

## Quick Summary

Given a signal  $\{s[n]\}$  (a wss r.p.)  
we have studied a method to estimate  
 $\{a_n\}$  and  $\zeta$  for a selected  $p$

based on minimizing  $\sum e^2[n]$

where

$$e[n] = s[n] - \sum_{k=1}^p a_k s[n-k]$$

We solved the following  $p$  eq<sup>n</sup>s in  $p$  unknowns for  $\{a_n\}$

$$r[i] = \sum_{k=1}^p a_k r[i-k], \quad i = 1, \dots, p$$

$$\nabla \quad r[0] = \sum_{k=1}^p a_k r[k] + \zeta^2 \quad \text{for 'G'}$$



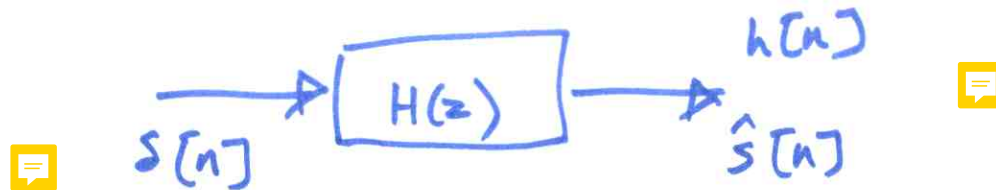
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$\{a_n\}$ ,  $G$ . Now we can define

$$H(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{k=1}^p a_k z^{-k}}$$

Consider a model:



Let us consider the properties of  $\{h[n]\}$  corresp. to the causal, stable c.r. of  $H(z)$ .

$$H(z) \left(1 - \sum_{k=1}^p a_k z^{-k}\right) = G$$
$$\Rightarrow h[n] = \sum_{k=1}^p a_k h[n-k] + G \delta[n] \quad \dots \textcircled{1}$$

$$\Rightarrow h[0] = G$$



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Define

$$\tilde{R}[m] = \sum_{n=0}^{\infty} h[n] h[n+m]$$

$$\tilde{R}[-m] = \sum_{n=0}^{\infty} h[n] h[n-m]$$

$$\text{We have } \tilde{R}[m] = \tilde{R}[-m] = \sum_{n=0}^{\infty} h[n] h[n-m]$$

$$= \sum_{n=0}^{\infty} h[n-m] \left( \sum_{k=1}^p a_k h[n-k] + G \delta[n] \right)$$

$$= \sum_{n=0}^{\infty} h[n-m] \cdot \sum_{k=1}^p a_k h[n-k] + \underbrace{\sum_{n=0}^{\infty} h[n-m] G \delta[n]}_{=0}$$

$$\tilde{R}[m] = \sum_{k=1}^p a_k \tilde{R}[m-k] + 0 \quad \rightarrow \quad \text{for } m=1, 2, \dots$$



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$$\Rightarrow \tilde{R}[m] = \sum_{k=1}^p a_k \tilde{R}[m-k], \quad m > 0$$

$$\& \quad \tilde{R}[0] = \sum_{k=1}^p a_k \tilde{R}[k] + \underbrace{\sum_{n=0}^{\infty} h[n] G \delta[n]}_{G \cdot h[0] = G^2}$$

$$\tilde{R}[0] = \sum_{k=1}^p a_k \tilde{R}[k] + G^2$$

(P)

Compare the above 2 eq<sup>ns</sup>  $\tilde{R}[m], m > 0$   
 $\tilde{R}[0]$

with (A)

We see that for the same  $\{a_k\}$ ,  $G$ , we must have  
the same  $r[\cdot] \& \tilde{R}[\cdot]$  for lags upto  $p$ .





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i.e.  $\tilde{r}[0], \dots, \tilde{r}[p]$  must be identical  
to  $r[0], \dots, r[p]$

or  $\tilde{r}[i] = r[i], \quad i = 0 \dots p.$

$\Rightarrow$  the act of  $\{s[n]\}$  & act of  $\{h[n]\}$   
are equal in the first  $(p+1)$  values of lag.

Now as  $p \rightarrow \infty$ , the resp. act match  $\forall$  lags

$\Rightarrow$  power spectra (F.T. of act) also match

$$\text{i.e. } \lim_{p \rightarrow \infty} |H(e^{j\omega})|^2 = |S(e^{j\omega})|^2$$



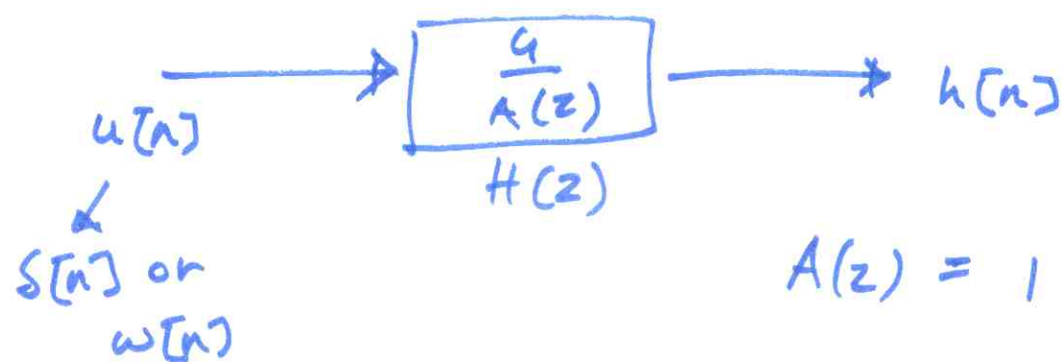




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Any  $\{s[n]\}$  can be modeled as the output of an all-pole system driven by either an impulse or by white noise. The approx<sup>n</sup> (model op) gets better as  $p \uparrow$ .

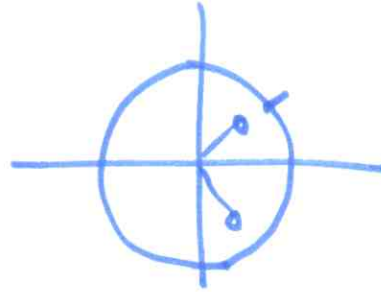
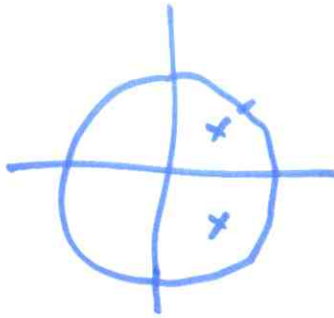


← AR modeling

$$A(z) = 1 - \sum_{k=1}^p a_k z^{-k}$$

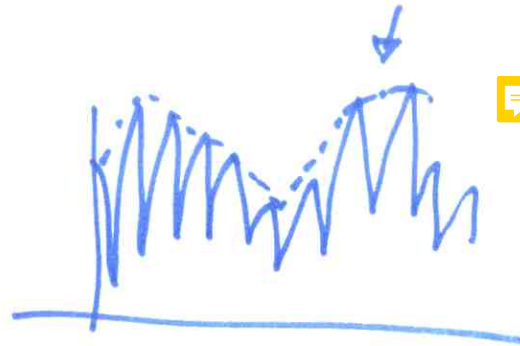
We saw earlier :

$$E^{(1)} \geq E^{(2)} \geq \dots \geq E^{(p)}$$



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Freq. domain interpretation of LP (error minimiz<sup>n</sup>)

$$E = \sum_n e^2[n]$$

$$e[n] = s[n] - \sum_{k=1}^p a_k s[n-k] \Rightarrow E(z) = S(z)A(z)$$

$$\Rightarrow E(e^{j\omega}) = S(e^{j\omega}) A(e^{j\omega})$$

By Parseval's thm:

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(e^{j\omega})|^2 |A(e^{j\omega})|^2 d\omega$$

$$\text{But } H(e^{j\omega}) = G/A(e^{j\omega})$$

$$\Rightarrow E = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{|S(e^{j\omega})|^2}{|H(e^{j\omega})|^2} d\omega = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} d\omega$$



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i.e. in the LS approach to all-pole modeling  
we're minimizing

$$E = \frac{G^2}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} d\omega$$

with  $G$  obtained by equating the energies of the  
2 spectra  $P(\omega)$  &  $\hat{P}(\omega)$

$$\text{i.e. } \frac{1}{2\pi} \int P(\omega) d\omega = r[0] = \tilde{R}[0]$$

$$\frac{1}{2\pi} \int \hat{P}(\omega) d\omega$$


As we saw, the optimal sol<sup>n</sup>  $\{a_n\}, G$   
at given  $p$  satisfies

$$P(\omega) \rightarrow \hat{P}(\omega) \text{ as } p \rightarrow \infty$$



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## Nature of spectrum approx<sup>n</sup> (Makhoul ref)

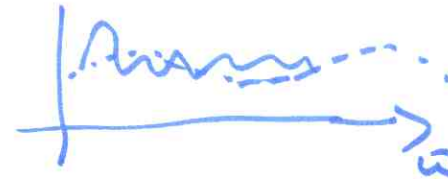
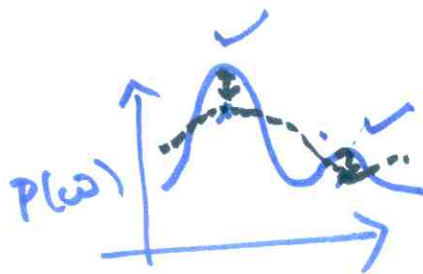
We had

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} d\omega \dots$$



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$$\log P(\omega) - \log \hat{P}(\omega)$$

... Global property

However, human perception is indeed less sensitive to similar mag differences at hi values of  $P(\omega)$  compared to at low values.

Local property:

$$E = \frac{G}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} d\omega$$

At the optimal sol<sup>n</sup>  $E = E_{\min}(G^2)$

$\therefore$  We have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(\omega)}{\hat{P}(\omega)} d\omega = 1$$

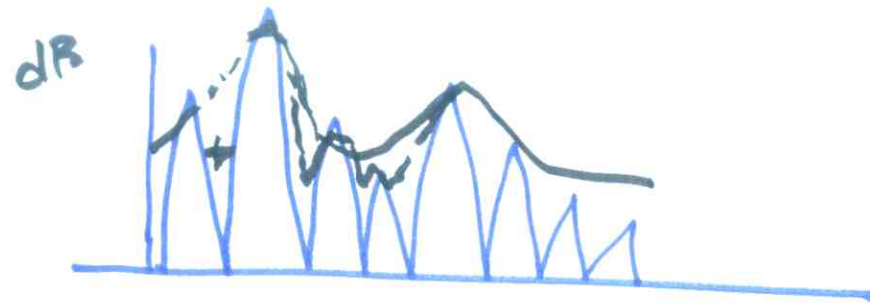
$\Rightarrow P(\omega)$  will be greater than  $\hat{P}(\omega)$  in some regions  
& less in other.

while  $\int P(\omega) d\omega = \int \hat{P}(\omega) d\omega = r[0]$



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$$\int \hat{P}(\omega) d\omega$$



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