**CS 726: Samples questions on Normalizing Flows**

1. Let $\mathbf{x} = [x_1, x_2, x_3]$ and $\mathbf{u} = [u_1, u_2, u_3]$ be real random variables where $p(\mathbf{u}) \sim N(0, I)$. Let $[x_1, x_2, x_3] = T(\mathbf{u}) = [3u_1 + 2, 4u_1 + u_2, u_1^2 u_2^2 + 2u_3 - 2]$.

   (a) Write the value of $\mathbf{g}(\mathbf{x}) = T^{-1}(\mathbf{x})$
   $[u_1, u_2, u_3] = [g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x})]$
   $= [(x_1 - 2)/3, x_2 - 4(x_1 - 2)/3, x_3/2 + 1 - \frac{(x_1-2)^2}{9*2}(x_2 - 4(x_1 - 2)/3)^2]$

   (b) Write the expression for $|J(g(\mathbf{x}))|$
   $\begin{bmatrix} \frac{\partial g_1(\mathbf{x})}{\partial x_1} & 0 & 0 \\ \frac{\partial g_2(\mathbf{x})}{\partial x_1} & \frac{\partial g_2(\mathbf{x})}{\partial x_2} & 0 \\ \frac{\partial g_3(\mathbf{x})}{\partial x_1} & \frac{\partial g_3(\mathbf{x})}{\partial x_2} & \frac{\partial g_3(\mathbf{x})}{\partial x_3} \end{bmatrix}$
   $= \begin{bmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ .. & .. & 1/2 \end{bmatrix}$ The determinant of this matrix is just the product of its diagonal which is 1/3*1*1/2.

   (c) Write the expression for $p(\mathbf{x})$ in terms of $p(\mathbf{u})$ using the change of variables formula.
   $p(x_1, x_2, x_3) = \frac{1}{(2\pi)^{3/2}} \exp(-\frac{1}{2}(g_1(\mathbf{x})^2 + g_2(\mathbf{x})^2 + g_3(\mathbf{x})^2)) * 1/6$ where the values of $g_1, g_2, g_3$ are as derived above.

   (d) What is the mode of the distribution $p(\mathbf{x})$? That is, for what $x_1, x_2, x_3$ is the value of $p(\mathbf{x})$ maximized. Solve for value for which $g_1, g_2, g_3$ are all zero.

2. In the real-NVP paper assume we use two layers of transformation as follows:

$$\mathbf{y} = T(\mathbf{z}) : Y_1 = Z_1; Y_2 = (Z_2 + t(Z_1)).\exp(s(Z_1)) \tag{1}$$
$$\mathbf{x} = H(\mathbf{y}) : X_2 = Y_2; X_1 = (Y_1 + t(Y_2)).\exp(s(Y_2)) \tag{2}$$
$$\tag{3}$$

   Assume $\mathbf{z}$ has $D = 4$ dimensions which we partition into $Z_1, Z_2$ of two dimensions each. Likewise for $X_1, X_2$ and $Y_1, Y_2$. Assume $p(\mathbf{z})$ follows a $D$-dimensional standard Gaussian distribution. Assume $s(u, v) = (u - v)^2, t(u, v) = uv$.

   Also, assume you sampled a value $\mathbf{z} = [0, 1, -1, 2]$. Make sure you know how to calculate the transformed sample in $\mathbf{x}$ space.

   Inversely, for a fixed $\mathbf{x}$, solve the above sub-questions of Q1 with these two stages of transformations.

$\boxed{\textbf{Total: 0}}$