Module-I

Numerical Solutions of Algebraic and Transcendental equations:

Algebraic equation: If $f(x)$ is a polynomial then the equation $f(x)=0$ is called an algebraic equation.

$x^3-3x+1=0,\ x^2+6x+8=0$ are some examples of algebraic equations.

Transcendental equation: Equations which involve transcendental functions like trigonometric functions, $\log x,\ e^x$ etc. are called transcendental equations.

$xe^x-3=0,\ x^2-\log_{10}x-12=0,\ x+\cos x-1=0$ are some examples of transcendental equations.

We can solve the given Algebraic or Transcendental equations numerically by using the following methods:

1. Secant Method
2. General Iteration Method (Successive approximation method)
3. Newton-Raphson Method

1. Secant Method:

Consider the function $f(x)=0$ where $f(x)$ is a continuous function. Choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs. Hence there exists root lying between a and b. In this method we approximate the curve of the function $y=f(x)$ by a chord. Equation of the chord joining the points $(a,\ f(a))$ and $(b,\ f(b))$ is given by

$$y-f(a)=\frac{f(a)-f(b)}{b-a}\,(x-a) \quad \text{----------------------- (1)}$$

The point of intersection of the chord with the x-axis is taken as the first approximation x_1 to the root, which is obtained by taking $y=0$ in (1).

Therefore, $0-f(a)=\dfrac{f(a)-f(b)}{b-a}\,(x_1-a)$ this gives $x_1=\dfrac{a\,f(b)-b\,f(a)}{f(b)-f(a)}$ -------- (2)

Now, if $f(a)$ and $f(x_1)$ are of opposite signs then the root lies between a and x_1. So, we replace b by x_1 in (2) and we get the next approximation x_2. But if $f(a)$ and $f(x_1)$ are of same sign, then $f(x_1)$ and $f(b)$ will be of opposite signs. Hence the root lies between x_1 and b. We replace a by x_1 in (2). We must repeat this process until the root is found to the desired accuracy.

Iterative formula is

$$x_{n+1}=\frac{x_n\,f(x_{n-1})-x_{n-1}\,f(x_n)}{f(x_{n-1})-f(x_n)} \quad (\ n=1,2,3,\ldots)$$

1. Find a positive real root of the equation $x^3 - 3x - 5 = 0$ correct up to two decimal places using Secant method.

   Solution: Let $f(x) = x^3 - 3x - 5$. Clearly $f(2) = -3 < 0,\ f(3) = 13 > 0$.

   Therefore, required root lies between 2 and 3.

   By the Secant method, we have $x_2 = \dfrac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)}$

   Taking $x_0 = 2$ and $x_1 = 3$.

   Therefore, $x_2 = 2.1875$ and $f(x_2) = -1.095 < 0$. Hence required root lies between 2.1875 and 3.

   We have $x_3 = \dfrac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$

   Taking $x_1 = 2.1875$ and $x_2 = 3$. Therefore, $x_3 = 2.2506$ and $f(x_3) = -0.3521 < 0$. Hence required root lies between 2.2506 and 3.

   We have $x_4 = \dfrac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)}$

   Taking $x_2 = 2.2506$ and $x_3 = 3$. Therefore, $x_4 = 2.2704$ and $f(x_3) = -0.1079 < 0$. Hence required root lies between 2.2704 and 3.

   We have $x_5 = \dfrac{x_4 f(x_3) - x_3 f(x_4)}{f(x_3) - f(x_4)}$

   Taking $x_3 = 2.2704$ and $x_4 = 3$. Therefore, $x_5 = 2.2764$.

   Therefore required root is 2.2764 corrected up to two decimal places.

2. Find the positive root of the equation $x^2 - \log_{10} x - 12 = 0$ correct up to three decimal places in between 3 and 4 using Secant method
3. Find the smallest positive root of the equation $x e^x - \cos x = 0$ using Secant method.
4. Find the negative root of the equation $x^3 - 4x + 1 = 0$ correct up to three decimal places using Secant method.
5. Find the positive root of the equation $x \log_{10} x = 1.2$ correct up to four decimal places using Secant method.
6. Find the positive root of the equation $x^3 - 3x + 1 = 0$ correct up to four decimal places using Secant method.

Note: i) It is guaranteed convergent method

ii) It is slowest convergent method(i.e., the rate of convergence is very low).

2. **General Iteration Method:**

Let $f(x)=0$ ---------------- (1)

be a given function.

Choose a and b such that $f(a)<0$ and $f(b)>0$. Therefore require root of (1) is lies between a and b. Let us take $I=(a,\ b)$.

Suppose the equation (1) can be expressed as $x=\phi(x)$ where $\phi(x)$ is a continuous function.

Let x_0 be an approximate value of the desired root of (1).

Then, we define $x_1=\phi(x_0)$, $x_2=\phi(x_1)$, $x_3=\phi(x_2)$, … , $x_n=\phi(x_{n-1})$,…

Therefore, $x_n=\phi(x_{n-1}),\ n=1,2,3,...$ ------------- (2)

is an iterative formula for solving the equation (1).

Note: The general iterative formula is converge if $|\phi'(x)|<1\ for\ all\ x\in I$.

1. Find a positive real root of the equation $x^3-3x+1=0$ using Iteration method.

   Solution: Let $f(x)=x^3-3x+1$. Clearly $f(1)=-1<0,\ f(2)=3>0$.
   Therefore, required root lies between 1 and 2.
   Now, $f(x)=0$ gives $x^3=3x-1$. Therefore, $x=(3x-1)^{\frac{1}{3}}=\phi(x)$ and hence $\phi'(x)=\dfrac{1}{(3x-1)^{\frac{2}{3}}}$. Clearly $|\phi'(x)|<1\ for\ all\ x\in(1,\ 2)$.
   Let $x_0=2$. Then $x_1=\phi(x_0)=1.7100$, $x_2=\phi(x_1)=1.6044$, $x_3=\phi(x_2)=1.5623$,
   $x_4=\phi(x_3)=1.5449$, $x_5=\phi(x_4)=1.5375$, $x_6=\phi(x_5)=1.5344$, $x_7=\phi(x_6)=1.5331$,
   $x_8=\phi(x_7)=1.5325$, $x_9=\phi(x_8)=1.5323$.
   Therefore, the required root is 1.5323 corrected up to three decimal places.

2. Find the real root of the equation $\cos x=3x-1$ using Successive approximation method.

   Solution: Let $f(x)=3x-1-\cos x$ Clearly $f(0)=-2<0,\ f\left(\dfrac{\pi}{2}\right)=3.7143>0$.

   Therefore, required root lies between and $\dfrac{\pi}{2}$.

   Now, $f(x)=0$ can be written as $x=\dfrac{1}{3}(\cos x+1)=\phi(x)$ and hence $\phi'(x)=\dfrac{\sin x}{3}$.

   Clearly $|\phi'(x)|<1\ \ for\ all\ \ x\in\left(0,\ \dfrac{\pi}{2}\right)$.

Let $x_0 = 0$. Then $x_1 = \phi(x_0) = 0.6667$, $x_2 = \phi(x_1) = 0.5953$, $x_3 = \phi(x_2) = 0.6093$,
$x_4 = \phi(x_3) = 0.6067$, $x_5 = \phi(x_4) = 0.6072$, $x_6 = \phi(x_5) = 0.6071$, $x_7 = \phi(x_6) = 0.6071$.
Therefore, approximate value of the required root is 0.6071

3. Find a positive real root of the equation $x^3 - 2x - 5 = 0$ using General iteration method.
4. Solve the equation $3x + \sin x = e^x$ using General iteration method.

3. Newton-Raphson Method:

Let x_0 be an approximate root of the equation $f(x) = 0$.

Let $x_1 = x_0 + h$ be the exact root of $f(x) = 0$ where h is very small, positive or negative.

Therefore, $f(x_1) = 0$. By the Taylor's series expansion, we have

$$f(x_1) = f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots$$

Since $f(x_1) = 0$ and h is very small, h^2 and higher powers of h terms can be neglected.

Hence $f(x_0) + \frac{h}{1!} f'(x_0) = 0$. Therefore $h = -\frac{f(x_0)}{f'(x_0)}$ *if* $f'(x_0) \neq 0$.

Thus $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is a first approximation to the root. Similarly starting with x_1 we get

the next approximation to the root given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$

and it is Known as Newton-Raphson's iteration formula or simply Newton-Raphson formula.

Note: i) It is a conditionally convergent method
ii) It is a very fastest convergent method (i.e., the rate of convergence is very high) and the order of the convergent is two.

1. Find a positive real root of the equation $x^3 - 2x - 5 = 0$ using Newton-Raphson method.

Solution: Let $f(x) = x^3 - 2x - 5$. Then $f'(x) = 3x^2 - 2$.

Clearly $f(2) = -1 < 0$, $f(3) = 16 > 0$. Therefore, required root lies between 2 and 3.

By the Newton-Raphson method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Taking $x_0 = 2$, we get $x_1 = 2 - \frac{(-1)}{10} = 2.1$.

Now, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.1 - \dfrac{0.061}{11.23} = 2.0946$,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0946 - \frac{0.0005}{11.1619} = 2.09456.$$

Therefore the approximate value of a positive real root is 2.09456

2. Find the real root of the equation $x\log_{10} x - 1.2 = 0$ correct to four decimal places using Newton-Raphson method.

Solution: Let $f(x) = x\log_{10} x - 1.2$.

Then $f'(x) = x\left(\dfrac{1}{\log_e 10}\left(\dfrac{1}{x}\right)\right) + \log_{10} x = \log_{10} e + \log_{10} x = 0.4343 + \log_{10} x$.

Clearly $f(1) = -1.2 < 0,\ f(2) = -0.598 < 0,\ f(3) = 0.2313$

Therefore, required root lies between 2 and 3.

By the Newton-Raphson method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Taking $x_0 = 2$, we get $x_1 = 2 - \dfrac{(-0.598)}{0.7353} = 2.8133$.

Now, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2.8133 - \dfrac{0.0637}{0.8835} = 2.7412$,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7412 - \frac{0.0004}{0.8722} = 2.7407,$$

$$x_4 = x_3 - \frac{f(x_2)}{f'(x_2)} = 2.7407.$$

Therefore the required root is 2.7407 correct up to four decimal places.

3. Derive an iterative formula to find $\sqrt[k]{N}\ (N > 0)$ using Newton-Raphson method and hence find $\sqrt[3]{24}$.

Solution: Let $x = \sqrt[k]{N}\ (N > 0)$. Then $x^k - N = 0$.

Let $f(x) = x^k - N$. Then $f'(x) = k\,x^{k-1}$.

By the Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\,k} - N}{k\,x_n^{\,k-1}} = \frac{1}{k}\left(\frac{k\,x_n^{\,k} - x_n^{\,k} + N}{x_n^{\,k-1}}\right)$$

Therefore, $x_{n+1} = \dfrac{1}{k}\left((k-1)x_n + \dfrac{N}{x_n^{\,k-1}}\right)\ (n = 0, 1, 2, \ldots)$

Now, to find $\sqrt[3]{24}$. Here $k=3$ and $N=24$.

By the above iterative formula, we have

$$x_1 = \frac{1}{k}\left((k-1)x_0 + \frac{N}{x_0^{\,k-1}} \right)$$

Taking $x_0 = 2.5$,

we get $x_1 = \frac{1}{k}\left((k-1)x_0 + \frac{N}{x_0^{\,k-1}} \right) = \frac{1}{3}\left(2(2.5) + \frac{24}{(2.5)^2} \right) = 2.9466$.

$$x_2 = \frac{1}{k}\left((k-1)x_1 + \frac{N}{x_1^{\,k-1}} \right) = \frac{1}{3}\left(2(2.9466) + \frac{24}{(2.9466)^2} \right) = 2.8857$$

$$x_3 = \frac{1}{k}\left((k-1)x_2 + \frac{N}{x_2^{\,k-1}} \right) = \frac{1}{3}\left(2(2.8857) + \frac{24}{(2.8857)^2} \right) = 2.8845$$

$$x_4 = \frac{1}{k}\left((k-1)x_3 + \frac{N}{x_3^{\,k-1}} \right) = \frac{1}{3}\left(2(2.8845) + \frac{24}{(2.8845)^2} \right) = 2.8844$$

Therefore the approximate value of $\sqrt[3]{24}$ is 2.8844

4. Derive an iterative formula to find $\frac{1}{N}$ using Newton-Raphson method and hence find $\frac{1}{18}$
5. Find the positive root of the equation $x e^x = 3$ using Newton-Raphson method.
6. Find a positive root of the equation $x^4 - 3x + 1 = 0$ using Newton-Raphson method.

Newton-Raphson method for Simultaneous Equations:

Consider the equations $f(x,\ y) = 0$ and $g(x,\ y) = 0$ ………………. (1)

Let $(x_0,\ y_0)$ be an initial approximate solution of (1). Let $x_1 = x_0 + h$ and $y_1 = y_0 + k$ be the next approximation. Expanding f and g by Taylor's series for a function of two variables around the point $(x_1,\ y_1)$, we have

$$f(x_1,\ y_1) = f(x_0 + h,\ y_0 + k) = f(x_0,\ y_0) + h\, f_x(x_0,\ y_0) + k\, f_y(x_0,\ y_0)$$

and $$g(x_1,\ y_1) = g(x_0 + h,\ y_0 + k) = g(x_0,\ y_0) + h\, g_x(x_0,\ y_0) + k\, g_y(x_0,\ y_0)$$

(neglecting higher powers of h and k).

If $(x_1,\ y_1)$ is a solution of (1), then $f(x_1,\ y_1) = 0$ and $g(x_1,\ y_1) = 0$.

Therefore, $f(x_0,\ y_0) + h\, f_x(x_0,\ y_0) + k\, f_y(x_0,\ y_0) = 0$

and $g(x_0,\ y_0) + h\, g_x(x_0,\ y_0) + k\, g_y(x_0,\ y_0) = 0$

$$h\,f_x(x_0,\ y_0)+k\,f_y(x_0,\ y_0)=-f(x_0,\ y_0)\ \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

and $$h\,g_x(x_0,\ y_0)+k\,g_y(x_0,\ y_0)=-g(x_0,\ y_0)\ \ldots\ldots\ldots\ldots\ldots \text{(3)}$$

If $\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} \neq 0$, then the equations (2) and (3) having a unique solution of h and k.

Now $x_1 = x_0 + h$ and $y_1 = y_0 + k$ gives a new approximation to the solution.

By repeating this process we obtain the required solution of desired accuracy.

1. Find a root of system of non-linear equations $x^2 + y = 11;\ y^2 + x = 7$ with $x_0 = 3.5$ and $y_0 = -1.8$ by Newton-Raphson method.

Solution: Let $f(x,\ y) = x^2 + y - 11 = 0$ and $g(x,\ y) = y^2 + x - 7 = 0$ ------- (1).

Given $x_0 = 3.5$ and $y_0 = -1.8$ as initial approximate solution of (1).

Let $x_1 = x_0 + h$ and $y_1 = y_0 + k$ be the next approximation.

Then we have

$$f(x_0,\ y_0)+h\,f_x(x_0,\ y_0)+k\,f_y(x_0,\ y_0)=0 \quad \text{---------------------- (2)}$$

and $$g(x_0,\ y_0)+h\,g_x(x_0,\ y_0)+k\,g_y(x_0,\ y_0)=0 \quad \text{------------------------------- (3)}$$

Now, $f(x_0,\ y_0) = -0.55$, $f_x(x_0,\ y_0) = 7$, $f_y(x_0,\ y_0) = 1$, $g(x_0,\ y_0) = -0.26$, $g_x(x_0,\ y_0) = 1$ and $g_y(x_0,\ y_0) = -3.6$

Therefore, from the equations (2) and (3) we have,

$$7h + k = 0.55 \quad \text{---------------------------- (4)}$$

and $$h - 3.6k = 0.26 \quad \text{--------------------------- (5)}$$

Solving the equations (4) and (5) , we get $h = 0.0855$ and $k = -0.0485$.

Therefore, $x_1 = x_0 + h = 3.5855$ and $y_1 = y_0 + k = -1.8485$.

Let $x_2 = x_1 + h$ and $y_2 = y_1 + k$ be the next approximation.

Then we have

$$f(x_1,\ y_1)+h\,f_x(x_1,\ y_1)+k\,f_y(x_1,\ y_1)=0 \quad \text{------------------------ (6)}$$

and $$g(x_1,\ y_1)+h\,g_x(x_1,\ y_1)+k\,g_y(x_1,\ y_1)=0 \quad \text{------------------------ (7)}$$

Now, $f(x_1, y_1) = 0.0073$, $f_x(x_1, y_1) = 7.171$, $f_y(x_1, y_1) = 1$, $g(x_1, y_1) = 0.0025$, $g_x(x_1, y_1) = 1$ and $g_y(x_1, y_1) = -3.697$.

Therefore, from the equations (6) and (7) we have,

$$7.171h + k = -0.0073 \quad \text{------------------------ (8)}$$

and

$$h - 3.697k = -0.0025 \quad \text{------------------------ (9)}$$

Solving the equations (8) and (9) , we get $h = -0.00106$ and $k = 0.00039$.

Therefore, $x_2 = x_1 + h = 3.5844$ and $y_2 = y_1 + k = -1.8481$.

Thus $x = 3.5844$ and $y = -1.8481$ is an approximate solution to the given system of equations.

Solve the following system of non-linear equations using Newton-Raphson method.

1. $x^2 + y^2 = 1$; $xy = x + y$ with $x_0 = 0.5$ and $y_0 = -1$
2. $x^2 - y^2 = 4$; $x^2 + y^2 = 16$ with $x_0 = 2.828$ and $y_0 = 2.828$
3. $2x^2 - 4xy - y^2 = 0$, $2y^2 + 10x - x^2 - 4xy - 5 = 0$ with $x_0 = y_0 = 1$
4. $\sin x - y + 1.32 = 0$, $x - \cos y - 0.85 = 0$ with $x_0 = 0.6$ and $y_0 = 1.9$