STA2202 - Time Series Analysis - Assignment 3 - THEORY

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Submission instructions:

Submit *two separate files* to A3 on Quercus - the deadline is 11:59PM on Monday, June 15.
- A PDF file with your Theory part answers.
- A PDF file with your Practice part report (w/ code in R Markdown chunks or in Appendix).

Theory

Question 1

1. Consider the causal representation of a VAR(p) model

$$\boldsymbol{X}_t = \overbrace{\sum_{j=1}^{p} \boldsymbol{\Phi}_j \boldsymbol{X}_{t-j} + \boldsymbol{W}_t}^{\text{VAR}(p)} = \overbrace{\sum_{j\geq 0} \boldsymbol{\Psi}_j \boldsymbol{W}_{t-j}}^{\text{causal repr.}}$$

for causal weight matrices $\{\boldsymbol{\Psi}_j\}$ and $\boldsymbol{W}_t \sim \text{WN}(\boldsymbol{0}, \boldsymbol{\Sigma}_w)$.

a. [3 marks] Prove equation (5.95) on SS p.280, which gives the series auto-covariance matrix at lag $h > 0$ as

$$\boldsymbol{\Gamma}(h) = \text{Cov}(\boldsymbol{X}_{t+h}, \boldsymbol{X}_t) = \sum_{j\geq 0} \boldsymbol{\Psi}_{j+h} \boldsymbol{\Sigma}_w \boldsymbol{\Psi}'_j$$

b. [2 marks] Show that $\boldsymbol{\Gamma}(h) = \boldsymbol{\Gamma}'(-h)$ for $h \geq 0$.

{*Solution.*}

item (a)

From the problem statement we have that

$$\boldsymbol{X}_t = \sum_{j=1}^{p} \boldsymbol{\Phi}_j \boldsymbol{X}_{t-j} + \boldsymbol{W}_t = \sum_{j\geq 0} \boldsymbol{\Psi}_j \boldsymbol{W}_{t-j} \tag{1}$$

Using the causal representation on (1), we have the auto-covariance matrix at lag $h \geq 0$ as follows:

$$\begin{aligned}\boldsymbol{\Gamma}(h) &= \text{Cov}(\boldsymbol{X}_{t+h}, \boldsymbol{X}_t)\\ &= \text{Cov}\left(\sum_{i\geq 0} \boldsymbol{\Psi}_i \boldsymbol{W}_{t+h-i}, \sum_{j\geq 0} \boldsymbol{\Psi}_j \boldsymbol{W}_{t-j}\right)\end{aligned}$$

This covariance is non-zero when $t+h-i=t-j$, i.e., when $i=j+h$, then considering this, we have:

$$\begin{aligned}
\boldsymbol{\Gamma}(h) &= \sum_{j=0}^{\infty} \operatorname{Cov}\big(\boldsymbol{\Psi}_{j+h}\boldsymbol{W}_{t-j}, \boldsymbol{\Psi}_{j}\boldsymbol{W}_{t-j}\big) \\
&= \sum_{j=0}^{\infty} \boldsymbol{\Psi}_{j+h}\operatorname{Cov}\big(\boldsymbol{W}_{t-j}, \boldsymbol{W}_{t-j}\big)\boldsymbol{\Psi}_{j}' \\
&= \sum_{j=0}^{\infty} \boldsymbol{\Psi}_{j+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{j}' \\
&= \sum_{j\geq 0} \boldsymbol{\Psi}_{j+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{j}'
\end{aligned}$$

$$\implies \boldsymbol{\Gamma}(h) = \sum_{j\geq 0} \boldsymbol{\Psi}_{j+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{j}' \tag{2}$$

item (b)

Using the expression on (2), for $h \geq 0$ we have that:

$$\boldsymbol{\Gamma}(-h) = \operatorname{Cov}(\boldsymbol{X}_{t-h}, \boldsymbol{X}_t) = \sum_{j\geq 0} \boldsymbol{\Psi}_{j-h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{j}' \tag{3}$$

By transposing the expression in (3) we have that

$$\boldsymbol{\Gamma}'(-h) = \sum_{j\geq 0} \boldsymbol{\Psi}_{j}\boldsymbol{\Sigma}_W'\boldsymbol{\Psi}_{j-h}' \tag{4}$$

Considering $\boldsymbol{\Sigma}_W$ is symmetric and by doing $i=j-h$ in the expression (4), we have:

$$\begin{aligned}
\boldsymbol{\Gamma}'(-h) &= \sum_{i=-h}^{\infty} \boldsymbol{\Psi}_{i+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{i}' \\
&= \sum_{i=0}^{\infty} \boldsymbol{\Psi}_{i+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{i}', \quad \text{for } h \geq 0 \\
&= \sum_{i\geq 0} \boldsymbol{\Psi}_{i+h}\boldsymbol{\Sigma}_W\boldsymbol{\Psi}_{i}' \\
&= \boldsymbol{\Gamma}(h), \quad \text{from equation (2)}
\end{aligned}$$

$$\implies \boldsymbol{\Gamma}'(-h) = \boldsymbol{\Gamma}(h). \tag{5}$$

Question 2

2. Consider the following bi-variate time series model:

$$\begin{cases} X_{1,t} = .5X_{1,t-1} + U_t \\ X_{2,t} = .5X_{2,t-2} + U_t + V_t \end{cases}$$

where U_t, V_t are *independent* WN$(0,1)$ sequences. Note that X_1 marginally follows AR(1) and X_2 marginally follows AR(2).

a. [2 marks] Write the model as a bi-variate VAR(p) model of the form

$$\boldsymbol{X}_t = \sum_{j=1}^{p} \boldsymbol{\Phi}_j \boldsymbol{X}_{t-j} + \boldsymbol{W}_t$$

where $\boldsymbol{W}_t \sim \text{WN}(\boldsymbol{0}, \boldsymbol{\Sigma}_w)$ for some variance-covariance matrix $\boldsymbol{\Sigma}_w$. Specify the values of the parameters $(\{\boldsymbol{\Phi}_j\}, \boldsymbol{\Sigma}_w)$

b. [4 marks] Find a closed form expression for the causal weight matrices $\{\boldsymbol{\Psi}_j\}_{j\geq 1}$, from the model's causal representation $\boldsymbol{X}_t = \sum_{j\geq 0} \boldsymbol{\Psi}_j \boldsymbol{W}_{t-j}$.

(*Hint*: you can use the recurrence equation $\boldsymbol{\Psi}_k = \sum_{j=1}^{\min\{p,k\}} \boldsymbol{\Psi}_{k-j}\boldsymbol{\Phi}_j$, where $\boldsymbol{\Psi}_0 = \boldsymbol{I}$)

c. [4 marks] Find the cross-covariance function $\gamma_{12}(h) = \text{Cov}(X_{1,t+h}, X_{2,t})$ for any h.

{*Solution.*}

item (a)

Using the result (1) we can write the bi-variate model as follows:

$$\underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_t}_{\boldsymbol{X}_t} = \underbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{\Phi}_1} \times \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{t-1}}_{\boldsymbol{X}_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}}_{\boldsymbol{\Phi}_2} \times \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{t-2}}_{\boldsymbol{X}_{t-2}} + \underbrace{\begin{bmatrix} W_1 = U \\ W_2 = U + V \end{bmatrix}_t}_{\boldsymbol{W}_t} \tag{6}$$

Now, calculating the parameter $\boldsymbol{\Sigma}_W$ we have that:

$$\begin{aligned} \boldsymbol{\Sigma}_W &= \text{Var}(\boldsymbol{W}_t) \\ &= \text{Cov}\left(\begin{bmatrix} U_t \\ U_t + V_t \end{bmatrix}, \begin{bmatrix} U_t & U_t + V_t \end{bmatrix} \right) \\ &= \begin{bmatrix} \text{Cov}(U_t, U_t) & \text{Cov}(U_t, U_t + V_t) \\ \text{Cov}(U_t + V_t, U_t) & \text{Cov}(U_t + V_t, U_t + V_t) \end{bmatrix} \\ &= \begin{bmatrix} \text{Var}(U_t) & \text{Var}(U_t) + \text{Cov}(U_t, V_t) \\ \text{Cov}(V_t, U_t) + \text{Var}(U_t) & \text{Var}(U_t + V_t) \end{bmatrix} \end{aligned}$$

Considering that:

- $Var(U_t) = Var(V_t) = 1$
- U_t and V_t are independent so $\text{Cov}(U_t, V_t) = \text{Cov}(V_t, U_t) = 0$

... then $\boldsymbol{\Sigma}_W$ can be written as follows:

$$\boldsymbol{\Sigma}_W = \begin{bmatrix} 1 & 1 \\ 1 & \text{Var}(U_t + V_t) \end{bmatrix} \tag{7}$$

Now, calculating $\text{Var}(U_t + V_t)$ we have

$$\begin{aligned}\text{Var}(U_t + V_t) &= \text{Cov}(U_t + V_t, U_t + V_t) \\ &= \text{Cov}(U_t, U_t) + \text{Cov}(U_t, V_t) + \text{Cov}(V_t, U_t) + \text{Cov}(V_t, V_t) \\ &= \text{Var}(U_t) + \text{Cov}(U_t, V_t) + \text{Cov}(V_t, U_t) + \text{Var}(V_t) \\ &= 1 + 0 + 0 + 1 \\ &= 2\end{aligned}$$

$$\implies \boldsymbol{\Sigma}_W = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{8}$$

item (b)

In this question we need to demonstrate that $\boldsymbol{X}_t$ can be written as a *Wold Process*, i.e.:

$$\boldsymbol{X}_t = \sum_{j \geq 0} \boldsymbol{\Psi}_j \boldsymbol{W}_{t-j} \tag{9}$$

... where $\boldsymbol{\Psi}$-matrices satisfy

$$\boldsymbol{\Psi}_k = \sum_{j=0}^{min(k,p)} \boldsymbol{\Psi}_{k-j} \boldsymbol{\Phi}_j$$

... and $\boldsymbol{\Psi}_0 = \boldsymbol{I}_p$. In our case $k = 2$ and $p = 2$ and we need to find $\boldsymbol{\Psi}_k$ such that

$$\boldsymbol{\Psi}_k = \sum_{j=0}^{2} \boldsymbol{\Psi}_{2-j} \boldsymbol{\Phi}_j, \quad \text{and } \boldsymbol{\Psi}_0 = \boldsymbol{I}_2 \tag{10}$$

Using the recurrence matrix in (10) and, for the sake of simplicity, we will omit the boring calculations involving 2×2 matrices and just expose the patterns encountered, which are:

Case **k is even**:

$$\boldsymbol{\Psi}_k = \begin{bmatrix} 0.5^k & 0 \\ 0 & 0.5^{k/2} \end{bmatrix}$$

Case **k is odd**:

$$\boldsymbol{\Psi}_k = \begin{bmatrix} 0.5^k & 0 \\ 0 & 0 \end{bmatrix}$$

Then we can express $\boldsymbol{X}_t$ as follows:

$$\begin{aligned}\boldsymbol{X}_t = \sum_{j=0}^{\infty} \Bigg(&\begin{bmatrix} 0.5^j & 0 \\ 0 & 0.5^{j/2} \end{bmatrix} \times \begin{bmatrix} U \\ U + V \end{bmatrix}_{t-j} \times \mathbb{1}(j : j = 2n, n \in \mathbb{N}) + \\ &\begin{bmatrix} 0.5^j & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} U \\ U + V \end{bmatrix}_{t-j} \times \mathbb{1}(j : j = 2n - 1, n \in \mathbb{N}) \Bigg)\end{aligned} \tag{11}$$

where $\mathbb{1}$ is an *indicator function.*

An alternative form can be written as

$$\boldsymbol{X}_t = \sum_{j=0}^{\infty} \left(\begin{bmatrix} 0.5^j & 0 \\ 0.5^{j/2} & 0.5^{j/2} \end{bmatrix} \times \begin{bmatrix} U \\ V \end{bmatrix}_{t-j} \times \mathbb{1}(j : j = 2n, n \in \mathbb{N}) + \begin{bmatrix} 0.5^j & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} U \\ V \end{bmatrix}_{t-j} \times \mathbb{1}(j : j = 2n-1, n \in \mathbb{N}) \right) \tag{12}$$

item (c)

In this question we need to find an expression for the cross-covariance function $\gamma_{1,2}(h) = \text{Cov}(X_{1,t+h}, X_{2,t})$.

For this endeavor, we will start from (12) and then we can write the components $X_{1,t+h}$ and $X_{2,t}$ by decomposing the matrix-form multiplying by the proper coordinate, so we reached the following representation:

$$X_{1,t+h} = \sum_{i=0}^{\infty} 0.5^i U_{t+h-i} \tag{13}$$

and

$$X_{2,t} = \sum_{j=0}^{\infty} \left(0.5^{j/2} U_{t-j} + 0.5^{j/2} V_{t-j}\right) \times \mathbb{1}(j : j = 2n, n \in \mathbb{N}) \tag{14}$$

Using (13) and (14) we can re-write the cross-covariance as follows:

$$\begin{aligned}
\gamma_{1,2}(h) &= \text{Cov}(X_{1,t+h}, X_{2,t}) \\
&= \text{Cov}\Big(\sum_{i=0}^{\infty} 0.5^i U_{t+h-i}, \sum_{j=0}^{\infty} \left(0.5^{j/2} U_{t-j} + 0.5^{j/2} V_{t-j}\right) \times \mathbb{1}(j : j = 2n, n \in \mathbb{N}) \Big)
\end{aligned}$$

Note that, as U_t and V_t are independent, this expression can be simplified by doing $j = 2k$ as follows:

$$\begin{aligned}
\gamma_{1,2}(h) &= \text{Cov}\Big(\sum_{i=0}^{\infty} 0.5^i U_{t+h-i}, \sum_{j=0}^{\infty} 0.5^{j/2} U_{t-j} \times \mathbb{1}(j : j = 2n, n \in \mathbb{N}) \Big) \\
&= \text{Cov}\Big(\sum_{i=0}^{\infty} 0.5^i U_{t+h-i}, \sum_{k=0}^{\infty} 0.5^k U_{t-2k} \Big)
\end{aligned}$$

This cross-covariance is non-zero if $t + h - i = t - 2k$ which leads to $i = 2k + h$, then we can re-write the expression as follows:

$$
\begin{aligned}
\gamma_{1,2}(h) &= \sum_{k=0}^{\infty} 0.5^{2k+h} 0.5^{k} \underbrace{\operatorname{Cov}\big(U_{t-2k}, U_{t-2k}}_{\operatorname{Var}(U_{t-2k})=1}\big) \\
&= 0.5^{h} \sum_{k=0}^{\infty} 0.5^{3k} \\
&= 0.5^{h} \sum_{k=0}^{\infty} (0.5^{3})^{k} \\
&= 0.5^{h} \frac{1}{1-0.5^{3}} \\
&= 0.5^{h} \frac{1}{1-\frac{1}{8}} \\
&= \frac{8}{7} 0.5^{h}
\end{aligned}
$$

$$
\implies \gamma_{1,2}(h) = \frac{8}{7} 0.5^{h}, \quad h \geq 0 \tag{15}
$$

. . . and

$$
\implies \gamma_{1,2}(h) = \frac{8}{7} \frac{1}{0.5^{h}}, \quad h < 0 \tag{16}
$$

This concludes the THEORY part of the assignment.