Constructing an UGM from a positive distribution using Local-CI

Definition: The Markov Blanket of a variable x_i, $\text{MB}(x_i)$ is the smallest subset of variables V that makes x_i CI of others given the Markov blanket.

$$x_i \perp\!\!\!\perp V - MB(x_i) - \{x_i\} | MB(x_i)$$

Theorem

The MB of a variable is always unique for a positive distribution.

Proof of uniqueness of MB

Proof.

(Not in Syllabus) We will prove by contradiction. Let x_i be a variable and M_1, M_2 be two MBs. Let $\alpha = M_1 - M_2$ and $\beta = M_2 - M_1$,
$M = M_1 \cap M_2, W = V - (M_1 \cup M_2)$
$x_i \perp\!\!\!\perp V - M_2 | M_2$, $x_i \perp\!\!\!\perp V - M_1 | M_1$,
This implies, $x_i \perp\!\!\!\perp W, \alpha | M, \beta$, $x_i \perp\!\!\!\perp W, \beta | M, \alpha$.
For positive distributions, this implies $x_i \perp\!\!\!\perp W, \alpha, \beta | M$ (Intersection property of distributions Sec 2.1)
This implies that M is also a MB. But then M_1, M_2 were supposed to be minimal — a contradiction.

□

Popular undirected graphical models

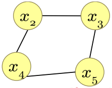
- Interacting atoms in gas and solids [1900]
- Markov Random Fields in vision for image segmentation
- Conditional Random Fields for information extraction
- Social networks
- Bio-informatics: annotating active sites in a protein molecules.

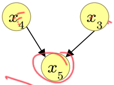
Lessons Learned

- BNs not great for representing symmetric interactions among variables. MRFs are better suited.
- Potentials are arbitrary scores not conditional probabilities
- We can draw MRFs for positive distributions by finding the Markov Blanket for each variable.
- In practice, MRFs are often constructed starting from potentials where we just connect together all variables that appear together in a potential.
- Easy to read-off all CIs using graph separability

Comparing directed and undirected graphs

- Some distributions can only be expressed in one and not the other.





$x_3 \perp\!\!\!\perp x_4$

$x_3 \not\perp\!\!\!\perp x_4 \mid x_5$

$X \perp\!\!\!\perp Y$

$\boxed{X \not\perp\!\!\!\perp Y \mid Z}$ cannot be represented by MRFs.

$x_2 \perp\!\!\!\perp x_5 \mid x_3, x_4$

$x_3 \perp\!\!\!\perp x_4 \mid x_2, x_5$

- Potentials
  - Directed: conditional probabilities, more intuitive
  - Undirected: arbitrary scores, easy to set.
- Dependence structure
  - Directed: Complicated d-separation test
  - Undirected: Graph separation: $A \perp\!\!\!\perp B \mid C$ iff C separates A and B in G.
- Often application makes the choice clear.
  - Directed: Causality
  - Undirected: Symmetric interactions.

Equivalent BNs

Two BN DAGs are said to be equivalent if they express the same set of CIs. (Examples)

$\boxed{x_1 \perp\!\!\!\perp x_3 \mid x_2 \,?}$

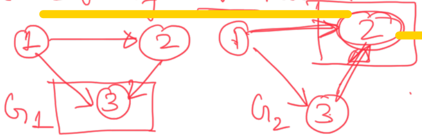
$P(x_1, x_2, x_3)$

1. $x_1 \longrightarrow x_2 \longrightarrow x_3$
2. $x_1 \longleftarrow x_2 \longleftarrow x_3$
3. $x_1 \longleftarrow x_2 \longrightarrow x_3$
4. $x_1 \longrightarrow x_2 \longleftarrow x_3$

One equivalence

G_1, G_2

Claim: Two BNs are equivalent if they have the same set of V-nodes and the same set of undirected edges.



Equivalent BNs

Theorem

Two BNs G_1, G_2 are equivalent iff they have the same skeleton and the same set of immoralities. (An immorality is a structure of the form $x \rightarrow y \leftarrow z$ with no edge between x and z)

DOUBT - immoraity also when Y has children?

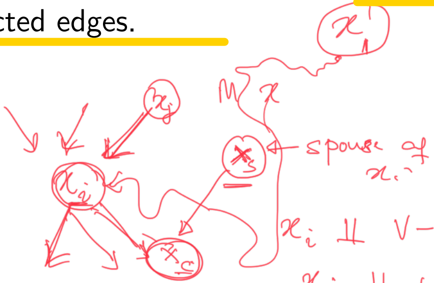
Converting BN to MRFs

Efficient: Using the Markov Blanket (MB) (also called the Local-CI) algorithm .

The MB of a x_i in a BN can be shown to be:

$$MB(x_i) = \boxed{Pa(x_i) \cup Ch(x_i) \cup \text{Spouse}(x_i)}$$

This is essentially obtained by moralizing a BN and removing all directed edges.



$x_i \not\perp\!\!\!\perp x_s \mid x_c$

$x_i \perp\!\!\!\perp V - MB(x_i) \mid MB(x_i)$

$ch(x_i)$

$x_i \perp\!\!\!\perp (\quad) \mid x_c$

For which BN can we create perfect MRFs?

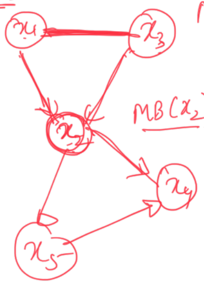
A BN which has no immorality will not require any new edges to be added when converting to MRF. Such networks will have a perfect MRF. (Formal proof in Proposition 4.9 of KF book)

MRF → is not perfect .

BN

$x_1 \perp x_3$

$x_1 \not\perp x_3$

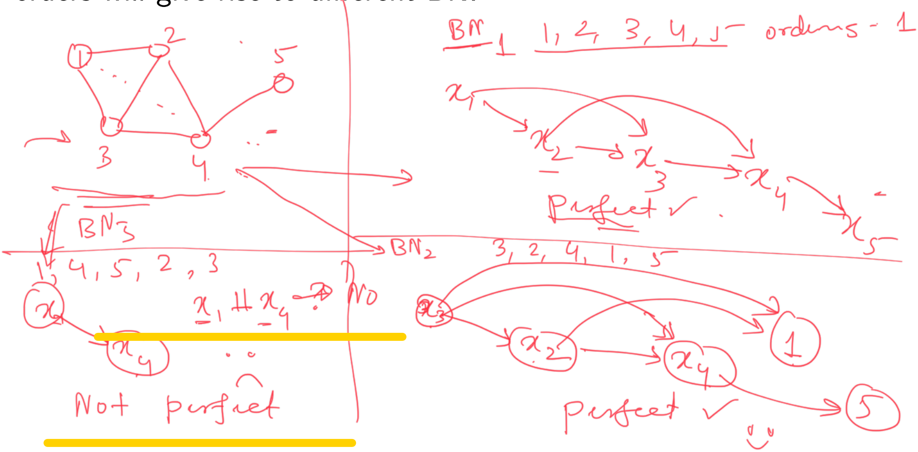


$MB(x_3) = \{x_1, x_2\}$

$MB(x_2) = \{x_1, x_3, x_5, x_4\}$

Converting MRFs to BNs

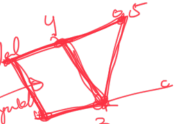
Use the BN construction algorithm (Discussed in the BN portion) starting with any random order of variables. Use the Global-CI on the MRF to answer the conditional independence queries. Different orders will give rise to different BN.



Which MRFs have perfect BNs

Chordal or triangulated graphs

A graph is chordal if it has no minimal cycle of length ≥ 4.



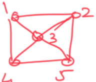
not chordal
not triangulated

C3 = {1,2,3,5,4,1}
not minimal

C1 (1,2,3,4,1)
minimal

C2 = (3,4,5,3)
minimal?

chordal



chordal?
Not.
(1,2,5,4,1)
minimal cycle > 3.

Theorem

A MRF can be converted perfectly into a BN iff it is chordal.

Proof.

Theorems 4.11 and 4.13 of KF book □

Algorithm for constructing perfect BNs from chordal MRFs to be discussed later.