# Problem Set 3, Answers

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**Problem 1.** This is a four-part problem.

1. Let $L$ be the set of leaves of the minimum-height protocol tree for $f$ that give output 1. For each $l \in L$, let $M_l$ be the matrix which is 1 on the input pairs whose path through the protocol tree ends at the lead $l$. By the definition of a protocol tree, $M_l$ can also be defined by a set $X$ and $Y$ with:
$$M_l[i, j] = 1 \iff i \in X, j \in Y \tag{1}$$
Thus, there are two possible rows in $M_l$, either all zeros or 1 precisely on indices in $Y$. Thus, the rank of $M_l$ is 1, regardless of what field the rank is taken over.

   Note that:
$$M_f = \sum_{l \in L} M_l \tag{2}$$
   Since rank is subadditive, we can conclude:
$$rank(M_f) \leq \sum_{l \in L} rank(M_l) = |L| \leq 2^{D(f)} \tag{3}$$
   The last inequality holds since $L$ is a subset of leaves of a tree of height $D(f)$. We can then conclude
$$log_2(rank(M_f)) \leq D(f) \tag{4}$$

2. There are $2^n$ rows of $M_f$. Consider them as vectors over the field $\mathbb{F}_2$, and say the rows span a subspace of dimension $d$, so $rank(M_f) = d$. Since the field has size 2, this subspace has exactly $2^d$ elements in it. Since the rows of $M_f$ are $2^n$ distinct vectors in this subspace of size $2^d$, we have $2^n \leq 2^d$. So $log_2(n) \leq log_s(d)$ and by the conclusion of the previous problem, $\log_2(n) \leq D(f)$.

3. TODO

4. TODO