Büchi automata

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ω - langages

Let Σ be a finite alphabet

An ω-word is an element of Σ^ω $(\omega = \mathbb{N})$

↳ an infinite sequence of symbols of Σ

↳ we denote u_i the i^{th} letter of the word u

An ω-word could represent the sequence of states a process goes through

Ex: a light + + + − − − − + − − − − + + + + − − − ...

(+ = on ; − = off)

a server r t t R r t t t R r t r t R t t r t R R r ...

r: reception of a request

R: response to a request

t: internal treatment

An ω-language is a subset of Σ^ω

An ω-language could represent:

- the possible executions of a process
- the expected executions of a process

E: - language of the possible executions for the light: $\{+,-\}^\omega$

- expected language for the light

→ does not stay "on" more than 20 "ticks"

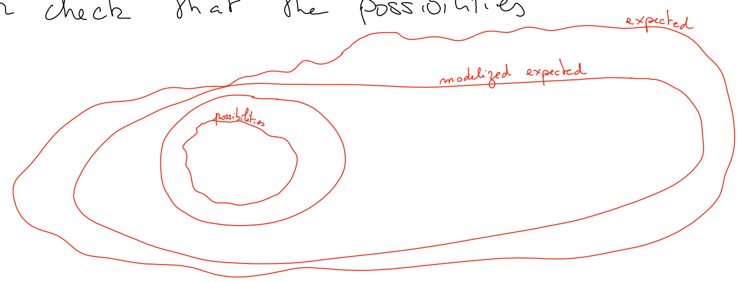
→ remains "off" 75% of the time (meaning?)

- language of the possible executions for the server:

→ the number of R so far is always less or equal to the number of r

- expected language for the server:
  → the R corresponding to each r is emitted at most 10 "ticks" after the arrival of the r

Verification problem: Prove or check that the possibilities language is included in the expected language



ω - Automata

Pour résoudre ce problème, il faut un moyen de décrire les ω - langages que l'on sachante manipuler

An ω - automaton is a tuple:

$$\mathcal{A} = \langle \Sigma, Q, s, \mathbb{A}, R \rangle$$

Σ : finite alphabet

Q : finite set of states

$s \in Q$, the initial state

$\mathbb{A}$ is an accepting condition

$R \subseteq Q \times \Sigma \times Q$, is the transition relation

A path of $\mathcal{A}$ is a sequence:

$$(q_i, l_i)_{i \in \omega} \in (Q \times \Sigma)^{\omega}$$

such that

$$q_0 = s \quad \text{and} \quad \forall i \in \omega \quad (q_i, l_i, q_{i+1}) \in R$$

We say that this path is labelled by $(l_i)_{i \in \omega}$

The accepting condition is a condition on the sequence of states $(q_i)_{i \in \omega}$. It has to be expressible by a finite presentation

Usual conditions are expressed using, if $x \in Q^\omega$, the set

$$Inf(x) = \{q \in Q \mid q \text{ occurs infinitly often in } x\}$$
$$= \{q \in Q \mid \forall i \in \omega \ \exists j > i \ \ x_j = q\}$$

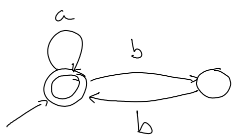
Name	Presentation	Condition
Büchi	$Q_A \subset Q$ accepting states	$Inf(x) \cap Q_A \neq \emptyset$ "an accepting state is visited infinitly often"
	$\pi : Q \to \mathbb{N}$ associates levels to states	$\min \pi(Inf(x)) \in 2\mathbb{N}$ "The level of states, visited infinitly often, of lowest level is even"
Muller	$\mathcal{Q}_A \subset 2^Q$	$Inf(x) \in \mathcal{Q}_A$ "The sets of infinitly often visited states is in $\mathcal{Q}_A$"

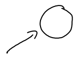
The ω-language accepted by the automaton $\mathcal{A}$ is the set of ω-words u such that <u>there exists</u> a path (x_i, u_i) of $\mathcal{A}$ where $(x_i)_{i\in\omega}$ is accepted by $\mathbb{A}$. We denote this language $\mathcal{L}(\mathcal{A})$

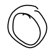
<u>Theorem</u>: Any language accepted by an ω-automaton $\mathcal{A}$ whose accepting condition is expressed using only the Inf predicate is accepted by an ω-automaton using the Büchi accepting condition

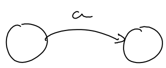
An ω-automaton using the Büchi accepting condition is called a Büchi automaton

Büchi automata are represented the same way finite automata do:



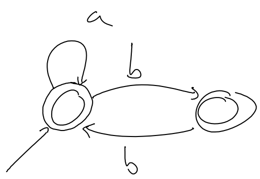
 : initial state

 : accepting state

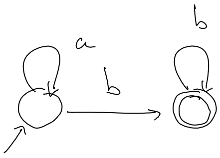
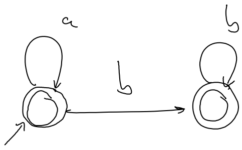
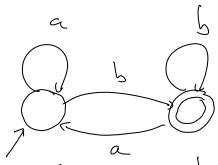
 : transition

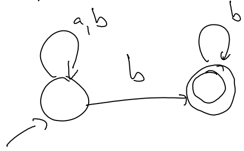
$(a+bb)^\omega$

{b comes by pairs}

This automaton is equivalent to 

Ex:

 ;  ; 



Deterministic ω-automata

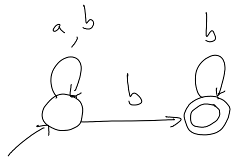
An ω-automaton $\mathcal{A}$ is deterministic if for all ω-word u there is at most one path of $\mathcal{A}$ labelled by u

An ω-automaton $\mathcal{A}$ is deterministic if and only if R is a partial function:

$$\forall \ell \in \Sigma, \forall q \in Q, \quad \left| \{ q' \in Q \mid (q, \ell, q') \in R \} \right| \leq 1$$

↳ for any letter ℓ, there is at most one arrow labelled by ℓ exiting each state

Theorem The deterministic Büchi automata accept strictly less languages than their non-deterministic counterparts.



ω-regular expressions

An ω-regular expression is recursively defined by

- where A is a regular expression $(\varepsilon \notin A$ and $A \neq \emptyset)$
- AB where A is a regular expression and B is an ω-regular expression
- $A+B$ where A and B are ω-regular expressions

$$AB = \{ uw \mid u \in A \text{ and } w \in B \}; \quad (uw)_i = \begin{cases} u_i & \text{if } i < |u| \\ w_{i-|u|} & \text{otherwise} \end{cases}$$

$$A + B = A \cup B$$

$$A^\omega = \{ w \mid \exists (u_i)_{i \in \omega}, \ \forall i \in \omega \ u_i \in A \wedge w = \bigodot_{i \in \omega} u_i \};$$

$$\left(\bigodot_{i \in \omega} u_i \right)_j = (u_n)_k \text{ when } n \text{ is the largest such that } \sum_{i=0}^{n-1} |u_i| \leq j \text{ and } k = j - \sum_{i=0}^{n-1} |u_i|$$

$$\bigodot_{i \in \omega} u_i = \underset{\in A}{u_0} \, \underset{\in A}{u_1} \, \underset{\in A}{u_2} \, \underset{\in A}{u_3} \cdots \underset{\in A}{u_n} \cdots$$

Theorem Let L be an ω-language

The three following propositions are equivalent:

- L is accepted by a Büchi automaton
- L is accepted by a deterministic Muller or Parity automaton
- L is expressible by an ω-regular expression

We call such ω-language as **ω-regular languages**.

Corollary

If A and B are ω-regular languages and C is a regular language then:

$$\underbrace{A \cup B \,;\, CA \,;\, C^\omega}_{\text{using } \omega\text{-regular expressions}} \,;\, \underbrace{\Sigma^\omega \setminus A}_{\text{using deterministic Muller automata}} \,;\, \underbrace{A \cap B}_{\complement(\complement A \cup \complement B)}$$

(inefficient for the number of states: double exponential blow-up)