Quick Summary

Given a signal $\{s[n]\}$ (a wss r.p.)
we have studied a method to estimate
$\{a_k\}$ and G for a selected p
based on minimizing $\sum e^2[n]$
where
$$e[n] = s[n] - \sum_{k=1}^{p} a_k \, s[n-k]$$

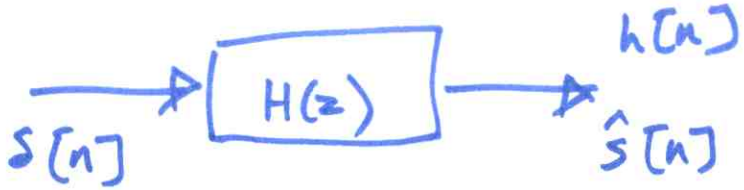
We solved the following p eq^{n}s in p unknowns for $\{a_k\}$

$$\left.\begin{aligned} r[i] &= \sum_{k=1}^{p} a_k \, r[i-k], \quad i = 1 \ldots p \\ \text{\&} \quad r[0] &= \sum_{k=1}^{p} a_k \, r[k] + G^2 \quad \text{for 'G'} \end{aligned}\right\} \; A$$

$\{a_k\}$, G. Now we can define

$$H(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

Consider a model:



Let us consider the properties of $\{h[n]\}$ corresp. to the causal, stable i.r. of $H(z)$.

$$H(z)\left(1 - \sum_{k=1}^{p} a_k z^{-k}\right) = G$$

$$\Rightarrow \quad h[n] = \sum_{k=1}^{p} a_k h[n-k] + G\,\delta[n] \quad \ldots \text{A}$$

$$\Rightarrow h[0] = G$$

Define

$$\tilde{R}[m] = \sum_{n=0}^{\infty} h[n]\, h[n+m]$$

$$\tilde{R}[-m] = \sum_{n=0}^{\infty} h[n]\, h[n-m]$$

We have

$$\tilde{R}[m] = \tilde{R}[-m] = \sum_{n=0}^{\infty} h[n]\, h[n-m]$$

$$= \sum_{n=0}^{\infty} h[n-m] \left(\sum_{k=1}^{p} a_k\, h[n-k] + G\,\delta[n] \right)$$

$$= \sum_{n=0}^{\infty} h[n-m] \cdot \sum_{k=1}^{p} a_k\, h[n-k] + \underbrace{\sum_{n=0}^{\infty} h[n-m]\, G\,\delta[n]}_{=0}$$

$$\tilde{R}[m] = \sum_{k=1}^{p} a_k\, \tilde{R}\,\cancel{[k]}\,(|m-k|) + 0 \rightarrow \text{for } m = 1, 2, \ldots$$

$$\Rightarrow \boxed{\tilde{R}[m] = \sum_{k=1}^{p} a_k \tilde{R}(|m-k|) \quad , \ m > 0}$$

& $$\tilde{R}[0] = \sum_{k=1}^{p} a_k \tilde{R}[k] + \underbrace{\sum_{n=0}^{\infty} h[n]\, G\, \delta[n]}_{G \cdot h[0] \,=\, G^2}$$

$$\boxed{\tilde{R}[0] = \sum_{k=1}^{p} a_k \tilde{R}[k] + G^2}$$

P

Compare the above 2 eq^{n}s $\tilde{R}[m]$, $m > 0$
$\tilde{R}[0]$

with A

we see that for the same $\{a_k\}$, G, we must have the same $r[\cdot]$ & $\tilde{R}[\cdot]$ for lags upto p.

i.e. $\tilde{R}[0], \ldots \tilde{R}[p]$ must be identical

to $r[0], \ldots r[p]$

or $\tilde{R}[i] = r[i]$, $i = 0 \ldots p$.

$\Rightarrow$ the acf of $\{s[n]\}$ & acf of $\{h[n]\}$

are equal in the first $(p+1)$ values of lag.

Now as $p \to \infty$, the resp. acf match $\forall$ lags

$\Rightarrow$ power spectra (F.T. of acf) also match

i.e. $\lim_{p \to \infty} |H(e^{j\omega})|^2 = |S(e^{j\omega})|^2$

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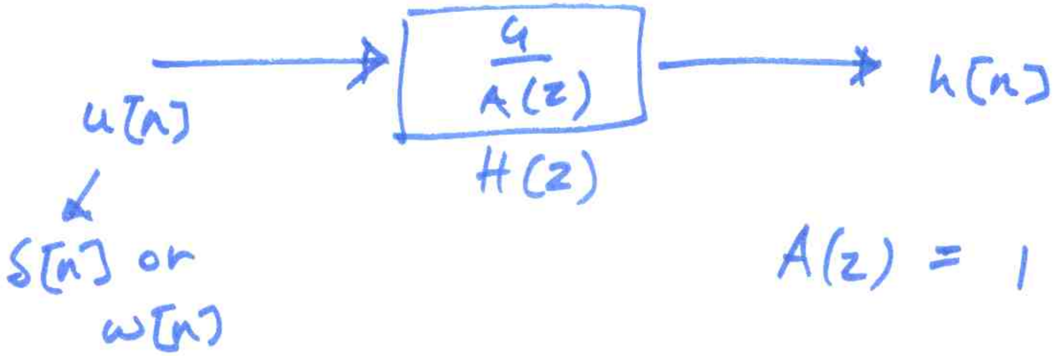
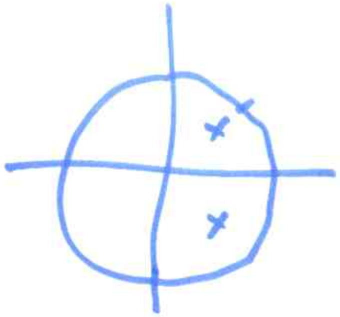
Any $\{s[n]\}$ can be modeled as the output of an all-pole system driven by either an impulse or by white noise. The approx^n (model o/p) gets better as $p \uparrow$.

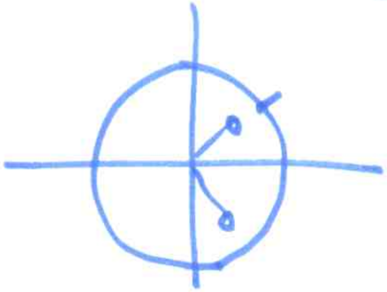


← AR modeling

$$A(z) = 1 - \sum_{k=1}^{p} a_k z^{-k}$$

We saw earlier : $E^{(1)} \geq E^{(2)} \geq \ldots\ldots \geq E^{(p)} \ldots$





Freq. domain interpretation of LP (m.s. error minimiz^n)

$$E = \sum_n e^2[n]$$

$$e[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k] \Rightarrow E(z) = S(z)A(z)$$

$$\Rightarrow E(e^{j\omega}) = S(e^{j\omega}) A(e^{j\omega})$$

$$H(z) = \frac{G}{A(z)}$$

By Parseval's thm:

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(e^{j\omega})|^2 \, |A(e^{j\omega})|^2 \, d\omega$$

But $H(e^{j\omega}) = G / A(e^{j\omega})$

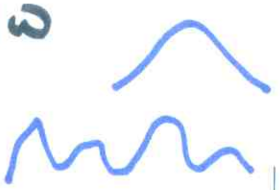
$$\Rightarrow E = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{|S(e^{j\omega})|^2}{|H(e^{j\omega})|^2} \, d\omega = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} \, d\omega$$

i.e. in the LS approach to all-pole modeling we're minimizing

$$E = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} \, d\omega$$

with G obtained by equating the energies of the 2 spectra $P(\omega)$ & $\hat{P}(\omega)$

i.e. $$\frac{1}{2\pi} \int P(\omega)\, d\omega = r[0] = \tilde{R}[0] = \frac{1}{2\pi} \int \hat{P}(\omega)\, d\omega$$



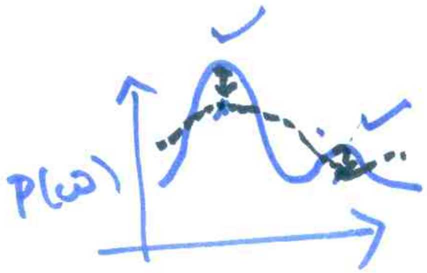
As we saw, the optimal sol^n $\{a_k\}$, G at given p satisfies

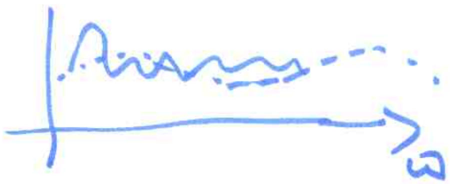
$P(\omega) \to \hat{P}(\omega)$ as $p \to \infty$

Nature of spectrum approx^n (Makhoul ref)

We had

$$E = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} \, d\omega \quad \ldots$$





$\underbrace{\log P(\omega) - \log \hat{P}(\omega)}$ . . . Global property

However, human perception is indeed less sensitive to similar mag differences at hi values of $P(\omega)$ compared to at low values.

Local property:

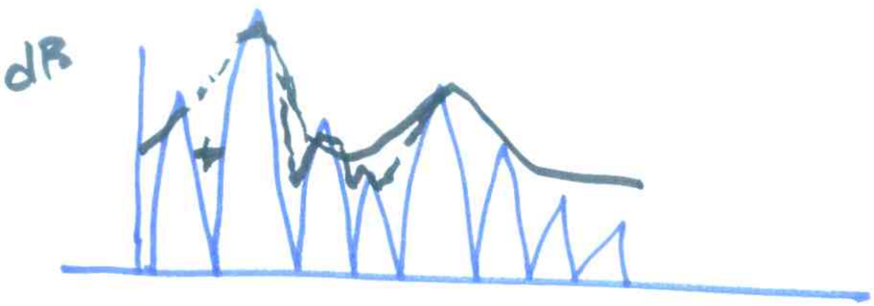
$$E = \frac{G}{2\pi}\int_{-\pi}^{\pi}\frac{P(\omega)}{\hat{P}(\omega)}\,d\omega$$

At the optimal sol^n $E = E_{min}$ $(= G^2)$

∴ We have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{P(\omega)}{\hat{P}(\omega)}\,d\omega = 1$$

=> $P(\omega)$ will greater than $\hat{P}(\omega)$ in <u>some</u> regions & less in other.

while $\int P(\omega)\,d\omega = \int \hat{P}(\omega)\,d\omega = r[0]$



$\int \hat{P}(\omega)\, d\omega$