Probability Theory

3.1. Essentials from Probability

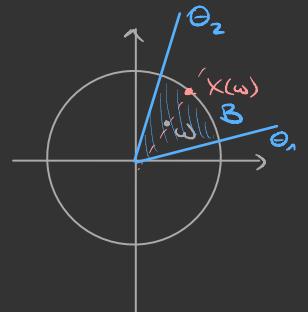
Example "dart": $X: \Omega \rightarrow \mathbb{R}$ random variable representing dart players
 $\omega \mapsto X(\omega) = \frac{\omega}{\|\omega\|_2} \in \mathbb{S}^1$



We define $P(A) = \frac{\text{Vol}(A)}{\text{Vol}(\Omega)}$ for $A \subseteq \Omega$.

Given two angles θ_1, θ_2 , we can define

$$\mathcal{B} = \{\omega \in \Omega \mid \theta_1 \leq X(\omega) \leq \theta_2\}.$$



Then

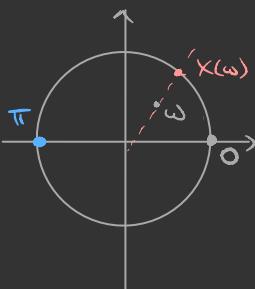
$$\text{Vol}(\mathcal{B}) = \int_{\theta_1}^{\theta_2} \int_0^1 r dr d\theta = \frac{\theta_2 - \theta_1}{2}.$$

and hence with $\text{Vol}(\mathcal{B}_1(0)) = 2\pi$, we get

$$P(X \in \mathcal{B}) = \frac{\theta_2 - \theta_1}{2\pi}.$$

What is the average outcome of such a player?

$$\begin{aligned} &= \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{i=1}^N X(\omega_i) \\ &= \int_{\Omega} X(\omega) dP(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \theta d\theta = \frac{(2\pi)^2}{4\pi} = \pi. \end{aligned}$$



Definition:

▷ probability space (Ω, Σ, P)

with $\Sigma \subseteq \mathcal{P}(\Omega)$ smallest σ -algebra on Ω
and P probability measure.

▷ σ -algebra $\Sigma \subseteq \mathcal{B}(\Omega)$

Def. $\left\{ \begin{array}{l} \bullet \Omega \in \Sigma \\ \bullet B \in \Sigma \Rightarrow B^c = \Omega \setminus B \in \Sigma \\ \bullet (B_i) \text{ a sequence in } \Sigma \Rightarrow \bigcup_{i=1}^{\infty} B_i \in \Sigma. \end{array} \right.$

▷ Probability measure

Def. $\left\{ \begin{array}{l} \bullet P(\Omega) = 1 \\ \bullet P(B) \geq 0 \quad \forall B \in \Sigma \\ \bullet A, B \in \Sigma, A \cap B = \emptyset : P(A \cup B) = P(A) + P(B) \end{array} \right.$

These imply that also

$$\bullet P(\emptyset) = 0$$

$$\bullet B \in \Sigma : P(B) = 1 - P(B^c)$$

$$\bullet A, B \in \Sigma, \text{ then } A \cup B \in \Sigma \text{ and}$$

$$P(A \cup B) \leq P(A) + P(B) \quad \text{"union bound"}$$

Remark: The function P behaves like an integral.

$$B \in \Sigma, \text{ then } P(B) = \int_B dP(\omega)$$

$$\emptyset \in \Sigma, \text{ then } P(\emptyset) = \int_{\emptyset} dP(\omega) = 0.$$

$$A, B \in \Sigma, \text{ then}$$

$$\begin{aligned} P(A \cup B) &= \int_{A \cup B} dP(\omega) = \int_A dP(\omega) + \int_B dP(\omega) \\ &= \text{only for } A \cap B = \emptyset. \end{aligned}$$

Instead of exploring P further, we will look at random variables and don't worry too much about (Ω, Σ, P) .

Definition: Let (Ω, Σ, P) be a probability space. Then, define a function

$$X: \Omega \rightarrow \mathbb{R} \quad \text{"Random number generator"}$$

Btw, can be constructed using intervals.

called random variable if $\forall A \in \mathcal{B}(\mathbb{R})$, the preimage of A is always contained in Σ , i.e.

$$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\} \in \Sigma .$$

↑
what we need

Definition: We call subsets in Σ events, and construct them as per below:

$$\Sigma \ni \{\omega \in \Omega \mid X(\omega) \leq t\} , t \in \mathbb{R}$$

$$\Sigma \ni \{\omega \in \Omega \mid t_1 \leq X(\omega) \leq t_2\} , t \in \mathbb{R}$$

We call the mapping $F_X: \mathbb{R} \rightarrow [0,1]$ with

$$F_X(t) = P(\{\omega \in \Omega \mid X(\omega) \leq t\})$$

the distribution function of the random variable X over (Ω, Σ, P) .

Property: Given a density function $\phi_X: \mathbb{R} \rightarrow \mathbb{R}_+$, we can calculate the probability $P(t_1 \leq X \leq t_2)$ as

$$P(\{\omega \in \Omega \mid t_1 \leq X(\omega) \leq t_2\}) = \int_{t_1}^{t_2} \phi_X(t) dt$$

for $t_1, t_2 \in \mathbb{R}$.

Exercise: $\phi_X(t) = \frac{d}{dt} F_X(t)$.

Definition: We define the expectation of X as

$$\mathbb{E} X = \int_{\Omega} X(\omega) dP(\omega) = \int_{-\infty}^{\infty} t \cdot \phi_X(t) dt$$

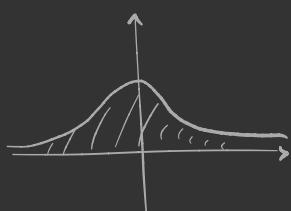
↑
density function

And for a function $g: \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} g(t) \phi_x(t) dt$$

with density function

Definition: (Ω, \mathcal{S}, P) , $X: \Omega \rightarrow \mathbb{R}$ random variable. Then X (Example RV) is called Gaussian random variable, if



$$\phi_x(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

Property: For X Gaussian RV, we have that

$$\mathbb{E}X = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} t \cdot e^{-\frac{t^2}{2}} dt = 0$$

Definition:

- p-moment of a RV $X: \Omega \rightarrow \mathbb{R}$. Then

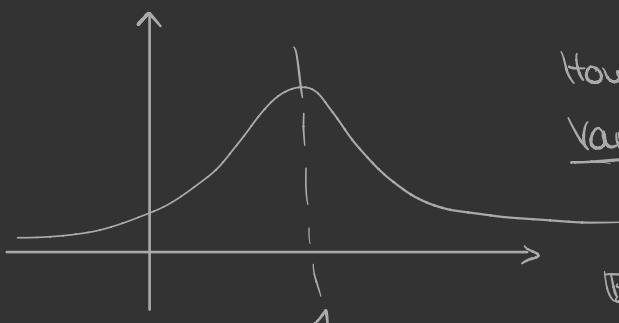
$$\mathbb{E} X^p = \int_0^{+\infty} t^p \phi_x(t) dt$$

} describe how "fat" a RV is.

- p-absolute moment of X

$$\mathbb{E}|X|^p = \int_{-\infty}^{+\infty} |t|^p \phi_x(t) dt$$

Definition: Let X be Gaussian with mean $\mathbb{E}X=1$.



concentrated around
the mean, hence
mean is informative

How close is the RV to its mean?

Variance:

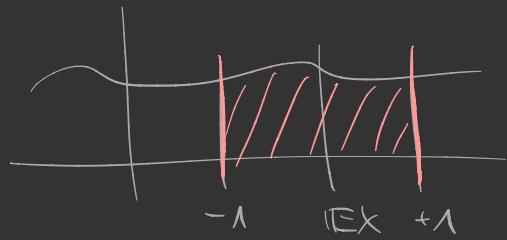
$$\mathbb{E}(X - \mathbb{E}X)^2 = \int |t-1|^2 \phi_x(t) dt$$

$$= \int_{-\infty}^{\infty} (t-1)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-1)^2}{2}} dt$$

Gaussian

$$s = t-1 \Rightarrow \int_{-\infty}^{\infty} s^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds = 1$$

Other RV might have a density like this



→ In this case the mean is not informative at all for the RV X

We can rewrite the variance for $\mu = \mathbb{E}X$ as

$$\begin{aligned}\mathbb{E}[(X - \mathbb{E}X)^2] &= \int_{-\infty}^{\infty} (t - \mu)^2 \phi_x(t) dt \\ &= \int_{-\infty}^{\infty} t^2 \phi_x(t) dt + \mu^2 \underbrace{\int_{-\infty}^{\infty} \phi_x(t) dt}_{=1} - 2\mu \underbrace{\int_{-\infty}^{\infty} t \phi_x(t) dt}_{=\mathbb{E}X} \\ &= \mathbb{E}X^2 + \mu^2 - 2\mu \cdot \mathbb{E}X \\ &= \mathbb{E}X^2 - (\mathbb{E}X)^2.\end{aligned}$$

Proposition 3.1: Cavalieri's formula

The absolute moments of a r.v. X can be expressed as

$$\mathbb{E}|X|^p = p \int_0^\infty P(|X| \geq t) t^{p-1} dt, \quad p > 0.$$

Proof: Recall that

$$\mathbb{1}_{\{|X(\omega)| \geq t\}} = \begin{cases} 1, & x \leq |X(\omega)|^p \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}|X|^p = \int_{\Omega} |X(\omega)|^p dP(\omega)$$

"useful integral trick" $\Rightarrow \int_{\Omega} \int_0^{|X(\omega)|^p} 1 dx dP(\omega)$

$$= \int_{\Omega} \int_0^{\infty} \mathbb{1}_{\{|X(\omega)|^p \geq x\}} dx dP(\omega)$$

for integrable functions $= \int_0^{\infty} \int_{\Omega} \mathbb{1}_{\{|X(\omega)|^p > x\}} dP(\omega) dx$

$$\begin{aligned}
 &= \int_0^\infty \mathbb{P}(|X|^p > x) dx \\
 \text{Substitution} \quad x = t^p \quad &\quad \text{d}x = p \cdot t^{p-1} dt \\
 &= \int_0^\infty \mathbb{P}(|X|^p > t^p) p \cdot t^{p-1} dt \\
 &= p \int_0^\infty \mathbb{P}(|X| > t) t^{p-1} dt. \quad \blacksquare
 \end{aligned}$$

Vorlesung 14

09.06.2020

Definition: let (Ω, Σ, P) a probability space, where Σ is a σ -algebra on the sample space and P a probability measure.

The probability of an event $B \in \Sigma$ is denoted by

$$P(B) = \int_B dP(\omega) = \int_{\Omega} \mathbb{1}_B(\omega) dP(\omega).$$

Then for events $B_l \in \Sigma$, $l \in [n]$, we have the union bound

$$P\left(\bigcup_{l=1}^n B_l\right) \leq \sum_{l=1}^n P(B_l).$$

Define a random variable $X: \Omega \rightarrow \mathbb{R}$ a real-valued measurable function on (Ω, Σ) .

X is called measurable if $\forall A \subset \mathbb{R}$ open, e.g. an open interval,

$$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\} \in \Sigma.$$

(In particular, all "reasonable" functions are measurable, such as for example continuous functions.)

Intuition: X as a random number generator, $X(\omega) \in \mathbb{R}$.

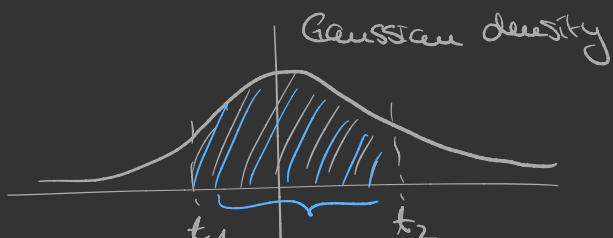
Definition: $F_X(t) = P(\{\omega \in \Omega \mid X(\omega) \leq t\})$, $t \in \mathbb{R}$
short $= P(X \leq t)$

is called the distribution function of X .

$\phi_X: \mathbb{R} \rightarrow \mathbb{R}_+$ with $t_1, t_2 \in \mathbb{R}$

$$P(\{\omega \in \Omega \mid t_1 \leq X(\omega) \leq t_2\}) = \int_{t_1}^{t_2} \phi_X(t) dt$$

is called density function.



Whenever we have a density, we can calculate

$$\int_{\Omega} g(X(\omega)) dP(\omega) = \int_{-\infty}^{\infty} g(t) \phi_x(t) dt$$

↑ Lebesgue-integral ↑ Riemann-integral

We define the expectation or mean of an r.v. as

$$E[X] = \int_{\Omega} X(\omega) dP(\omega).$$

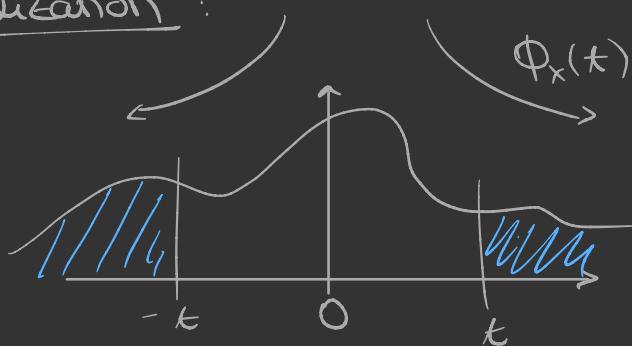
If we have a density function, for $g: \mathbb{R} \rightarrow \mathbb{R}$, we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(t) \phi_x(t) dt.$$

We call $E[X^p]$, $p > 0$, moments of X and $E[|X|^p]$ absolute moments.

The quantity $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$ is called variance.

Visualization:



no tails described by

$$P(|X| \geq t) = P(\{\omega \in \Omega \mid \text{either } X(\omega) \geq t \text{ or } X(\omega) \leq -t\})$$

contained
in Prop. 3.1. {

$$= \int_{\mathbb{R} \setminus [-t, t]} \phi_x(s) ds.$$

$$E[|X|^p] = p \cdot \int_0^{+\infty} \left(\int_{\mathbb{R} \setminus [-t, t]} \phi_x(s) ds \right) t^{p-1} dt.$$

Corollary 3.2: Relationship between the mean and the tails of a random variable.

Let X be a random variable. Then

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq t) dt - \int_0^\infty \mathbb{P}(X \leq -t) dt.$$

Proof: $X = X_+ - X_-$ with

$$0 \leq X_+ = \max\{0, X\} = X \cdot \mathbf{1}_{\{X \in (0, \infty)\}}$$

and

$$0 \leq X_- = \max\{0, -X\} = -X \cdot \mathbf{1}_{\{-X \in (0, \infty)\}}.$$

$$\begin{aligned} \mathbf{1}_A(\omega) \\ &= \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} \end{aligned}$$

We have that $|X_+| = X_+$ and $|X_-| = X_-$.

Then:

$$\mathbb{E}[X] = \mathbb{E}[|X_+|] - \mathbb{E}[|X_-|].$$

Applying Prop 3.1 gives the assertion. \square

Remark: knowing the tails

$$t \mapsto \mathbb{P}(X \geq t)$$

tells us about the moments.

Theorem 3.3: Markov inequality / Chebyshev inequality

let X be a random variable, then

$$\underbrace{\mathbb{P}(|X| \geq t)}_{\text{tail}} \leq \frac{\mathbb{E}[|X|]}{t}, \quad t > 0$$

(i.e. estimate the tails by using the moments).

$$\text{Proof: } \mathbb{P}(|X| \geq t) = \int_{\Omega} \mathbb{1}_{\{|X(\omega)| \geq t\}} d\mathbb{P}(\omega)$$

$$= \mathbb{E} \left[\mathbb{1}_{\{\omega \in \Omega \mid |X(\omega)| \geq t\}} \right]$$

We have that

$$t \cdot \mathbb{1}_{\{\omega \in \Omega \mid |X(\omega)| \geq t\}} = \begin{cases} t \cdot 1 & , |X(\omega)| \geq t \\ 0 & , |X(\omega)| < t \end{cases}$$

$$\leq |X(\omega)|.$$

applying

$$\mathbb{E}[\cdot] \Rightarrow \mathbb{E} \left[t \cdot \underbrace{\mathbb{1}_{\{\omega \in \Omega \mid |X(\omega)| \geq t\}}}_{\text{from above}} \right] \leq \mathbb{E}[|X(\omega)|]$$

$$\Leftrightarrow t \cdot \mathbb{P}(|X| \geq t) \leq \mathbb{E}[|X(\omega)|]. \quad \blacksquare$$

Remark:

For other exponents

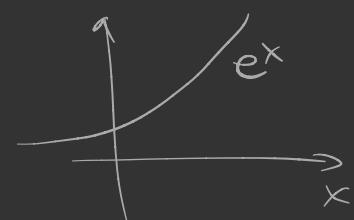
$$\mathbb{P}(|X| \geq t) = \mathbb{P}(|X|^p \geq t^p) \leq \frac{\mathbb{E}[|X|^p]}{t^p}.$$

For non-increasing functions and $\theta > 0$

$$\mathbb{P}(X \geq t) = \mathbb{P}(\theta X \geq \theta t)$$

$$= \mathbb{P}(\exp(\theta X) \geq \exp(\theta t))$$

$$\leq \frac{\mathbb{E}[\exp(\theta X)]}{\exp(\theta t)}$$



Definition: The function $\theta \mapsto \mathbb{E}[\exp(\theta X)]$ is called the Laplace Transform of X or moment generating function.

Remark:

$$\begin{aligned}\mathbb{E}[\exp(\theta X)] &= \int_{\Omega} \exp(\theta X(\omega)) dP(\omega) \\ &= \int_{-\infty}^{\infty} \exp(\theta t) \cdot \phi_X(t) dt.\end{aligned}$$

is known as Laplace Transformation.

Definition: A normal distributed r.v. or Gaussian r.v. X has the probability density function

$$\phi(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right).$$

It's often denoted by $N(\mu, \sigma^2)$, where $\mathbb{E}[X] = \mu$, $\text{Var}(X) = \sigma^2$ for $X \sim N(\mu, \sigma^2)$.

Standard Gaussian is given by $N(0, 1)$.

Remark: We know this distribution from the Law of Large Numbers.

Let X_1, \dots, X_n be iid random variables, i.e.

- » independent: they don't effect each other
- » identically distributed: $\phi_{X_1} = \phi_{X_2} = \dots = \phi_{X_n}$.

\Rightarrow we can consider them as copies of X , i.e. $X_i \sim X$.

Then, the deviation of the average to the mean is a r.v. itself and we have

$$Y_n(\omega) = \frac{1}{n} \underbrace{\sum_{i=1}^n X_i(\omega)}_{\substack{\text{new random variable} \\ \text{average over all variables}}} - \underbrace{\mathbb{E}X}_{\substack{\text{their common mean}}} \sim \mathcal{N}(0, \sigma_n)$$

\uparrow

\uparrow

\uparrow

independently of the distribution of X

$\text{and } \sigma_n \xrightarrow{n \rightarrow \infty} 0$.

gets closer and closer to zero

Definition: A random vector $X = (X_1, \dots, X_n)^T \in \mathbb{R}^n$ is a collection of n random variables on a common prob. space (Ω, \mathcal{E}, P) .

Its expectation is $\mathbb{E}[X] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_n])^T$.

Their joint-distribution function is defined as

$$F(t_1, \dots, t_n) = P(X_1 \leq t_1, \dots, X_n \leq t_n)$$

and their joint density function $\phi_X : \mathbb{R}^n \rightarrow [0,1]$ is defined for all $D \subseteq \mathbb{R}^n$ by

$$\mathbb{P}(X \in D) = \int_D \phi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

Definition: We call X_1, \dots, X_n r.v.'s stochastically independent, if, for all $t_1, \dots, t_n \in \mathbb{R}$

$$\underbrace{\mathbb{P}(X_1 \leq t_1, \dots, X_n \leq t_n)}_{\text{Joint event}} = \prod_{i=1}^n \underbrace{\mathbb{P}(X_i \leq t_i)}_{\text{Individual events}}$$

Prop: As a consequence, we get

$$\mathbb{E}\left[\prod_{i=1}^n X_i\right] = \prod_{i=1}^n \mathbb{E}[X_i].$$

If we have a joint density function ϕ , then it factorizes as

$$\phi(t_1, \dots, t_n) = \phi_1(t_1) \cdot \dots \cdot \phi_n(t_n).$$

Vorlesung 15

10.06.2020

Prop: The sum $X+Y$ of two independent random variables X, Y having prob. density functions ϕ_x, ϕ_y , has prob. density function ϕ_{x+y} given by the convolution

$$\begin{aligned}\phi_{x+y}(t) &= (\phi_x * \phi_y)(t) \\ &= \int_{-\infty}^{+\infty} \phi_x(u) \phi_y(t-u) du.\end{aligned}$$

Visualization:

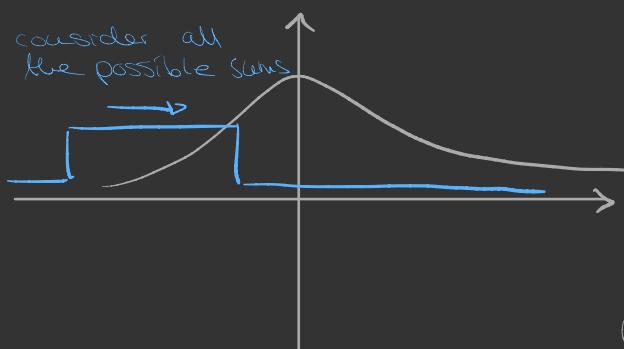


► Define $Z(\omega) = X(\omega) + Y(\omega)$, then the frequency of a specific value $c = X(\omega_0) + Y(\omega_0)$ for fixed $\omega_0 \in \Omega$ is given by

$$P(Z(\omega_0) = c) = P(X = X(\omega_0)) \cdot P(Y = Y(\omega_0))$$

(due to independence).

► For a fixed value of the sum $Z = X+Y$, we have



$$\begin{aligned}&\phi_x(u) \cdot \phi_y(t-u) \\&= \underbrace{\phi_x(u)}_{\text{frequency of } X(\omega)=u} \cdot \underbrace{\phi_y(t-u)}_{\text{frequency of } Y(\omega)=t-u}\end{aligned}$$

consider all possible results
 \downarrow
 \rightarrow

$$\Rightarrow \phi_Z(t) = \int_{-\infty}^{+\infty} \phi_x(u) \cdot \phi_y(t-u) dt$$

Definition: We call a vector $x \in \mathbb{R}^n$ a standard Gaussian vector, if its components are independent standard normal distributed random variable.

More generally: A random vector $X \in \mathbb{R}^n$ is said to be a multivariate normal distributed vector if $\exists A \in \mathbb{R}^{n \times k}$ s.t.

$$X = Ag + \mu \quad \text{generates the vector's variance}$$

where $g \in \mathbb{R}^k$ is a standard Gaussian vector and $\mu \in \mathbb{R}^n$ is the mean of X .

$$\Sigma = AA^H = \mathbb{E}[(X - \mu)(X - \mu)^T] \quad \text{Covariance matrix}$$

If AA^H is positive definite, hence invertible, then X has a joint density function of the form

$$\phi(x) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \langle x - \mu, \Sigma^{-1}(x - \mu) \rangle\right)$$

i.e. is multivariate normal distributed.

Prop: For X_1, \dots, X_n independent and normal dist. random variables with mean μ_1, \dots, μ_n and variance $\sigma_1^2, \dots, \sigma_n^2$. Then $S_n = X_1 + \dots + X_n$ is again normally distributed with $\mu = \sum_{i=1}^n \mu_i$ and $\sigma^2 = \sum_{i=1}^n \sigma_i^2$.

Proof: Exercise 6.7.

Prop: Jensen's inequality (not used by us)

let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, and let $X \in \mathbb{R}^n$ be a random vector. Then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$