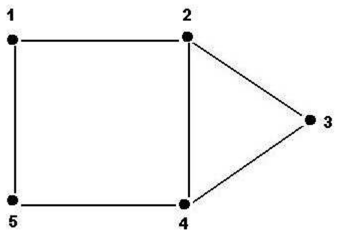
CS 726: Problems on Inference (Graded)

Write your answers in the space provided. You are expected to solve each question on your own. Do not try to search the answers from any external sources, like the web. You are allowed to discuss a few questions with your classmates provided you mention their names.

---

1. The maximum independent set (MIS) of a graph is defined as the largest set of vertices such that there is no edge between them. For example, in this graph, the MIS is two consisting of $\{3, 5\}$ or $\{3, 1\}$ or $\{1, 4\}$ (there are more).

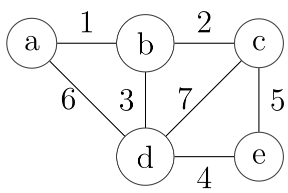


Encode the problem of finding the MIS of a graph H over n nodes as a problem of MAP inference in a suitably constructed undirected graphical model (UGM) $P(X_1, \ldots X_m)$. Clearly show the graph of the UGM and the potentials such that $\text{argmax}_{X_1,\ldots,X_m} P(X_1, \ldots X_m)$ will give us a solution to the original MIS problem. (Higher marks for a positive distribution.)

For this question, I am providing the solution so that serves as a template for helping you solve the following question.

Answer: Let G be the required UGM. The nodes and edges in G are identical to those in H. Each variable in G is binary and all the node potential (in log form) are defined as follows. $\theta(1) = 1, \theta(0) = 0$. Edge potentials between any two nodes: $\theta(1, 1) = -2n, \theta(0, 1) = \theta(1, 0) = \theta(0, 0) = 0$. The exponent of all these potentials is positive. Thus, we get a positive distribution. The MAP solution gives as the maximum independent set those vertices that have been assigned a label 1. It is easy to see that the MAP will always give rise to an independent set because the $\theta(1, 1) = -2n$ ensures that any solution where adjacent nodes are labeled 1 can always be improved by flipping a node to 0.

2. Now you need to solve a related problem. A Hamiltonion cycle in an undirected graph G is a cycle that visits each vertex exactly once. For example, in the graph below a Hamiltonian cycle comprises of edges 1,2,5,4,6. This is a well-known NP-hard problem. Reduce it to an



inference problem on a suitably constructed undirected graphical model H. The number of nodes in H should be polynomial in the number of nodes and edges in G and the size of the largest potential should be a constant. [Hint: The inference task will check if the maximum probability assignment (MAP) in H has score greater than a threshold.] Specify each of the steps below to provide your reduction. ..6

(a) Nodes in H along with the set of values each node is allowed to take. There might be many correct solutions, here is one of them.

Each edge in G is a node in H. Each edge variable can take one of $n+1$ labels where n is the number of nodes in G. Label 0 denotes the edge is not selected in Hamiltonian cycle. Other labels denote the order of selected edges around the cycle.

(b) Edges in H. Fully connected graph, that is, for each edge pair in G, there is an edge in H.

(c) Potentials in H over its nodes and cliques [Need not be maximal cliques] For each node e in H, $\psi_e(0) = 1$ and $\psi_e(l) = 3$ for $l > 0$.

Let $\text{adj}(l, l')$ denote the condition that l, l' are consecutive in a circular sense (for e.g. n and 1 are consecutive and so are 1 and 2, 4 and 5 etc.).

For each pair of nodes e_i, e_j in H, define a potential that is takes a value of 1 in all but the following three cases.

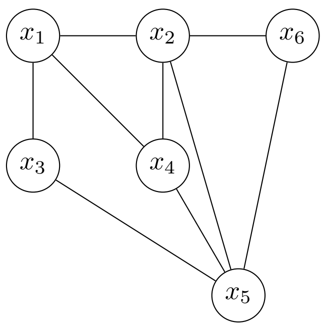
$\psi_{i,j}(l, l) = 0$ if $l > 0$. This ensures that each label between 1 to n is assigned to exactly one edge.

$\psi_{i,j}(l, l') = 0$ if $l, l' > 0$ and not $\text{adj}(l, l')$ and i, j are co-incident on a node. This condition makes sure that the two edges selected around each node have consecutive labels denoting its position in the Hamiltonian cycle.

$\psi_{i,j}(l, l') = 0$ if $l, l' > 0$ and $\text{adj}(l, l')$ and i, j are not co-incident on a node. This makes sure that when two edges have adjacent labels, they have to be co-incident on a node. Thus, we cannot get disconnected cycles as solutions.

(d) Specify the test you will do on the MAP assignment of the above graphical model, to check if you found a Hamiltonion path in G or not. If the MAP score is equal to 3^n we found a Hamiltonian cycle, otherwise not.

3. For the undirected H below, perform the following operations



(a) Triangulate H ..1

(b) Identify the maximal cliques using the method discussed in class. [Just listing the maximal cliques will carry no marks.] ..2

(c) Draw the weighted graph over clique nodes and find its maximum spanning tree to get the junction tree of H. ..2

(d) Let C be set of variables that form a leaf node of a junction tree (JT), and let $S \subset C$ be the separator set that connects C to the rest of the JT. What is the minimum amount of work required to compute $\Pr(C - S|S)$. Let n be the number of variables, m the cardinality of each variable, and k the size of the largest clique in JT. ..3

$\Pr(C - S|S) = \frac{\Pr(C)}{S} = \frac{\sum_{\mathbf{x}_{V-C}} \prod_F \psi_F(\psi_F(\mathbf{x}_F)}{\sum_{\mathbf{x}_{V\S}} \prod_F \psi_F(\psi_F(\mathbf{x}_F)}$ By the running intersection property, all factors $F \neq C$ contain no variable from $C - S$. Thus, we can rewrite the numerator and denominator and obtain that $\Pr(C - S|S) = \frac{\psi_C(\mathbf{x}_C)}{\sum_{\mathbf{x}_{C-S}} \psi_C(\mathbf{x}_C)}$

4. Let G be a chordal graph with a junction tree T. We add an edge between two non-adjacent vertices x_i, x_j in G to get a new graph G'.

   (a) Show how you will modify T to get the junction tree T' for G' under the following cases. Justify your answer.

      i. x_i, x_j are simplicial and their neighbors in G are the same. ..2
      let ci and cj be the cliques which contain xi and xj respectively and let sij be common neighbors. Add xj to ci. Let Pij be the path between ci and cj. Connect all nbrs of cj that are not on Pij, to node ci.

      ii. There is no path between x_i and x_j in G. ..2
      Easy. Just create a new clique node C with xixj and connect C to a clique that contains xi and another clique that contains xj.

   (b) Show a case where T and T' have no cliques in common and T has at least three nodes. ..3

   If G is an open chain and we add an edge connecting the two ends. Then triangulation will lead to creating cliques over three variables at a time. The original junction tree is only over variable pairs in a chain.