

Analytic expression of potential operators for Helmholtz equation in circular and spherical geometries

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Analytic solutions in 2-D

In this section, we give detailed expression of the boundary integral operators associated to a circle in 2-D. Here we consider an Helmholtz equation $-\Delta u - \kappa^2 u = 0$ with outgoing radiation condition. The corresponding Green kernel is given by

$$\begin{cases} -\Delta \mathcal{G} - \kappa^2 \mathcal{G} = \delta_0 & \text{in } \mathbb{R}^2 \\ \lim_{\rho \rightarrow +\infty} \int_{\partial D} |\partial_\rho \mathcal{G} - \imath \kappa \mathcal{G}|^2 d\sigma = 0 \\ \text{where } \mathcal{G}(\mathbf{x}) = \frac{\imath}{4} H_0^{(1)}(\kappa |\mathbf{x}|) \end{cases}$$

Let us denote $D \subset \mathbb{R}^2$ the disk with center 0 and radius ρ , and $\Gamma = \partial D$. We denote $\gamma_D : H^1(D) \rightarrow H^{1/2}(\Gamma)$ the interior Dirichlet trace defined by $\gamma_D(u) := u|_\Gamma$ for any $u \in \mathcal{C}^0(\overline{D})$, and $\gamma_N : H^1(\Delta, D) \rightarrow H^{-1/2}(\Gamma)$ the interior Neumann trace defined by $\gamma_N(u) := \mathbf{n} \cdot \nabla u|_\Gamma = \partial_r u|_\Gamma$ where \mathbf{n} refers to the normal vector field directed toward the exterior of D . We define $\gamma_{D,c}, \gamma_{N,c}$ in the same manner, except that the traces are taken from the exterior of D . Finally, we set

$$\{\gamma_D\} := (\gamma_D + \gamma_{D,c})/2 \quad \{\gamma_N\} := (\gamma_N + \gamma_{N,c})/2$$

$$[\gamma_D] := \gamma_D - \gamma_{D,c} \quad [\gamma_N] := \gamma_N - \gamma_{N,c}$$

Let us introduce the layer potentials associated to the interior of the disc D . For any trace $v \in H^{1/2}(\Gamma), p \in H^{-1/2}(\Gamma)$, their explicit expression is given by:

$$\begin{aligned} \text{SL}(p)(\mathbf{x}) &:= \int_\Gamma \mathcal{G}(\mathbf{x} - \mathbf{y}) p(\mathbf{y}) d\sigma(\mathbf{y}), \\ \text{DL}(p)(\mathbf{x}) &:= \int_\Gamma \mathbf{n}(\mathbf{y}) \cdot (\nabla \mathcal{G})(\mathbf{x} - \mathbf{y}) p(\mathbf{y}) d\sigma(\mathbf{y}). \end{aligned} \tag{1}$$

We are going to provide a completely explicit expression of these operators in terms of Fourier harmonics and Bessel functions. Set $\mathbf{e}_n(\theta) = \exp(\imath n\theta)$. We have

$$\text{SL}(\mathbf{e}_n)(r, \theta) = \begin{cases} \imath \rho \frac{\pi}{2} H_{|n|}^{(1)}(\kappa |\mathbf{x}|) J_{|n|}(\kappa \rho) \mathbf{e}_n(\theta) & \text{for } |\mathbf{x}| < \rho \\ \imath \rho \frac{\pi}{2} J_{|n|}(\kappa |\mathbf{x}|) H_{|n|}^{(1)}(\kappa \rho) \mathbf{e}_n(\theta) & \text{for } |\mathbf{x}| > \rho \end{cases}$$

and

$$\text{DL}(\mathbf{e}_n)(r, \theta) = \begin{cases} -\imath \kappa \rho \frac{\pi}{2} H_{|n|}^{(1)'}(\kappa \rho) J_{|n|}(\kappa |\mathbf{x}|) \mathbf{e}_n(\theta) & \text{for } |\mathbf{x}| < \rho \\ -\imath \kappa \rho \frac{\pi}{2} J_{|n|}'(\kappa \rho) H_{|n|}^{(1)}(\kappa |\mathbf{x}|) \mathbf{e}_n(\theta) & \text{for } |\mathbf{x}| > \rho \end{cases}$$

From these formulas, it is clear that $\int_{\Gamma} \bar{\mathbf{e}}_p \gamma_N \text{SL}(\mathbf{e}_n) d\sigma = \int_{\Gamma} \bar{\mathbf{e}}_p \gamma_D \text{SL}(\mathbf{e}_n) d\sigma = 0$ as well as $\int_{\Gamma} \bar{\mathbf{e}}_p \gamma_N \text{DL}(\mathbf{e}_n) d\sigma = \int_{\Gamma} \bar{\mathbf{e}}_p \gamma_D \text{DL}(\mathbf{e}_n) d\sigma = 0$ for $p \neq n$. In the case $p = n$, we have on the one hand

$$\begin{aligned} \int_{\Gamma} \bar{\mathbf{e}}_n \{\gamma_D\} \cdot \text{SL}(\mathbf{e}_n) d\sigma &= \imath \rho^2 \pi^2 H_{|n|}^{(1)}(\kappa \rho) J_{|n|}(\kappa \rho) \\ \int_{\Gamma} \bar{\mathbf{e}}_n \{\gamma_N\} \cdot \text{DL}(\mathbf{e}_n) d\sigma &= -\imath \kappa^2 \rho^2 \pi^2 H_{|n|}^{(1)'}(\kappa \rho) J_{|n|}'(\kappa \rho) \end{aligned}$$

and on the other hand

$$\begin{aligned} \int_{\Gamma} \bar{\mathbf{e}}_n \{\gamma_N\} \cdot \text{SL}(\mathbf{e}_n) d\sigma &= +\rho^2 \imath \kappa \frac{\pi^2}{2} \left(H_{|n|}^{(1)}(\kappa \rho) J_{|n|}'(\kappa \rho) + H_{|n|}^{(1)'}(\kappa \rho) J_{|n|}(\kappa \rho) \right) \\ \int_{\Gamma} \bar{\mathbf{e}}_n \{\gamma_D\} \cdot \text{DL}(\mathbf{e}_n) d\sigma &= -\imath \kappa \rho^2 \frac{\pi^2}{2} \left(H_{|n|}^{(1)}(\kappa) J_{|n|}'(\kappa) + H_{|n|}^{(1)'}(\kappa) J_{|n|}(\kappa) \right) \end{aligned}$$

Analytic solutions in 3-D

Now we consider the same problem but, this time, for an Helmholtz equation in 3-D with a boundary $\Gamma = \mathbb{S}^2$ the unit sphere. The corresponding outgoing radiating kernel satisfies the equations

$$\begin{cases} -\Delta \mathcal{G} - \kappa^2 \mathcal{G} = \delta_0 & \text{in } \mathbb{R}^3 \\ \lim_{\rho \rightarrow +\infty} \int_{\partial B_\rho} |\partial_\rho \mathcal{G} - \imath \kappa \mathcal{G}|^2 d\sigma_\rho = 0 \\ \text{where } \mathcal{G}(\mathbf{x}) = \exp(\imath |\mathbf{x}|) / (4\pi |\mathbf{x}|) \end{cases}$$

Let $B \subset \mathbb{R}^3$ refer to the ball of center $\mathbf{0}$ and radius 1, and $\Gamma = \partial B$. We use the same notations regarding trace operators compared to the previous section. in particular, \mathbf{n} shall refer to the unit vector normal to Γ :directed toward the exterior of B . Once again, we define the potential operators by (1)

Let us recall that any function $u \in L^2(\Gamma)$ can be decomposed on the orthonormal basis of spherical harmonics $Y_l^m, l \geq 0, |m| \leq l$ as follows

$$\begin{aligned} u(\theta, \varphi) &= \sum_{l=0}^{+\infty} \sum_{-l \leq m \leq l} u_{l,m} Y_l^m(\theta, \varphi) \quad \text{where} \quad u_{l,m} := \int_{\Gamma} u(\sigma) \bar{Y}_l^m(\sigma) d\sigma \\ Y_l^m(\theta, \varphi) &:= (-1)^m \sqrt{\frac{l+1/2}{2\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) \exp(\imath m \varphi). \end{aligned}$$

Here the $P_l^m(z)$ are associated Legendre functions. Define the spherical Bessel and Hankel functions $j_l, h_l^{(1)}$ like in paragraph 10.47 of [1]. The single layer and double layer potentials admit explicit expressions in terms of the spherical harmonics. On the one hand

$$\text{SL}(Y_l^m)(\rho, \sigma) = \begin{cases} \imath \kappa j_l(\kappa \rho) h_l^{(1)}(\kappa) Y_l^m(\theta, \varphi) & \text{for } \rho < 1 \\ \imath \kappa h_l^{(1)}(\kappa \rho) j_l(\kappa) Y_l^m(\theta, \varphi) & \text{for } \rho > 1. \end{cases}$$

And on the other hand we have

$$\text{DL}(\mathbf{Y}_l^m)(\rho, \sigma) = \begin{cases} -\imath \kappa^2 j_l(\kappa \rho) h_l^{(1)'}(\kappa) \mathbf{Y}_l^m(\theta, \varphi) & \text{for } \rho < 1 \\ -\imath \kappa^2 h_l^{(1)}(\kappa \rho) j_l'(\kappa) \mathbf{Y}_l^m(\theta, \varphi) & \text{for } \rho > 1. \end{cases}$$

Note that, like in the 2-D problem of the previous paragraph, we have $\int_{\Gamma} \bar{\mathbf{Y}}_p^q \gamma_D \text{SL}(\mathbf{Y}_l^m) d\sigma = \int_{\Gamma} \bar{\mathbf{Y}}_p^q \gamma_N \text{SL}(\mathbf{Y}_l^m) d\sigma = 0$ for $p \neq l$ or $q \neq m$, and similar identity holds for DL. Hence we have the following expressions for the entries of the Calderon projector

$$\begin{aligned} \int_{\Gamma} \bar{\mathbf{Y}}_l^m \{\gamma_D\} \text{SL}(\mathbf{Y}_l^m) d\sigma &= \imath \kappa j_l(\kappa) h_l^{(1)}(\kappa), \\ \int_{\Gamma} \bar{\mathbf{Y}}_l^m \{\gamma_N\} \text{DL}(\mathbf{Y}_l^m) d\sigma &= -\imath \kappa^3 j_l'(\kappa) h_l^{(1)'}(\kappa). \end{aligned}$$

As well as

$$\begin{aligned} \int_{\Gamma} \bar{\mathbf{Y}}_l^m \{\gamma_N\} \text{SL}(\mathbf{Y}_l^m) d\sigma &= +\imath \frac{\kappa^2}{2} (j_l'(\kappa) h_l^{(1)}(\kappa) + j_l(\kappa) h_l^{(1)'}(\kappa)), \\ \int_{\Gamma} \bar{\mathbf{Y}}_l^m \{\gamma_D\} \text{DL}(\mathbf{Y}_l^m) d\sigma &= -\imath \frac{\kappa^2}{2} (j_l'(\kappa) h_l^{(1)}(\kappa) + j_l(\kappa) h_l^{(1)'}(\kappa)). \end{aligned}$$

References

- [1] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).

A Details