Problem Set 3, Answers

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Problem 1. This is a four-part problem.

1. Let L be the set of leaves of the minimum-height protocol tree for f that give output 1. For each $l \in L$, let M_l be the matrix which is 1 on the input pairs whose path through the protocol tree ends at the lead l. By the definition of a protocol tree, M_l can also be defined by a set X and Y with:
$$M_l[i,j] = 1 \iff i \in X, j \in Y \tag{1}$$
Thus, there are two possible rows in M_l, either all zeros or 1 precisely on indices in Y. Thus, the rank of M_l is 1, regardless of what field the rank is taken over.

   Note that:
$$M_f = \sum_{l \in L} M_l \tag{2}$$

   Since rank is subadditive, we can conclude:
$$rank(M_f) \leq \sum_{l \in L} rank(M_l) = |L| \leq 2^{D(f)} \tag{3}$$

   The last inequality holds since L is a subset of leaves of a tree of height $D(f)$. We can then conclude
$$log_2(rank(M_f)) \leq D(f) \tag{4}$$