Rate of Convergence

Fixed pt iteration: An iterative method is said to be of order p or has the rate of convergence p, if p is the largest +ve real no. for which $\exists$ a finite constant $C \neq 0$ s.t.

$$|\epsilon_{i+1}| \leq C |\epsilon_i|^p$$

where $\epsilon_i = x_i - \alpha$ is the error in the i^{th} iteration and α is ~~approximate~~ exact root of the funct $f(x)=0$

Similarly we can write for $(i+1)^{th}$ iteration

$$x_{i+1} = \alpha + \epsilon_{i+1}$$

$$x_i = \alpha + \epsilon_i$$

For fixed point iterative method we know

$$x_{i+1} = \phi(x_i)$$

$$= \phi(\epsilon_i + \alpha)$$

Expand using Taylor series expansion

$$x_{i+1} = \phi(\alpha) + \phi'(\alpha)\epsilon_i + \frac{\phi''(\alpha)}{2!}\epsilon_i^2 + \cdots$$

$$x_{i+1} - \phi(\alpha) = \phi'(\alpha)\epsilon_i + \frac{\phi''(\alpha)}{2!}\epsilon_i^2 + \cdots$$

$$x_{i+1} - \alpha = \phi'(\alpha)\cdot\epsilon_i + \frac{\phi''(\alpha)}{2!}\epsilon_i^2 + \cdots$$

$$\epsilon_{i+1} = \phi'(\alpha)\epsilon_i + \frac{\phi''(\alpha)}{2!}\cdot\epsilon_i^2 + \cdots$$

Convergence criteria of fixed point iteration method

$$|\phi'(x)| < 1$$

$$|\epsilon_{i+1}| = |\phi'(\xi)|\, \epsilon_i^{(1)}$$

$\Rightarrow$ order of convergence is 1

Secant method

We know the formula

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad (1)$$

Let α is the root of $f(x)=0 \Rightarrow f(\alpha)=0$

$x_i = \alpha + \epsilon_i$

$x_{i+1} = \alpha + \epsilon_{i+1}$, $x_{i-1} = \alpha + \epsilon_{i-1}$

Substitute in (1)

$$\epsilon_{i+1} = \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1}) f(\alpha+\epsilon_i)}{f(\alpha+\epsilon_i) - f(\alpha+\epsilon_{i-1})}$$

Expanding $f(\alpha+\epsilon_i)$ and $f(\alpha+\epsilon_{i-1})$ in Taylor Series about pt. α

$$f(\alpha+\epsilon_i) = \underset{0}{f(\alpha)} + \epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2!} f''(\alpha) + \cdots$$

& $$f(\alpha+\epsilon_{i-1}) = \underset{0}{f(\alpha)} + \epsilon_{i-1} f'(\alpha) + \frac{\epsilon_{i-1}^2}{2!} f''(\alpha) + \cdots$$

$$\Rightarrow \quad \epsilon_{i+1} = \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1})\left[\epsilon_i f'(\alpha) + \frac{1}{2}\epsilon_i^2 f''(\alpha) + \cdots\right]}{\left[\epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2!} f''(\alpha) + \cdots\right] - \left[\epsilon_{i-1} f'(\alpha) + \frac{\epsilon_{i-1}^2}{2!} f''(\alpha) + \cdots\right]}$$

$$= \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1})\left[\epsilon_i f'(\alpha) + \frac{1}{2}\epsilon_i^2 f''(\alpha) + \cdots\right]}{(\epsilon_i - \epsilon_{i-1}) f'(\alpha) + \frac{1}{2}(\epsilon_i^2 - \epsilon_{i-1}^2) f''(\alpha) + \cdots}$$

$$= \epsilon_i - \frac{\cancel{(\epsilon_i - \epsilon_{i-1}) f'(\alpha)}\left[\epsilon_i + \frac{1}{2}\epsilon_i^2 \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]}{\cancel{(\epsilon_i - \epsilon_{i-1}) f'(\alpha)}\left[1 + \frac{1}{2}(\epsilon_i + \epsilon_{i-1}) \frac{f''(\alpha)}{f'(\alpha)}\right]}$$

$$= \epsilon_i - \left[\epsilon_i + \frac{1}{2}\epsilon_i^2 \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 + \frac{1}{2}(\epsilon_i + \epsilon_{i-1}) \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$\epsilon_{i+1} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} \epsilon_i \epsilon_{i-1} + O(\epsilon_i^2 \epsilon_{i-1} + \epsilon_i \epsilon_{i-1}^2)$$

$$\epsilon_{i+1} = c\,\epsilon_i\,\epsilon_{i-1}$$ Neglecting higher powers of ϵ_i

& $c = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$

Standard form of error eqn $\boxed{\epsilon_{i+1} = A\,\epsilon_i^{p}}$

Similarly we can write

$$\epsilon_i = A\,\epsilon_{i-1}^{p}$$

or $\epsilon_{i-1} = A^{-1/p}\,\epsilon_i^{1/p}$ → Substitute ϵ_{i+1} & ϵ_{i-1}

$$A\,\epsilon_i^{p} = c\,\epsilon_i\,A^{-1/p}\,\epsilon_i^{1/p}$$

$$\epsilon_i^{p} = c\,A^{-1-1/p}\,\epsilon_i^{1+1/p}$$

Comparing the powers of ϵ_i

$$p = 1 + \frac{1}{p} \Rightarrow p^2 - p - 1 = 0$$

$$\Rightarrow p = \frac{1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

neglecting the -ve sign

$$p = \frac{1+\sqrt{5}}{2} = 1.618$$

⇒ rate of convergence of secant method is $\simeq 1.62$

Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad -①$$

$\because$ $x_i = \alpha + \epsilon_i$

$x_{i+1} = \alpha + \epsilon_{i+1}$ } substitute in eq ①

$$\epsilon_{i+1} = \epsilon_i - \frac{f(\alpha+\epsilon_i)}{f'(\alpha+\epsilon_i)}$$

Expand $f(\alpha+\epsilon_i)$ and $f'(\alpha+\epsilon_i)$ in Taylor series about α

$$\epsilon_{i+1} = \epsilon_i - \frac{[\overset{=0}{f(\alpha)} + \epsilon_i f'(\alpha) + \frac{1}{2}\epsilon_i^2 f''(\alpha) + \cdots]}{[f'(\alpha) + \epsilon_i f''(\alpha) + \frac{\epsilon_i}{2!} f'''(\alpha) + \cdots]}$$

$$= \epsilon_i - \frac{\cancel{f'(\alpha)}[\epsilon_i + \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\epsilon_i^2 \cdots]}{\cancel{f'(\alpha)}[1 + \frac{f''(\alpha)}{f'(\alpha)}\epsilon_i + \cdots]}$$

$$\epsilon_{i+1} = \epsilon_i - \left[\epsilon_i + \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\epsilon_i^2 + \cdots\right]\left[1 + \frac{f''(\alpha)}{f'(\alpha)}\epsilon_i + \cdots\right]^{-1}$$

$$\epsilon_{i+1} = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\epsilon_i^2 + O(\epsilon_i^3)$$

neglecting ϵ_i^3 and higher power of ϵ_i.

$$\epsilon_{i+1} = c\,\epsilon_i^2 \qquad \text{where } c = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$\Rightarrow$ N-R method has 2nd order convergence