Analytic expression of potential operators for Helmholtz equation in circular and spherical geometries

Analytic solutions in 2-D

In this section, we give detailed expression of the boundary integral operators associated to a circle in 2-D. Here we consider an Helmholtz equation $-\Delta u - \kappa^2 u = 0$ with outgoing radiation condition. The corresponding Green kernel is given by

$$\begin{cases} -\Delta\mathscr{G} - \kappa^2\mathscr{G} = \delta_0 \quad \text{in } \mathbb{R}^2 \\ \lim_{\rho\to+\infty}\int_{\partial\mathrm{D}_\rho} |\partial_\rho\mathscr{G} - \imath\kappa\mathscr{G}|^2 d\sigma_\rho = 0 \\ \text{where} \quad \mathscr{G}(\boldsymbol{x}) = \frac{\imath}{4}H_0^{(1)}(\kappa|\boldsymbol{x}|) \end{cases}$$

Let us denote $\mathrm{D}\subset\mathbb{R}^2$ the disk with center 0 and radius 1, and $\Gamma = \partial\mathrm{D}$. We denote $\gamma_D : \mathrm{H}^1(\mathrm{D}) \to \mathrm{H}^{1/2}(\Gamma)$ the interior Dirichlet trace defined by $\gamma_D(u) := u|_\Gamma$ for any $u \in \mathscr{C}^0(\overline{\mathrm{D}})$, and $\gamma_N : \mathrm{H}^1(\Delta,\mathrm{D}) \to \mathrm{H}^{-1/2}(\Gamma)$ the interior Neumann trace defined by $\gamma_N(u) := \boldsymbol{n}\cdot\nabla u|_\Gamma = \partial_r u|_\Gamma$ where $\boldsymbol{n}$ refers to the normal vector field ditrected toward the exterior of D. We define $\gamma_{D,c}, \gamma_{N,c}$ in the same manner, except that the traces are taken from the exterior of D. Finally, we set

$$\{\gamma_D\} := (\gamma_D + \gamma_{D,c})/2 \quad \{\gamma_N\} := (\gamma_N + \gamma_{N,c})/2$$

$$[\gamma_D] := \gamma_D - \gamma_{D,c} \qquad [\gamma_N] := \gamma_N - \gamma_{N,c}.$$

Let us introduce the layer potentials associated to the interior of the disc D. For any trace $v \in \mathrm{H}^{1/2}(\Gamma), p \in \mathrm{H}^{-1/2}(\Gamma)$, their explicit expression is given by:

$$\begin{aligned} \mathrm{SL}(p)(\boldsymbol{x}) &:= \int_\Gamma \mathscr{G}(\boldsymbol{x}-\boldsymbol{y})p(\boldsymbol{y})d\sigma(\boldsymbol{y}), \\ \mathrm{DL}(p)(\boldsymbol{x}) &:= \int_\Gamma \boldsymbol{n}(\boldsymbol{y})\cdot(\nabla\mathscr{G})(\boldsymbol{x}-\boldsymbol{y})p(\boldsymbol{y})d\sigma(\boldsymbol{y}). \end{aligned} \tag{1}$$

We are going to provide a completely explicit expression of these operators in terms of Fourier harrmonics and Bessel functions. Set $\mathfrak{e}_n(\theta) = \exp(\imath n\theta)$. We have

$$\mathrm{SL}(\mathfrak{e}_n)(r,\theta) = \begin{cases} \imath\frac{\pi}{2}H_{|n|}^{(1)}(\kappa)J_{|n|}(\kappa r)\mathfrak{e}_n(\theta) & \text{for} \quad |\boldsymbol{x}|<1 \\ \imath\frac{\pi}{2}J_{|n|}(\kappa)H_{|n|}^{(1)}(\kappa r)\mathfrak{e}_n(\theta) & \text{for} \quad |\boldsymbol{x}|>1 \end{cases}$$

and

$$\mathrm{DL}(\mathfrak{e}_n)(r,\theta) = \begin{cases} -\imath\kappa\frac{\pi}{2}H_{|n|}^{(1)\prime}(\kappa)J_{|n|}(\kappa r)\mathfrak{e}_n(\theta) & \text{for} \quad |\boldsymbol{x}|<1 \\ -\imath\kappa\frac{\pi}{2}J_{|n|}'(\kappa)H_{|n|}^{(1)}(\kappa r)\mathfrak{e}_n(\theta) & \text{for} \quad |\boldsymbol{x}|>1 \end{cases}$$

From these formulas, it is clear that $\int_\Gamma \bar{\mathfrak{e}}_p \gamma_N \mathrm{SL}(\mathfrak{e}_n) d\sigma = \int_\Gamma \bar{\mathfrak{e}}_p \gamma_D \mathrm{SL}(\mathfrak{e}_n) d\sigma = 0$ as well as $\int_\Gamma \bar{\mathfrak{e}}_p \gamma_N \mathrm{DL}(\mathfrak{e}_n) d\sigma = \int_\Gamma \bar{\mathfrak{e}}_p \gamma_D \mathrm{DL}(\mathfrak{e}_n) d\sigma = 0$ for $p \neq n$. In the case $p = n$, we have on the one hand

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_D\} \cdot \mathrm{SL}(\mathfrak{e}_n) d\sigma = \imath\pi^2 H^{(1)}_{|n|}(\kappa) J_{|n|}(\kappa)$$
$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_N\} \cdot \mathrm{DL}(\mathfrak{e}_n) d\sigma = -\imath\kappa^2\pi^2 H^{(1)\prime}_{|n|}(\kappa) J'_{|n|}(\kappa)$$

and on the other hand

$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_N\} \cdot \mathrm{SL}(\mathfrak{e}_n) d\sigma = +\imath\kappa \frac{\pi^2}{2} \big(\, H^{(1)}_{|n|}(\kappa) J'_{|n|}(\kappa) + H^{(1)\prime}_{|n|}(\kappa) J_{|n|}(\kappa) \,\big)$$
$$\int_\Gamma \bar{\mathfrak{e}}_n \{\gamma_D\} \cdot \mathrm{DL}(\mathfrak{e}_n) d\sigma = -\imath\kappa \frac{\pi^2}{2} \big(\, H^{(1)}_{|n|}(\kappa) J'_{|n|}(\kappa) + H^{(1)\prime}_{|n|}(\kappa) J_{|n|}(\kappa) \,\big)$$

Analytic solutions in 3-D

Now we consider the same problem but, this time, for an Helmholtz equation in 3-D with a boundary $\Gamma = \mathbb{S}^2$ the unit sphere. The corresponding outgoing radiating kernel satisfies the equations

$$\begin{cases} -\Delta \mathscr{G} - \kappa^2 \mathscr{G} = \delta_0 \quad \text{in } \mathbb{R}^3 \\ \lim_{\rho\to+\infty} \int_{\partial \mathrm{B}_\rho} |\partial_\rho \mathscr{G} - \imath\kappa \mathscr{G}|^2 d\sigma_\rho = 0 \\ \text{where} \quad \mathscr{G}(\boldsymbol{x}) = \exp(\imath|\boldsymbol{x}|)/(4\pi|\boldsymbol{x}|) \end{cases}$$

Let $\mathrm{B} \subset \mathbb{R}^3$ refer to the ball of center $\mathbf{0}$ and radius 1, and $\Gamma = \partial\mathrm{B}$. We use the same notations regarding trace operators compared to the previous section. in particular, $\boldsymbol{n}$ shall refer to the unit vector normal to Γ :directed toward the exterior of B. Once again, we define the potential operators by (1)

Let us recall that any function $u \in \mathrm{L}^2(\Gamma)$ can be decomposed on the orthonormal basis of spherical harmonics $\mathrm{Y}_l^m, l \geq 0, |m| \leq l$ as follows

$$u(\theta,\varphi) = \sum_{l=0}^{+\infty} \sum_{-l\leq m\leq +l} u_{l,m} \mathrm{Y}_l^m(\theta,\varphi) \quad \text{where} \quad u_{l,m} := \int_\Gamma u(\sigma) \overline{\mathrm{Y}}_l^m(\sigma) d\sigma$$
$$\mathrm{Y}_l^m(\theta,\varphi) := (-1)^m \sqrt{\frac{l+1/2}{2\pi} \frac{(l-m)!}{(l+m)!}}\, \mathrm{P}_l^m(\cos\theta) \exp(\imath m\varphi).$$

Here the $\mathrm{P}_l^m(z)$ are associated Legendre functions. Define the spherical Bessel and Hankel functions $j_l, h_l^{(1)}$ like in paragraph 10.47 of [1]. The single layer and double layer potentials admit explicit expressions in terms of the spherical harmonics. On the one hand

$$\mathrm{SL}(\mathrm{Y}_l^m)(\rho,\sigma) = \begin{cases} \imath\kappa\, j_l(\kappa\rho) h_l^{(1)}(\kappa) \mathrm{Y}_l^m(\theta,\varphi) \quad \text{for} \quad \rho < 1 \\ \imath\kappa\, h_l^{(1)}(\kappa\rho) j_l(\kappa) \mathrm{Y}_l^m(\theta,\varphi) \quad \text{for} \quad \rho > 1. \end{cases}$$

And on the other hand we have

$$\mathrm{DL}(\mathrm{Y}_l^m)(\rho,\sigma) = \begin{cases} -\imath\kappa^2\, j_l(\kappa\rho) h_l^{(1)'}(\kappa)\mathrm{Y}_l^m(\theta,\varphi) & \text{for } \rho < 1 \\ -\imath\kappa^2\, h_l^{(1)}(\kappa\rho) j_l'(\kappa)\mathrm{Y}_l^m(\theta,\varphi) & \text{for } \rho > 1. \end{cases}$$

Note that, like in the 2-D problem of the previous paragraph, we have $\int_\Gamma \overline{\mathrm{Y}}_p^q \gamma_D \mathrm{SL}(\mathrm{Y}_l^m) d\sigma = \int_\Gamma \overline{\mathrm{Y}}_p^q \gamma_N \mathrm{SL}(\mathrm{Y}_l^m) d\sigma = 0$ for $p \neq l$ or $q \neq m$, and similar identity holds for DL. Hence we have the following expressions for the entries of the Calderon projector

$$\int_\Gamma \overline{\mathrm{Y}}_l^m \{\gamma_D\} \mathrm{SL}(\mathrm{Y}_l^m) d\sigma = \imath\kappa\, j_l(\kappa) h_l^{(1)}(\kappa),$$
$$\int_\Gamma \overline{\mathrm{Y}}_l^m \{\gamma_N\} \mathrm{DL}(\mathrm{Y}_l^m) d\sigma = -\imath\kappa^3\, j_l'(\kappa) h_l^{(1)'}(\kappa).$$

As well as

$$\int_\Gamma \overline{\mathrm{Y}}_l^m \{\gamma_N\} \mathrm{SL}(\mathrm{Y}_l^m) d\sigma = +\imath\frac{\kappa^2}{2}\big(j_l'(\kappa) h_l^{(1)}(\kappa) + j_l(\kappa) h_l^{(1)'}(\kappa) \big),$$
$$\int_\Gamma \overline{\mathrm{Y}}_l^m \{\gamma_D\} \mathrm{DL}(\mathrm{Y}_l^m) d\sigma = -\imath\frac{\kappa^2}{2}\big(j_l'(\kappa) h_l^{(1)}(\kappa) + j_l(\kappa) h_l^{(1)'}(\kappa) \big).$$

References

[1] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).