CS 726: Sample questions on Gaussian Processes

1. Consider a Gaussian Process $f(x) \sim GP(0, K)$ where K is the RBF kernel $K(x_1, x_2) = \exp(-(x_1 - x_2)^2/2)$, where $x \in R$ and mean is 0. We have one data point $D = \{x_0 = 0, Y_0 = -1\}$. Answer the following :

   (a) What is $-2\mu(x) + 4\sigma(x)^2$, where $\mu(x), \sigma(x)^2$ are mean and variance of the posterior distribution of $f(x)|D$ when

      i. $x = \sqrt{\log(4)}$
      We know the equation

      $$f(x_*)|f(x_0), \ldots, f(x_n) \sim \mathcal{N}(\mathbf{k}_*^T \Sigma^{-1} y, K(x_*, x_*) - \mathbf{k}_*^T \Sigma^{-1} \mathbf{k}_*$$

      We have just 1 data point here so only x_0 is there. Σ is [1] as $K(0,0) = K(x_*, x_*) = 1$ and $\mathbf{k}_* = [\exp(-x^2/2)]$. Substituting $x = \sqrt{\log(4)}$ we get $\mu = -1/2, \sigma^2 = 3/4$ and hence the answer is 4

      ii. $x \to \inf$
      As $x \to \inf$ we get $\mu = 0, \sigma^2 = 1$, which can be checked by by applying limit in the above equations. As $x \to \inf$ the effect of the observed variable vanishes due to the RBF kernel and the distribution is again a standard normal distribution. Answer = 4

      iii. $x = 0$
      At $x = 0$ we get $\mu = -1, \sigma^2 = 0$, which can be checked by the above equations. Since we have already observed $f(0)$ there is no variance in our distribution with our mean coinciding with the observed value. Answer = 2.

      iv. JUST FOR THIS PART, Suppose the kernel is re-defined as $K(x_1, x_2) = 1$ if $x_1 = x_2$ *else* 0 and $x = \sqrt{\log(4)}$
      The given kernel models all the points independently, i.e. $f(x_*)|f(x_i) \sim f(x_*)$ if $x_* \neq x_i$. Hence the distribution is $\mu = 0, \sigma^2 = 1$, which can be checked by following steps in (a). Only difference being that $\mathbf{k}_* = [0]$. Answer = 4.

   (b) We want to search for a new point x' for minimisation of f. We choose x' using the acquire function of Lower Confidence Bound, that is $x' = \text{argmin}_x \mu(x) - \kappa\sigma(x)$. What is $5\exp(-x'^2)$ when $\kappa = 1$?
   For a point x' the mean and variance, conditioned on the sampled observation as derived in part (a) is

   $$\mu(x) = -\exp(-x^2/2), \sigma(x) = \sqrt{1 - \exp(-x^2)}$$

   We wish to minimize $\mu(x) - \sigma(x)$ over all x. Taking gradient and solving gives us that $\exp(-x^2) = 1/2$ for the minima. Hence answer = 2.5.

   (c) Let $Y_1 = f(x = \sqrt{\log(4)})$ and $Y_2 = f(x = -\sqrt{\log(4)})$ be 2 Random Variables. What is $y_1 + y_2$ where $y1, y2 = \text{argmax } \Pr(Y_1 = y_1, Y_2 = y_2|D)$?

Maximum probability density of a multivariate Gaussian is at its mean. The mean of the Gaussian in this case can be found by using the formula

$$\mu_{A|B} = \mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(Y_B - \mu_B)$$

where $Y_B = [Y_0]$,$Y_A = [Y_1, Y_2]$. $\Sigma_{BB} = [1]$, $\mu_A = \mu_B = 0$ (zero mean gaussian process considered), $\Sigma_{AB} = [\exp(-x_1^2/2), \exp(-x_2^2/2)]$, where $x_1 = \sqrt{\log(4)}$ and $x_2 = -\sqrt{\log(4)}$ which gives $\mu_{A|B} = [-1/2, -1/2] = y_1, y_2$. Answer = -1