CS 726: Homework 1 (Graded)

Write your answers in the space provided. You are expected to solve each question on your own. Do not try to search the answers from any external sources, like the web.

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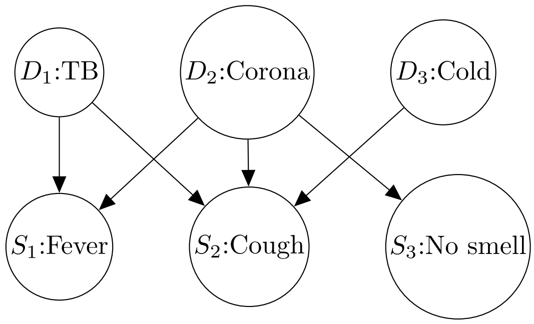
1. Consider a special QMR Bayesian network comprising of disease nodes $D_1, \ldots, D_n$ and symptom nodes $S_1, \ldots S_k$ An example appears below.



Assume all variables are binary. Let probability of a disease D_i occurring be denoted as $P(D_i = 1) = \beta_i$ and α_{ij} denote the probability of a patient showing symptom S_j if he only has disease D_i. Also, assume the probability of a symptom not occurring conditioned on its parent nodes is $\Pr(S_j = 0|Pa(S_j)) = \prod_{i \in Pa(S_j)}(1-\alpha_{ij})^{D_i}$. For example, if someone has D_1, D_2 but not D_3 then the probability of his not showing symptom S_2 is $(1-\alpha_{12})(1-\alpha_{22})$

(a) In a study over 1000 random Indians, 20 had Cold, 2 had TB, and 1 had Corona. Doctors have estimated that Corona causes each of these symptoms with probability 0.2, Flu causes cough with probability 0.7, and TB causes fever with probability 0.8, and cough with probability 0.3. Write exact numerical values for the following potentials as a table:

i. $P(D_1)$ ..1

$D_1 =$	0	1
	0.998	0.002

ii. $P(S_3|Pa(S_3))$ ..2

$D_3 =$	0	1
$P(S_3) = 0$	1	0.3
$P(S_3) = 1$	0	0.7

iii. $P(S_1|Pa(S_1))$ ..2

$D_1, D_2 =$	0,0	0,1	1,0	1,1
$P(S_3) = 0$	1	0.2	0.8	0.16

(b) What is the marginal probability $P(S_1 = 1)$ in the above example graphical model? Express only in terms of α_{ij}s and β_is. ..2 $\sum_{D_1,D_2} \Pr(S_i = 0|D_1, D_2)P(D_1)P(D_2) = (1-\alpha_{21})(1-\alpha_{11})\beta_1\beta_2 + (1-\alpha_{11})\beta_1(1-\beta_2) + (1-\alpha_{21})(1-\beta_1)\beta_2 + (1-\beta_1)(1-\beta_2)$

(c) If someone has Fever, No sense of smell, and No cough, what is the most likely disease that the person has assuming he has at most one disease? [Give a brief explanation for the answer.] ..2 TB:2*0.8*(1-0.3)

2. Consider the reduction of 2-SAT to Bayesian network inference discussed in class. Consider an instance of 2-SAT on the following set of clauses over three Boolean variables.

- $C_1 = x_1 \vee x_3$

- $C_2 = \bar{x_1} \vee \bar{x_2}$
- $C_3 = x_2 \vee \bar{x_3}$

where we are interested in the truth of $S = C_1 \wedge C_2 \wedge C_3$

(a) For this 2-SAT problem, draw the corresponding Bayesian network (BN) over nodes $x_1, x_2, x_3, C_1, C_2, C_3, S$. Do not introduce any other nodes. For node C_2 show the potentials as a conditional probability table. ..2

(b) Moralize the above BN ..1

(c) Triangulate the above graph by adding extra edges as required. ..2

(d) Identify the maximal cliques in the graph ..2

(e) Draw the junction tree (JT) of the graph above and assign potentials to the cliques from the CPDs. ..3

(f) What is the space required to store the clique potentials of the above JT? ..1

3. Consider the undirected graphical model below over binary variables.

$$P(x_1, \dots, x_5) = \frac{1}{Z}\psi_a(x_1, x_3)\psi_b(x_2, x_3)\psi_c(x_3, x_5)\psi_d(x_4, x_5)$$

where the potentials are defined as follows. For example, the first entry says that $\psi_a(x_1 = 1, x_3 = 1) = 2^3$ and $\psi_d(x_4 = 0, x_5 = 1) = 2^2$

$\psi_a(x_1, x_3)$	$\psi_b(x_2, x_3)$	$\psi_c(x_3, x_5)$	$\psi_d(x_4, x_5)$
$\begin{bmatrix} 1 & 2^2 \\ 2 & 2^3 \end{bmatrix}$	$\begin{bmatrix} 2 & 2^2 \\ 2^3 & 1 \end{bmatrix}$	$\begin{bmatrix} 2^4 & 2^3 \\ 2^2 & 2^3 \end{bmatrix}$	$\begin{bmatrix} 2 & 2^2 \\ 2^4 & 1 \end{bmatrix}$

(a) Draw the junction tree for the above distribution calling the cliques as C_a, C_b, C_c, C_d corresponding to the four potentials above. ..1

(b) Our goal is to find the assignment of values to $x_1, \dots, x_5$ for which the probability is maximum. For this we run the max-product message passing algorithm. Compute the minimal messages needed to be sent so that we can compute the maximizing assignment at clique node $C_d = (4, 5)$. You need to show the values of the minimal set of intermediate messages to be computed before this step can be executed by filling in the values for the question marks below. [Remember that these are messages for computing Max not Sum over variables. Also plugin numerical values for messages and not just formula.] ..4

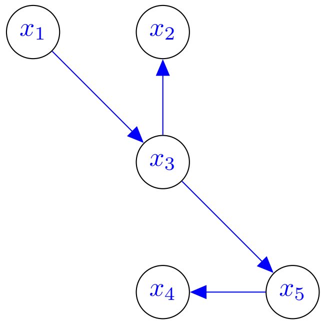
   i. Step 1: $m_{? \to ?}(?) = ?$
   $m_{a \to c}(x_3) = [2, 8]$

   ii. Step 2: $m_{? \to ?}(?) = ?$
   $m_{b \to c}(x_3) = [8, 4]$

   iii. Step 3: $m_{C_c \to C_d}(x_5) = ?$
   $\max_{x_3} m_{a \to c}(x_3) m_{b \to c}(x_3) \psi_c(x_3, x_5) = [2^8, 2^8]$

   iv. Step 4: Multiply message $m_{C_c \to C_d}(x_5)$ with potential ψ_4 at clique C_d and report $\text{argmax} P(x_1, \dots, x_4)$. State the maximizing assignment on all five variables after this step.
   $\text{argmax}_{x_4, x_5} m_{c \to d}(x_5)\psi_d(x_4, x_5) = (x_4 = 1, x_5 = 0)$ maximizing assignment after retracing becomes: $x_5 = 0, x_4 = 1, x_3 = 0, x_1 = 2, x_2 = 1$

4. Draw a *perfect* Bayesian network for P with variable x_1 as the first variable and a simplicial order after that.[Hint: Draw the undirected graphical model starting from the above potentials and convert that into a BN.] ..3



This network does not have any v-nodes and if you elininate the edge directions you will get the undirected graph that corresponds to the pontentials above.

5. In terms of above potentials and sum-product messages $m_{i\to j}(X_{s_{ij}})$ write the CPDs for node x_1, and node x_3 in this Bayesian network. ..3

$\Pr(x_1) = \sum_{x_3} \psi_1(x_1, x_3) m_{c_3\to c_1}(x_3)$

$\Pr(x_3|x_1) = \frac{\psi_1(x_1,x_3) m_{c_3\to c_1}(x_3)}{\sum_{x_3} \psi_1(x_1,x_3) m_{c_3\to c_1}(x_3)}$