PHS 307: THEORETICAL PHYSICS II

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PHS 307: Theoretical Physics II

This is the second in the series of three mathematical courses designed specially for physics students. You have studied PHS 209: Theoretical Physics I, and beyond this course, there is PHS 308: Theoretical Physics III. As you already know, a physics students must be well grounded in Mathematics, as it is the language of Physics.

PHS 307 will give you the opportunity of understanding important topics like Numerical analysis, specifically, finding the equation that best fits a given set of data, perhaps the readings obtained in a laboratory experiment. Vector spaces, are a generalisation of the concept of Euclidean vectors that you know so well. The knowledge of this important topic will enable you to expand any given vector in terms of a given set of vectors, known as basis vectors. In turn this expansion, along with the knowledge of orthogonal functions gives you an idea of what proportion of a system is any of a possible set of states, one of the most important concepts in Quantum Mechanics. Fourier series enables you to expand a given function as a sum of simple sinusoidal function. Conversely, you can compose a complex function from a set of simple functions. Fourier transform is a powerful tool that enables you to transform a function of time to a function of frequency, or a function of the frequency to a function of the wave number. You will also get to know the connection between Fourier series and Fourier transform.

Complex analysis will deepen the knowledge you acquired in PHS 209, and provide you with the knowledge of integrating a function of a complex variable. You will also get to integrate, with the help of complex integration, functions of a real variable that would have proven too difficult to do ordinarily. Fourier transform, as well as Laplace transform will prove useful in solving problems from diverse areas of Physics.

Eigenvalue problems crop up virtually in all areas of Physics. You will get to know how to find the eigenvalues and the eigenvectors of a given operator, and the physical meaning of each of these terms. You will also learn how to diagonalise a matrix, and with the help of this, find the eigenvalues of a given matrix.

From the foregoing, you can see that this is one of the most important courses in your programme of study. As such, whatever time you can put into understanding the topics will be time well spent. To enhance your understanding of the course, each study session has in-text questions as well as self-assessment questions. Try your hands on them and solve some more problems from the recommended textbooks at the end of each study session.

We wish you the best.

Study Session 1 A Review of Some Basic Mathematics

Introduction

It has been observed that some students take basic mathematics for granted. It is always a good idea to review some mathematical topics that will be relevant to our study. This study unit takes you through such topics as basic trigonometry, exponential functions, differentiation and integration.

Learning Outcomes for Study Unit 1

By the time you are through with this study unit, you should be able to the following:

1.1 Recall the formula for the trigonometric ratios (SAQ 1.1)

1.2 Remember how to work with exponential functions (SAQ 1.2)

1.3 Recall the differentiation of a function of a single variable (SAQ 1.3)

1.4 Integrate simple mathematical functions (SAQ 1.4)

1.5 Work with angular measures: degree and radian (SAQ 1.5)

1.1 Trigonometry

1.1.1 The Right Angle Triangle

The nomenclature for a right angle triangle is shown in Fig. 1.1, where O, A and H are the opposite (the angle of interest), the adjacent and the hypotenuse of the triangle.

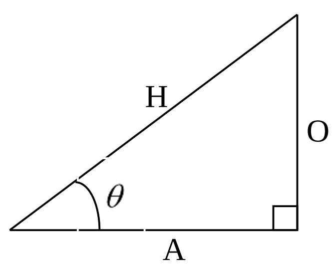

Fig. 1.1: The Right Angle Triangle

By Pythagoras Theorem,

$$H^2 = O^2 + A^2 \qquad 1.1$$

In addition,

$$\sin\theta = \frac{O}{H} \qquad 1.2$$

$$\cos\theta = \frac{A}{H} \qquad 1.3$$

$$\tan\theta = \frac{O}{A}. \qquad 1.4$$

Dividing Equation 1.2 by Equation 1.3,

$$\frac{\sin\theta}{\cos\theta} = \frac{(O/H)}{(A/H)} = \frac{O}{A} = \tan\theta$$

Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad 1.5$$

Also, squaring and adding equations (1.2) and (1.3),

$$\sin^2\theta + \cos^2\theta = \frac{O^2}{H^2} + \frac{A^2}{H^2} = \frac{O^2 + A^2}{H^2} \qquad 1.6$$

Using Pythagoras Theorem, we conclude that

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1.7$$

Though we have proved equations 1.5 and 1.6 for an angle in a right-angle triangle, they are indeed true for any angle θ.

We recall that

$$\sin(A+B) = \sin A\cos B + \sin B\cos A \qquad 1.8$$

$$\sin(A-B) = \sin A\cos B - \sin B\cos A \qquad 1.9$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad 1.10$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad 1.11$$

- Write $\sin(90+\theta)$ in terms of $\cos\theta$.

- ❑ $\sin(90^0 + \theta) = \sin 90^0 \cos\theta + \cos 90^0 \sin\theta$

  $= 1\cdot\cos\theta + 0\cdot\sin\theta$

Next, we draw the graphs of the trigonometric functions (Fig. 1.2):

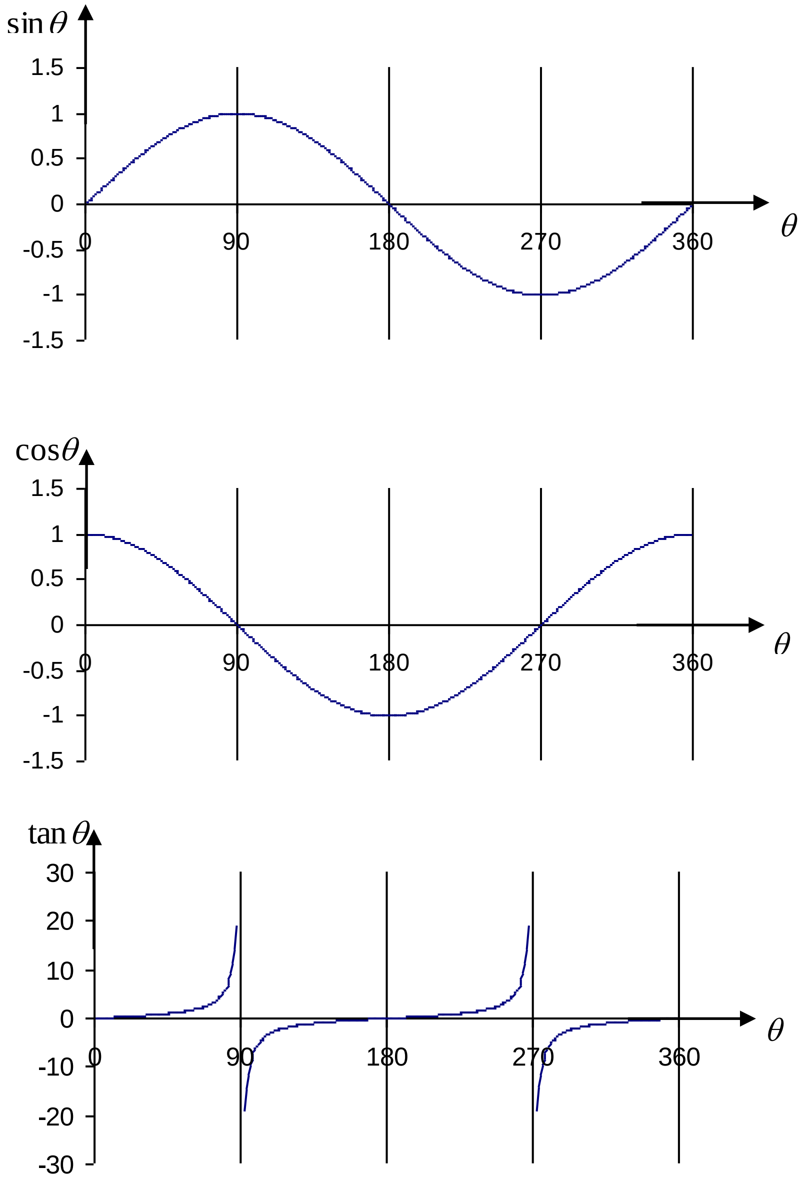

Fig. 1.2: The graphs of sine, cosine and tangent of θ, $0 \leq \theta \leq 360^0$

If the y-axis of the sine function is shifted forward by 90^0, we shall obtain the graph for the cosine function. We can therefore say that

$$\sin(90^0 + \theta) = \cos\theta \qquad 1.12$$

Similarly, we infer that

$$\cos(90 + \theta) = -\sin\theta \qquad 1.13$$

From equation 1.12, we note that $\cos\theta$ is ahead of $\sin\theta$, or in other words, $\sin\theta$ lags $\cos\theta$ by 90^{0}.

From Fig. 1.2, note that

$$\tan\left(180^{0}+\theta\right)=\tan\theta \qquad 1.14$$

$$\cos\left(360^{0}+\theta\right)=\cos\theta \qquad 1.15$$

$$\sin\left(360^{0}+\theta\right)=\sin\theta \qquad 1.16$$

In other words, the Sine and Cosine functions repeat after every 360^{0}, while the Tangent function repeats itself after every 180^{0}. We therefore say that Cosine and Sine functions have a period of 360^{0}, while the Tangent function has a period of 180^{0}.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \qquad 1.17$$

From which it readily follows that

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \qquad 1.18$$

As an example,

$$\tan\left(90^{0}+\theta\right)=\frac{\tan 90^{0}+\tan\theta}{1-\tan 90^{0}\tan\theta}$$

Therefore,

$$\tan(90^{0}+\theta)=-\cot\theta \qquad 1.19$$

1.2 Exponential Functions

We limit our discussion to the number *e*, which is approximately 2.178. The Maclaurin series expansion of the exponential function e^{x} is

$$e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots\cdot \qquad 1.23$$

To get the expansion for e^{-x}, replace *x* by –*x* in equation 1.23

$$e^{-x}=1-x+\frac{x^{2}}{2!}-\frac{x^{3}}{3!}+\cdots\cdot \qquad 1.24$$

- Show that $\dfrac{e^{x}+e^{-x}}{2}=\cosh x$, given that the Maclaurin series for cosh *x* is $1+\dfrac{x^{2}}{2!}+\dfrac{x^{4}}{4!}+\cdots\cdot$

- Adding equations 1.23 and 1.24 and dividing the result by 2 gives

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad 1.25$$

But the right hand side of equation 1.25 is the Maclaurin series expansion of $\cosh x$. Hence,

$$\frac{e^x + e^{-x}}{2} = \cosh x \qquad 1.26$$

The square root of the number –4 is

$$\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i$$

where $\sqrt{-1} = i$. i is an example of an imaginary number.

$$i^2 = (\sqrt{-1})^2 = -1; \; i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1 \cdot$$

Higher powers of i just repeat this set of four numbers. Hence, we can write a sequence

$$i, \; -1, \; -i, \; 1, \; i, \ldots.$$

We then notice that this sequence looks like that of $\sin\theta$, or $\cos\theta$, taken in steps 90^0 from 0^0 to 360^0. Respectively, these are:

0, 1, 0, -1, 0, ………

and

1, 0, -1, 0, 1, ……….

Thus, we expect that there might be a relationship between the number i, and $\sin\theta$ and $\cos\theta$. This is indeed so.

Replacing x by ix in the expansion for e^x, we get

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \cdots \qquad 1.27$$

Similarly,

$$e^{-ix} = 1 + (-ix) + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \cdots = 1 - ix - \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \qquad 1.28$$

From equations 1.29 and 1.30, we obtain

$$\frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad 1.29$$

and

$$\frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

1.30

The right hand sides of equations 1.29 and 1.30 are, respectively, the series expansion for $\cos\theta$ and $\sin\theta$.

Therefore,

$$\frac{e^{ix}+e^{-ix}}{2}=\cos x \tag{1.31}$$

$$\frac{e^{ix}-e^{-ix}}{2i}=\sin x \tag{1.32}$$

Similarly,

$$\frac{e^{iax}+e^{-iax}}{2}=\cos\ ax \tag{1.33}$$

$$\frac{e^{iax}-e^{-iax}}{2i}=\sin\ ax \tag{1.34}$$

1.3 Differentiation

1.3.1 Polynomials

Given the function $f(x)=ax^n$, where a is a constant, the differential of $f(x)$ with respect to x is nax^{n-1}. We write

$$\frac{d(ax^n)}{dx}=nax^{n-1} \tag{1.35}$$

▪ Find the differential of $3x^2$ with respect to x.

❑ $\frac{d}{dx}(3x^2)=2\times 3\times x^{2-1}=6x$

1.3.2 Exponential and Trigonometric functions

We can differentiate the expansion for an exponential function to get the differential of the function. For instance,

Let

$$f(x)=e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\cdot \tag{1.36}$$

Then, differentiating term by term, we get

$$f'(x) = 0 - 1 + x - \frac{3x^2}{3!} + \cdots \qquad 1.37$$

$$= -\left(1 - x + \frac{x^2}{2!} + \cdots\right) = -e^{-x} \qquad 1.38$$

Indeed, the Maclaurin series for $f(x) = e^{g(x)}$ is

$$e^{g(x)} = 1 + g(x) + \frac{[g(x)]^2}{2!} + \frac{[g(x)]^3}{3!} + \cdots \qquad 1.39$$

$$f'(x) = g'(x) + g(x)g'(x) + \frac{[g(x)]^2 g(x)}{2!} + \cdots \text{ (where } g'(x) = \frac{dg(x)}{dx}\text{)} \qquad 1.40$$

$$= g'(x)\left(1 + g(x) + \frac{[g(x)]^2}{2!} + \cdots\right) \qquad 1.41$$

$$= g'(x)e^{g(x)} \qquad 1.42$$

It should now be easy to differentiate trigonometric functions.

▪ Differentiate $\cos ax$ with respect to x, where a is a constant.

❑ $$\frac{d}{dx}\cos ax = \frac{d}{dx}\left(\frac{e^{iax} + e^{-iax}}{2}\right) = \frac{iae^{iax} + (-ia)e^{-iax}}{2}$$

$$= ia\frac{e^{iax} - e^{-iax}}{2} = -a\frac{e^{iax} - e^{-iax}}{2i}$$

since $i \times i = -1$, or $i = -\frac{1}{i}$

Hence,

$$\frac{d}{dx}\cos ax = -a\sin ax \qquad 1.43$$

▪ Find the differential of $\sin \omega t$ with respect to t.

❑ $$\frac{d}{dt}\sin \omega t = \frac{d}{dt}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) = \frac{1}{i}\left(\frac{iae^{i\omega t} - (-ia)e^{-i\omega t}}{2}\right)$$

$$\frac{d}{dt}\sin \omega t = \frac{d}{dt}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) = \frac{1}{i}\left(\frac{i\omega e^{i\omega t} - (-i\omega e^{-i\omega t})}{2}\right)$$

$$= \frac{i\,\omega}{i}\left(\frac{e^{i\omega t}+e^{i\omega t}}{2}\right) = \omega\cos\omega t$$

We conclude that,

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t \qquad 1.44$$

1.4 Integration

1.4.1 Polynomials

We have seen that $\frac{d}{dx}ax^n = nax^{n-1}$, where a and n are constants. Multiplying both sides by dx gives

$$dax^n = nax^{n-1}dx$$

Integrating both sides,

$$\int dax^n = \int nax^{n-1}dx + c\,,$$

where c is an arbitrary constant, called the constant of integration.

Thus,

$$ax^n = na\int x^{n-1}dx + c$$

That is,

$$\int x^{n-1}dx = \frac{x^n}{n} + c\,,\; n \neq 0 \qquad 1.45$$

This can also be written as $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$. Notice that we still put *c*, and not – *c*. This is because c is arbitrary.

▪ Integrate $4x^5$ with respect to x.

❑ $\int 4x^5 dx = \frac{4x^{5+1}}{5+1} + c = \frac{2}{3}x^6 + c$

1.4.2 Exponential and Trigonometric functions

Recall that

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)} \qquad 1.46$$

Then,

$$\int de^{f(x)} = \int f'(x)e^{f(x)}dx + c$$

Therefore,

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c \qquad 1.47$$

▪ Integrate e^{2x} with respect to x.

❑ $\int 2e^{2x}dx = e^{2x} + c$

A way of carrying out this integration is to start with

$$y = e^{2x}$$

Then,

$$\frac{dy}{dx} = 2e^{2x}$$

Hence,

$$dy = 2e^{2x}dx$$

Integrating both sides,

$$\int dy = \int 2e^{2x}dx + c$$

$$y = e^{2x} = 2\int e^{2x}dx + c$$

Dividing both sides by 2 and rearranging,

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

Note that we could have got the same answer just by inspection.

$$\int e^{2x}dx = \frac{1}{2}\int 2e^{2x}dx = \frac{1}{2}\int f'(x)e^{f(x)}dx = \frac{1}{2}e^{f(x)} + c = \frac{1}{2}e^{2x} + c$$

where $f(x) = e^{2x}$, and $f'(x) = 2e^{2x}$.

Trigonometric functions, being sums of exponential functions are easy to integrate as all we need to do is to integrate the terms in the sum one by one.

▪ Integrate $\sin \omega t$ with respect to t.

❑ $$\int \sin\omega t \ dt = \int \frac{e^{i\omega t} - e^{-i\omega t}}{2i} dt$$
$$= \frac{1}{2i}\left(\frac{1}{i\omega}e^{i\omega t} - \frac{1}{(-i\omega)}e^{-i\omega t}\right) + c$$
$$= -\frac{1}{\omega}\frac{\left(e^{i\omega t} + e^{-i\omega t}\right)}{2} + c = -\frac{1}{\omega}\cos\omega \ t + c$$

1.5 The radian measure

We are quite familiar with the idea of the circumference of a circle of radius r being $2\pi r$. Thus, 360^0 is equivalent to $2\pi r$.

180 degrees is equal to π radians, written as π^c. Thus, $1^c = \frac{180^0}{\pi} = 57.3^0$. Conversely, $1^0 = \frac{\pi^c}{180} = 0.01745^c$. Thus, we could measure angles in either degrees or radians. What advantage does measuring in radians have over measuring in degrees? Back to the circumference of a circle.

$$360^0 \rightarrow 2\pi r$$

$$\theta^0 \rightarrow \frac{2\pi r}{360^0} \times \theta^0 = \frac{\theta^0}{360^0} \times 2\pi r = \text{length of the arc subtended by angle } \theta^0$$

But suppose we measure angles in radians. Then,

$$\theta^c \rightarrow \frac{2\pi r}{2\pi} \times \theta^c = \frac{\theta^c}{2\pi} \times 2\pi r = r\theta^c = \text{length of the arc subtended by angle } \theta^c$$

Thus, when we measure angles in radians, the length of the arc is just the radius multiplied by the angle subtended, hence $s = r\theta^c$ in Fig. 3.3.

▪ Convert 5 rad/s to rev/s.

❑ 1 rev/s = $1 \times 2\pi^c / s = 2\pi$ rad/s

1 rad/s = $\frac{1}{2\pi}$ rev/s

5 rad/s = $5 \times \frac{1}{2\pi}$ rev/s = 0.7958 rad/s

Summary of Study Session 1

In Study Session 1, you have reviewed:

1. Basic ideas of trigonometry
2. Exponential functions
3. Differentiation
4. Integration
5. The radian measure.

References

Ayres, F., Schmidt, P. A. (1958). Theory and Problems of College Mathematics, Schaum's Outline Series.

Self-Assessment Questions (SAQs) for Study Session 1

Having completed this study session, you may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 1.1 (tests Learning Outcome 1.1)

Show that $\tan\left(90^0 - \theta\right) = -\tan\left(90^0 + \theta\right)$

SAQ 1.2 (tests Learning Outcome 1.2)

Using the Maclaurin series for e^x, show that $\dfrac{e^x - e^{-x}}{2} = \sinh x$.

SAQ 1.3 (tests Learning Outcome 1.3)

Differentiate $t^4 - 3t^{-1}$ with respect to t.

SAQ 1.4 (tests Learning Outcome 1.4)

Integrate $\sin \omega t$ with respect to t.

SAQ 1.5 (tests Learning Outcome 1.5)

Convert $15\, rev/\min$ to rad/sec.

Solutions to SAQs

SAQ 1.1

You can easily show that

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

from which it follows immediately that

$$\tan\left(90^0-\theta\right)=\cot\theta$$

From equations 0.5 and 0.6, we conclude that

$$\tan\left(90^0-\theta\right)=-\tan\left(90^0+\theta\right)$$

SAQ 1.2

Equation 1.23 – equation 1.24 yields

$$\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$$

or

$$\frac{e^x-e^{-x}}{2}=\sinh x$$

SAQ 1.3

$$\frac{d}{dt}(t^4-3t^{-1})=4\times1\times t^{4-1}+(-1)\times3\times t^{-1-1}=4t^3-3t^{-2}$$

SAQ 1.4

$$\begin{aligned}\int\sin\omega t\ \ dt\ &=\int\frac{e^{i\omega t}-e^{-i\omega t}}{2i}dt\\&=\frac{1}{2i}\left(\frac{1}{i\omega}e^{i\omega t}-\frac{1}{(-i\omega)}e^{-i\omega t}\right)+c\\&=-\frac{1}{\omega}\frac{\left(e^{i\omega t}+e^{-i\omega t}\right)}{2}+c=-\frac{1}{\omega}\cos\omega\ \ t+c\end{aligned}$$

SAQ 1.5

$$\frac{rev}{\min}=\frac{2\pi^c}{60s}=0.1047\ \ rad/s$$

$$15\,rev/\min=15\times0.1047\,rad/s=1.5705\,rad/s$$

Study Session 2 Numerical Analysis

2.0 Introduction

In most experiments as a physicist, you would be expected to plot some graphs. This chapter explains in details, how you can interpret the equation governing a particular phenomenon, plot the appropriate graph with the data obtained, to illustrate the inherent physical features, and deduce the values of some physical quantities. The process of fitting a curve to a set of data is called **curve-fitting**. We shall now take a look at the possible cases that could arise in curve-fitting.

Learning Outcomes for Study Session 2

When you have studied this session, you should be able to:

2.1 Define and correctly use all the key words printed in **bold** (SAQ 2.1)

2.2 Linearise some expressions in Physics (SAQ 2.2-2.4)

2.3 Use the method of Least Squares to derive the equation of a line (SAQ 2.2-2.4).

2.4 Use the method of Group Averages to derive the equation of a line (SAQ 2.2-2.4).

2.1 Linear Graph

The law governing the physical phenomenon under investigation could be linear, of the form $y = mx + c$. It follows that a graph could be plotted of the points (x_i, y_i), $i = 1, \ldots, n$, where n is the number of observations (or sets of data). We could obtain the line of best fit via any of a number of methods.

2.2 Linearisation

A nonlinear relationship can be linearised and the resulting graph analysed to bring out the relationship between variables. We shall consider a few examples:

▪ Put the equation $y = ae^x$ in the form of $f(z) = dz + b$, where d and b are constants. Deduce the slope and the intercept of the resulting linear graph.

❑ (i) We could take the logarithm of both sides of equation 2.1 to base e,

$$\ln y = \ln(ae^x) = \ln a + x$$

since $\ln e^x = x$. Thus, a plot of $\ln y$ against x gives a linear graph with slope unity and a y-intercept of ln a.

(ii) We could also have plotted y against e^x. The result is a linear graph through the origin, with slope equal to a.

▪ Linearise the expression $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Hence, deduce what functions should be plotted to get a linear graph. How can you recover the acceleration due to gravity from the graph?

- We can write this expression in three different ways:

(i) $\ln T = \ln(2\pi) + \frac{1}{2}\ln\left(\frac{l}{g}\right) = \ln(2\pi) + \frac{1}{2}(\ln l - \ln g)$

Rearranging, we obtain,

$$\ln T = \frac{1}{2}\ln l + \left(\ln(2\pi) - \frac{1}{2}\ln g\right)$$

writing this in the form $y = mx + c$, we see that a plot of $\ln T$ against $\ln l$ gives a slope of 0.5 and a ln T intercept of $\left(\ln(2\pi) - \frac{1}{2}\ln g = \ln\left(\frac{2\pi}{g^{1/2}}\right)\right)$. Once the intercept is read of the graph, you can then calculate the value of g. Note that we could also have deduced this right away from the original equation:

$$T = \left(\frac{2\pi}{g^{1/2}}\right)l^{1/2}$$

$$\ln T = \frac{1}{2}\ln l + \ln\left(\frac{2\pi}{g^{1/2}}\right)$$

(ii) $T = \frac{2\pi}{\sqrt{g}}\sqrt{l}$

A plot of T versus $\sqrt{l}$ gives a linear graph through the origin (as the intercept is zero). The slope of the graph is $\frac{2\pi}{\sqrt{g}}$, from which the value of g can be recovered.

(iii) Squaring both sides,

$$T^2 = \frac{4\pi^2}{g}l$$

A plot of T^2 versus l gives a linear graph through the origin. The slope of the graph is $\frac{4\pi^2}{g}$, and the value of g can be obtained appropriately.

- Given the expression $N = N_0 e^{-\lambda t}$, what functions would you plot to achieve a linear graph?

- The student can show that a plot of $\ln N$ versus t will give a linear graph with slope $-\lambda$, and $\ln N$ intercept is $\ln N_0$.

- If $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, what functions would you plot to get a linear graph, and how would you deduce the focal length of the mirror from your graph?

- We rearrange the equation:

$$\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$$

A plot of v^{-1} (y-axis) versus u^{-1} (x-axis) gives a slope of -1 and a vertical intercept of $\frac{1}{f}$.

- A student took the following reading with a mirror in the laboratory.

U	10	20	30	40	50
V	-7	-10	-14	-15	-17

Linearise the relationship $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$. Plot the graph of v^{-1} versus u^{-1} and draw the line of best fit by eye judgment. Hence, find the focal length of the mirror. All distances are in cm.

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Table 2.1

u	v	1/u	1/v
10	-7	0.100	-0.143
20	-10	0.050	-0.100
30	-14	0.033	-0.071
40	-15	0.025	-0.067
50	-17	0.020	-0.059

The graph is plotted in Fig. 2.1.

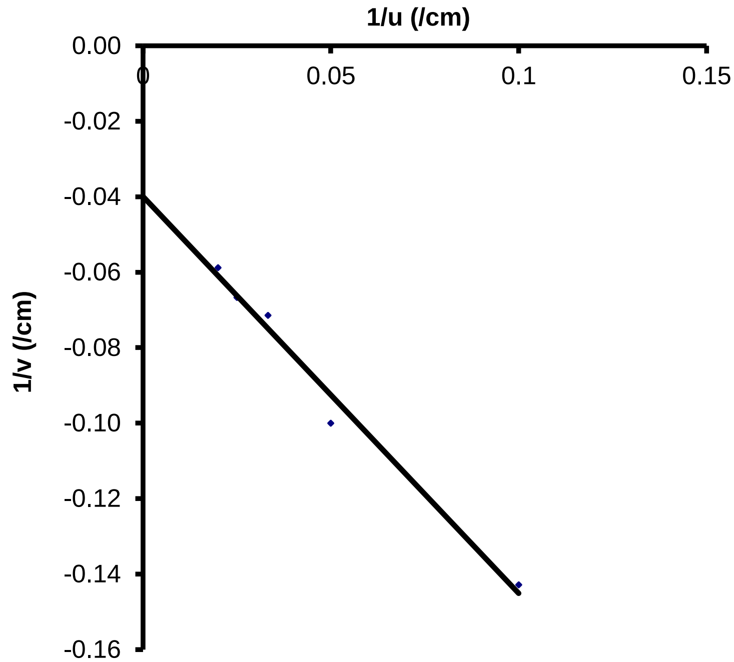

Fig. 2.1: Linear graph of the function $\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$

The slope is -1.05 and the intercept -0.04. From $\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$, we see that the intercept is $\frac{1}{f} = -0.04$, or $f = -\frac{1}{0.04} = -25$ cm.

2.3 Curve Fitting

What we did in Section 2.2, generally, was to plot the values of dependent variable against the corresponding values of the independent variable. With this done, we got the line of best fit. The latter could have been obtained by eye judgment. There are some other ways of deducing the relationship between the variables. We shall consider the ones based on linear relationship, or the ones that can be somehow reduced to such relationships.

2.3.1 Method of Least Squares

Suppose x_i, $i = 1, \cdots, n$ are the points of the independent variable where the dependent variable having respective values y_i, $i = 1, \cdots, n$ is measured. Consider Fig. 2.2, where we have assumed a linear graph of equation $y = mx + c$. Then at each point x_i, $i = 1, \cdots, n$, $y_i = mx_i + c$.

The **least square method** entails minimizing the sum of the squares of the difference between the measured value and the one predicted by the assumed equation.

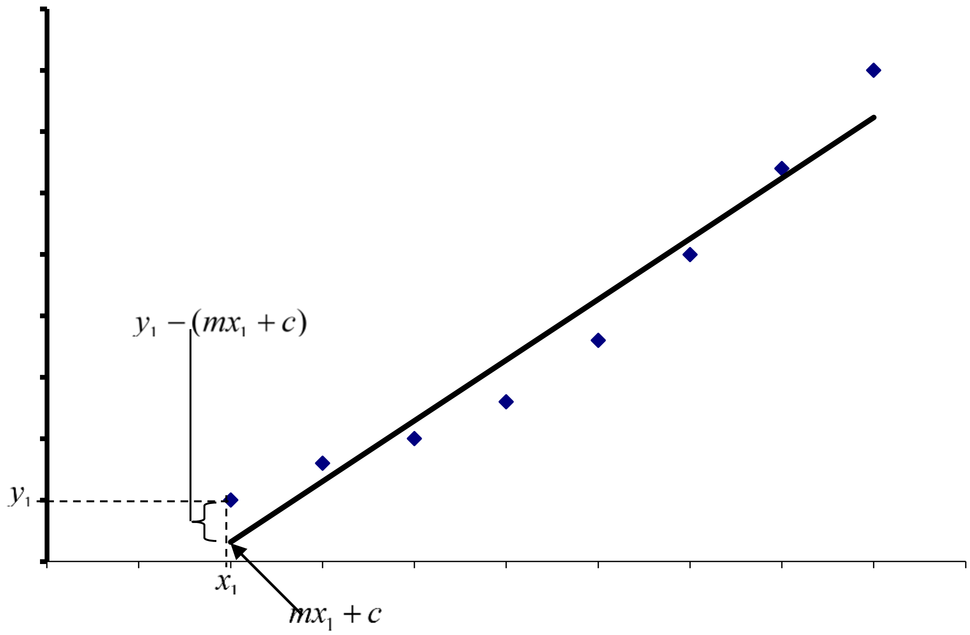

Fig. 2.2: Illustration of the error in representing a set of data with the line of best fit

$$S = \sum_{i=1}^{n} [y_i - (mx_i + c)]^2 \tag{2.1}$$

We have taken the square of the difference because taking the sum alone might give the impression that there is no error if the sum of positive differences is balanced by the sum of negative differences, just as in the case of the relevance of the variance of a set of data.

Now, S is a function of m and c, that is, $S = S(m,c)$. This is because we seek a line of best fit, which will be determined by an appropriate slope and a suitable intercept. In any case, x_i and y_i are not variables in this case, having been obtained in the laboratory, for instance.

You have been taught at one point or another that for a function of a single variable $f(x)$, the extrema are the points where $\frac{df}{dx} = 0$. However, for a function of more than one variable, partial derivatives are the relevant quantities. Thus, since $S = S(m,c)$, the condition for extrema is

$$\frac{\partial S}{\partial m} = 0 \text{ and } \frac{\partial S}{\partial c} = 0 \tag{2.2}$$

$$\frac{\partial S}{\partial m} = 2\sum_{i=1}^{n}[y_i - (mx_i + c)](-x_i) = 0 \qquad 2.3$$

$$\frac{\partial S}{\partial c} = 2\sum_{i=1}^{n}[y_i - (mx_i + c)](-1) = 0 \qquad 2.4$$

From equation 2.3,

$$\sum_{i=1}^{n} x_i y_i - m\sum_{i=1}^{n} x_i^{\,2} - \sum_{i=1}^{n} cx_i = 0 \qquad 2.5$$

and from equation 2.4,

$$\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} c = 0 \qquad 2.6$$

It follows from the fact that $\bar{x}_i = \frac{\sum_{i=1}^{n} x_i}{n}$ and similar expressions, that equations 2.5 and 2.6 give, respectively,

$$\overline{xy} - m\overline{x^2} - c\bar{x} = 0 \qquad 2.7$$

$$\bar{y} - m\bar{x} - c = 0 \qquad 2.8$$

Multiplying equation 2.8 by $\bar{x}$ gives

$$\bar{x}\,\bar{y} - m\bar{x}^2 - c\bar{x} = 0 \qquad 2.9$$

Finally, from equations 2.7 and 2.9,

$$m = \frac{\overline{xy} - \bar{x}\,\bar{y}}{\overline{x^2} - \bar{x}^2} \qquad 2.10$$

and from equation 2.8,

$$c = \bar{y} - m\bar{x} \qquad 2.11$$

- A student obtained the following data in the laboratory. By making use of the method of least squares, find the relationship between *x* and *t*.

- Thus, for the following set of readings:

Table 2.2

t	5	12	19	26	33
x	23	28	32	38	41

The table can be extended to give

Table 2.3

t	5	12	19	26	33	Σ=95	$\bar{t}$ =19
x	23	28	32	38	41	Σ=162	$\bar{x}$ =32.4
tx	115	336	608	988	1353	Σ=3400	$\overline{tx}$ =680
t^2	25	144	361	676	1089	Σ=2295	$\overline{x^2}$ =459

$$m = \frac{\overline{tx} - \bar{t}\,\bar{x}}{\overline{t^2} - \bar{t}^2} = \frac{680 - 19 \times 32.4}{459 - 19^2} = 0.6571 \qquad 2.12$$

$$c = \bar{x} - m\bar{t} = 32.4 - 0.6571 \times 19 = 19.9151 \qquad 2.13$$

Hence, the relationship between x and t is,

$$x = 0.6571t + 19.9151$$

2.3.2 Method of group averages

As the name implies, a set of data is divided into two groups, each of which is assumed to have a zero sum of residuals. Thus, given the equation

$$y = mx + c \qquad 2.14$$

we would like to fit a set of n observations as close as possible.

The error in the measured value of the variable and the value predicted by the equation is (as we have seen in Fig. 2.2):

$$\varepsilon_i = y_i - (mx_i + c) \qquad 2.15$$

The fitted line requires two unknown quantities: m and c. Thus, two equations are needed. We would achieve these two equations by dividing the data into two, one of size l and the other of size n-l, where n is the total number of observations.

The assumption that the sum of errors for each group is zero, requires that

$$\sum_{i=1}^{l} [y_i - (mx_i + c)] = 0 \qquad 2.16$$

and

$$\sum_{l+1}^{n} [y_i - (mx_i + c)] = 0 \qquad 2.17$$

From equation 2.16,

$$\sum_{i=1}^{l} y_i = m\sum_{i=1}^{l} x_i + lc \qquad 2.18$$

and equation 2.17 yields

$$\sum_{i=l+1}^{n} y_i = m\sum_{i=l+1}^{n} x_i + (n-l)c \qquad 2.19$$

the latter equation being true since *n* – *l* is the number of observations that fall into that group.

Dividing through by *l* and *n* – *l*, respectively, equation 2.18 and 2.19 give, respectively,

$$\frac{1}{l}\sum_{i=1}^{l} y_i = m\frac{1}{l}\sum_{i=1}^{l} x_i + c \qquad 2.20$$

$$\frac{1}{n-l}\sum_{i=l+1}^{n} y_i = m\frac{1}{n-l}\sum_{i=l+1}^{n} x_i + c \qquad 2.21$$

Thus,

$$\bar{y}_1 = m\bar{x}_1 + c \qquad 2.22$$

$$\bar{y}_2 = m\bar{x}_2 + c \qquad 2.23$$

Subtracting equation 2.23 from equation 2.22,

$$\bar{y}_1 - \bar{y}_2 = m(\bar{x}_1 - \bar{x}_2) \qquad 2.24$$

$$m = \frac{\bar{y}_1 - \bar{y}_2}{\bar{x}_1 - \bar{x}_2} \qquad 2.25$$

and

$$c = \bar{y}_1 - m\bar{x}_1 \qquad 2.26$$

▪ Solve the problem in Table 2.2 using the method of group averages.

t	5	12	19	26	33
x	23	28	32	38	41

❑ We shall divide the data into two groups, such as:

Table 2.4

t	5	12	19
x	23	28	32

And

Table 2.5

t	26	33
x	38	41

The tables can be extended to give, for Table 2.4:

t	5	12	19	Σ=36	$\bar{t}_1$ =12
x	23	28	32	Σ=83	$\bar{x}_1$ =27.666667

and for Table 2.5:

t	26	33	Σ=59	$\bar{t}_2$ =29.5
x	38	41	Σ=79	$\bar{x}_2$ =39.5

$$m = \frac{\bar{x}_1 - \bar{x}_2}{\bar{t}_1 - \bar{t}_2} = \frac{27.666667 - 39.5}{12 - 29.5} = 0.67619 \qquad 2.39$$

and

$$c = \bar{x}_1 - m\bar{t}_1 = 27.666667 - (0.67619 \times 12) = 19.552387 \qquad 2.40$$

Thus, the equation of best fit is, putting equations 2.39 and 2.40 into the equation $y = mx + c$,

$$x = 0.67619t + 19.552387 \qquad 2.41$$

Summary of Study Session 2

In Study Session 2, you have learnt the following:

1. How to linearise a given relationship.
2. How to fit a line of best fit to a set of data using the Least squares method.
3. How to fit a line of best fit to a set of data using the method of group averages.

References

1. Conte, S. D. and de Boor, C. (1965). Elementary Numerical Analysis, an algorithmic approach. McGraw-Hill International Student Edition, McGraw-Hill.

2. Grewal, B. S. (1997). Numerical Methods in Engineering and Science. Khanna Publishers.

3. Bradie, B. (2006). A Friendly Introduction to Numerical Analysis. Pearson Education Inc., New Jersey, USA.

Self Assessment Questions (SAQs) for Study Session 2

Having completed this study session, you may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the Solutions to the Self-Assessment Questions at the end of this study session.

SAQ 2.1 (tests Learning Outcome 2.1)

Explain the following terms:

(i) Curve-fitting

(ii) Least squares method of curve-fitting.

(iii) The method of group averages.

SAQ 2.2 (tests Learning Outcomes 2.2 and 2.3)

The current flowing in a particular R-C circuit is tabulated against the change in the time $t - t_0$, such that at time $t = t_0$, the current is 1.2 A. Using the least-squares method, find the slope and the intercept of the linear function relating the current *i* to the time *t*. Hence, determine the time-constant of the circuit.

t	2	2.2	2.4	2.6	2.8	3
i	0.20	0.16	0.13	0.11	0.09	0.07

SAQ 2.3 (tests Learning Outcome 2.4)

Solve the problem in SAQ 2.2 with the method of group averages by dividing into two groups of three data sets each.

t	2	2.2	2.4
i	0.20	0.16	0.13

and

t	2.6	2.8	3
i	0.11	0.09	0.07

SAQ 2.4 (tests Learning Outcomes 2.2, 2.3 and 2.4)

A student performing the simple pendulum experiment obtained the following table, where t is the time for 50 oscillations.

l (cm)	50	45	40	35	30	25	20	15
t (s)	71	69	65	61	56	52	48	43

Find the acceleration due to gravity at the location of the experiment, using

(i) the method of least squares, and

(ii) the method of group averages.

Solutions to SAQs

SAQ 2.1

(i) Curve-fitting is the process of fitting a curve or a mathematical function to a set of data.

(ii) This is a method of curve-fitting based on minimising the sum of the squares of the difference in the tabulated (observed) values of the dependent variable and the fitted values predicted by the fitted equation.

(iii) The method of group averages divides the data to be fitted into a curve into two groups, each of which is assumed to have a zero sum of residuals; one of size *l* and the other of size *n*-*l*, where *n* is the total number of observations. Two variables are needed, the slope and the intercept of the line of fit, two equations are needed, one each from each group of data.

SAQ 2.2

The current flowing in a particular R-C circuit is tabulated against the change in the time $t - t_0$, such that at time $t = t_0$, the current is 1.2 A. Using the least-squares method, find the slope and the intercept of the linear function relating the current *i* to the time *t*. Hence, determine the time-constant of the circuit.

t	2	2.2	2.4	2.6	2.8	3
i	0.20	0.16	0.13	0.11	0.09	0.07

Taking logs: $i = i_0 e^{-t/RC}$. $\log i = \log i_0 + \log(e^{-t/RC}) = \log i_0 - \dfrac{t}{RC}$. A plot of $\log i$ against *t* gives slope $-\dfrac{1}{RC}$ and intercept $\log i_0$.

	t	I	tsquare	log I	tlogI	
	2.0	0.200000	4	-0.69897	-1.39794	-0.7022
	2.2	0.160000	4.84	-0.79588	-1.75094	-0.78902
	2.4	0.130000	5.76	-0.88606	-2.12654	-0.87584
	2.6	0.110000	6.76	-0.95861	-2.49238	-0.96266
	2.8	0.090000	7.84	-1.04576	-2.92812	-1.04948
	3.0	0.070000	9	-1.1549	-3.46471	-1.1363
Sum	15		38.2	-5.54017	-14.1606	
Average	2.5		6.3666667	-0.92336	-2.3601	
			Slope	-0.4431		
			Intercept	0.1844		

$$m = \frac{-2.3601 - (2.5 \times -0.92336)}{6.3666667 - 2.5^2} = -0.4431$$

$$c = \overline{\log I} - m\bar{t} = 0.1844$$

$$m = -\frac{1}{RC}, \text{ or } RC = -\frac{1}{m} = 2.2568 = \text{time constant of the circuit.}$$

SAQ 2.3

Solve the problem in SAQ 2.2 with the method of group averages by dividing into two groups of three data sets each.

t	2	2.2	2.4
i	0.20	0.16	0.13

and

t	2.6	2.8	3
i	0.11	0.09	0.07

Group 1

	t	*i*	log i
	2.0	0.20	-0.69897
	2.2	0.16	-0.79588
	2.4	0.13	-0.88606
Sum	6.6		-2.38091
Average	2.2		-0.79364

Group 2

	t	*i*	log i
	2.6	0.11	-0.95861
	2.8	0.09	-1.04576
	3.0	0.07	-1.15490
Sum	8.4		-3.15927
Average	2.8		-1.05309

$$m = \frac{\bar{y}_1 - \bar{y}_2}{\bar{x}_1 - \bar{x}_2} = \frac{-.79364 - (-1.05309)}{2.2 - 2.8} = -0.4324$$

$$c = \bar{y}_1 - m\bar{x}_1 = -0.79364 - (-0.4324 \times 2.2) = 0.1576$$

SAQ 2.4

A student performing the simple pendulum experiment obtained the following table, where t is the time for 50 oscillations.

l (cm)	50	45	40	35	30	25	20	15
t (s)	71	69	65	61	56	52	48	43

Find the acceleration due to gravity at the location of the experiment, using

(i) the method of least squares, and

(ii) the method of group averages.

Method of least squares (taking logs)

$\log T = \log\left(\dfrac{2\pi}{\sqrt{g}}\right) + \dfrac{1}{2}\log l$: A plot of $\log T$ against $\log l$ gives slope 0.5 and intercept c = $\log \dfrac{2\pi}{\sqrt{g}}$, from which the value of g is $\left(\dfrac{2\pi}{\log^{-1}(c)}\right)^2$.

(log *l*)*(log *l*)	(log *l*)*(log *T*)	
0.090619058	-0.04584	0.2966771
0.120261561	-0.04851	0.3165404
0.158356251	-0.04534	0.3387458
0.207873948	-0.03937	0.3639201
0.273402182	-0.02574	0.3929817
0.362476233	-0.01026	0.4273542
0.488559067	0.012392	0.4694229
0.678825613	0.053967	0.5236588
2.380373913	-0.1487	
0.297546739	-0.01859	
Slope =	0.429391	
Intercept =	0.282157	
2 pi =	6.283185	
log 2 pi =	0.798236	
log 2 pi –intercept =	0.516079	
2(log 2pi-inter)	1.032159	= log g
	10.84	= g

Method of least squares (taking squares)

$T^2 = \frac{4\pi^2}{g} l$. A plot of T^2 against l gives a line through the origin with slope $m = \frac{4\pi^2}{g}$, from which $g = \frac{4\pi^2}{m}$:

l	t	L (m)	$T = t/20$	T^2	l^2	$T^2 l$		
50	71	0.50	1.42	2.01640	0.2500	1.00820		
45	69	0.45	1.38	1.90440	0.2025	0.85698		
40	65	0.40	1.30	1.69000	0.1600	0.67600		
35	61	0.35	1.22	1.48840	0.1225	0.52094		
30	56	0.30	1.12	1.25440	0.0900	0.37632		
25	52	0.25	1.04	1.08160	0.0625	0.27040		
20	48	0.20	0.96	0.92160	0.0400	0.18432		
15	43	0.15	0.86	0.73960	0.0225	0.11094		
	Sum	2.6		11.0964	0.9500	4.00410		
	Average	0.325		1.38705	0.11875	0.50051		
					Slope =	3.78829		
					Intercept =	0.15586	g =	10.42

Method of group averages (taking logs)

Group 1				
L	t	*T*	log l	log T
0.50	71	1.42	-0.3010	0.1523
0.45	69	1.38	-0.3468	0.1399
0.40	65	1.30	-0.3979	0.1139
0.35	61	1.22	-0.4559	0.0864
	Sum		-1.5017	0.4925
	Average		-0.3754	0.1231

Group 2				
l	*t*	*T*	log l	log T
0.30	56	1.12	-0.5229	0.0492
0.25	52	1.04	-0.6021	0.0170
0.20	48	0.96	-0.6990	-0.0177
0.15	43	0.86	-0.8239	-0.0655
	Sum		-2.6478	-0.0170
	Average		-0.6620	-0.0043

			Slope =	0.4445
			Intercept =	0.29
			g =	10.38

Method of group averages (taking squares)

Group 1

l	*t*	*T*	*l*	T^2
0.50	71	1.42	0.50	2.0164
0.45	69	1.38	0.45	1.9044
0.40	65	1.30	0.40	1.6900
0.35	61	1.22	0.35	1.4884
	Sum		1.70	7.0992
	Average		0.43	1.7748

Group 2

l	*t*	*T*	*l*	T^2
0.30	56	1.12	0.30	1.2544
0.25	52	1.04	0.25	1.0816
0.20	48	0.96	0.20	0.9216
0.15	43	0.86	0.15	0.7396
	Sum		0.9	3.9972
	Average		0.225	0.9993

			Slope =	3.8775
			Intercept =	1.0205
			g =	10.18

Study Session 3 Vector Spaces I

Introduction

Vectors are of utmost importance in Physics, appearing in practically all areas of Physics. As is usual with Mathematics, the idea of Euclidean vectors will be extended to cover mathematical structures you never could picture as having anything to do with vectors. You shall come to know that some sets also behave like vectors, and hence, the idea of vectors has been expanded to include these structures, and generally we call such sets vector spaces.

Learning Outcomes of Study Session 3

At the end of this study session, you should be able to do the following:

3.1 Understand and correctly use all the keywords in **bold** print (SAQ 3.1).

3.2 Identify structures that are vector spaces and those that are not (SAQ 3.2).

3.3 Prove whether a set of vectors is linearly independent or linearly dependent (SAQ 3.3-3.7)

3.4 Check if a given set of vectors is a basis for a given vector space (SAQ 3.8).

3.5 Find the components of a given vector in a given basis (SAQ 3.9).

3.1 Definition: Vector Space

Given a set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\} = S$. If

(i) $\mathbf{v}_i + \mathbf{v}_j \in S \quad \forall i, j = 1, 2, ..., n$ (additivity) 3.1

(ii) $\alpha \mathbf{v}_i \in S \;\; \forall\; i, = 1, 2, .., n$; (homogeneity) 3.2

(iii) $v_1 + v_2 = v_2 + v_1$ (commutativity) 3.3

(iv) There is zero vector $\mathbf{0} \in S$**,** such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ 3.4

(v) There is an additive inverse $\mathbf{v}'$, such that $\mathbf{v} + \mathbf{v}' = \mathbf{0}$ 3.5

(vi) $\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3$, $\forall\; \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in S$ (associativity) 3.6

(vii) $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$, $\forall\; \mathbf{v} \in S$ 3.7

(viii) $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$, $\forall\; \mathbf{v} \in S$ 3.8

(ix) $1\mathbf{v} = \mathbf{v}$ 3.9

(x) $\alpha(\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ $\mathbf{v}_1, \mathbf{v}_2 \in S$ and $\alpha \in K$. 3.10

$\alpha, \beta \in K$, where K is a field, e.g., the real number line (R) or the complex plane (C).

then, S is called a **vector space** or **linear space**. The vector space is a **real vector space** if $K \equiv R$, and a **complex vector space** if $K \equiv C$.

The vector spaces we shall come across in our study are such that we shall be bothered with only the first two properties of a vector space, that is, additivity and homogeneity. You might as well take this slogan to heart: ‘additivity plus homogeneity equals linearity.’

- Show that the set of Cartesian vectors in 3-dimensions, V_3, is a real vector space.

- $\mathbf{a}, \mathbf{b} \in V_3$, $\lambda \in R$.

  The sum of two vectors in V_3 is also in V_3:

  (i) $\mathbf{a} + \mathbf{b} \in V_3$ 3.11

  The product of λ, a real number, and any vector $\mathbf{a}$ V_3 is also in V_3:

  (ii) $\lambda\mathbf{a} \in V_3$ 3.12

  It is clear that all other conditions are also satisfied by vectors in V_3. Indeed, the zero vector is the three-dimensional vector $\mathbf{0}$ = (0, 0, 0) which is also in V_3. The vector $-\mathbf{a}$ is the additive inverse of the vector **a**.

- Prove that the set of $m \times n$ matrices under matrix addition and scalar multiplication, M_{mn}, is a vector space over the real numbers or the complex plane.

- $A, B \in M_{mn}$, $\lambda \in R$ or C

  The matrix sum of two $m \times n$ is also an $m \times n$ matrix:

  (i) $A + B \in M_{mn}$ 3.13

  The product of λ in the field of real numbers or the complex plane is, respectively, still in M_{mn}:

  (ii) $\lambda A_{mn} \in M_{mn}$ 3.14

  Other conditions are also satisfied by vectors in M_{mn}. Indeed, the zero vector is O_{mn}, the $m \times n$ matrix that has all its entries zero which is also in M_{mn}. The vector $-M$ is the additive inverse of the vector *M*; it is also in M_{mn}.

- A set $\{f(x), g(x), ...\} = F$ of square integrable functions of x over the interval $[a,b]$, $f(x), g(x) \in F$, $\lambda \in R$ or C. That is, real- or complex-valued functions, such that $\int_a^b | f(x) |^2 \, dx < \infty$.

- The sum of two such functions is also a square integrable function of *x* over the interval:

  (i) $f(x) + g(x) \in F$ 3.15

  The scalar multiple of such a function is also a square integrable function over the interval:

(ii) $\lambda f(x) \in F$ 3.16

All other conditions are satisfied by vectors in F. The zero vector is the vector $r(x) \equiv 0$, the zero-valued function which is also in F. The vector $-s(x)$ is the additive inverse of the vector $s(x)$, and is also in F.

3.2 Linear Independence

Given a set $\{\mathbf{v}_i\}_{i=1}^{n}$. If we can write

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0 \qquad 3.17$$

and this implies $a_1 = a_2 = \cdots = a_n = 0$, then we say $\{\mathbf{v}_i\}_{i=1}^{n}$ is a linearly independent set.

If even just one of them is non-zero, then the set is linearly dependent.

- Check if the set $\{\mathbf{i}, 2\mathbf{i}, \mathbf{j}\}$ is linearly independent.

- We form the expression

  $$c_1\phi_1 + c_2\phi_2 + c_3\phi_3 = 0 \qquad 3.18$$

  where $\phi_1 = \mathbf{i}$, $\phi_2 = 2\mathbf{j}$ and $\phi_3 = \mathbf{j}$

  Thus,

  $$\mathbf{i}c_1 + 2\mathbf{i}c_2 + \mathbf{j}c_3 = 0 \qquad 3.19$$

  or

  $$\mathbf{i}(c_1 + 2c_2) + \mathbf{j}c_3 = 0 \qquad 3.20$$

  which implies

  $$c_1 + 2c_2 = 0 \qquad 3.21$$

  and

  $$c_3 = 0 \qquad 3.22$$

  We see that

  $$c_1 = -2c_2 \qquad 3.23$$

  $$c_3 = 0 \qquad 3.24$$

  c_1 and c_2 do not necessarily have to be zero.

  The vectors are linearly dependent.

- Is the set $\{1, x, x^2\}$ linearly independent?

- Let the elements of the set be represented as 'vectors' $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, then,

$$a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + a_3\mathbf{p}_3 = \mathbf{0} \quad 3.25$$

$$a_1(1) + a_2 x + a_3 x^2 = 0 \quad 3.26$$

Equating coefficients of x on both sides, we see that this equation can only be satisfied if $a_1 = a_2 = a_3 = 0$.

The vectors are linearly independent.

If we wish to write any vector in 1 (say, x) direction, we need only one unit vector. Any two vectors in the x direction must be linearly dependent, for we can write one as $a_1\mathbf{i}$ and the other $a_2\mathbf{i}$, where a_1 and a_2 are scalars.

- Show that any two vectors in the vector space of one-dimensional vectors are linearly dependent.
- Let the two vectors be $a_1\mathbf{i}$ and the other $a_2\mathbf{i}$.

We form the linear combination

$$c_1(a_1\mathbf{i}) + c_2(a_2\mathbf{i}) = 0 \quad 3.27$$

where a_1 and a_2 are scalar constants.

Obviously, c_1 and c_2 need not be identically zero for the expression to hold, for

$$c_1 = -c_2 \frac{a_2}{a_1} \quad 3.28$$

would also satisfy the expression.

We conclude therefore that the vectors must be linearly dependent.

In general any $n+1$ vectors in an n-dimensional space must be linearly dependent.

3.3 Span

If we can express any vector in a vector space as a linear combination of a set of vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, then we say the vector space V is **spanned** by the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$. Thus the set of all linear combinations of vectors in the $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ set is said to be **span** of the vector space V. $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is said to be a **basis** for V. Thus, a set of n linearly independent vectors that span V is a basis for V. However, we can also span the same vector space with another set of $(n+k)$ vectors with k an integer greater than zero. However, the vectors will not be linearly independent as we have seen in ITQ 3. Indeed, the coefficients c_i of the sum $\sum_{i=1}^{n} c_i\mathbf{e}_i$ will not be unique.

3.4 Basis Vector

Let V be an n - dimensional vector space. Any set of n linearly independent vectors $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ forms a basis for V. Then, each vector $\mathbf{e}_i$ is called a basis vector for the vector space. Given such a basis, any vector $\mathbf{v} \in V$ can be expressed, *uniquely*, as a linear combination of the vectors $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, i.e.,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots x_n\mathbf{e}_n \qquad 3.29$$

Note that the basis vectors do not have to be unit vectors, neither must they be mutually at right angle.

Summary of Study Session 3

In Study Session 3, you have learnt the following:

1. The properties of a vector space.
2. How to show that a set is a vector space.
3. The definition of linear independence.
4. How to show that a set of vectors is linearly independent.
5. What we mean by a basis, and what a basis vector is.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.
4. Hefferson, J. (2012). Linear Algebra, http://joshua.smcvt.edu/linearalgebra/book.pdf
5. Hefferson, J. (2012). Answers to Exercises, http://joshua.smcvt.edu/linearalgebra/jhanswer.pdf

Self-Assessment Questions (SAQs) for Study Session 3

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 3.1 (tests Learning Outcome 3.1).

Define the following terms:

(i) Vector spaces (ii) Linear independence (iii) Span (iv) Basis vectors

SAQ 3.2 (tests Learning Outcome 3.2)

Show that the following are vector spaces over the indicated field:

(i) The set of real numbers over the field of real numbers.

(ii) The set of complex numbers over the field of real numbers.

(iii) The set of quadratic polynomials over the complex field.

SAQ 3.3 (tests Learning Outcome 3.3)
Show that $\{\mathbf{i}, 2\mathbf{k}, \mathbf{j}\}$ is a linearly independent set.

SAQ 3.4 (tests Learning Outcome 3.3)
Show that the set $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ is linearly independent

SAQ 3.5 (tests Learning Outcome 3.3)
Show that any set of $n+1$ vectors in n-dimensional space is linearly dependent.

SAQ 3.6 (tests Learning Outcome 3.3)
Show whether or not the set $\{\begin{bmatrix}i & 0\\2 & -i\end{bmatrix}, \begin{bmatrix}1 & 0\\i & -i\end{bmatrix}, \begin{bmatrix}2i & 0\\-i & -i\end{bmatrix}, \begin{bmatrix}i & 1\\2 & -1\end{bmatrix}\}$ is linearly independent?

SAQ 3.7 (tests Learning Outcome 3.3)
Check whether the following vectors are linearly independent.

(i) $2i+3j-k$, $-i+j+3k$ and $-3i+2j+k$

(ii) $\begin{bmatrix}i & 1\\-2 & 2i\end{bmatrix}, \begin{bmatrix}2 & 1\\-i & 2i\end{bmatrix}, \begin{bmatrix}-1 & 2\\3 & -i\end{bmatrix}$ and $\begin{bmatrix}-i & 2i\\i & -2\end{bmatrix}$

SAQ 3.8 (tests Learning Outcome 3.4)
Show whether or not the set $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}$ is a basis for the two-dimensional Euclidean space.

SAQ 3.9 (tests Learning Outcome 3.5)
Find the coordinates of the vector $\begin{bmatrix}1 & 2\\-2 & i\end{bmatrix}$ with respect to the basis

$$\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\}.$$

Solutions to SAQs

SAQ 3.1

In the text.

SAQ 3.2

(i) The set of real numbers over the field of real numbers.

Let the set be *R*, the set of real numbers, then,

$$a + b \in R \qquad \forall \ a, b \in R$$

and $\lambda \ a \in R \qquad \forall \ a \in R, \ \lambda \in R$

(iii) The set of complex numbers over the field of real numbers.

Let the set be *C* be the set of complex numbers, then,

$$c_1 + c_2 \in C \qquad \forall \ a, b \in C$$

and $\alpha \ c \in R \qquad \forall \ c \in C, \ \alpha \in C$

(iii) The set of quadratic polynomials over the complex field.

Let this set be P. Then $P_1 = a_1 x^2 + b_1 x^2 + c_1$ and $P_2 = a_2 x^2 + b_2 x^2 + c_2$ are in P, where $a_1, a_2, b_1, b_2, c_1, c_2$ are constants.

$$a_1 x^2 + b_1 x^2 + c_1 + a_2 x^2 + b_2 x^2 + c_2$$

$$= (a_1 + a_2) x^2 + (b_1 + b_2) x + (c_1 + c_2) \in \text{P} \ \forall \ P_1, \ P_2 \in \text{P}$$

$\lambda \ (a_1 x^2 + b_1 x + c_1) \in \text{P} \ \forall \ P_1 \in \text{P}, \ \lambda \in$ the complex field.

SAQ 3.3

$$\mathbf{i} c_1 + 2\mathbf{k} c_2 + \mathbf{j} c_3 = \mathbf{0}$$

$$c_1 = 0, \ c_2 = 0, \ c_3 = 0$$

The set is linearly independent.

Note that we have made use of the fact that

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{0}$ implies $x = 0, \ y = 0, \ z = 0$

SAQ 3.4

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$$

from which we obtain

$c_1 + c_2 + c_3 = 0$ (i)

$-c_2 + 2c_3 = 0$ (ii)

$c_1 + c_3 = 0$ (iii)

From (iii), $c_1 = -c_3$ (iv)

and from (ii) (v)

Putting (iv) and (v) in (i), gives

$-c_3 - 2c_3 + c_3 = 0$

$-2c_3 = 0$ or $c_3 = 0$

$c_1 = -c_3 = 0,\ c_2 = -2c_3 = 0$

$c_1 = c_2 = c_3 = 0$

Hence, we conclude the set is linearly independent.

Note that we could have written the set of three vectors as $\{\mathbf{i}+\mathbf{k}, \mathbf{i}+\mathbf{j}, \mathbf{i}+\mathbf{j}+\mathbf{k}\}$

$(\phi_1, \phi_2) = 1,\ (\phi_2, \phi_3) = 3,\ (\phi_1, \phi_3) = 2$

These vectors are not mutually orthogonal, yet, since they are linearly independent, we can write any vector in 3-dimensional Euclidean space as a linear combination of the members of the set.

Let us take the determinant of the matrix formed from the three equations (i) to (iii).

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1(-1-0) - 1(0-2) + 1(0+1) = -1+2+1 = 2 \neq 0$$

Thus, if the determinant of the system formed by the equations is non-zero, the sets are linearly independent. If the determinant is zero, then the vectors are linearly dependent. Of course, you know that if one row in a matrix is a multiple of another, then the determinant of the matrix is zero.

SAQ 3.5

Let $n+1$ vectors be linearly independent in an n dimensional space. Then, we can write

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n + a_{n+1}\mathbf{v}_{n+1} = 0 \qquad \text{(i)}$$

But we can also write any vector in terms of the n basis vectors:

$$\mathbf{v}_{n+1} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n \qquad \text{(ii)}$$

Then, putting (ii) in (i), and setting $\beta_k = -a_{n+1}\alpha_k$

$$(a_1 - \beta_1)\mathbf{v}_1 + (a_2 - \beta_2)\mathbf{v}_2 + \ldots + (a_n - \beta_n)\mathbf{v}_n = 0$$

Then, unless $\beta_i = 0$ (equivalent to $\alpha_i = 0$) for all i, in which case the vector $\mathbf{v}_{n+1} = 0$, or the set $\{\mathbf{v}_i\}_{i=1}^{n}$ is not a linearly dependent set, a contradiction. Thus, we conclude that any $n+1$ vectors in an n-dimensional space must be linearly dependent.

SAQ 3.6

$$\begin{bmatrix} i & 0 \\ 2 & -i \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix}, \begin{bmatrix} 2i & 0 \\ -i & -i \end{bmatrix}, \begin{bmatrix} i & 1 \\ 2 & -1 \end{bmatrix}$$

$$\alpha\begin{bmatrix} i & 0 \\ 2 & -i \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix} + \gamma\begin{bmatrix} 2i & 0 \\ -i & -i \end{bmatrix} + \delta\begin{bmatrix} i & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It follows that

$$i\alpha + \beta + 2i\gamma + \delta i = 0 \qquad \text{(i)}$$

$$2\alpha + i\beta - i\gamma + 2\delta = 0 \qquad \text{(ii)}$$

$$\delta = 0 \qquad \text{(iii)}$$

$$-i\alpha - i\beta - i\gamma - \delta = 0 \qquad \text{(iv)}$$

In view of (iii),

$$i\alpha + \beta + 2i\gamma = 0 \qquad \text{(v)}$$

$$2\alpha + i\beta - i\gamma = 0 \qquad \text{(vi)}$$

$$-i\alpha - i\beta - i\gamma = 0 \qquad \text{(vii)}$$

Adding (vi) and (vii),

$$\alpha(2-i) = 2i\gamma$$

$$\text{or } \alpha = \frac{2i\gamma}{2-i} \qquad \text{(viii)}$$

Eliminate β with (v) and (vi)

$$\alpha - i\beta + 2\gamma = 0$$

$$2\alpha + i\beta - i\gamma = 0$$

$$3\alpha = (i-2)\gamma$$

$$\alpha = \frac{(i-2)\gamma}{3} = \frac{2i\gamma}{2-i}$$

Therefore, $-(i-2)^2\gamma = 6i\gamma$

$$\gamma\left[6i + (-1 - 2i + 4)\right] = 0 \Rightarrow \gamma = 0 \Rightarrow \alpha = 0 \Rightarrow \beta = 0$$

The set is linearly independent.

SAQ 3.7

(i) $2i+3j-k$, $-i+j+3k$ and $-3i+2j+k$

$$a\begin{pmatrix}2\\3\\-1\end{pmatrix}+b\begin{pmatrix}-1\\1\\3\end{pmatrix}+c\begin{pmatrix}-3\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$2a-b-3c=0$
$3a+b+2c=0$
$-a+3b+c=0$

The solution set is (0, 0, 0), i.e., $a = b = c = 0$.
The vectors are linearly independent.

Alternatively,

$$\begin{vmatrix}2 & -1 & -3\\3 & 1 & 2\\-1 & 3 & 1\end{vmatrix}=2(1-6)+1(3+2)-3(9+1)$$

$$=2(-5)+5-30=-35\neq 0$$

(ii) $\begin{bmatrix}i & 1\\-2 & 2i\end{bmatrix}$, $\begin{bmatrix}2 & 1\\-i & 2i\end{bmatrix}$, $\begin{bmatrix}-1 & 2\\3 & -i\end{bmatrix}$ and $\begin{bmatrix}-i & 2i\\i & -2\end{bmatrix}$

$$a\begin{bmatrix}i & 1\\-2 & 2i\end{bmatrix}+b\begin{bmatrix}2 & 1\\-i & 2i\end{bmatrix}+c\begin{bmatrix}-1 & 2\\3 & -i\end{bmatrix}+d\begin{bmatrix}-i & 2i\\i & -2\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

Expanding,

$ai+2b-c-id=0$ (i)
$a+b+2c+2id=0$ (ii)
$-2a-ib+3c+id=0$ (iii)
$2ia+2ib-ic-2d=0$ (iv)

Multiplying (i) by 2 and adding to (ii),

$a(1+2i)+5b=0$ (v)

Multiplying (iii) by i and adding to (iv) gives

$(1+2i)b+2ic-3d=0$ (vi)

Multiplying (ii) by 2 and adding to (iii),

$(2-i)b+7c+5id=0$ (vii)

Multiplying (vi) by 5i and (vii) by 3 and adding,

$5i(1+2i)b-10c-15id=0$ (vi)

$$3(2-i)b+21c+15id=0 \qquad \text{(vii)}$$

$$(5i-10)b-10c-15id=0$$

$$(6-3i)b+21c+15id=0$$

$$(-4+2i)b+11c=0 \qquad \text{(viii)}$$

From (v) and (viii), $b=-\frac{a(1+2i)}{5}=\frac{11}{2i-4}c$

Hence,

$$c=\frac{a(1+2i)(2i-4)}{55}=-\frac{8+6i}{55}a$$

Substituting for b and c in equation (vi),

$$(1+2i)\left(-\frac{(1+2i)}{5}a\right)+2i\left(-\frac{8+6i}{55}a\right)-3d=0$$

$$\frac{(3-4i)}{5}a+-\frac{16i-12}{55}a=3d$$

$$\frac{33-44i-16i+12}{165}a=\frac{45-60i}{165}a=\frac{9-12i}{55}a=d \qquad \text{(ix)}$$

Putting *b*, *c*, *d* in (i),

$$ai+2\left(-\frac{a(1+2i)}{5}\right)+\frac{8+6i}{55}a+i\frac{12i-9}{55}a=0$$

$$ai-\frac{2a}{5}-\frac{4ai}{5}+\frac{8a}{55}+\frac{6ai}{55}-\frac{12a}{55}-\frac{9ia}{55}=0$$

$$ai\left(1-\frac{4}{5}+\frac{6}{55}-\frac{9}{55}\right)+a\left(\frac{8}{55}-\frac{2}{5}-\frac{12}{55}\right)=0$$

$$\frac{55-44+6-9}{55}ai+a\frac{8-22-12}{55}=0$$

$a\left(\frac{18}{55}i-\frac{26}{55}\right)=0$. Hence, a = 0, meaning that *b*, *c*, and *d* are also zero.

$$ai+2b-c-id=0 \qquad \text{(i)}$$

$$a+b+2c+2id=0 \qquad \text{(ii)}$$

$$-2a-ib+3c+id=0 \qquad \text{(iii)}$$

$$2ia+2ib-ic-2d=0 \qquad \text{(iv)}$$

$$\text{Check if } \begin{vmatrix} i & 2 & -1 & -i \\ 1 & 1 & 2 & 2i \\ -2 & -i & 3 & i \\ 2i & 2i & -i & -2 \end{vmatrix} \neq 0$$

SAQ 3.8

For the set to be a basis, the vectors must be linearly independent.

$$a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a - b = 0,\ a - b = 0$$

$$a = b$$

a and *b* do not have to be zero. Hence, the vectors are not linearly independent. Sketch the vectors and satisfy yourself that they are indeed linearly dependent: they are parallel.

Alternatively,

$$\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0$$

SAQ 3.9

The set is,

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Forming a linear combination,

$$\begin{bmatrix} 1 & 2 \\ -2 & i \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$1 = a + d \qquad \text{(i)}$$

$$2 = b - ic \qquad \text{(ii)}$$

$$-2 = b + ci \qquad \text{(iii)}$$

$$i = a - d \qquad \text{(iv)}$$

Adding (i) and (iv):

$$\frac{1+i}{2} = a$$

(i) – (iv):

$$\frac{1-i}{2} = d$$

(ii) + (iii):

$$0 = b$$

(iii) – (ii):

$$-\frac{2}{i} = 2i = c$$

Hence,

$$\begin{bmatrix} 1 & 2 \\ -2 & i \end{bmatrix} = \frac{1+i}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \frac{1-i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Study Session 4 Vector Spaces II

4.1 Introduction

In Study Session 3, you learnt about the properties of a vector space, as well as linear independence of a set of vectors. In this study session, you will learn about the inner or dot or scalar product of two vectors in a vector space, as well as the norm of a vector.

Learning Outcomes of Study Session 4

By the end of this study session, you would be able to do the following:

4.1 Define and correctly use all the keywords in **bold** print (SAQ 4.1).

4.2 State the properties of an inner product (SAQ 4.2).

4.3 Find the inner product of two vectors in a vector space (SAQs 4.3 and 4.5).

4.4 State the properties of a norm (SAQ 4.4).

4.5 Find the norm of a vector (SAQ 4.5, 4.7).

4.6 Normalise a given vector (SAQ 4.6, 4.8)

4.7 Prove that a given function is an inner product (SAQ 4.9).

4.8 Prove that a given function is a norm (SAQ 4.10).

4.1 Inner or Scalar Product

Properties of the Inner Product

Let V be a vector space, real or complex. Then, the inner product of $\mathbf{v}$, $\mathbf{w} \in V$,written as $(\mathbf{v},\mathbf{w})$, has the following properties:

(i) $(\mathbf{v},\mathbf{v}) \geq 0$ 4.1

(ii) $(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$ 4.2

(iii) $(\mathbf{v},\mathbf{w}) = (\mathbf{w},\mathbf{v})$ (Symmetry) 4.3

(iv) $(c\mathbf{v},\mathbf{w}) = c^*(\mathbf{v},\mathbf{w})$; $(\mathbf{v},c\mathbf{w}) = c(\mathbf{v},\mathbf{w})$ 4.4

(v) $(\mathbf{v},\mathbf{w}+\mathbf{z}) = (\mathbf{v},\mathbf{w}) + (\mathbf{v},\mathbf{z})$ 4.5

(vi) $(\mathbf{v},\mathbf{w}) \leq \|\mathbf{v}\|\|\mathbf{w}\|$ Cauchy-Schwarz inequality 4.6

❑ Prove the Cauchy-Schwarz inequality for any inner product.

❑ To prove the Cauchy-Schwarz inequality, we write, for an arbitrary $t \in \Re$, a real number,

$$0 \leq \|\mathbf{v} + t\,\mathbf{w}\|^2 = (\mathbf{v}, t\,\mathbf{w})^2 = (\mathbf{v},\mathbf{v}) + (t\mathbf{w}, t\mathbf{w}) + 2t\,(\mathbf{v},\mathbf{w}) = r(t)$$

$$0 \leq \|\mathbf{v}\|^2 + t^2\,\|\mathbf{w}\|^2 + 2t(\mathbf{v},\mathbf{w})$$

Differentiating the right side with respect to t and equating to zero to find the value of t that gives the minimum value of $r(t)$ being a quadratic function increasing upward, since $p(t) \geq 0$,

$$0 = 2t \parallel \mathbf{w} \parallel^2 + 2(\mathbf{v}, \mathbf{w})$$

$$t = -\frac{(\mathbf{v}, \mathbf{w})}{\parallel \mathbf{w} \parallel^2}$$

Hence,

$$0 \leq \parallel \mathbf{v} \parallel^2 + \left(-\frac{(\mathbf{v}, \mathbf{w})}{\parallel \mathbf{w} \parallel^2} \right)^2 \parallel \mathbf{w} \parallel^2 + 2\left(-\frac{(\mathbf{v}, \mathbf{w})}{\parallel \mathbf{w} \parallel^2} \right)(\mathbf{v}, \mathbf{w})$$

$$0 \leq \parallel \mathbf{v} \parallel^2 + \frac{(\mathbf{v}, \mathbf{w})^2}{\parallel \mathbf{w} \parallel^2} - 2\frac{(\mathbf{v}, \mathbf{w})^2}{\parallel \mathbf{w} \parallel^2} = \parallel \mathbf{v} \parallel^2 - \frac{(\mathbf{v}, \mathbf{w})^2}{\parallel \mathbf{w} \parallel^2}$$

Therefore,

$$(\mathbf{v}, \mathbf{w})^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

Taking the positive square root of both sides,

$$(\mathbf{v}, \mathbf{w}) \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

Note that we did not have to make recourse to the exact definition of the inner product. As such, every inner product satisfies the Cauchy-Schwarz inequality because, as you can see, we only made use of the definition of the inner product $(\mathbf{v}, \mathbf{w})$.

- Given the vectors **a** and **b** in 3-dimensions, i.e., V_3, we define the inner product as

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \mathbf{b} \qquad 4.7$$

where $\mathbf{a}^T$ is the transpose of the line matrix representing **a**. Find the inner product of **a** and **b**, where **a** = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and **b** = $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

- ❑ $(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 3$

- Show that $(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v_i w_i$ is an inner product on $\Re^n$.

- ❑ (i) $(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^{n} v_i v_i = \sum_{i=1}^{n} v_i^2 \geq 0$.

  (ii) $(\mathbf{v}, \mathbf{v})$ is indeed 0 if and only if $\mathbf{v} = \mathbf{0}$, the zero vector.

  (ii) $(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} v_i w_i = \sum_{i=1}^{n} w_i v_i = (\mathbf{w}, \mathbf{v})$

(iii) $(c\mathbf{v},\mathbf{w}) = \sum_{i=1}^{n} c^* v_i w_i = c^* \sum_{i=1}^{n} v_i w_i = c^*(\mathbf{v},\mathbf{w})$

$$(\mathbf{v},c\mathbf{w}) = \sum_{i=1}^{n} v_i (cw_i) = c\sum_{i=1}^{n} v_i w_i = c(\mathbf{v},\mathbf{w})$$

(iv) $(\mathbf{v},\mathbf{w}+\mathbf{z}) = \sum_{i=1}^{n} v_i (\mathbf{w}+\mathbf{z})_i = \sum_{i=1}^{n} v_i (w_i + z_i) = \sum_{i=1}^{n} v_i w_i + \sum_{i=1}^{n} v_i z_i = (\mathbf{v},\mathbf{w}) + (\mathbf{v},\mathbf{z})$

Hence, $(\mathbf{v},\mathbf{w}) = \sum_{i=1}^{n} v_i w_i$ is an inner product on $\Re^n$.

- In the space of $m \times n$ matrices, the inner product of **A** and $\mathbf{B} \in M_{mn}$ is defined as

$(\mathbf{A},\mathbf{B}) = Tr(\mathbf{A}^{+}\mathbf{B})$ 4.8

where $\overline{\mathbf{A}^T}$, the complex conjugate of the transpose of **A**. (Tr (**P**) is the trace of the matrix **P**, the sum of the diagonal elements of **P**.) Let $\mathbf{A} = \begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}$. Find (**A**,**B**).

❑

$$\mathbf{A}^T = \begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$$

$$\overline{\mathbf{A}^T} = \begin{bmatrix} -i & 1 \\ 0 & 1 \end{bmatrix}$$

$$\overline{\mathbf{A}^T}\mathbf{B} = \begin{bmatrix} -i & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Tr(\mathbf{A}^{+}\mathbf{B}) = 1 + 0 = 1$$

- Find the inner product of $\mathbf{A}$, $\mathbf{B} \in M_{mn}$ if $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$.

❑

$$\mathbf{A}^T\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -5 & 3 \\ 2 & 8 & 5 \\ 0 & 2 & 3 \end{bmatrix}$$

Hence,

$$(\mathbf{A},\mathbf{B}) = Tr\begin{bmatrix} -3 & -5 & 3 \\ 2 & 8 & 5 \\ 0 & 2 & 3 \end{bmatrix} = 8$$

In the case of the space of square integrable complex valued functions, F_s, over the interval (a,b), i.e., $f(x) \in F_s$ implies that $\int_a^b |f(x)|^2 dx < \infty$, we define the inner product on this space as

$$(f,g) = \int_a^b f^*(x)g(x)dx \qquad 4.9$$

where $f^*(x)$ is the complex conjugate of $f(x)$.

- Given that $f(x) = 2x + i$ and $g(x) = 5x - 2ix$ both belong to the vector space of real valued functions of x, $0 \le x \le 2$,
  (i) Normalise $f(x)$.
  (ii) Find the inner product of the functions.

❑

(i) $\int_0^2 Af^*(x)Af(x)dx = A^2\int_0^2 (2x-i)(2x+i)dx = A^2\int_0^2 (4x^2+1)dx$

$$= A^2\left(\frac{4}{3}x^3 + x\right)\Big|_0^2 = A^2\left(\frac{4}{3}(2)^3 + 2\right) = A^2\left(\frac{32}{3} + 2\right) = \frac{38}{3}A^2 = 1$$

Hence, the normalisation constant is $A = \sqrt{\frac{3}{38}}$

Therefore, the normalised function is, $\sqrt{\frac{3}{38}}(2x+i)$.

(ii) $\int_0^2 f^*(x)g(x) = \int_0^2 (2x+i)(5x-2ix) = \int_0^2 (10x^2 - 4ix + 5ix + 2i^2x)dx$

$$= \int_0^2 (10x^2 + ix - 2x)dx = \frac{10}{3}x^3 + i\frac{x^2}{2} - x^2\Big|_0^2$$

$$= \frac{10}{3}(2^3) + i\frac{2^3}{2} - 2^2 = \frac{68}{3} + 4i$$

4.3 Norm

Let X be a vector space over K, real or complex number field. A real valued function $\|\cdot\|$ on X is a norm on X (i.e., $\|\cdot\| : X \to R$) if and only if the following conditions are satisfied:

(i) $\|\mathbf{x}\| \ge 0$ 4.10

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ 4.11

(iii) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| \quad \forall\, \mathbf{x},\ \mathbf{y} \in X$ (Triangle inequality) 4.12

(iv) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\| \quad \forall\, \mathbf{x} \in X$ and $\alpha \in C$ (Absolute homogeneity) 4.13

The norm of a vector is its "distance" from the origin.

$\|\mathbf{x}\|$ is called the norm of **x**.

There are many ways of defining a norm on a vector space. In the case where $X = R$, the real number line, the **absolute-value norm** is the absolute value, $|x|$. This is the distance from the origin ($x = 0$). The **Euclidean norm** on the n-dimensional Euclidean space is $\|\mathbf{x}\| = \sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}$. More generally, the $\mathbf{L^p}$ **norm** is defined on a class of spaces called L^p spaces, and is defined as $\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$. You can then see that the Euclidean norm is the L^2-space and the L^2 norm is $\|\mathbf{x}\|_2 = \left(|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2\right)^{1/2}$. The n-dimensional complex space has a norm defined as $\|\mathbf{z}\| = \sqrt{|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \ldots + z_n\bar{z}_n}$. The **max norm** on a finite-dimensional space is defined as $\|\mathbf{x}\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$, which is the largest absolute value largest component of **x**, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

- Prove that the triangle inequality holds for any norm.

- $\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x},\mathbf{y}) \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2|(\mathbf{x},\mathbf{y})|$

  But from Cauchy-Schwarz inequality, $(\mathbf{x},\mathbf{y}) \le \|\mathbf{x}\|\|\mathbf{y}\|$. Hence,

  $\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x},\mathbf{y}) \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2|(\mathbf{x},\mathbf{y})|$

  $$\le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x},\mathbf{y}) \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|$$
  $$= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$

  Taking the positive square root of both sides as a norm must be non-negative,

  $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$

If the norm of **v** in the vector space V is unity, such a vector is said to be normalised. In any case, even if a vector is not normalised, we can normalise it by dividing by its norm.

In Physics, the most commonly used norm is the one induced by the inner product and this is the one we shall be concerning ourselves with. The Euclidean norm is an example of such a norm. In that case, $\|\mathbf{A}\| = \sqrt{(\mathbf{A},\mathbf{A})}$, the square root of the inner product of **A** with itself. We shall illustrate with three cases:

Case 1: Given the vector **a** in V_3, the norm of **a** is

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a},\mathbf{a})} \qquad 4.14$$

▪ If **a** = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, find its norm, and hence normalise it.

❑

$$(\mathbf{a},\mathbf{a}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a},\mathbf{a})} = \sqrt{2}$$

We see that **a** is not normalised.

However, $\mathbf{c} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is normalised. (Check)

Case 2: The space of $m \times n$ matrices:

Given the $m \times n$ matrix **A**, then the norm of **A** is defined as

$$\|\mathbf{A}\| = \sqrt{Tr(\mathbf{A},\mathbf{A})} \qquad 4.15$$

▪ Normalise **A** = $\begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix}$.

❑

$$\mathbf{A}^T = \begin{bmatrix} i & 1 \\ 0 & 1 \end{bmatrix}$$

$$\overline{\mathbf{A}^T} = \begin{bmatrix} -i & 1 \\ 0 & 1 \end{bmatrix}$$

$$\overline{\mathbf{A}^T}\mathbf{A} = \begin{bmatrix} -i & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Tr(\mathbf{A}^+\mathbf{A}) = 2 + 1 = 3$$

Therefore, $\|\mathbf{A}\| = \sqrt{3}$.

A is not normalised, but **C** = $\dfrac{\mathbf{A}}{\|\mathbf{A}\|} = \dfrac{1}{\sqrt{3}}\begin{bmatrix} i & 0 \\ 1 & 1 \end{bmatrix}$ is normalised.

Case 3: The space of square integrable complex valued functions, F_s, over the interval (a,b). Let $f(x) \in F_s$, then we define

$$\|f\| = \sqrt{(f, f)} \qquad 4.16$$

where $(f, f) = \int_a^b |f(x)|^2 \, dx$

It is now obvious that we have to deal with a square integrable set of functions:

f might not be normalised, but $h = \dfrac{f}{\sqrt{(f, f)}}$ is normalised. f is said to be normalisable. If for instance the function is not square integrable, that would mean that we could be dividing f by an infinite number. We say such a function is not normalisable.

▪ Find the norm of $g(x) = 5x - 2ix$. Hence, normalise the function.

❑ The norm of $g(x)$ is $\|g(x)\| = \sqrt{\int_0^2 g^*(x)g(x)dx} = \sqrt{\int_0^2 (5x - 2ix)(5x + 2ix)dx}$

$$= \int_0^2 29x^2 dx = \frac{29}{3}x^3\Big|_0^2 = \frac{29}{3}(2^3) = \frac{232}{3}$$

The normalised function is

$$\frac{g(x)}{\|g(x)\|} = \frac{5x - 2ix}{\sqrt{232/3}} = \sqrt{\frac{3}{232}}(5x - 2ix)$$

Summary of Study Session 4

In Study Session 4, you learnt the following:

1. How to calculate the inner product of two vectors in a given vector space.
2. How to find the norm of a vector in a given vector space.
3. How to normalise a given vector in a given vector space.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.
4. Hefferson, J. (2012). Linear Algebra, http://joshua.smcvt.edu/linearalgebra/book.pdf
5. Hefferson, J. (2012). Answers to Exercises, http://joshua.smcvt.edu/linearalgebra/jhanswer.pdf

Self-Assessment Questions (SAQs) for Study Session 4

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 4.1 (tests Learning Outcome 4.1)

State each of the following norms:

(i) L^p-norm (ii) max norm (iii) absolute-value norm

SAQ 4.2 (tests Learning Outcome 4.2)

State the properties of an inner product.

SAQ 4.3 (tests Learning Outcome 4.3)

Find the inner product of the following vectors:

(i) $\begin{pmatrix} i \\ -2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

(ii) ix^2+2 and $2x-3i \quad 0 \le x \le 2$.

(iii) $\mathbf{A}$, $\mathbf{B} \in M_{mn}$ if $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$.

SAQ 4.4 (tests Learning Outcome 4.4)

State the properties of a norm.

SAQ 4.5 (tests Learning Outcomes 4.3 and 4.5)

Given that $\mathbf{u} = \begin{bmatrix} 2+3i & -2 \\ i & -3+i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 & -4-i \\ -i & 3 \end{bmatrix}$ find

(i) $(\mathbf{u}, \mathbf{u})$, and hence, the norm of $\mathbf{u}$.

(ii) $(\mathbf{u}, \mathbf{v})$

SAQ 4.6 (tests Learning Outcome 4.6)

Normalize the wave function $\psi = A\sin\dfrac{2\pi nx}{L}$, where $0 \le x \le L$ and n can take positive integral values.

SAQ 4.7 (tests Learning Outcome 4.5)

Find the norm of each of the following vectors:

(i) $\begin{pmatrix} 2i \\ -1 \\ 3 \end{pmatrix}$ (ii) $ix^2 + 2,\ 0 \le x \le 1$ (iii) $\mathbf{D} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

SAQ 4.8 (tests Learning Outcome 4.6)

Normalise each vector in the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

SAQ 4.9 (tests Learning Outcome 4.7)

Prove that the function defined on $\Re^2$ by $(\mathbf{x}, \mathbf{y}) = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$ is an inner product.

SAQ 4.10 (tests Learning Outcome 4.8)

Define $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$ with the inner product defined in SAQ 4.9. Show that $\|\mathbf{x}\|$ is a norm.

Solutions to SAQs

SAQ 4.1 (tests Learning Outcome 4.1)

(i) The $\mathbf{L^p}$ **norm** is defined on a class of spaces called L^p spaces, and is defined as $\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$.

(ii) The **max norm** on a finite-dimensional space is defined as $\|\mathbf{x}\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right)$, which is the largest absolute value largest component of **x**, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(iii) In the case where $X = R$, the real number line, the **absolute-value norm** is the absolute value, $|x|$. This is the distance from the origin (x = 0).

SAQ 4.2

The inner product of $\mathbf{v}$, $\mathbf{w} \in V$,written as (**v**,**w**), has the following properties:

(i) (**v**,**v**) ≥ 0 4.1

(ii) (**v**,**v**) = 0 if and only if **v** = 0 4.2

(iii) (**v**,**w**) = (**w**,**v**) (Symmetry) 4.3

(iv) $(c\mathbf{v}, \mathbf{w}) = c^*(\mathbf{v}, \mathbf{w})$; $(\mathbf{v}, c\mathbf{w}) = c(\mathbf{v}, \mathbf{w})$ 4.4

(v) $(\mathbf{v}, \mathbf{w} + \mathbf{z}) = (\mathbf{v}, \mathbf{w}) + (\mathbf{v}, \mathbf{z})$ 4.5

(vi) $(\mathbf{v}, \mathbf{w}) \leq \|\mathbf{v}\|\|\mathbf{w}\|$ Cauchy-Schwarz inequality 4.6

SAQ 4.3

(i) $\begin{pmatrix} i \\ -2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} -i & -2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = -2i + 2 + 6 = 8 - 2i$$

(ii) $ix^2 + 2$ and $2x - 3i$ $0 \leq x \leq 2$.

$$\int_0^2 (ix^2 + 2)^* \times (2x - 3i)dx = \int_0^2 (-ix^2 + 2)^* \times (2x - 3i)dx$$

$$= \int_0^2 (2ix^3 - 3x^2 + 4x - 6i)dx$$

$$= \left[i\frac{x^4}{2} - x^3 + 2x^2 - 6ix \right]_0^2$$

$$= 8i - 8 + 8 - 12i$$

$$-4i$$

(iii) $\mathbf{A}$, $\mathbf{B} \in M_{mn}$ if $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$.

$$(A,B) = Tr(A^{+}B) = Tr\left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 2 \\ 1 & 3 & 1 \end{bmatrix}\right) = Tr\begin{bmatrix} -2-1 & -2-3 & 4-1 \\ -1+3 & -1+9 & 2+3 \\ -1+1 & -1+3 & 2+1 \end{bmatrix}$$

$$= Tr\begin{bmatrix} -3 & -5 & 3 \\ 2 & 8 & 5 \\ 0 & 2 & 3 \end{bmatrix} = 8$$

SAQ 4.4

A real valued function $\|\cdot\|$ on X is a norm on X (i.e., $\|\cdot\| : X \to R$) if and only if the following conditions are satisfied:

(i) $\|\mathbf{x}\| \geq 0$ 4.10

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ 4.11

(iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ $\forall \mathbf{x}, \mathbf{y} \in X$ (Triangle inequality) 4.12

(iv) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ $\forall \mathbf{x} \in X$ and $\alpha \in C$ (Absolute homogeneity) 4.13

SAQ 4.5

(i) $$(\mathbf{u},\mathbf{u}) = \mathbf{u}^{+}\mathbf{u} = \mathrm{Tr}\left(\begin{bmatrix} 2+3i & -2 \\ i & -3+i \end{bmatrix}^{+}\right)\begin{bmatrix} 2+3i & -2 \\ i & -3+i \end{bmatrix}^{+}\Bigg)$$

$$= \mathrm{Tr}\left(\begin{bmatrix} 2-3i & -i \\ -2 & -3-i \end{bmatrix}\begin{bmatrix} 2+3i & -2 \\ i & -3+i \end{bmatrix}^{+}\right)$$

$$= \mathrm{Tr}\begin{bmatrix} (2-3i)\times(2+3i)+(-i)\times(i) & (2-3i)\times(-2)+(-i)\times(-3+i) \\ (-2)\times(2+3i)+(-3-i)\times(i) & (-2)\times(-2)+(-3-i)\times(-3+i) \end{bmatrix}$$

$$= \mathrm{Tr}\begin{bmatrix} 4+9+1 & -4-6i+3i+1 \\ -4-6i-3i+1 & 4+9+1 \end{bmatrix}$$

$$= \mathrm{Tr}\begin{bmatrix} 14 & -3-3i \\ -3-9i & 14 \end{bmatrix}$$

$= 14 + 14$

$= 28$

The norm of $\mathbf{u}$ is $\sqrt{(\mathbf{u},\mathbf{u})} = \sqrt{28}$

(ii) $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^{+}\mathbf{v} = \begin{bmatrix} 2+3i & -2 \\ i & -3+i \end{bmatrix}^{+} \begin{bmatrix} 3 & -4-i \\ -i & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2-3i & -i \\ -2 & -3-i \end{bmatrix} \begin{bmatrix} 3 & -4-i \\ -i & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (2-3i)\times 3+(-i)\times(-i) & (2-3i)\times(-4-i)+(-i)\times 3 \\ (-2)\times 3+(-3-i)\times(-i) & (-2)\times(-4-i)+(-3-i)\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-9i-1 & -8+10i-3i \\ -6+3i-1 & 8+2i-9-3i \end{bmatrix}$$

$$= \mathrm{Tr}\begin{bmatrix} 5-9i & -8+7i \\ -7+3i & -1-i \end{bmatrix}$$

$$= (5-9i)+(-1-i)$$

$$= 4-10i$$

SAQ 4.6

$$\psi = A\sin\frac{2\pi n x}{L}, \text{ where } 0 \le x \le L$$

$$(\psi,\psi) = A^2\int_0^L \sin^2\frac{2\pi n x}{L}\,dx$$

$$= A^2\int_0^L \frac{1}{2}\left[1-\cos\frac{2\pi n x}{L}\right]dx$$

$$= A^2\frac{1}{2}x\Big|_0^L = A^2\frac{L}{2} = 1$$

Therefore, $A = \sqrt{\dfrac{2}{L}}$

SAQ 4.7

(i) $\sqrt{\begin{pmatrix} -2i & -1 & 3 \end{pmatrix}\begin{pmatrix} 2i \\ -1 \\ 3 \end{pmatrix}} = \sqrt{4+1+9} = \sqrt{14}$

(ii) $\int_0^1 (ix^2+2)^*(ix^2+2)dx = \int_0^1 (-ix^2+2)(ix^2+2)dx$

$$= \int_0^1 (4+x^4)dx = \left[4x+\frac{x^5}{5}\right]_0^1 = 4+\frac{1}{5} = \frac{21}{5}$$

Norm = $\sqrt{\dfrac{21}{5}}$

SAQ 4.8

Norm of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is $\sqrt{1+4+9} = \sqrt{14}$

The normalised vector is $\dfrac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Similarly, $\dfrac{1}{\sqrt{20}}\begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$ and $\dfrac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are normalised.

SAQ 4.9

(i) $(\mathbf{x},\mathbf{x}) = 2x_1x_1 - x_1x_2 - x_2x_1 + x_2x_2 = 2x_1{}^2 + x_2{}^2 - 2x_1x_2$

$$= x_1{}^2 + (x_1{}^2 + x_2{}^2 - 2x_1x_2)$$

$$= x_1{}^2 + (x_1 - x_2)^2 \geq 0$$

since the sum of two squares (of real numbers) cannot be negative.

(ii) $(\mathbf{x},\mathbf{x}) = 0$, if and only if **x** = 0, and this is true only if $x_1 = x_2 = 0$. This is true of the expression $x_1{}^2 + (x_1 - x_2)^2$

(iii) $(\mathbf{x},\mathbf{y}) = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$

(iv) $(c\mathbf{x},\mathbf{y}) = 2c*x_1y_1 - c*x_1y_2 - c*x_2y_1 + c*x_2y_2 = c*(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2)$

$(\mathbf{x},c\mathbf{y}) = 2cx_1y_1 - cx_1y_2 - cx_2y_1 + cx_2y_2 = c(2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2)$

(v) $(\mathbf{x},\mathbf{y}+\mathbf{z}) = 2x_1(\mathbf{y}+\mathbf{z})_1 - x_1(\mathbf{y}+\mathbf{z})_2 - x_2(\mathbf{y}+\mathbf{z})_1 + x_2(\mathbf{y}+\mathbf{z})_2$

$$= 2x_1(y_1+z_1) - x_1(y_2+z_2) - x_2(y_1+z_1) + x_2(y_2+z_2)$$

$$= (2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2) + (2x_1z_1 - x_1z_2 - x_2z_1 + x_2z_2)$$

$$= (\mathbf{x},\mathbf{y}) + (\mathbf{x},\mathbf{z})$$

SAQ 4.10

(i) $\|\mathbf{x}\| = \sqrt{x_1{}^2 + (x_1 - x_2)^2} \geq 0$

(ii) $\|\mathbf{x}\| = 0$, if and only if **x** = 0, and this is true only if $x_1 = x_2 = 0$.

(iii) $\|\alpha\mathbf{x}\| = \sqrt{(\alpha x_1)^2 + (\alpha x_1 - \alpha x_2)^2} = \sqrt{\alpha^2 x_1{}^2 + \alpha^2(x_1 - x_2)^2} = |\alpha|\sqrt{x_1{}^2 + (x_1 - x_2)^2}$

The last equality being due to positivity of the norm.

Study Session 5 Orthogonal Matrices; Change of Basis

Introduction

In this study session, you shall learn about orthogonal matrices, because as you shall soon see among some other desirable properties, the columns of an orthogonal matrix form an orthonormal set. So also do the rows. The orthogonal matrix is so important in Physics because it gives the rotation matrix of a coordinate system about one of the axes. These special matrices also have some interesting properties that will really pique your interest. Rotation of a coordinate system ultimately results in a change of basis, although this is not the only way you can achieve a change of basis. Generally therefore, we shall be looking at the matrix of transformation from one basis to another. This is so important because the basis for a particular vector space is not unique.

Learning Outcomes for Study Session 5

By the time you are through with this study session, you should be able to:

5.1 Define and use correctly all the key words printed in **bold**. (SAQ 5.1).
5.2 Determine whether a matrix is an orthogonal matrix or not (SAQ 5.2)
5.3 Determine if an orthogonal matrix is a proper or an improper orthogonal matrix (SAQ 5.3).
5.4 Find the angle of rotation of a matrix (SAQ 5.4).
5.5 Write the matrix of transformation between two bases for a vector space (SAQ 5.5 – 5.7).
5.6 Find the components of a vector in the coordinate system you have changed to (SAQ 5.5 – 5.7).

5.1 Orthogonal Matrices

A matrix Q such that $(Q\mathbf{a},\cdot Q\mathbf{b}) = (\mathbf{a},\mathbf{b}) \quad \forall \quad \mathbf{a},\mathbf{b} \in E$, the set of n-dimensional Euclidean vectors is called an **orthogonal matrix**. In other words, an orthogonal matrix preserves the inner product of two Euclidean vectors. You should find this interesting: it means the inner product remains the same in the two different coordinate systems, if one of them is obtained by rotating the other one about one axis. We will soon illustrate what we said earlier, that the rotation matrix is an orthogonal matrix.

Since $(Q\mathbf{a}, Q\mathbf{b}) = (\mathbf{b}, \{Q^T(Q\mathbf{a})\} = (\mathbf{b}, (Q^TQ)\mathbf{a}\}$, a necessary and sufficient condition for Q to be orthogonal is

$$QQ^T = I \qquad 5.1$$

or equivalently, by multiplying on the left by Q^{-1},

$$Q^T = Q^{-1} \qquad 5.2$$

Note that

$$\det(QQ^T) = \det(Q)\det(Q^T) \quad \text{since } \det(AB) = (\det A)(\det B) \qquad 5.3$$

$$= \det(Q)\det(Q) \quad \text{since } \det A^T = \det A \qquad 5.4$$

$$= (\det(Q))^2 = \det(I) = 1$$

Hence,

$$\det(Q) = \pm 1 \tag{5.5}$$

Q is said to be a **proper orthogonal matrix** if $\det(Q) = 1$ and an **improper orthogonal matrix** if $\det(Q) = -1$.

If $\det(Q) = 1$, then

$$\det(Q - I) = \det(Q - I)\det(Q^T) \quad (\text{since } \det(Q) = 1) \tag{5.6}$$
$$= \det(QQ^T - Q^T) \tag{5.7}$$
$$= \det(I - Q^T) \qquad (\text{since } QQ^T = I) \tag{5.9}$$
$$= \det(I^T - Q^{TT}) \qquad (\det(A^T) = \det A \tag{5.10}$$
$$= +\det(I - Q) = -\det(Q - I) \quad (\text{since } \det(A - B) = -\det(B - A)) \tag{5.11}$$
$$= 0 \qquad (\text{since } x = -x \text{ implies } x = 0)$$

Therefore, 1 is an eigenvalue so that there exists $\mathbf{e}_3$ such that $Q\mathbf{e}_3 = 1\mathbf{e}_3 = \mathbf{e}_3$.

Let us choose $\mathbf{e}_1$, $\mathbf{e}_2$ to be orthonormal to $\mathbf{e}_3$. $\mathbf{e}_3$ could be the unit vector **k**, then $\mathbf{e}_1$ and $\mathbf{e}_2$ could be **i** and **j**, respectively. In terms of this basis, we have

$$Q = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.12}$$

$$Q^T = \begin{bmatrix} a & c & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.13}$$

$$QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd & 0 \\ ca + bd & c^2 + d^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.14}$$

$$a^2 + b^2 = 1 = c^2 + d^2 \tag{5.15}$$
$$ac + bd = 0 = ca + bd \tag{5.16}$$

Also,

$$\det(Q) = 1 = ad - bc \tag{5.17}$$

From equation (ii),

$$b = -\frac{ac}{d} \tag{5.18}$$

Putting this in (iii) gives

$$ad + \frac{ac^2}{d} = 1 \tag{5.19}$$

$\Rightarrow a(c^2 + d^2) = d \Rightarrow a = d$. Use $a = d$ in (5.18) to get $c = -b$.

Therefore,

$$Q = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 5.20$$

with

$$a^2 + b^2 = 1 \qquad 5.21$$

Thus, $\exists\, \theta$, $\ni$ $a = \cos\theta$, $b = \sin\theta$,
so

$$Q = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 5.22$$

Interpretation:

This corresponds to a rotation about an axis perpendicular to $\mathbf{e}_3$ or **k**.

Improper orthogonal matrices have determinant -1, and are equivalent to a reflection combined with a proper rotation.

Note that only proper orthogonal matrices represent rotations. In addition, the direction and axis of rotation is not as simple in all cases as in the example. We state without proof, the Rotation Matrix Theorem.

5.1.1 Rotation Matrix Theorem

i. Rotation matrices are orthogonal matrices with unit determinant, that is, with determinant equal to +1.

ii. The rotation matrix A that is not the identity matrix, i.e., a non-trivial rotational matrix has eigenvalues 1, $e^{i\theta}$ and $e^{-i\theta}$.

iii. The eigenvector corresponding to the eigenvalue 1 is a line. This is the eigenvector that determines the axis of rotation as it is invariant with respect to the rotation, hence, $A\mathbf{e} = 1\mathbf{e} = \mathbf{e}$.

iv. The axis of rotation is the line in (3), and is determined from $(\mathbf{A} - \mathbf{I})\mathbf{u} = \mathbf{0}$, since this is the matrix equation for eigenvalue 1, and the angle of rotation is the angle measured anticlockwise when $\mathbf{v} \times \mathbf{A}\mathbf{v}$ points in your direction, where **v** is any vector in the plane perpendicular to the axis of rotation.

v. The angle of rotation can be obtained from $\cos\theta = \frac{1}{2}\left[tr(A) - 1\right]$, up to sign.

❑ Show that rule (iv) trivially gives us the same angle we deduced in the matrix in equation 5.22.

❑

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{2}\left[tr(A) - 1\right] = \frac{1}{2}\left[(2\cos\theta + 1) - 1\right] = \cos\theta$$

meaning that the angle of rotation is the same as θ in the matrix.

▪ Confirm that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a rotation matrix. Also show that irrespective of the angle θ, the rotation angle is π.

❑

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of the matrix is $-1 \times (-\cos^2\theta - \sin^2\theta) = 1$ (using the last row because of the ease it offers us, having only one non-zero element). Thus, the matrix is orthogonal and has determinant 1. It is a proper orthogonal matrix, a rotation matrix.

To find the axis of rotation, we solve

$$(A - I)\mathbf{u} = \left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 1-\cos\theta & -\sin\theta & 0 \\ -\sin\theta & 1+\cos\theta & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The axis is given by,

$u_1(1 - \cos\theta) - u_2 \sin\theta = 0$ or $-u_1 \sin\theta + u_2(1 + \cos\theta) = 0$

The axis is $(\sin\theta, 1 - \cos\theta, 0)$ or $(1 + \cos\theta, \sin\theta, 0)$. Notice that this is not the z-axis. This is partly because $A_{33} \neq 1$.

The angle of rotation is $\frac{1}{2}\left[tr(A)-1\right]=\frac{1}{2}\left[(\cos\theta-\cos\theta-1)-1\right]=-1$. Hence, θ is not the angle of rotation. Irrespective of the angle θ, the angle of rotation is $\cos^{-1}(-1)=\pi$.

▪ The matrix $\frac{1}{25}\begin{bmatrix} 9 & 12 & -20 \\ -20 & 15 & 0 \\ 12 & 16 & 15 \end{bmatrix}$ is a proper orthogonal matrix. Find the axis and angle of rotation.

❑ $A=\frac{1}{25}\begin{bmatrix} 9 & 12 & -20 \\ -20 & 15 & 0 \\ 12 & 16 & 15 \end{bmatrix}$

The axis is given by

$$(A-I)\mathbf{u}=\begin{bmatrix} \frac{9}{25}-1 & \frac{12}{25} & -\frac{20}{25} \\ -\frac{20}{25} & \frac{15}{25}-1 & 0 \\ \frac{12}{25} & \frac{16}{25} & \frac{15}{25}-1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-I)\mathbf{u}=\begin{bmatrix} -\frac{16}{25} & \frac{12}{25} & -\frac{20}{25} \\ -\frac{20}{25} & -\frac{10}{25} & 0 \\ \frac{12}{25} & \frac{16}{25} & \frac{10}{25} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-16u_1+12u_2-20u_3=0$$
$$-20u_1-10u_2=0$$
$$12u_1+16u_2+10u_3=0$$

$$-4u_1+3u_2-5u_3=0$$
$$2u_1+u_2=0$$
$$6u_1+8u_2+5u_3=0$$

or

$$u_2=-2u_1$$
$$-4u_1+3(-2u_1)-5u_3=0$$
$$-5u_3=10u_1$$

$u_3 = -2u_1$

Hence, the axis of rotation is $(1,-2,-2)$.

The angle of rotation is given by $\cos\theta = \frac{1}{2}\left[tr(A)-1\right] = \frac{1}{2}\left(\frac{39}{25}-1\right) = \frac{1}{2}\frac{14}{25} = \frac{7}{25}$.

$\theta = 73.74^0$

Motivation

If Q is an orthogonal matrix, then, its rows are an orthonormal set. In addition, its columns are also an orthonormal set.

❑ Show that the following is an orthogonal matrix.

$$Q = \begin{pmatrix} \sqrt{2}/2 & \sqrt{6}/6 & -\sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \end{pmatrix}$$

❑

$$Q^T = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{6}/6 & \sqrt{6}/3 & -\sqrt{6}/6 \\ -\sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{pmatrix}$$

$$QQ^T = \begin{pmatrix} \sqrt{2}/2 & \sqrt{6}/6 & -\sqrt{3}/3 \\ 0 & \sqrt{6}/3 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & \sqrt{3}/3 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ \sqrt{6}/6 & \sqrt{6}/3 & -\sqrt{6}/6 \\ -\sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, Q is an orthogonal matrix.

You can verify that the columns are orthonormal. You can also show that the rows are orthonormal.

5.2 Change of Basis

We are quite familiar with the usual basis in the Euclidean plane: **i** and **j**. We can write these as

$$\begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix}$$

We can also show that,

$$\begin{pmatrix}1\\1\end{pmatrix} \text{ and } \begin{pmatrix}1\\-1\end{pmatrix}$$

is also a basis for the Euclidean plane. These vectors are, respectively, $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}$. We see that the basis for a vector space is not unique. We can easily construct a linear map (matrix) that takes a basis vector in one basis to another.

Let us consider R^n as a vector space.

Let $\{\mathbf{u}_i\}_{i=1}^{n}$ be a basis in the vector space. We can write any vector **a** in the vector space as $\mathbf{a} = \begin{pmatrix}a_1\\a_2\\.\\.\\a_n\end{pmatrix}$. Then, we can write

$$\begin{pmatrix}a_1\\a_2\\.\\.\\a_n\end{pmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = c_1\begin{pmatrix}u_{11}\\u_{12}\\.\\.\\u_{1n}\end{pmatrix} + c_2\begin{pmatrix}u_{21}\\u_{22}\\.\\.\\u_{2n}\end{pmatrix} + \ldots + c_n\begin{pmatrix}u_{n1}\\u_{n2}\\.\\.\\u_{nn}\end{pmatrix} \qquad 5.23$$

It follows that

$$\begin{aligned}
a_1 &= c_1u_{11} + c_2u_{21} + \ldots + c_nu_{n1}\\
a_2 &= c_1u_{12} + c_2u_{22} + \ldots + c_nu_{n2}\\
&.\\
&.\\
&.\\
a_n &= c_1u_{1n} + c_2u_{2n} + \ldots + c_nu_{nn}
\end{aligned} \qquad 5.24$$

We can write this compactly as

$$\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{21} & . & . & u_{n1} \\ u_{12} & u_{22} & . & . & u_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{1n} & u_{2n} & . & . & u_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} \tag{5.25}$$

or
$$\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = B \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} \tag{5.26}$$

where B is a matrix formed by arranging the vectors (in columns) $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ in order.

It follows immediately that we can write (by multiplying both sides by B^{-1}):

$$\begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} = B^{-1} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \tag{5.27}$$

But we might as well have written **a** rightaway in another basis $\{\mathbf{v}_j\}_{j=1}^{n}$, as

$$\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_n\mathbf{v}_n = \begin{pmatrix} v_{11} & v_{21} & . & . & v_{n1} \\ v_{12} & v_{22} & . & . & v_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ v_{1n} & v_{2n} & . & . & v_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ . \\ . \\ d_n \end{pmatrix} = D \begin{pmatrix} d_1 \\ d_2 \\ . \\ . \\ d_n \end{pmatrix} \tag{5.28}$$

where

$$D = \begin{pmatrix} v_{11} & v_{21} & . & . & v_{n1} \\ v_{12} & v_{22} & . & . & v_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ v_{1n} & v_{2n} & . & . & v_{nn} \end{pmatrix} \tag{5.29}$$

Therefore, equation 5.26 gives

$$\begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} = B^{-1} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = B^{-1} D \begin{pmatrix} d_1 \\ d_2 \\ . \\ . \\ d_n \end{pmatrix} \qquad 5.30$$

Conversely,

$$\begin{pmatrix} d_1 \\ d_2 \\ . \\ . \\ d_n \end{pmatrix} = D^{-1} B \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} \qquad 5.31$$

- Given the basis {(2, 3), (1, 4)}, write the matrix that will transform this basis to the basis {(0, 2), (-1, 5)}? What are the coordinates of the vector $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ in the new basis?

❑

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix},\ D = \begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix},\ B^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix},\ D^{-1} = \frac{1}{2}\begin{pmatrix} 5 & 1 \\ -2 & 0 \end{pmatrix}$$

$$B^{-1}D = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & -9 \\ 4 & 13 \end{pmatrix}$$

The matrix of transformation is,

$$D^{-1}B = \frac{1}{2}\begin{pmatrix} 5 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 13 & 9 \\ -4 & -2 \end{pmatrix}$$

Applying this to the vector $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$,

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = D^{-1}B\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 13 & 9 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} -2 \\ 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 28 \\ -4 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix}$$

Check! Let us go back the way we came. We should recover our original coordinates in that basis. Now, we make use of the matrix $B^{-1}D$.

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^{-1}D\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & -9 \\ 4 & 13 \end{pmatrix}\begin{pmatrix} 14 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -10 \\ 30 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

We recovered the original coordinates in the first basis.

Summary of Study Session 5

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.
4. Hefferson, J. (2012). Linear Algebra, http://joshua.smcvt.edu/linearalgebra/book.pdf
5. Hefferson, J. (2012). Answers to Exercises, http://joshua.smcvt.edu/linearalgebra/jhanswer.pdf

Self-Assessment Questions for Study Session 5

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 5.1 (tests Learning Outcome 5.1)

What are orthogonal matrices, and why are they so important in the theory of vector spaces?

SAQ 5.2 (tests Learning Outcome 5.2)

Which of the following is(are) orthogonal matrices?

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (iii) $\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

SAQ 5.3 (tests Learning Outcome 5.3)

Check if any of the matrices in SAQ 4.1 is a proper orthogonal matrix.

SAQ 5.4 (tests Learning Outcome 5.4)

Find the axis and the angle of rotation of the matrix $A = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$.

SAQ 5.5 (tests Learning Outcomes 5.5 and 5.6)

Find the matrix of transformation between the bases $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$. Hence, express the vector $\begin{pmatrix}3\\4\end{pmatrix}$ in the two different bases.

SAQ 5.6 (tests Learning Outcomes 5.5 and 5.6)

Write the matrix of transformation between the basis S_a to basis S_b in R^3, the 3-dimensional Euclidean plane.

$$S_a = \left\{\begin{pmatrix}1\\0\\2\end{pmatrix}\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}0\\3\\5\end{pmatrix}\right\};\ S_b = \left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix},\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$$

SAQ 5.7 (tests Learning Outcomes 5.5 and 5.6)

Given the basis $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$, can we write the expression for a transformation to the basis $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}-2\\0\\4\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$. What would the vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$ be in the new basis?

Solutions to SAQs

SAQ 5.1

A matrix Q such that $(Q\mathbf{a}, Q\mathbf{b}) = (\mathbf{a}, \mathbf{b}) \;\; \forall \;\; \mathbf{a}, \mathbf{b} \in E$, the set of *n*-dimensional Euclidean vectors is called an **orthogonal matrix**. In other words, an orthogonal matrix preserves the inner product of two Euclidean vectors. As such, the determinant of an orthogonal matrix is either 1 (proper orthogonal matrix) or -1 (improper orthogonal matrix).

They are so important in the theory of vector spaces because their columns and rows are orthonormal vectors.

SAQ 5.2

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ The matrix is not an orthogonal matrix.

(ii) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The matrix is an orthogonal matrix.

(iii) $\dfrac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}\dfrac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} = \dfrac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The matrix is an orthogonal matrix.

(iv) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ The matrix is an orthogonal matrix.

SAQ 5.3

The matrices (ii) and (iii) are proper orthogonal matrices, since each has a determinant of unity. The matrix (i) is not even an orthogonal matrix (SAQ 4.1). Matrix (iv) is an orthogonal matrix, but its determinant is -1. Hence, it is not a proper orthogonal matrix, but an improper orthogonal matrix.

SAQ 5.4

- The axis is given by,

  $$(A - I)\mathbf{u} = \mathbf{0}$$

or

$$\left[\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}-\frac{1}{3}\begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}\right]\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\frac{1}{3}\begin{pmatrix}-4 & 2 & 2\\ 2 & -4 & 2\\ 2 & 2 & -4\end{pmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-4u_1+2u_2+2u_3=0$$

$$2u_1-4u_2+2u_3=0$$

$$2u_1+2u_2-4u_3=0$$

$$-2u_1+u_2+u_3=0 \qquad \text{(i)}$$

$$u_1-2u_2+u_3=0 \qquad \text{(ii)}$$

$$u_1+u_2-2u_3=0 \qquad \text{(iii)}$$

From (ii) and (iii), $-3u_2+3u_3=0$, or $u_2=u_3$

Putting this in (i) gives,

$$-2u_1+u_2+u_2, \text{ or } u_1=u_2$$

We conclude that $u_1=u_2=u_3$. Hence, the axis of rotation is $(1,1,1)$, or the line $x=y=z$.

The angle of rotation is given by,

$$\cos\theta=\frac{1}{2}(tr(A)-1)=\frac{1}{2}\left(-\frac{3}{3}-1\right)=\frac{1}{2}(-1-1)=-1$$

Hence,

$$\theta=180^0$$

SAQ 5.5

The matrix from basis S_U is $B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, and the matrix from basis S_1 is $D=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$,

$$B^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ D^{-1}=\frac{1}{-2}\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

The matrix of transformation from S_U to S_1 is $B^{-1}D=D=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

The matrix of transformation from S_1 to S_U is $D^{-1}B=D^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

So, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in S_U transforms to $D^{-1}B\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -2 \end{bmatrix}$ in S_1.

Crosscheck! Does this transform into $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ the other way?

$\begin{bmatrix} 7/2 \\ -2 \end{bmatrix}$ in S_1 transforms to $B^{-1}D\left(\frac{1}{2}\begin{bmatrix} 7 \\ -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in S_U.

SAQ 5.6

The matrix related to S_a is $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 5 \end{bmatrix}$, while the one related to S_b is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

We need to get the inverse of D, since we need $D^{-1}B$. The inverse of a matrix is the matrix of cofactors divided by the determinant. First, we evaluate the determinant of *D*.

Determinant of *D* is

$$1(1-1) - 2(-2-1) + 1(2+1) = 9$$

The inverse of *D* is the transpose of the matrix of cofactors divided by the determinant:

$$D^{-1} = \frac{1}{9}\begin{bmatrix} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & -2 & 1 \\ 3 & 1 & -5 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & -2 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

$$\frac{1}{9}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 \\ 3 & -2 & 1 \\ 3 & 1 & -5 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have got the inverse right, $DD^{-1} = I$.

The matrix of transformation from S_a to S_b is

$$D^{-1}B = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & -2 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 5 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 6 & 3 & 24 \\ 5 & 4 & -1 \\ -7 & 7 & -12 \end{bmatrix}$$

SAQ 5.7

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & 2 \\ 3 & 4 & 1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & 1 \end{pmatrix} \qquad D^{-1} = \frac{1}{4}\begin{pmatrix} 4 & -3 & 2 \\ -2 & 1 & 0 \\ -4 & 5 & -2 \end{pmatrix}$$

The transformation matrix is therefore,

$$D^{-1}B = \frac{1}{4}\begin{pmatrix} 4 & -3 & 2 \\ -2 & 1 & 0 \\ -4 & 5 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 & 1 & 0 \\ -2 & -1 & 0 \\ -6 & 1 & 4 \end{pmatrix}$$

We might need this just to confirm our calculations later:

$$B^{-1}D = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & 2 \\ 3 & 4 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 2 & 0 \\ -4 & -12 & 0 \\ 4 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -6 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

Now, what would the vector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ be in the new basis?

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = D^{-1}B\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 6 & 1 & 0 \\ -2 & -1 & 0 \\ -6 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

We could also transform back to the original basis. Obviously, we should get the vector we started with:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = B^{-1}D\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -6 & 0 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Study Session 6 Orthogonality and Orthonormality

Introduction

You are quite familiar with the usual basis vectors in the Euclidean plane, **i** and **j**. No doubt, you are aware that they are at right angle to each other. This is an example of orthogonality. Moreover, you know that each of them has a unit length. But then, that reminds you of normalised vectors. You can safely say that the two vectors are orthonormal: they are orthogonal to each other, and each is normalised. We shall, as usual, be extending this idea to some other vector spaces. Why is it so important to have orthogonality and at times orthonormality? It is easy to see that it is so for the usual vectors you are familiar with. You shall also learn how important it is that we have an orthonormal basis in quantum mechanics, even though the vectors in this case are in the space of square integrable functions. Once you have an orthonormal basis, you can define an orthonormal basis, and you can find the coefficients of expansion of any wavefunction as a linear combination of the vectors in the basis (eigenstates). Moreover, if the wavefunction itself is normalised, you can then find the probability that the particle described by the wavefunction is in a particular eigenstate. In addition, you shall learn to construct an orthonormal set for a vector space from a given set of vectors in the space.

Learning Outcomes of Study Session 6

At the end of this study session, you should be able to:

6.1 Understand and correctly use all the key words printed in **bold** (SAQ 6.1).

6.2 Determine whether two vectors in a vector space are orthogonal or not (SAQs 6.2 and 6.6).

6.3 Determine whether or not two vectors in a vector space are orthonormal (SAQs 6.3 and 6.6).

6.4 Expand a given vector in a vector space as a linear combination of an orthonormal set (SAQ 6.3 and 6.6)

6.5 Recover the coefficient of a particular basis vector in the expansion of a vector (SAQ 6.3-6.5)

6.6 Construct an orthonormal set from a set of vectors in a vector space using the Gram-Smichdt orthogonalisation procedure (SAQ 6.2).

6.1 Orthogonality and Orthonormality

We say $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V are **orthogonal** if

$$(\mathbf{v}_1, \mathbf{v}_2) = 0 \qquad 6.1$$

Suppose there exists a linearly independent set $\{\phi_i\}_{i=1}^n$, i.e., $\{\phi_1, \phi_2, \cdots, \phi_n\}$, such that

$$(\phi_i, \phi_j) = 0, \; i \neq j \qquad 6.2$$

then, $\{\phi_i\}_{i=1}^n$ is an orthogonal set.

If in addition to the condition above (equation 6.2),

$$(\phi_i, \phi_i) = 1 \qquad 6.3$$

$\{\phi_i\}_{i=1}^n$ is an **orthonormal set**.

For an orthonormal set, therefore, we can write

$$(\phi_i, \phi_j) = \delta_{ij} \tag{6.4}$$

where δ_{ij} is the Kronecker delta, equal to 0 if $i \neq j$ and equal to 1 if $i = j$.

▪ Show that the usual basis in the 2-dimensional Euclidean plane is an orthonormal set.

❑ $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1$, and $\mathbf{i} \cdot \mathbf{j} = 0$
Hence, the set is an orthonormal set.

▪ Show that the set $\{\mathbf{i}+\mathbf{j}, \mathbf{i}-\mathbf{j}\}$ is an orthogonal set, but not an orthonormal set.

❑ $\mathbf{i}+\mathbf{j} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{i}-\mathbf{j} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$(\mathbf{i}+\mathbf{j}, \mathbf{i}-\mathbf{j}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

But the vectors are not normalised. They are an orthogonal set, but not an orthonormal set.

If any vector in the *n*-dimensional vector space, V, can be written as a linear combination of *n* vectors

$$\mathbf{v} = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n = \sum_{i=1}^{n} c_i\phi_i \tag{6.5}$$

such that the c_i's are constants in the underlying field of the vector space and $(\phi_i, \phi_j) = \delta_{ij}$ = 1 if $i = j$ and zero otherwise, then we say that $\{\phi_i\}_{i=1}^n$ is a the complete orthonormal basis for *V*.

If $\{\phi_i\}_{i=1}^n$ is an orthonormal set, It follows that

$$(\phi_j, \mathbf{v}) = (\phi_j, \sum_{i=1}^{n} c_i\phi_i) = \sum_{i=1}^{n} c_i(\phi_j, \phi_i) = \sum_{i=1}^{n} c_i\delta_{ij} = c_j \tag{6.6}$$

Moreover,

$$(\mathbf{v}, \mathbf{v}) = (\sum_{k=1}^{n} c_k\phi_k, \sum_{i=1}^{n} c_i\phi_i) = \sum_{k=1}^{n} c_k{}^* \sum_{i=1}^{n} c_i(\phi_j, \phi_i) = \sum_{i=1}^{n} |c_i|^2 \tag{6.7}$$

If, in addition, the vector **v** is normalised, then

$$(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^{n} |c_i|^2 = 1 \tag{6.8}$$

Thus, we can interpret $|c_i|^2$ as the probability that the system which has n possible states, assumes state i with probability $|c_i|^2$. In other words, the probability that the system is in state i is $|c_i|^2$.

- The usual basis in 2-dimensional Euclidean plane is **i and j**. The basis $\{\mathbf{i},\mathbf{j}\} \equiv \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is an orthonormal basis. (i)Expand the vector $\begin{pmatrix}2\\-3\end{pmatrix}$ in terms of this basis, and recover the coefficient of each basis vector in the expansion.

  (i) Show that the squares of the coefficients are not useful in finding the probability that the system described by vector $\begin{pmatrix}2\\-3\end{pmatrix}$ is in either state $\begin{pmatrix}0\\1\end{pmatrix}$ or state $\begin{pmatrix}0\\1\end{pmatrix}$.

  (ii) Normalise $\begin{pmatrix}2\\3\end{pmatrix}$, and then find the coefficient of the expansion of the vector in terms of the basis vectors.
  Show that in this case, the coefficient of the expansion can be used as the probability that the system described by the vector $\begin{pmatrix}2\\3\end{pmatrix}$ is in the corresponding state.

  (iii) Derive the result in (ii) from that in (i).

❑

(i)

$$\begin{pmatrix}2\\-3\end{pmatrix} = a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\-1\end{pmatrix}$$

Clearly, $a_1 = 2$ and $a_2 = -3$. Notice that $|a_1|^2 + |a_2|^2 = 2^2 + (-3)^2 = 13 \neq 1$.

This is because the vector $\begin{pmatrix}2\\-3\end{pmatrix}$ is not normalised.

(ii) Normalising it, we get the vector $\frac{1}{\sqrt{13}}\begin{pmatrix}2\\-3\end{pmatrix}$. Then,

$$\frac{1}{\sqrt{13}}\begin{pmatrix}2\\-3\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix}$$

$c_1 = \frac{2}{\sqrt{13}}$, and $c_2 = -\frac{3}{\sqrt{13}}$

Then,

$$| c_1 |^2 + | c_2 |^2 = \frac{4}{13} + \frac{9}{13} = 1$$

We can now interpret $| c_1 |^2$ and $| c_2 |^2$ respectively as the probability of finding the system described by the vector $\frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that the wavefunction you are expanding must be normalised, and you must have an orthonormal set.

(ii) Notice that you might also have achieved the same answer by dividing the squared coefficients of the expansion of the unnormalised vector by dividing by the norm of the vector:

$$| c_1 |^2 = \frac{| a_1 |^2}{| a_1 |^2 + | a_2 |^2} = \frac{4}{13};\ | c_2 |^2 = \frac{| a_2 |^2}{| a_1 |^2 + | a_2 |^2} = \frac{9}{13}$$

6.2.1 Gram-Schmidt Orthogonalisation Procedure

This provides a method of constructing an orthogonal set from a given set. Normalising each member of the set then provides an orthonormal set. The method entails setting up the first vector, and then constructing the next member of the orthogonal set by making it orthogonal to the first member of the set under construction. Then the next member of the set is constructed in a way to be orthogonal to the two preceding members. This procedure can be continued until the last member of the set is constructed.

Example from function vector space

Do you recall that the inner product of two functions $f(x)$ and $g(x)$ in the vector space of square-integrable functions over an interval [*a*, *b*] is given by

$$(f(x), g(x)) = \int_a^b f^*(x) g(x) dx$$

Now, we wish to construct a set of functions $\{\phi_1(x), \phi_2(x), \ldots\}$ from the given set $\{f_1(x), f_2(x), \ldots\}$, such that each pair of the functions will be orthogonal. So, we pick the first function $f_1(x)$ as the first element of the orthogonal set $\phi_1(x)$, i.e., $\phi_1(x) = f_1(x)$. Then, we form a linear combination of the next function and a scalar multiple of the first element of the orthogonal set that we just obtained $\phi_2(x) = f_2(x) + \alpha\phi_1(x)$. This forms the second element of the orthogonal set. Then, we need to determine the constant multiplying the first element of the set, α, so as to get the second function of the orthogonal set. We obtain this constant by applying the fact that the inner product of the two new functions must be orthogonal, i.e., $\int_a^b \phi_1^*(x)\phi_2(x)dx = 0$. Then, we get the third element as the third function, plus a linear multiple of the first two elements of the orthogonal set, $\phi_3(x) = f_2(x) + \alpha\phi_1(x) + \beta\phi_2(x)$. Do note that the constant α in this case is not the same as the one associated with the previous stage of finding $\phi_2(x)$. $\int_a^b \phi_1^*(x)\phi_2(x)dx$. Thus, we need to determine two constants. We equate the two inner products, $\int_a^b \phi_1^*(x)\phi_3(x)dx$ and $\int_a^b \phi_2^*(x)\phi_3(x)dx$ to zero, as each of ϕ_1 and ϕ_2 must be orthogonal to ϕ_3. This process is continued until all the required vectors are got. Then, normalizing each element of the orthogonal set, we get the elements of the orthonormal set. An example of this process is given in Example …

▪ Construct an orthonormal set from the set $\{1, x, x^2, \ldots\}$ over the interval $-1 \le x \le 1$.

❑ Thus, given the set $\{f_1, f_2, f_3, \ldots\}$, we want to construct an orthogonal set $\{\phi_1, \phi_2, \phi_3, \ldots\}$, i.e.,

$$\int_{-1}^{1} \phi_i(x)\phi_j(x)dx = 0, \text{ if } i \neq j \qquad 6.9$$

Then we normalise each member of the set.

Let $\phi_1 = f_1 = 1$, and $\phi_2 = f_2 + \alpha\phi_1 = x + \alpha$

Then, we determine α, subject to

$$(\phi_1, \phi_2) = 0 \qquad 6.10$$

$$\int_{-1}^{1} 1 \cdot (x+\alpha)dx = \left.\frac{x^2}{2}\right|_{-1}^{1} + \alpha x\Big|_{-1}^{1} = 0$$

$$\Rightarrow \alpha = 0 \qquad 6.11$$

Thus, $\phi_2 = x$

Let $\phi_3 = f_3 + \alpha\phi_2 + \beta\phi_1 = x^2 + \alpha x + \beta$, subject to $(\phi_1, \phi_3) = 0$ and $(\phi_2, \phi_3) = 0$

The first condition gives:

$$\int_{-1}^{1} 1 \cdot (x^2 + \alpha x + \beta)dx = 0 \qquad 6.12$$

or

$$\left.\frac{x^3}{3}\right|_{-1}^{1} + \left.\frac{\alpha x^2}{2}\right|_{-1}^{1} + \beta x\Big|_{-1}^{1} = \left.\frac{2x^2}{3} + 2\beta x\right|_{0}^{1} = 0$$

$$\frac{2}{3} + 2\beta = 0$$

or

$$\beta = -\frac{1}{3} \qquad 6.13$$

The second condition gives

$$\int_{-1}^{1} x\cdot(x^2+\alpha x+\beta)dx = \int_{-1}^{1}(x^3+\alpha x^2+\beta x)dx \qquad 6.14$$

or

$$\left.\frac{x^4}{4}\right|_{-1}^{1} + \left.\frac{\alpha x^3}{3}\right|_{-1}^{1} + \left.\frac{\beta x^2}{2}\right|_{-1}^{1} = 0$$

$$\left.\frac{2\alpha x^3}{3}\right|_{0}^{1} = 0$$

or

$$\alpha = 0 \qquad 6.14$$

From equations (6.13) and (6.14),

$$\phi_3 = x^2 - \frac{1}{3} \qquad 6.15$$

ϕ_4, ϕ_5, etc. can be got in a similar fashion.

To normalise ϕ_j, we multiply the function by a normalisation constant, A, say, and invoke the relation

$$\int_{-1}^{1} A^2 \phi_j{}^2(x)dx = 1$$

For ϕ_1, this becomes

$$\int_{-1}^{1} A^2 1^2\, dx = 2\int_{0}^{1} A^2 dx = 1$$

from which

$$2A^2 = 1$$

or

$$A = \sqrt{\frac{1}{2}} \qquad 6.16$$

The normalised function

$$\psi_1 = A\phi_1 = \frac{1}{\sqrt{2}}\times 1 = \frac{1}{\sqrt{2}} \qquad 6.17$$

Similarly,

$$\int_{-1}^{1} A^2 x^2 dx = 1$$

$$A^2 \left.\frac{x^3}{3}\right|_{-1}^{1} = A^2\left[\frac{1}{3}+\frac{1}{3}\right] = 2\frac{A^2}{3} = 1$$

Thus,

$$A = \sqrt{\frac{3}{2}}$$

Hence, the normalised function

$$\psi_2 = A\phi_2 = \sqrt{\frac{3}{2}}x \qquad 6.18$$

In like manner,

$$\int_{-1}^{1} A^2\left(x^2 - \frac{1}{3}\right)^2 dx = 2\int_0^1 A^2\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx = 1$$

from which

$$2A^2 \left. \frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right|_0^1 = 1$$

or

$$2A^2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) = 1$$

Therefore,

$$\frac{80}{45}A^2 = 1$$

The normalised function

$$\psi_2 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \qquad 6.19$$

Example from the n-dimensional Euclidean space, R^n

Suppose we are given two vectors **u** and **v**. The projection of **v** onto **u** (see Fig. 6.1) is indeed,

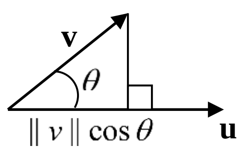

Fig. 6.1: The projection of vector **v** onto vector **u**.

$$\mathrm{Pr}_{\mathbf{u}}\, \mathbf{v} = (\|\mathbf{v}\|\cos\theta)\hat{\mathbf{u}}$$

where θ is the angle between the two vectors, and $\hat{\mathbf{u}}$ is the unit vector in the direction of **u**, and. Therefore, $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$. But $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$. So, $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$

Hence,

$$\Pr_{\mathbf{u}} \mathbf{v} = \left(\|\mathbf{v}\| \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \right) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{(\mathbf{u},\mathbf{v})}{(\mathbf{u},\mathbf{u})}\mathbf{u} \qquad 6.20$$

since $(\mathbf{u},\mathbf{u}) = |\mathbf{u}|^2$.

We proceed as we did in the case of the example from function space.

$$\mathbf{u}_1 = \mathbf{v}_1 \qquad 6.21$$

The projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$ is $\Pr_{\mathbf{u}_1} \mathbf{v}_2$. This is the component of $\mathbf{v}_2$ in the direction of $\mathbf{u}_1$. The component of $\mathbf{v}_2$ perpendicular (i.e., orthogonal) to $\mathbf{u}_1$ is (vector subtraction):

$$\mathbf{u}_2 = \mathbf{v}_2 - \Pr_{\mathbf{u}_1} \mathbf{v}_2 \qquad 6.22$$

Similarly the components of $\mathbf{v}_3$ perpendicular to both $\mathbf{u}_1$ and $\mathbf{u}_2$ is,

$$\mathbf{u}_3 = \mathbf{v}_3 - \Pr_{\mathbf{u}_1} \mathbf{v}_3 - \Pr_{\mathbf{u}_2} \mathbf{v}_3 \qquad 6.23$$

.

.

.

$$\mathbf{u}_n = \mathbf{v}_n - \sum_{i=1}^{n-1} \Pr_{\mathbf{u}_i} \mathbf{v}_n \qquad 6.24$$

We can then normalise each vector

$$\mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|} \qquad 6.25$$

Note that $\Pr_{\mathbf{u}} \mathbf{v}$ projects vector **v** orthogonally onto vector **u**.

- You are given the basis $\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix} \right\}$. With the standard Euclidean inner product, construct an orthonormal basis for the 3-dimensional Euclidean space.

❑

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2,\mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$$

In this case,

$$(\mathbf{v}_2,\mathbf{u}_1) = \begin{pmatrix}1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 1 \qquad\qquad \|\mathbf{u}_1\|^2 = \begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 2$$

Hence,

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2,\mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1/2\\1\\-1/2\end{pmatrix}$$

In this case,

$$\mathbf{u}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad\qquad \mathbf{u}_2 = \begin{pmatrix}1/2\\1\\-1/2\end{pmatrix}$$

$$(\mathbf{v}_3,\mathbf{u}_1) = \begin{pmatrix}1 & 2 & 1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = 2$$

$$\|\mathbf{u}_1\|^2 = 2$$

$$(\mathbf{v}_3,\mathbf{u}_2) = \begin{pmatrix}1 & 2 & 1\end{pmatrix}\begin{pmatrix}1/2\\1\\-1/2\end{pmatrix} = \frac{1}{2} + 2 - \frac{1}{2} = 2$$

$$\|\mathbf{u}_2\|^2 = \begin{pmatrix}1/2 & 1 & -1/2\end{pmatrix}\begin{pmatrix}1/2\\1\\-1/2\end{pmatrix} = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$$

Hence,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{(\mathbf{v}_3,\mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{(\mathbf{v}_3,\mathbf{u}_2)}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{2}{3/2}\begin{pmatrix}1/2\\1\\-1/2\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1/2\\1\\-1/2\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}2/3\\4/3\\-2/3\end{pmatrix} = \begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix} = \frac{2}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Check!

$$(\mathbf{u}_1,\mathbf{u}_2)=\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}1/2\\ 1\\ -1/2\end{pmatrix}=0$$

$$(\mathbf{u}_1,\mathbf{u}_3)=\frac{2}{3}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}=0$$

$$(\mathbf{u}_2,\mathbf{u}_3)=\frac{2}{3}\begin{pmatrix}1/2 & 1 & -1/2\end{pmatrix}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}=\frac{2}{3}\left(-\frac{1}{2}+1-\frac{1}{2}\right)=0$$

You get the orthonormal set by dividing each vector by its norm. We already have, $\|\mathbf{u}_1\|=\sqrt{2}$, $\|\mathbf{u}_2\|=\sqrt{3/2}$. You can easily show that $\|\mathbf{u}_3\|=\sqrt{4/3}=2/\sqrt{3}$.

$$\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)=\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\sqrt{\frac{2}{3}}\begin{pmatrix}1/2\\ 1\\ -1/2\end{pmatrix},\frac{\sqrt{3}}{2}\times\frac{2}{3}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}\right\}=\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\sqrt{\frac{2}{3}}\times\frac{1}{2}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}\right\}$$

$$=\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix},\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}\right\}$$

$$=\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\frac{\sqrt{6}}{6}\begin{pmatrix}1\\ 2\\ -1\end{pmatrix},\frac{\sqrt{3}}{3}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}\right\}$$

Summary of Study Session 6

In Study Session 6, you learnt the following:

1. Two vectors in a vector space are orthogonal if their inner product is zero.
2. Two vectors in a vector space are orthonormal if their inner product is zero, and each vector is normalised, that is if its norm is unity.
3. Once a basis is given for a vector space, we can expand any vector in the vector space as a linear combination of the basis vectors.
4. The coefficients of expansion can be obtained by taking the inner product of the respective basis vector and the vector expanded in terms of the basis vectors.
5. If the basis is an orthonormal set, and the vector expanded in terms of the basis is normalised, then the squares of the magnitude of the coefficients are the probabilities of finding the system described by the function in the respective states.
6. How to construct an orthogonal and hence, orthonormal set from a given set of vectors using the Gram-Smichdt orthogonalisation procedure.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.
4. Hefferson, J. (2012). Linear Algebra, http://joshua.smcvt.edu/linearalgebra/book.pdf
5. Hefferson, J. (2012). Answers to Exercises, http://joshua.smcvt.edu/linearalgebra/jhanswer.pdf

Self Assessment Questions (SAQs) for Study Session 6

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 6.1 (tests Learning Outcome 6.1)

Define the following:

(i) Orthogonal set (ii) Orthonormal set (iii) Gram-Schmidt orthogonalisation

SAQ 6.2 (tests Learning Outcomes 6.2 and 6.6)

Show that the set $\left\{\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}3\\7\\-1\end{pmatrix}\right\}$ is a basis for the 3-dimensional Euclidean space. Show that the set is not orthogonal. Hence, construct an orthogonal set from the vectors.

SAQ 6.3 (tests Learning Outcomes 6.3, 6.4 and 6.5)

Show that the set you obtained in SAQ 6.2 is not an orthonormal set, even though it is an orthogonal set. Normalise each vector in the set. Hence, expand the normalised vector $(-1 \quad 2 \quad 1)/\sqrt{6}$ in terms of this basis. Calculate the probability that the system represented by the vector is in each of these states.

SAQ 6.4 (tests Learning Outcomes 6.4 and 6.5)

A particle trapped in the well

$$V = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{elsewhere} \end{cases}$$

is found to have a wavefunction

$$\frac{i}{2}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{2}{3a}}\sin\left(\frac{2\pi x}{a}\right)-\sqrt{\frac{1}{2a}}\sin\left(\frac{3\pi x}{a}\right)$$

Given that the allowable wavefunctions are of the form $\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$, what is the probability of obtaining each result?

SAQ 6.5 (tests Learning Outcomes 6.4 and 6.5)

A particle in a one-dimensional box $0 \le x \le a$ is in state:

$$\psi(x) = \frac{1}{\sqrt{5a}} \sin \frac{\pi x}{a} + \frac{A}{\sqrt{a}} \sin \frac{2\pi x}{a} + \frac{3}{\sqrt{6a}} \sin \frac{3\pi x}{a}$$

(a) Find A so that $\psi(x)$ is normalized.

(b) What are the possible results of measurements of the energy, and what are the respective probabilities of obtaining each result?

SAQ 6.6 (tests Learning Outcomes 6.2, 6.3 and 6.4)

You are given the set $S_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

(a) Are the vectors linearly independent? If so, does the set form a basis for the 2-dimensional Euclidean plane?

(b) Are the vectors orthogonal?

(c) Are they normalised? If not, normalise them.

(d) Write the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ in the usual basis, S_U, in the Euclidean plane in terms of the vectors of the basis S_1.

Solutions to SAQs

SAQ 6.1

A set $\{\mathbf{v}_i\}$ is said to be an orthogonal set if for any pair $\mathbf{v}_k$, $\mathbf{v}_m$, the inner product $(\mathbf{v}_k, \mathbf{v}_m)$ is zero for $k \neq m$.

(i) A set $\{\mathbf{v}_i\}$ is an orthonormal set if for any pair $\mathbf{v}_k$, $\mathbf{v}_m$, the inner product $(\mathbf{v}_k, \mathbf{v}_m) = \delta_{km}$, that is, zero for $k \neq m$ and 1 for $k = m$.

(ii) The Gram-Schmidt orthogonalisation procedure is one by which vectors in a vector space on which an inner product is defined can be made mutually orthogonal.

SAQ 6.2

The set must be linearly independent:

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 0 & 7 \\ -1 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 7 \\ 0 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 2(7) - 1(1) + 3(1) = 16 \neq 0$$

Since we have 3 linearly independent vectors in the 3-dimensional Euclidean space, the vectors form a basis for the space.

The set is not orthogonal:

$$\begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2 \qquad \begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} = -1 \quad \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4$$

To construct an orthogonal set: $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} \right\}$

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2, \mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$$

In this case,

$$(\mathbf{v}_2, \mathbf{u}_1) = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2 \qquad\qquad \|\mathbf{u}_1\|^2 = \begin{pmatrix} 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 5$$

Hence,

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{(\mathbf{v}_2, \mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 4/5 \\ -2/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix}$$

In this case,

$$\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad \mathbf{u}_2 = \begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix}$$

$$(\mathbf{v}_3, \mathbf{u}_1) = \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = -1$$

$$\|\mathbf{u}_1\|^2 = 5$$

$$(\mathbf{v}_3, \mathbf{u}_2) = \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}\begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix} = \frac{3}{5} + \frac{14}{5} + 1 = \frac{22}{5}$$

$$\|\mathbf{u}_2\|^2 = \begin{pmatrix} 1/5 & 2/5 & -1 \end{pmatrix}\begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix} = \frac{1}{25} + \frac{4}{25} + 1 = \frac{30}{25} = \frac{6}{5}$$

Hence,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{(\mathbf{v}_3, \mathbf{u}_1)}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{(\mathbf{v}_3, \mathbf{u}_2)}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} - \left(-\frac{1}{5}\right)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{22/5}{6/5}\begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} - \left(-\frac{1}{5}\right)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{11}{3}\begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix} + \begin{pmatrix} 2/5 \\ -1/5 \\ 0 \end{pmatrix} - \begin{pmatrix} 11/15 \\ 22/15 \\ -11/3 \end{pmatrix} = \begin{pmatrix} 40/15 \\ 80/15 \\ 8/3 \end{pmatrix}$$

$$= \begin{pmatrix} 8/3 \\ 16/3 \\ 8/3 \end{pmatrix}$$

The orthogonal set is,

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/5 \\ 2/5 \\ -1 \end{pmatrix}, \begin{pmatrix} 8/3 \\ 16/3 \\ 8/3 \end{pmatrix} \right\}$$

SAQ 6.3

The norms of the vectors in the set are, respectively, $\sqrt{5}$, $\sqrt{6/5}$ and $\sqrt{128/3}$. Hence, the normalised set is,

$$\left\{\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{6/5}}\begin{pmatrix}1/5\\2/5\\-1\end{pmatrix}, \frac{1}{\sqrt{128/3}}\begin{pmatrix}8/3\\16/3\\8/3\end{pmatrix}\right\}$$

$$\frac{c_1}{\sqrt{5}}\begin{pmatrix}2\\-1\\0\end{pmatrix} + \sqrt{\frac{5}{6}}c_2\begin{pmatrix}1/5\\2/5\\-1\end{pmatrix} + \sqrt{\frac{3}{128}}c_3\begin{pmatrix}8/3\\16/3\\8/3\end{pmatrix}$$

$$c_1 = (\phi_1, \psi) = \frac{1}{\sqrt{6}\times\sqrt{5}}\begin{pmatrix}2 & -1 & 0\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix} = -\frac{4}{\sqrt{30}}$$

$$c_2 = (\phi_2, \psi) = \frac{\sqrt{5}}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix}1/5 & 2/5 & -1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix} = -\frac{\sqrt{5}}{6}\left(-\frac{1}{5}+\frac{4}{5}-1\right)$$

$$c_3 = (\phi_3, \psi) = \frac{\sqrt{3}}{\sqrt{128}}\frac{8}{3\sqrt{6}}\begin{pmatrix}1 & 2 & 1\end{pmatrix}\begin{pmatrix}-1\\2\\1\end{pmatrix} = \frac{\sqrt{3}}{4\sqrt{8}}\frac{8}{3\sqrt{6}}4 = \frac{\sqrt{8}}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{8}}{3\sqrt{2}} = \frac{2}{3}$$

$$|c_1|^2 = \frac{16}{30} = \frac{8}{15} \qquad |c_2|^2 = \frac{1}{45} \qquad |c_3|^2 = \frac{4}{9}$$

Adding to check if we are right,

$$|c_1|^2 + |c_2|^2 + |c_3|^2 = \frac{16}{30} + \frac{1}{45} + \frac{4}{9} = \frac{48+2+40}{90} = \frac{90}{90} = 1$$

We are!

You know we could have expanded the vector in terms of the basis vectors, but we chose to make use of the fact that,

$$c_i = (\phi_i, \psi)$$

provided $\{\phi_k\}_{k=1}^{n}$ is an orthonormal set, and

$$\sum_{i=1}^{n} |c_i|^2 = 1$$

provided, in addition, the vector we expanded in terms of the orthonormal basis is itself normalised, as it is the case in this SAQ.

SAQ 6.4

The allowable wavefunctions are of the form $\left\{ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \right\}_{n=1}^{3}$. Hence the expansion in terms of these eigenfunctions is,

$$\frac{i}{2}\sin\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}+\frac{1}{\sqrt{3}}\sin\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}-\frac{1}{2}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}$$

Since $c_1 = i/2$, $c_2 = 1/\sqrt{2}$, $c_3 = 1/2$, the probability, respectively, that the particle will be found in these states (n = 1, 2, 3) is:

$$|c_1|^2 = \frac{i}{2}\times\frac{-i}{2}=\frac{1}{4},\ |c_2|^2 = \frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{1}{2},\ |c_3|^2=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$

SAQ 6.5

We first put ψ in the form of the allowable eigenfunctions:

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{5a}}\sin\frac{\pi x}{a}+\frac{A}{\sqrt{a}}\sin\frac{2\pi x}{a}+\frac{3}{\sqrt{6a}}\sin\frac{3\pi x}{a}\\ &= \frac{1}{\sqrt{5a}}\sqrt{\frac{2}{2}}\sin\frac{\pi x}{a}+\frac{A}{\sqrt{a}}\sqrt{\frac{2}{2}}\sin\frac{2\pi x}{a}+\frac{3}{\sqrt{6a}}\sqrt{\frac{2}{2}}\sin\frac{3\pi x}{a}\\ &= \frac{1}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}+\frac{A}{\sqrt{2}}\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}+\frac{3}{2\sqrt{6}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\\ &= \frac{1}{\sqrt{10}}\phi_1+\frac{A}{\sqrt{2}}\phi_2+\frac{3}{2\sqrt{6}}\phi_3\end{aligned}$$

For ψ to be normalized,

$$(\psi(x),\psi(x))=1$$

or

$$\left(\frac{1}{\sqrt{10}}\phi_1+\frac{A}{\sqrt{2}}\phi_2+\frac{3}{2\sqrt{6}}\phi_3,\frac{1}{\sqrt{10}}\phi_1+\frac{A}{\sqrt{2}}\phi_2+\frac{3}{2\sqrt{6}}\phi_3\right)=1$$

$\Rightarrow$

$$\frac{1}{10}+\frac{A^2}{2}+\frac{9}{24}=1$$

or

$$A^2 = 2\left(1-\frac{9}{24}-\frac{1}{10}\right)=2\left(\frac{120-45-12}{120}\right)=\frac{63}{60}.$$

Hence, $A=\sqrt{\frac{63}{60}}$

The normalised ψ is therefore,

$$\psi(x) = \frac{1}{\sqrt{10}}\phi_1 + \sqrt{\frac{63}{120}}\phi_2 + \frac{3}{2\sqrt{6}}\phi_3$$

The possible values of the energy are:

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2}, \; E_2 = \frac{2\hbar^2\pi^2}{ma^2}, E_3 = \frac{9\hbar^2\pi^2}{2ma^2},$$

and the probabilities, respectively, are $|c_1|^2 = \frac{1}{10}$, $|c_2|^2 = \frac{63}{120}$, $|c_3|^2 = \frac{9}{24}$

SAQ 6.6

(a) Given a set $\{\mathbf{v}_i\}_{i=1}^n$, if we can write $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$ and this implies $a_1 = a_2 = \cdots = a_n = 0$, then we say $\{\mathbf{v}_i\}_{i=1}^n$ is a linearly independent set.

$$\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\-1\end{pmatrix}$$

To check if they are linearly independent.

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Hence, $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. From the last equation, $c_1 = c_2$. Putting this in the first equation, $c_1 + c_1 = 0$, or $c_1 = 0$. Consequently, $c_2 = 0$. Set is linearly independent. Since there are two linearly independent vectors in two-dimensional space (plane), they form a basis.

(b) To check orthogonality, $(\mathbf{a},\mathbf{b}) = \mathbf{a}^T\mathbf{b} = (1 \quad 1)\begin{pmatrix}1\\-1\end{pmatrix} = 1 - 1 = 0$

They are orthogonal.

(c) Are they normalised? $(\mathbf{a},\mathbf{a}) = (1 \quad 1)\begin{pmatrix}1\\1\end{pmatrix} = 2$, or $\|\mathbf{a}\| = \sqrt{2}$. $(\mathbf{b},\mathbf{b}) = (1 \quad -1)\begin{pmatrix}1\\-1\end{pmatrix} = 2$.

Also, $\|\mathbf{b}\| = \sqrt{2}$.

They are not normalised.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \text{ are normalised.}$$

The set $\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\}$ forms an orthonormal basis for R^2.

In the usual basis S_U,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\mathbf{i} + 4\mathbf{j} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the basis S_1

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{\alpha}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\beta}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence, $\alpha + \beta = 3\sqrt{2}$ and $\alpha - \beta = 4\sqrt{2}$

$$2\alpha = 7\sqrt{2} \text{ and } 2\beta = -\sqrt{2}$$

Therefore,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{7\sqrt{2}}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\sqrt{2}}{2\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Study Session 7 Fourier Series

Introduction

Would it not be nice to be able to decompose a signal into its constituent frequencies? On the other hand, you would be glad if you could compose a complex signal or waveform from simple sinusoids. This you achieve by understanding Fourier Series. Moreover, from your knowledge of odd and even functions in Study Session 6, you should also expect that the Fourier series of an even function should consist only of a constant and some cosine functions. Also, the Fourier series of an odd function would definitely be a composition of only sine functions. These you will learn eventually as Fourier cosine series and Fourier sine series respectively.

Learning Outcomes for Study Session 6

At the end of this study session, you should be able to the following:

7.1 Understand and be able to correctly use, all keywords in **bold** print (SAQ 7.1).
7.2 Write a function as a sum of odd and even functions (SAQ 7.1).
7.3 Simply deduce the value of, or simplify an integral based on whether the integrand is even or odd (SAQ 7.2).
7.4 Calculate the Fourier coefficients in the Fourier expansion of a periodic function of x $-\pi \le x \le \pi$ (SAQ 7.5-7.8).
7.5 Write the Fourier expansion for a periodic function in the interval $-\pi \le x \le \pi$ and in a general interval (SAQ 7.9).
7.6 State the condition under which the Fourier series of a function exists (SAQ 7.10).
7.7 Calculate the half-range Fourier series of a given function (SAQ 7.11).
7.8 Develop a series for the irrational number π (SAQ 7.5).

7.1 Odd and Even Functions

Examine the following integrals:

(i) $\int_{-a}^{a} x^{2n+1} dx = 0$, $n = 0, 1, 2, \ldots$

(ii) $\int_{-a}^{a} x^{2n} dx = 2\int_{0}^{a} x^{2n} dx$, $n = 0, 1, 2, \ldots$

$f(x) = x^{2n+1}$ is an **odd function**

$f(x) = x^{2n}$ is an **even function**.

Figure 7.1 (a) and (b) shows examples, respectively, of odd and even functions.

(a) Odd function (b) Even function

Fig. 7.1: Examples of odd and even functions

An even function is symmetrical about the y axis. In other words, a plane mirror placed on the axis will produce an image that is exactly the function across the axis. An odd function will need to be mirrored twice, once along the y axis, and once along the x axis to achieve the same effect.

A function $f(x)$ of x is said to be an odd function if $f(-x) = -f(x)$, e.g., $\sin x,\ x^{2n+1}$

A function $f(x)$ of x is said to be an even function if $f(-x) = f(x)$, e.g., $\cos x,\ x^{2n}$

where $n = 0,1,2,\ldots$.

Some real-valued functions are odd, some are even; the rest are neither odd nor even. However, we can write any real-valued function as a sum of an odd and an even function. Let the function be $h(x)$, then we can write

$$h(x) = f(x) + g(x) \qquad 7.1$$

where $f(x)$ is odd and $g(x)$ is even. Then, $f(-x) = -f(x)$ and $g(-x) = g(x)$

$$h(-x) = f(-x) + g(-x) = -f(x) + g(x) \qquad 7.2$$

Adding equations (7.1) and (7.2) gives

$$h(x) + h(-x) = 2g(x)$$

Subtracting equation (7.2) from equation (7.1) gives

$$h(x) - h(-x) = 2f(x)$$

It follows, therefore, that

$$f(x) = \frac{h(x) - h(-x)}{2} \qquad 7.3$$

and

$$g(x) = \frac{h(x) + h(-x)}{2} \qquad 7.4$$

▪ Write the function $h(x) = e^{2x}\sin x$ as a sum of odd and even functions.

❑

$$h(x) = e^{2x}\sin x,\ h(-x) = e^{-2x}\sin(-x) = -e^{-2x}\sin x$$

Therefore, the odd function is

$$f(x) = \frac{h(x) - h(-x)}{2} = \frac{e^{2x}\sin x + e^{-2x}\sin x}{2} = \frac{e^{2x} + e^{-2x}}{2}\sin x$$
$$= \cosh 2x \sin x$$

The even function is

$$g(x) = \frac{h(x) + h(-x)}{2} = \frac{e^{2x}\sin x - e^{-2x}\sin x}{2} = \frac{e^{2x} - e^{-2x}}{2}\sin x$$
$$= \sinh 2x \sin x$$

It is obvious that the odd function is a product of an odd function and an even function. Likewise, the even function is a product of two odd functions.

$\text{Even} \times \text{Even} = \text{Even}$
$\text{Even} \times \text{Odd} = \text{Odd}$
$\text{Odd} \times \text{Odd} = \text{Even}$
The integral
$\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is odd.
$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if $f(x)$ is even.

Recall that the inner product in the space of twice integrable complex valued functions of two complex valued functions $f(x)$ and $g(x)$ over the interval $a \le x \le b$ is defined as
$(f, g) = \int_{a}^{b} f^*(x)g(x)dx$.

Two functions $f(x)$ and $g(x)$ are said to be **orthogonal** over an interval $a \le x \le b$ if their inner product is zero.

- Show that $\sin mx$ and $\sin nx$ are orthogonal, $m \ne n$, $-\pi \le x \le \pi$.

- The inner product is

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m-n)x - \cos(m+n)x]dx$$
$$= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x\bigg|_{-\pi}^{\pi} + \frac{1}{m+n}\sin(m+n)x\bigg|_{-\pi}^{\pi}\right] = 0$$

Similarly, the you can show that
(i) $\sin mx$ and $\cos nx$ are orthogonal, $-\pi \le x \le \pi$.
(ii) $\sin mx$ and $\sin nx$ are orthogonal, $m \ne n$, $-\pi \le x \le \pi$.

7.2 Fourier Series

Suppose we can write the function $f(x)$ as a function of x defined between $-\pi$ and π as

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + ... + b_1 \sin x + b_2 \sin 2x + ...$$
$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad 7.5$$

the right hand side is called the Fourier series for the function $f(x)$. Of course, we expect the Fourier coefficients to be unique, i.e., $f(x) \ne g(x)$ implies they do not have identical Fourier coefficients: $a_0, a_1, ..., b_1, b_2,...$

Now, integrate equation 7.5 from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{1}{2}a_0 dx + \int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty} a_n \cos nx\right]dx + \int_{\pi}^{\pi}\left[\sum_{n=1}^{\infty} b_n \sin nx\right]dx = \frac{1}{2}a_0 x\Big|_{-\pi}^{\pi}$$

since each of the last two terms on the right is zero.

Therefore,

$$\int_{-\pi}^{\pi} f(x)dx = a_0\pi$$

from which we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = a_0 \qquad 7.6$$

Multiplying $f(x)$ by $\cos mx$ and integrating from $-\pi$ to π,

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mx dx + \int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty} a_n \cos mx \cos nx dx\right] + \int_{-\pi}^{\pi}\left[\sum_{n=1}^{\infty} b_n \sin nx \cos mx dx\right]$$

Only the second term on the right survives, and this only when $m = n$. Then,

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi} a_m \cos^2 mx dx = \frac{1}{2}a_m \int_{-\pi}^{\pi}(1+\cos 2mx)dx = \frac{1}{2}x\Big|_{-\pi}^{\pi} \frac{1}{2}a_m(\pi - (-\pi)) = a_m\pi$$

implying that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx \qquad 7.7$$

The student can show that, in a similar vein,

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \qquad 7.8$$

▪ Find the Fourier coefficients of the function $f(x) = 1; -\pi \le x \le \pi$ (Fig. 7.2).

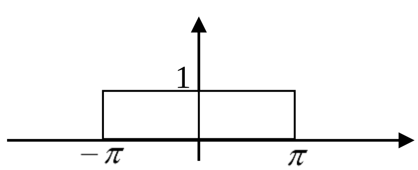

Fig. 7.2

❑

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

From equations 7.2 through 7.4,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} 1dx = \frac{2x}{\pi}\bigg|_0^{\pi} = 2$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1.\cos mxdx = \frac{2}{\pi}\int_0^{\pi} \cos mxdx$$

$$= \frac{2}{\pi m}\sin mx\bigg|_0^{\pi} = 0$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1.\sin mxdx = -\frac{1}{m\pi}\cos mx\bigg|_{-\pi}^{\pi} = 0$$

Therefore,

$$f(x) = 1 = \frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$$

$$= \frac{1}{2}\times 2 = 1$$

- Find the Fourier series for $f(x) = x$ if x lies between $-\pi$ and π. Hence, derive the formula,

  $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

❑

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = 0$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos mxdx = 0$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin mxdx$$

We integrate the right hand side by parts, with $x = u,\ du = dx;\ \sin mxdx = dv,\ v = \int \sin mxdx = -\frac{1}{m}\cos mx$

from which $I = uv - \int vdu$ gives

$$b_m = -\frac{1}{\pi}\left[\frac{x}{m}\cos mx\bigg|_{-\pi}^{\pi} - \frac{1}{m}\int_{-\pi}^{\pi}\cos mxdx\right] = -\frac{1}{\pi}\left[\frac{\pi}{m}\cos m\pi - \frac{-\pi}{m}\cos(-m\pi)\right]$$

$$= \frac{-2}{m}\cos m\pi = \frac{2}{m} \text{ for } m \text{ odd and } -\frac{2}{m} \text{ for } m \text{ even.}$$

Hence, the Fourier series is,

$$f(x) = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x + \frac{2}{4}\sin 4x + \ldots = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots\right)$$

To derive the formula, we set $x = \pi/2$. Then,

$$f(\pi/2) = 2\left(1 - \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\right)$$
$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right) = \frac{\pi}{2}, \text{ since } f(x) = x$$

Hence,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

- Find the Fourier coefficients of the function (Fig. 7.3),

$$f(x) = \begin{cases} -1; -\pi \le x \le 0 \\ 1 \; ;0 \le x \le \pi \end{cases}.$$

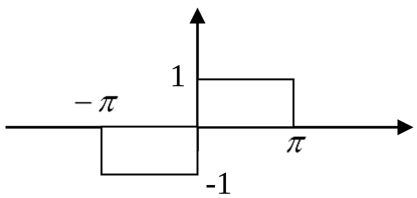

Fig. 7.3

❑

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -1dx + \frac{1}{\pi}\int_{0}^{\pi} 1dx = \frac{1}{\pi}\left(-x\Big|_{-\pi}^{0}\right) + \frac{1}{\pi}\left(x\Big|_{0}^{\pi}\right) = -\frac{1}{\pi}[0-(-\pi)] + \frac{1}{\pi}(\pi - 0) = -1 + 1 = 0$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{0} -1\times\cos mx dx + \frac{1}{\pi}\int_{0}^{\pi} 1\times\cos mx dx$$
$$= \frac{1}{m\pi}\sin mx\Big|_{-\pi}^{0} + \frac{1}{m\pi}\sin mx\Big|_{0}^{\pi} = 0$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{0} -1\times\sin mx dx + \frac{1}{\pi}\int_{0}^{\pi} 1\times\sin mx dx$$
$$= \frac{1}{m\pi}\cos mx\Big|_{-\pi}^{0} - \frac{1}{m\pi}\cos mx\Big|_{0}^{\pi} = \frac{1}{m\pi}[\cos 0 - \cos m\pi] - \frac{1}{m\pi}[\cos m\pi - \cos 0]$$
$$= \frac{2}{m\pi}\left\{[1-(-1)^m\right\}$$
$$= \frac{4}{m\pi} \text{ for } m \text{ odd,}$$

= 0 for m even.

Hence, the Fourier series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{m} = \frac{4}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right\}$$

Notice that for even functions, only the $a's$ are non-zero. For the odd functions, however, only the $b's$ survive.

Now, what value does the Fourier series attain at a discontinuity, say at $x=0$ in the Fourier series for

$$f(x)=\begin{cases}-1; -\pi \le x \le 0\\ 1;\ \ 0 \le x \le \pi\end{cases}$$

This is the average of the two values at that point, in this case, $(1-1)/2=0$. Let us check: at x = 0, the Fourier series gives $f(x)=0$.

Some useful results

a. $\int_{-\pi}^{\pi}\sin nxdx=0$ 7.9

b. $\int_{-\pi}^{\pi}\cos nxdx=0$ 7.10

c. $\int_{-\pi}^{\pi}\cos^2 nxdx=\int_{-\pi}^{\pi}\sin^2 xdx=\pi$ 7.11

d. $\int \cos mx\cos nxdx=\begin{cases}0,\ m\neq n\\ \pi,\ m=n\end{cases}$ 7.12

e. $\int \sin mx\sin nxdx=\begin{cases}0,\ m\neq n\\ \pi,\ m=n\end{cases}$ 7,13

f. $\int \sin mx\cos nxdx=0$ 7.14

7.2.1 Conditions that must be fulfilled in order that a function $f(x)$ may be expanded as a Fourier series

(i) The function must be periodic.
(ii) It must be single-valued and continuous, except possibly at a finite number of finite discontinuities.
(iii) It must have only a finite number of maxima and minima within one period.
(iv) The integral over one period of $|f(x)|$ must converge.

If these conditions are satisfied, then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous.

Note: At a point of finite discontinuity, the Fourier series converges to the value half-way between the upper and the lower values.

7.3 Fourier Series of Odd and Even Functions

7.3.1 Fourier Sine Series

You would recall the series for the odd function $\sin x$ has only the odd powers of *x*. Moreover, in Study Session 6, we found that an odd function has no even part. This should give you an idea that perhaps the Fourier series of an odd function should have only the sine terms. For such a function, it suffices to find the coefficients b_n, and we can write the Fourier sine series as,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad 7.15$$

with the coefficients given by,

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx \qquad 7.16$$

But since $f(x)$ is odd, the integrand in equation (7.16) is even, and we can write,

$$b_m = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin mx dx \qquad 7.17$$

7.3.1 Fourier Cosine Series

Just as in the case for the Fourier sine series, the Fourier series of an even function would involve only the constant part (don't forget a constant function is an even function, if the range includes the origin), as well as the cosine terms. Consequently, we can write the Fourier cosine series as,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad 7.18$$

with

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad 7.19$$

and

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx \qquad 7.20$$

$f(x)$ is even, the integrand in equation (7.20) is even, and we can write,

$$a_m = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos mx dx \qquad 7.21$$

7.5 The General form of Fourier Series

The general form of the Fourier series for a function $f(x)$ with period L is

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right)\right] \qquad 7.22$$

where

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx \qquad 7.23$$

and

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx \qquad 7.24$$

where x_0 is arbitrary, but is often taken as 0 or $-L/2$.

7.6 Half-Range Fourier Series

Suppose the function we are interested in is defined only in the half range $0 \le x \le L/2$. Could we write it in the form of Fourier series and if we could, what would the series look like? To start with, Fourier series is for periodic functions. As such, we extend the function to the left of $x = 0$, either as an odd or an even function, so that the function is now defined over the interval $-L/2 \le x \le L/2$. We can now assume (with the extension giving an odd or an even function) that if the extension is an odd one, then we get the Half-range Fourier sine series, and if even, it is a half-range Fourier cosine series.

For a Fourier sine series, we then make use of equations (7.15) and (7.17) and for a Fourier cosine series, we make use of equations (7.18), (7.19) and (7.21).

▪ Calculate the half-range Fourier sine series for the function $f(x) = \cos x$; $0 \le x \le \pi$. Sketch the sum of the first three terms and comment on your result.

❑ This function is illustrated in Fig. 7.4.

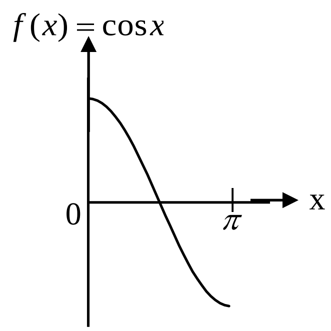

Fig. 7.4: Graph of cos x between 0 and π

For the half-range Fourier sine series, we need to expand the function into an odd function (Fig. 7.5).

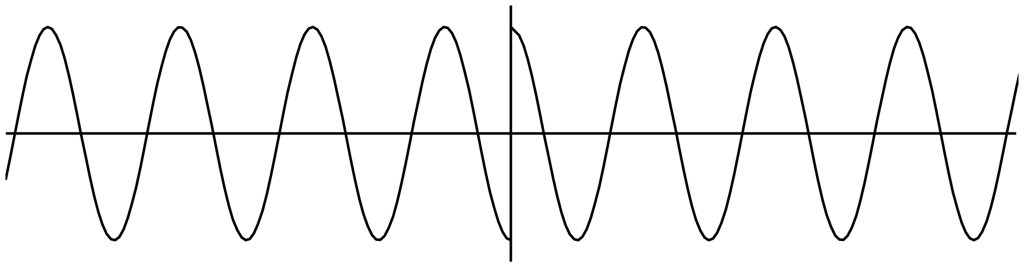

Fig. 7.5: Expansion of the function in Fig. 7.4 into an odd function

$$b_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin mx dx = \frac{2}{\pi}\int_0^{\pi}\cos x\sin mx dx = \frac{2}{\pi}\frac{1}{2}\left[\int_0^{\pi}\sin(m+1)x+\sin(m-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{m+1}\cos(m+1)x-\frac{1}{m-1}\cos(m-1)x\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}(\cos(m+1)\pi-1)+\frac{1}{m-1}(\cos(m-1)\pi-1)\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^{m+1}-1]+\frac{1}{m-1}[(-1)^{m-1}-1]\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^{m}(-1)-1]+\frac{1}{m-1}[(-1)^{m}\frac{1}{(-1)}-1]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^{m}+1]+\frac{1}{m-1}[(-1)^{m}+1]\right]$$

$$= \frac{1}{\pi}\left[\frac{(m-1)[(-1)^{m}+1]+(m+1)[(-1)^{m}+1]}{m^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{m(-1)^{m}+m-(-1)^{m}-1+m(-1)^{m}+m+(-1)^{m}+1}{m^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{m(-1)^{m}+m+m(-1)^{m}+m}{m^2-1}\right] = \frac{2m[(-1)^{m}+1]}{\pi(m^2-1)}\ ;\ m\neq 1$$

$$= \begin{cases} 0, & m\,\text{odd} \\ \dfrac{4m}{\pi(m^2-1)}, & m\,\text{even} \end{cases}$$

For $m=1$, $b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x\sin x dx = 0$

Hence, the half-range Fourier sine series for $\cos x$ is,

$$\cos x = \sum_{n=1}^{\infty} b_n \sin nx = \frac{4}{\pi}\left[\frac{2}{3}\sin 2x+\frac{4}{15}\sin 4x+\frac{6}{35}\sin 6x+\ldots\right]$$

$$= \frac{8}{\pi}\left[\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \ldots\right]$$
$$= \frac{8}{\pi}\left[\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \ldots\right]$$

The sketch of the sum of the first three terms is shown in Fig. 7.8.

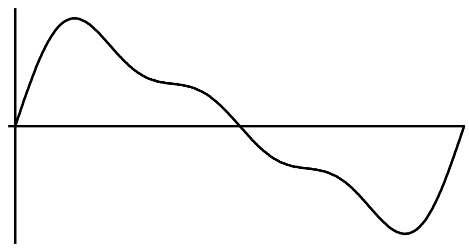

Fig. 7.8: The sum of the first three terms of half-range Fourier sine series for cos x

Even though each of the functions in the brackets is a sinusoidal function, the sum of the terms as more terms are added tends towards the function $f(x) = \cos x$.

7.7 Complex Fourier Series

The complex Fourier series is given by

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi irx/L} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r x} \qquad 7.12$$

where we have made use of the fact that $e^{irx} = \cos rx + i\sin rx$.

The Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-i\omega x}dx \qquad 7.13$$

Summary of Study Session 7

In Study Session 7, you learnt how to do the following:

1. Write a given real-valued function as a sum of odd and even functions.
2. Deduce or simplify an integral based on whether the integrand is odd or even.
3. Expand a periodic function in terms of the Fourier coefficients and calculate the coefficients.
4. Give condition under which the Fourier series of a function exists.
5. Calculate the half-range Fourier series of a given function.
6. Develop a series for the irrational number π.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self Assessment Questions for Study Session 7

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 7.1 (tests Learning Outcome 7.1)

(a) What are even and odd functions?

(b) Which of the following functions are even and which ones are odd?

(i) $x^2 \sin x \cosh x$ (ii) $|e^x| \cosh 2x$ (iii) $\sec x$

SAQ 7.2 (tests Learning Outcome 7.2)

Write the following as a sum of odd and even functions.

(i) $e^{-x} \cosh x$ (ii) $x \ln x$

SAQ 7.3 (tests Learning Outcome 7.2)

Evaluate the following integrals

(i) $\int_{-a}^{a} x^{2n+1} dx$, $n = 0,1,2,....$ (ii) $\int_{-a}^{a} x^{2n} dx$, $n = 0,1,2,....$

SAQ 7.4 (tests Learning Outcome 7.2)

Show that

(i) $\sin mx$ and $\cos nx$ are orthogonal, $-\pi \le x \le \pi$.

(ii) $\sin mx$ and $\sin nx$ are orthogonal, $m \ne n$, $-\pi \le x \le \pi$.

SAQ 7.5 (tests Learning Outcome 7.4)

Find the Fourier series for $f(x) = 1 + x$ on $[-\pi, \pi]$. Hence, prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + ...$

SAQ 7.6 (tests Learning Outcome 7.4)

Find the Fourier series for $f(x) = x$ on [0, 1].

SAQ 7.7 (tests Learning Outcome 7.4)

Find the Fourier series for $|x|, -\pi < x < \pi$

SAQ 7.8 (tests Learning Outcome 7.4)

a. Given that $F(x) = \int_0^x (2f(x) - a_0)\, dx$, show that $F(\pi) - F(-\pi) = \int_{-\pi}^{\pi} (2f(x) - a_0)\, dx$.

Hence, show that $F(\pi) = F(-\pi)$.

b. Find the Fourier coefficients of the function $x^2(2\pi^2 - x^2)$ over the interval $-\pi \le x \le \pi$.

SAQ 7.9 (tests Learning Outcome 7.5)

(a) Write the Fourier series for a periodic function of x between $-\pi$ and π .

(b) Write the general and the complex forms of Fourier series.

SAQ 7.10 (tests Learning Outcome 7.6)
What are the conditions a function must satisfy for it to be expandable in the form of a Fourier series?

SAQ 7.11 (test Learning Outcome 7.7)
Calculate the half-range Fourier cosine series for $f(x) = \sin x$; $0 \le x \le \pi$.

Solutions to SAQs

SAQ 7.1

(a) An even function is one such that $f(x) = f(-x)$, while an odd function is such that $f(x) = -f(-x)$.

(b) (i) is odd, being the product of two even functions and an odd function.

(ii) is an even function, a product of two even function.

(iii) is an even function:

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x$$

SAQ 7.2

(i) $h(x) = e^{-x}\cosh x$, $h(-x) = e^{x}\cosh(-x) = e^{x}\cosh x$

$$f(x) = \frac{1}{2}[h(x) - h(-x)] = \frac{e^{-x}\cosh x - e^{x}\cosh x}{2} = -\cosh x\left[\frac{e^{x} - e^{-x}}{2}\right]$$

$$= -\cosh x \sinh x$$

$$g(x) = \frac{1}{2}[h(x) + h(-x)] = \frac{e^{-x}\cosh x + e^{x}\cosh x}{2} = \cosh x\left[\frac{e^{x} + e^{-x}}{2}\right]$$

$$= \cosh^2 x$$

SAQ 7.3

(i) $\int_{-a}^{a} x^{2n+1}dx = 0$, the integrand being an odd function.

(ii) $$\int_{-a}^{a} x^{2n}dx = 2\int_{0}^{a} x^{2n}dx = 2\left.\frac{x^{2n+1}}{2n+1}\right|_{0}^{a} = 2\,\frac{a^{2n+1}}{2n+1}$$

SAQ 7.4

(i) $\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = 0$, the integrand is an odd function

(ii) $$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x\Big|_{-\pi}^{\pi} - \frac{1}{m+n}\sin(m+n)x\Big|_{-\pi}^{\pi}\right] = 0$$

SAQ 7.5

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi x}{L}dx,\ n = 0,\ 1,\ 2,\ \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi x}{L}dx,\ n = 1, 2, \ldots$$

The n = 0 case is not needed since the integrand in the formula for b_0 is $\sin 0$.

In the present problem,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\cos\frac{n\pi x}{\pi}dx = \frac{1}{\pi}\left(\left[\frac{(1+x)\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{(1+x)\cos nx}{n^2}dx\right) = 0$$

But since the right hand side is not defined if n = 0, the 0 index for a_0 will have to be calculated separately.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)dx = \frac{1}{\pi}\left[x + \frac{1}{2}x^2\right]_{-\pi}^{\pi} = 2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\sin nx dx = \frac{1}{\pi}\left(-\frac{1}{n}\left[(1+x)\cos(nx)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}(1+\pi-1+\pi)\cos(nx) + \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{\pi}\right) = \frac{2(-1)^{n+1}}{n}$$

So the Fourier series is

$$f(x) = 1 + x \sim 1 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx) \text{ for } [-\pi, \pi].$$

When $x = \pi/2$, then,

$$f(\pi/2) = 1 + \pi/2 = 1 + 2\left(\frac{1}{1} - \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\right)$$

$$= 1 + \frac{\pi}{2} = 1 + 2\left(1 - \frac{1}{3} + \frac{1}{5} + \ldots\right)$$

Hence,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots$$

Thus, the sum of the infinite series on the right is $\pi/4$, or we say the series converges to $\pi/4$.

On the other hand, we can say that the value of the irrational number π is given by the infinite series

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right)$$

SAQ 7.6

A general formula for the Fourier series of a function on an interval $[c, c+T]$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}\right)$$

$$a_n = \frac{2}{T}\int_c^{c+T} f(x)\cos\frac{2n\pi x}{T}dx$$

$$b_n = \frac{2}{T}\int_c^{c+T} f(x)\sin\frac{2n\pi x}{T}dx$$

In the current problem, $c = 0$ and $T = 1$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2n\pi x + b_n \sin 2n\pi x)$$

The function $f(x)$ is odd, so the cosine coefficients will all equal zero. Nevertheless, a_0 should still be calculated separately.

$$a_0 = 2\int_0^1 xdx = 1$$

$$b_n = 2\int_0^1 x\sin 2n\pi x dx$$

$$= 2\left(\left[\frac{-x\cos 2n\pi x}{2n\pi}\right]_0^1 + \int_0^1 \frac{\cos 2n\pi x}{2n\pi}dx\right)$$

$$= \frac{-1}{n\pi} + \left[\frac{\sin 2n\pi x}{(2n\pi)^2}\right]_0^1 = -\frac{1}{n\pi}$$

So the Fourier series for $f(x)$ is

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n\pi}$$

SAQ 7.7

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left(\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right) = \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2(-1)^n}{\pi n^2} - \frac{2}{\pi n^2} = \frac{2((-1)^n - 1)}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} - x\sin nx dx + \frac{1}{\pi}\int_0^{\pi} x\sin nx dx$$

$$= \frac{1}{\pi}\left(\left[\frac{x\cos(nx)}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos(nx)}{n}dx + \frac{1}{n}\left[\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}dx\right]\right)$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n} - \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right]\right]$$

$$= \frac{1}{\pi}\frac{\pi(-1)^n}{n} + \frac{1}{\pi}\frac{-\pi(-1)^n}{n} = 0$$

So,

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{-2}{(2n+1)^2}\cos((2n+1)x)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}(\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \ldots$$

SAQ 7.8

a. $F(x) = \int_0^x [2f(x) - a_0]dx$

$$F(\pi) = \int_0^{\pi}[2f(x) - a_0]dx$$

$$F(-\pi) = \int_0^{-\pi}[2f(x) - a_0]dx$$

$$F(\pi) - F(-\pi) = \int_0^{\pi}[2f(x) - a_0]dx - \int_0^{-\pi}[2f(x) - a_0]dx$$

$$= \int_0^{\pi}[2f(x) - a_0]dx + \int_{-\pi}^{0}[2f(x) - a_0]dx$$

$$= \int_{-\pi}^{\pi}[2f(x) - a_0]dx$$

$$= \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}[a_n\cos nx + b_n\sin nx]dx$$

$$= 0$$

b. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2(2\pi^2 - x^2)\cos nx dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2(2\pi^2 - x^2)\cos nx dx$$

$$= 4\pi\int_0^{\pi} x^2 \cos nx dx - \frac{2}{\pi}\int_0^{\pi} x^4 \cos nx dx$$

$$= 0 + 0 + \frac{2x}{n^2}\cos nx\Big|_0^{\pi} + \left(-\frac{1}{n^2}\right)\int_0^{\pi} 2\cos nx dx$$

$$-\frac{2}{\pi}\left[0 - \left(-\frac{4x^3}{n}\right)\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi} - \left(-\frac{1}{n^2}\right)\int_0^{\pi} 12x^2 \cos nx dx\right]$$

$$= \frac{2\pi}{n^2}(-1)^n - \frac{4\pi^3}{n^2}(-1)^n + \frac{12}{n^2}\frac{2\pi}{n^2}(-1)^n$$

$$= \frac{2\pi}{n^2}\left\{1 - 2\pi^2 + \frac{12}{n^2}\pi\right\}(-1)^n$$

SAQ 7.9

(a) $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

From equations 7.6 through 7.8,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} 1dx = \frac{2x}{\pi}\Big|_0^{\pi} = 2$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1.\cos mx dx = \frac{2}{\pi}\int_0^{\pi} \cos mx dx$$

$$= \frac{2}{\pi m}\sin mx\Big|_0^{\pi} = 0$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1.\sin mx dx = -\frac{1}{m\pi}\cos mx\Big|_{-\pi}^{\pi} = 0$$

(b) The general form of the Fourier series for a function $f(x)$ with period L is

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right)\right]$$

where

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx,$$

and

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx$$

where x_0 is arbitrary, but is often taken as 0 or $-L/2$.

The complex Fourier series is given by

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi irx/L} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r x}$$

where we have made use of the fact that $e^{irx} = \cos rx + i\sin rx$.

The Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-i\omega x}dx$$

SAQ 7.10

Conditions that must be fulfilled in order that a function $f(x)$ may be expanded as a Fourier series:

(i) The function must be periodic.

(ii) It must be single-valued and continuous, except possibly at a finite number of finite discontinuities.

(iii) It must have only a finite number of maxima and minima within one period.

(iv) The integral over one period of $|f(x)|$ must converge.

If these conditions are satisfied, then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous. At a point of finite discontinuity, the Fourier series converges to the value half-way between the upper and the lower values.

SAQ 7.11

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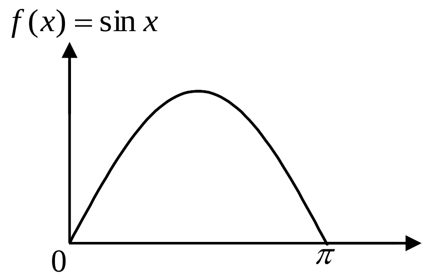


Fig. 7.6: Graph of $f(x) = \sin x; 0 \le x \le \pi$

Since what we need is the half-range Fourier cosine series, we expand the function into an even function as shown in Fig. 7.7.

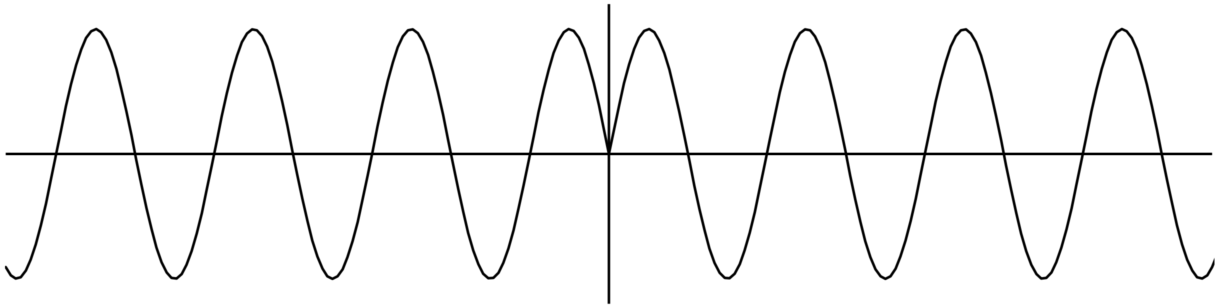


Fig. 7.7: Expansion of the function in Fig. 7.6 into an even function

For $m = 0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\sin x dx = \frac{2}{\pi}[-\cos x]_0^{\pi} = -\frac{2}{\pi}[\cos\pi - \cos 0] = \frac{2(1+1)}{\pi} = \frac{4}{\pi}$$

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\cos mx dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos mx dx = \frac{2}{\pi}\frac{1}{2}\left[\int_0^{\pi}\sin(m+1)x + \sin(1-m)x\right]dx$$

Even though we could have carried out the integration in the usual way, we have decided to make use of the procedure in ITQ, by writing the second term in the integrand as $\sin(1-m)x = -\sin(m-1)x$.

Hence,

$$a_m = \frac{2}{\pi}\frac{1}{2}\left[\int_0^{\pi}\sin(m+1)x - \sin(m-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{m+1}\cos(m+1)x + \frac{1}{m-1}\cos(m-1)x\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}(\cos(m+1)\pi - 1) - \frac{1}{m-1}(\cos(m-1)\pi - 1)\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^{m+1} - 1] - \frac{1}{m-1}[(-1)^{m-1} - 1]\right]$$

$$= -\frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^m(-1) - 1] - \frac{1}{m-1}[(-1)^m\frac{1}{(-1)} - 1]\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{m+1}[(-1)^m + 1] - \frac{1}{m-1}[(-1)^m + 1]\right]$$

$$= \frac{1}{\pi}\left[\frac{(m-1)[(-1)^m + 1] - (m+1)[(-1)^m + 1]}{m^2 - 1}\right]$$

$$= \frac{1}{\pi}\left[\frac{m(-1)^m + m - (-1)^m - 1 - m(-1)^m - m - (-1)^m - 1}{m^2 - 1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2(-1)^m - 2}{m^2 - 1}\right] = -\frac{2[(-1)^m + 1]}{\pi(m^2 - 1)}\ ;\ m \neq 1.$$

$$= \begin{cases} 0, & m \text{ odd} \\ -\dfrac{4}{\pi(m^2-1)}, & m \text{ even} \end{cases}$$

Hence, the half-range Fourier cosine series for $\sin x$ is,

$$\sin x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{\pi} \;\; + \frac{4}{\pi}\left[-\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x - \frac{1}{35}\cos 6x - \ldots\right]$$

$$\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos 2x + \frac{2}{15}\cos 4x + \frac{3}{35}\cos 6x + \ldots\right]$$

Observation

The graphs for the sum of the first three (thin curve) and the first five (thick curve) terms are displayed in Fig. 7.8.

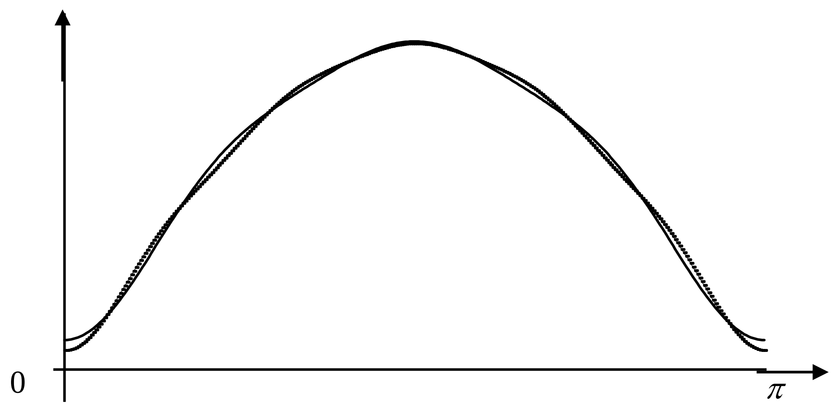


Fig. 7.8: The graph of the sum of the first 3 (thin) and the sum of the first 5 terms (thick)

Observation: As the number of terms becomes large, the function approximates a sine function as shown in Fig. 7.9 for the sum of the first 120 terms.

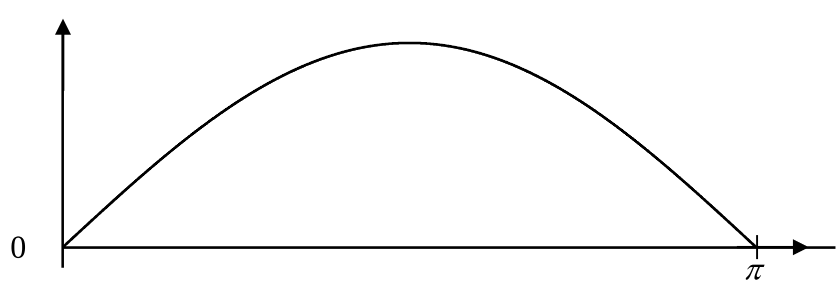


Fig. 7.9: The graph of the sum of the first 120 terms

Study Session 8 Fourier Transform

Introduction

In Study Session 7, we explored Fourier series, and mentioned that Fourier series works only with periodic functions. What happens if the function of interest is not periodic? In that case, we make recourse to Fourier transform. Thus, **Fourier transform** is the generalisation of Fourier series to include non-periodic functions. In the forced harmonic oscillator, Fourier series comes to the rescue when non-sinusoidal driving forces are involved. The driving forces are decomposed by Fourier methods into sine waves of separate frequency and phase. The system response is then determined for each frequency. Finally the inverse Fourier transform is used to determine the overall response. On the other hand, Engineers often use the Laplace Transform for the same purpose.

Learning Outcomes for Study Session 8

At the end of this study session, you should be able to:

8.1 Write the expression for the Fourier transform of a function (SAQ 8.2-8.5).

8.2 Find the Fourier transform of a given function (SAQ 8.2-8.5).

8.3 State the properties of a Fourier transform (SAQ 8.9).

8.4 Know and be able to apply the properties of the Fourier transform (SAQ 8.3 and 8.4).

8.5 Evaluate the Dirac delta function of a general function (SAQ 8.1 and 8.6)

8.6 List the properties of the Dirac delta function (SAQ 8.7).

8.7 Evaluate some definite integrals with the help of Fourier transform (SAQ 8.8).

8.1 The Fourier Transform

We learnt in Study Session 7 that the Fourier series of a function of x can be written as,

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r x/L} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r x} \qquad 8.1$$

The Fourier coefficients are given by

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-i\omega x}dx \qquad 8.2$$

If we allow T to tend to infinity, and recall that $\Delta\omega = \dfrac{2\pi}{T}$, $\Delta\omega$ tends to zero, and we can replace the summation in Equation (8.1) with an integral. Thus, the infinite sum of terms in the Fourier series becomes an integral, and the coefficients c_r become functions of the continuous variable ω, as follows:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \qquad 8.3$$

and the Fourier inverse transform as

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \qquad 8.4$$

The Fourier transform exists if the following conditions are satisfied:

1. $f(t)$ is piecewise continuous on every finite interval
2. $f(t)$ is absolutely integrable on the t-axis.

Just as we have the pair (t,ω), we could also get involved in the pair (x,k).

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \qquad 8.5$$

and the Fourier inverse transform as

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk \qquad 8.6$$

▪ Find the Fourier transform of the constant $f(t)=c$, given that $-a\le t\le a$.

❑

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} ce^{-i\omega t}dt=\frac{c}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega t}dt=\frac{c}{\sqrt{2\pi}}\frac{1}{-i\omega}\left[e^{-i\omega t}\right]_{-a}^{a}$$

$$= c\sqrt{\frac{2}{\pi}}\frac{\sin\omega a}{\omega}$$

▪ Let us find the Fourier transform of the box-car function, $f(t)=\begin{cases}1, |x|\le a\\ 0, \text{otherwise}\end{cases}$

❑

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1e^{-ikx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{-ik}e^{-ikx}\Big|_{-a}^{a}=\frac{1}{\sqrt{2\pi}}\frac{1}{k}\left(e^{-ika}-e^{-ika}\right)=\frac{2}{\sqrt{2\pi}}\frac{\sin ka}{k}$$

$$=\sqrt{\frac{2}{\pi}}\frac{\sin ka}{k}$$

We can rewrite this as,

$$F(k)=\sqrt{\frac{2}{\pi}}a\frac{\sin ka}{ka}$$

The graph of $f(x)$ and its Fourier transform, $F(k)$ are shown in Fig. 8.1.

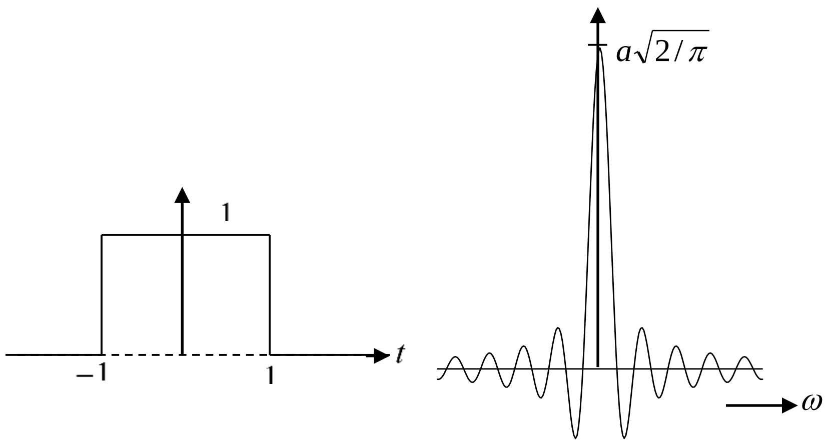


Fig. 8.1: The Box-car function and its Fourier transform

The inverse Fourier transform of $\sqrt{\frac{2}{\pi}}\frac{\sin ka}{k}$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin ka}{k}e^{ikx}dk = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}\sin ka}{k}dk$$

But this should be equal to the original function whose Fourier transform we took. It follows, therefore, that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}\sin ka}{k}dk = \begin{cases} 1, |x| < a \\ \frac{1}{2}, x = \pm a \\ 0, |x| > a \end{cases}$$

The value assumed at the discontinuity, very much as is the case in Fourier series, is halfway up the discontinuity, $\frac{1}{2}(1-0) = \frac{1}{2}$.

We might have written $e^{ika} = \cos ka + i\sin ka$. Then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\cos kx + i\sin kx)\sin ka}{k}dk = \begin{cases} 1, |x| < a \\ \frac{1}{2}, x = \pm a \\ 0, |x| > a \end{cases}$$

Thus, the real parts equated gives,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos kx\sin ka}{k}dk=\begin{cases}1,|x|<a\\ \frac{1}{2},x=\pm a\\ 0,|x|>a\end{cases}$$

and the imaginary part gives,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos kx\sin ka}{k}dk=0$$

This has provided us with a way of integrating the functions $\frac{\cos kx\sin ka}{k}$ and $\frac{\sin ka\sin ka}{k}$ with respect to k, functions which ordinarily could not have been otherwise integrated.

8.2 The Dirac Delta Function

The Dirac delta function centred on $x=0$ is defined as

$$\delta(x)=\begin{cases}0,x\neq 0\\ \infty,x=0\end{cases} \tag{8.7}$$

Note that the **Dirac delta function** is not truly a function because it lacks continuity. Rather, it is a distribution in the language of Mathematics.

The Dirac delta function centred on $x=x_0$, is (Fig. 8.2)

$$\delta(x-x_0)=\begin{cases}0,x\neq x_0\\ \infty,x=x_0\end{cases} \tag{8.8}$$

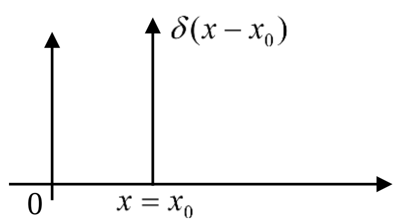


Fig. 8.2: The Dirac delta function located at $x=x_0$

If centred at the origin,

$$x_0=0\text{, and }\delta(x)=\begin{cases}0,x\neq 0\\ \infty,x=0\end{cases} \tag{8.9}$$

The Dirac delta function satisfies the conditions:

$$\int_{-\infty}^{\infty}\delta[x-x_0]dx=1 \tag{8.10}$$

This means that even though the delta function is a ‘spike’ of zero width and infinite height, the integral is taken to be unity.

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \text{ (The sifting property)} \qquad 8.11$$

Notice that the Dirac delta function sifts out $f(x)$ at the point where the ‘spike’ is located, that is at $x = x_0$.

8.3.1 Generalisation of the Dirac Delta Function

The general Dirac delta functions is $\delta[g(x)]$. Let us take a look at

$$\int_{-\infty}^{\infty} \delta[g(x)]dx \qquad 8.12$$

Let $g(x) = y$, then $g'(x)dx = dy$

In addition, let $g'(x)\big|_y \equiv g'(x)$ as a function of y

Then, (8.11) becomes

$$\int_{-\infty}^{\infty} \delta[y]\frac{dy}{g'(x)\big|_y} = \int_{-\infty}^{\infty} G(y)\delta(y)dy = G(y=0) = G(0) = \left[\frac{1}{g'(x)\big|_y}\right]_{y=0} \qquad 8.13$$

i.e., $\dfrac{1}{g'(x)}$ as a function of y, evaluated at $y = 0$.

- Evaluate $\int_{-\infty}^{\infty} \delta(x^2 - a^2)dx$, where a is a constant.

- In comparison with equation (8.12), $g(x) = x^2 - a^2 = y$

$$g'(x) = 2x$$

$$g'(x)\big|_y = 2\sqrt{y + a^2}$$

$$g'(x)\big|_{y=0} = 2a$$

Therefore,

$$\int_{-\infty}^{\infty} \delta(x^2 - a^2)dx = \frac{1}{2a}$$

8.3.2 The Relationship between the Dirac Delta function and Fourier transforms

Putting Equation 8.3 in Equation 8.4,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega t}\int_{-\infty}^{\infty}duf(u)e^{-i\omega u} \qquad 8.14$$

where we have changed the dummy variable t to u in the expression for $F(\omega)$.

This can be written as

$$f(t)=\int_{-\infty}^{\infty}duf(u)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega(t-u)}\right\} \qquad 8.15$$

Comparing with the equation defining the sifting property (Equation 8.11),

$$\int_{-\infty}^{\infty}dxf(x)\delta(x-x_0)=f(x_0) \quad \text{(The sifting property)}$$

we see that we can write (from equations 8.11 and 8.15)

$$\delta(t-u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(t-u)}d\omega \qquad 8.16$$

▪ Find the Fourier transform of the box-car function,

$$f(t)=\begin{cases}1, & -1\le t\le 1\\ 0, & otherwise\end{cases}$$

This is equivalent to a signal turned on at time t = -1 and off at time t = 1.

❑

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}1e^{-i\omega t}dt=-\frac{1}{i\omega}\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\Big|_{-1}^{1}=\frac{1}{\sqrt{2\pi}}\frac{2i}{i\omega}\left[\frac{e^{i\omega}-e^{-i\omega}}{2i}\right]=\frac{2}{\sqrt{2\pi}}\frac{\sin\omega}{\omega}$$

This is the sinc function. The box-car function as well as its Fourier transform (the sinc function) are shown in Fig. 8.2.

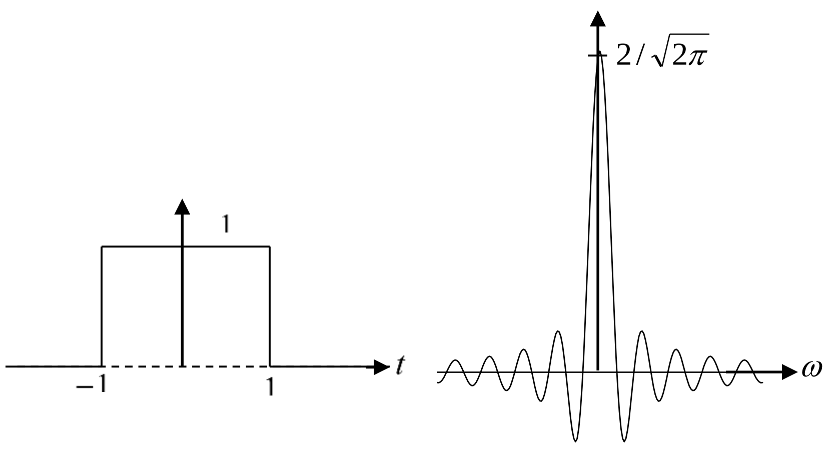


Fig. 8.2: The Box-car function and its Fourier transform.

- What is the inverse Fourier transform of the box-car function

$$f(t)=\begin{cases}1, & -1\le t\le 1\\ 0, & otherwise\end{cases}$$

- ❑

$$\frac{1}{\sqrt{2\pi}}\int_{-d}^{d}1\cdot e^{i\omega t}d\omega=\frac{1}{\sqrt{2\pi}\,i}\frac{e^{i\omega t}}{t}\bigg|_{-d}^{d}=\frac{2d}{\sqrt{2\pi}}\frac{e^{itd}-e^{-itd}}{2idt}=\frac{2d}{\sqrt{2\pi}}\frac{\sin dt}{dt}$$

The graphs of the function and its inverse Fourier transform are shown in Fig. 8.3. It is clear that as d tends to infinity, the peak at $t=0$ tends to infinity and the function becomes narrower at $t=0$.

We conclude that the Dirac delta function can be viewed as

$$\lim_{d\to\infty}\left(\frac{\sin dt}{dt}\right) \qquad 8.17$$

The Fourier transform of the function on the right is the function on the left in Fig. 8.3.

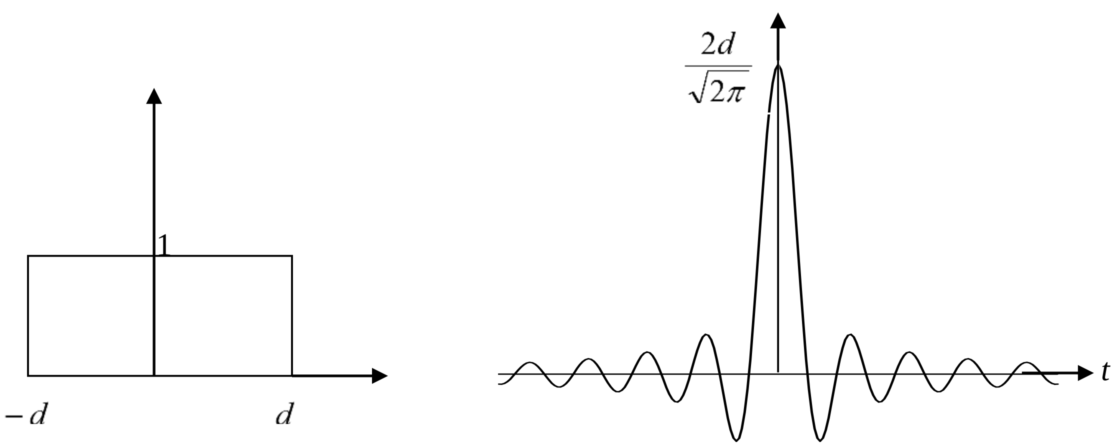


Fig. 8.3: The Fourier transform of the sinc function is the box-car function

8.3 Properties of the Fourier transform

Let the Fourier transform of $f(t)$ be $F(\omega)$. Then,

(i) Differentiation

$$F\left[f'(t)\right]=i\omega F(\omega) \qquad 8.18$$

$$F\left[f''(t)\right]=-\omega^2F(\omega) \qquad 8.19$$

(ii) Integration

$$F\left[\int f(s)ds\right]=\frac{1}{i\omega}F(\omega)+2\pi c\delta(\omega) \qquad 8.20$$

where the term $2\pi c\delta(\omega)$ is the Fourier transform of the constant of integration associated with the indefinite integral.

(iii) Scaling

$$F\left[f(at)\right]=\frac{1}{a}F(\omega) \qquad 8.21$$

(iv) Exponential multiplication

$$F\left[e^{\alpha t}f(t)\right]=F(\omega+i\alpha) \qquad 8.22$$

where α may be real, imaginary or complex.

(v) Duality

Let us build a new time-domain function $g(t)=f(t)$ by exchanging the roles of time and frequency.

Then,

$$G(\omega)=\sqrt{2\pi}F(-\omega)$$

▪ Find the Fourier transform of the function,

$$f(t)=\begin{cases}0, \; t<0\\ e^{-\alpha t}, t\geq 0\end{cases}$$

This is equivalent to switching on a signal at time $t=0$, with the signal decaying exponentially with time thereafter.

❑

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\alpha t}e^{-i\omega t}dt=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(i\omega+\alpha)t}dt$$

$$=-\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha+i\omega}e^{-(i\omega+\alpha)t}\Big|_0^\infty=\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha+i\omega}$$

We can write this expression as an amplitude multiplied by the phase, as we would any complex number.

Since modulus of $\alpha+i\omega$ is $\sqrt{\alpha^2+\omega^2}$, and its phase $\phi(\omega)=\tan^{-1}\frac{\omega}{\alpha}$, then we can write,

$$F(\omega)=\frac{1}{\sqrt{2\pi}\sqrt{\alpha^2+\omega^2}}e^{i\phi(\omega)}=A(\omega)e^{i\phi(\omega)}$$

$F(\omega)$ is a complex number with amplitude $A(\omega)$ and phase angle $\phi(\omega)=\tan^{-1}\dfrac{\omega}{\alpha}$.

- Find the Fourier transform of the box-car function,

  $$f(t)=\begin{cases}1, & -1\le t\le 1\\ 0, & otherwise\end{cases}$$

This is equivalent to a signal turned on at time t = -1 and off at time t = 1.

❑

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}1e^{-i\omega t}dt=-\frac{1}{i\omega}\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\Big|_{-1}^{1}=\frac{1}{\sqrt{2\pi}}\frac{2i}{i\omega}\left[\frac{e^{i\omega}-e^{-i\omega}}{2i}\right]=\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

This is the sinc function. The box-car function as well as its Fourier transform are shown in Fig 8.2, but now with d being equal to 1.

- Find an expression for the Fourier transform of the displacement of the forced damped harmonic oscillator,

  $$\frac{d^2x(t)}{dt^2}+2\beta\frac{dx(t)}{dt}+\omega_0{}^2x(t)=f(t)$$

  where all symbols have their usual meanings, with $f(t)$ an input signal. If the forcing term is a delta function located at $t=t_0$, write an integral expression for $x(t)$.

❑ Taking the Fourier transform (using the differentiation property) of both sides,

$$-\omega^2X(\omega)+2\beta i\omega X(\omega)+\omega_0{}^2X(\omega)=F(\omega)$$

Solving for $X(\omega)$,

$$X(\omega)=\frac{F(\omega)}{\omega_0{}^2-\omega^2+2i\beta}$$

If for example, $f(t)$ is a delta function, then, $f(t)=\delta(t-t_0)$, which corresponds to an impulsive force at time $t=t_0$. We assume the system is at equilibrium and at rest at time t = 0, meaning $x(0)=0=(dx/dt)_{t=0}$. With $t_0>0$, then,

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t-t_0)e^{-i\omega t}dt=\frac{1}{\sqrt{2\pi}}e^{-i\omega t_0}\ \text{(using the sifting property)}$$

and

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t_0}}{{\omega_0}^2 - \omega^2 + 2i\beta}$$

Taking the inverse Fourier transform,

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega t_0}}{{\omega_0}^2 - \omega^2 + 2i\beta} \right) e^{i\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-t_0)}}{{\omega_0}^2 - \omega^2 + 2i\beta} d\omega$$

We have in principle, solved the problem. We have turned the ordinary differential equation into an integral, although this integral is by no means easy to solve.

Summary of Study Session 8

In Study Session 8, you learnt the following:

1. How to write the expression for the Fourier transform of a function.
2. How to find the Fourier transform of a given function.
3. The properties of the Dirac delta function.
4. How to evaluate the Dirac delta function of a function of a single variable.
5. How to evaluate some definite integrals with the help of Fourier transform.
6. The properties of the Fourier transform.
7. How to evaluate the Dirac delta function of a general function.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self-Assessment Questions for Study Session 8

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the Solutions to the Self-Assessment Questions at the end of this study session.

SAQ 8.1 (tests Learning Outcome 8.4)

Evaluate $\int_{-\infty}^{\infty} \delta(3x-2)x^2 dx$.

SAQ 8.2 (tests Learning Outcomes 8.1 and 8.2)

Find the Fourier transform of $f(t) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 t^2}$.

SAQ 8.3 (tests Learning Outcomes 8.1 and 8.2)

Find the Fourier transform of $tf(t)$.

SAQ 8.4 (tests Learning Outcome 8.3)

Find the Fourier transform of $e^{|t|}$.

SAQ 8.5 (tests Learning Outcome 8.3)

Find the Fourier transform of the delta function.

SAQ 8.6 (tests Learning Outcome 8.4)

Evaluate

$\int_{-\infty}^{\infty} \delta(2x^3 + a^2)dx$, where a is a constant.

SAQ 8.7 (tests Learning Outcome 8.5)

State the properties of the Dirac delta function.

SAQ 8.8 (tests Learning Outcome 8.6)

Find the Fourier transform of the function,

$$f(x) = \begin{cases} e^{-ax}, x > 0 \\ 0, \quad x < 0 \end{cases}$$

given that $a > 0$. Hence, deduce the value of the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{a+ik} dk$.

SAQ 8.9 (tests Learning Outcome 8.7)

State the properties of the Fourier transform.

Solutions to SAQs

SAQ 8.1

$$\int_{-\infty}^{\infty}\delta(3x-2)x^2dx$$

We need to find the zero of $f(x)=3x-2$. This is $x=2/3$. Also, $f'(x)=3$.

Therefore, $\int_{-\infty}^{\infty}\delta(3x-2)x^2dx = \int_{-\infty}^{\infty}\frac{1}{3}\delta\left(x-\frac{2}{3}\right)x^2dx = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27}$

SAQ 8.2

$$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\frac{\alpha}{\sqrt{\pi}}e^{-\alpha^2t^2}e^{-i\omega t} = \frac{\alpha}{\pi\sqrt{2}}\int_{-\infty}^{\infty}dte^{-(\alpha\ t^2+i\omega t)}$$

We complete the square to evaluate the integral.

$$\alpha^2t^2+i\omega t=\alpha^2t^2+i\omega t+\gamma-\gamma=(\alpha t+\beta)^2-\gamma$$

$$2\alpha\beta t=i\omega t\rightarrow\beta=\frac{i\omega}{2\alpha}$$

$$\gamma=\beta^2=-\frac{\omega^2}{4\alpha^2}$$

Thus,

$$F(\omega)=\frac{\alpha}{\pi\sqrt{2}}e^{-\frac{\omega^2}{4\alpha^2}}\int_{-\infty}^{\infty}dte^{-\left(\omega t+\frac{i\omega}{2\alpha}\right)^2}$$

Let $x=\alpha t+\frac{i\omega}{2\alpha}\Rightarrow dx=\alpha dt$

$$F(\omega)=\frac{\alpha}{\pi\sqrt{2}}e^{-\frac{\omega^2}{4\alpha^2}}\frac{1}{\alpha}\int_{-\infty}^{\infty}dxe^{-x^2}=\frac{1}{\pi\sqrt{2}}e^{-\frac{\omega^2}{4\alpha^2}}\times\sqrt{\pi}=\frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{4\alpha^2}}$$

SAQ 8.3

$$\begin{aligned}
F[tf(t)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}tf(t)e^{-i\omega t}dt\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)(i)\frac{d}{d\omega}(e^{-i\omega t})dt\\
&= i\frac{d}{d\omega}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt\right]\\
&= i\frac{d}{d\omega}F[f(t)]
\end{aligned}$$

SAQ 8.4

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|t|}e^{-i\omega t}dt$$

But $|t|$ is v-shaped and centred at $t=0$. $-|t|$ is a tent in the negative vertical axis with the peak at $t = 0$. The function will then be t from ∞ to 0, and $-t$ from 0 to ∞. Hence,

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{t}e^{-i\omega t}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-t}e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(1-i\omega)t}dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(-1-i\omega)t}dt$$

$$= \frac{1}{(1-i\omega)\sqrt{2\pi}} e^{(1-i\omega)t}\Big|_{-\infty}^{0} + \frac{1}{(-1-i\omega)\sqrt{2\pi}} e^{(1-i\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{(1-i\omega)\sqrt{2\pi}} + \frac{1}{(1+i\omega)\sqrt{2\pi}} = \frac{1+i\omega+1-i\omega}{(1+\omega^2)\sqrt{2\pi}} = \frac{2}{(1+\omega^2)\sqrt{2\pi}}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{(1+\omega^2)}$$

SAQ 8.5

Fourier Transform of the delta function:

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dte^{-i\omega t}\delta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dte^{-i\omega t}\delta(t-0) = \frac{1}{\sqrt{2\pi}}e^{-i\omega(0)} = \frac{1}{\sqrt{2\pi}}$$

The inverse transform is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dte^{-i\omega t}\frac{1}{\sqrt{2\pi}} = \delta(t)$$

Now, $FT\left(\frac{df}{dt}\right) = i\omega FT(f) = i\omega F(\omega)$

Therefore, for the square pulse,

$$\frac{df}{dt} = \delta(t+T/2) - \delta(t-T/2) \quad \rightarrow \quad F\left(\frac{df}{dt}\right) = F(\delta(t+T/2) - \delta(t-T/2))$$

But $FT(f(t-t_0)) = FT(f(t))e^{-i\omega t_0}$

Thus, $FT\left(\frac{df}{dt}\right) = (e^{-i\omega(-T/2)} - e^{-i\omega T/2})FT(\delta(t))$

$$= i\sqrt{\frac{2}{\pi}}\sin\frac{\omega T}{2} = i\omega F(\omega)$$

Hence, $F(\omega) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin(\omega T/2)}{\omega}\right]$

SAQ 8.6

$$g(x) = 2x^3 + a^2 = y$$

$$g'(x) = 6x^2$$

$$g'(x)|_y = 6\left(\frac{y-a^2}{2}\right)^2$$

Thus, $g'(x)\big|_{y=0} = \frac{6a^{1/3}}{4^{1/3}}\ g'(x)\big|_{y=0} = -\frac{6a^{1/3}}{4^{1/3}}$

Therefore, $\int_{-\infty}^{\infty}\delta(2x^3 + a^2)dx = -\frac{4^{1/3}}{6a^{1/3}}$

SAQ 8.7

$$\int_{-\infty}^{\infty}\delta[x - x_0]dx = 1$$

$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$ (The sifting property)

SAQ 8.8

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-ax}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a+ik)x}dx = -\frac{1}{\sqrt{2\pi}}\frac{1}{a+ik}e^{-(a+ik)x}\Big|_0^{\infty}$$

$$= -\frac{1}{\sqrt{2\pi}}\frac{1}{a+ik}(0-1)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{a+ik}$$

The inverse Fourier transform is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{a+ik}e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{a+ik}dk$$

Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{a+ik}dk=\begin{cases}e^{-ax}, x>0\\ \frac{1}{2}, \quad x=0\\ 0, \quad x<0\end{cases}$$

Yet again, we are able to evaluate an otherwise intractable definite integral.

SAQ 8.9

Properties of the Fourier transform

Let the Fourier transform of $f(t)$ be $F(\omega)$. Then,

(i) Differentiation

$$F\left[f'(t)\right]=i\omega F(\omega)$$

$$F\left[f''(t)\right]=-\omega^2 F(\omega)$$

(ii) Integration

$$F\left[\int f(s)ds\right]=\frac{1}{i\omega}F(\omega)+2\pi c\delta(\omega)$$

where the term $2\pi c\delta(\omega)$ is the Fourier transform of the constant of integration associated with the indefinite integral.

(iii) Scaling

$$F\left[f(at)\right]=\frac{1}{a}F(\omega)$$

(iv) Exponential multiplication

$$F\left[e^{\alpha t}f(t)\right]=F(\omega+i\alpha)$$

where α may be real, imaginary or complex.

(v) Duality

Let us build a new time-domain function $g(t)=f(t)$ by exchanging the roles of time and frequency.

Then,

$$G(\omega)=\sqrt{2\pi}F(-\omega)$$

Study Session 9 Laplace Transform and Application I

Introduction

This study session is a useful concept in Physics in that it converts differential, integral and integro-differential equations into algebraic equations. In particular, we can transform any of these equations (which ordinarily at times might look intractable) into an algebraic equation, or a set of algebraic equations, solve the equations, and then transform the solution to obtain the solution of the appropriate original equation. You can then truly appreciate the importance of this study session.

Learning Outcomes of Study Session 9

At the end of this study session, you should be able to:

9.1 Write the expression for the Laplace transform of a given function of a variable (SAQ 9.3, 9.6).

9.2 Find the Laplace transform of some functions (SAQ 9.2-9.4, 9.7, 9.8).

9.3 State the properties of the Laplace transform (SAQ 9.2, 9.3-9.5).

9.4 Apply the properties of Laplace transform (SAQ 9.2-9.4)

9.5 Find the inverse Laplace transform of some functions (SAQ 9.7,9.8).

9.6 Find the Laplace transform of a periodic function (SAQ 9.6).

9.7 Laplace transform a differential equation (SAQ 9.5, 9.7).

9.8 Apply Laplace transform in solving an ordinary differential equation with constant coefficients (SAQ 9.7).

9.9 Apply Laplace transform in solving an integral equation (SAQ 9.8).

9.2 Laplace Transform

We are familiar with the differential operator, $\frac{d}{dx}$ and the integral operator, $\int f(x)dx$.

The general form of an integral transform of $f(x)$ is

$$F(s) = \int_{-\infty}^{\infty} K(s,t) f(t)dt \qquad 9.1$$

$K(s,t)$ is the kernel of the transformation.

Different forms of $K(s,t)$ give different transformations.

(i) Fourier Transform

$s \equiv \omega$, $t \equiv t$

$$K(\omega,t) = e^{-i\omega t} / \sqrt{2\pi} \qquad 9.2$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad 9.3$$

(ii) Laplace Transform

$s \equiv p \quad , \; t = t$

$$K(s,t) = \begin{matrix} e^{-pt};\; 0 \le t \le \infty \\ 0; \quad otherwise \end{matrix} \qquad 9.4$$

$$L\{f(t)\} = \int_0^{\infty} e^{-pt} f(t) dt \qquad 9.5$$

9.2 Rules of Laplace Transform

(i) Linearity of Laplace transform and inverse Laplace transform:

$$L(af(t) + bg(t)) = aL(f(t)) + bL(g(t)) \qquad 9.6$$

where a and b are constants

(ii) The time-shift or t-shift rule

$$L\{f(t - t_0)\} = e^{-pt_0} F(p) \qquad 9.7$$

(iii) Frequency shift

$$L(e^{at} f(t)) = F(p - a) \qquad 9.8$$

(iv) Time scaling

$$L\{f(at)\} = \frac{1}{|a|} F\left(\frac{p}{a}\right) \qquad 9.9$$

(v) Differentiation Rule

$$L\{f^n(t)\} = p^n L\{f(t)\} - p^{n-1} f(0) - p^{n-2} f'(0) - \cdots - p^0 f^{(n-1)}(0) \qquad 9.10$$

(vi) Integration rule

$$L\left\{\int_0^t f(t) dt\right\} = \frac{F(p)}{p} \qquad 9.11$$

More generally,

$$L\left\{\int f(t) dt\right\} = \frac{F(p)}{p} + \frac{1}{p}\left[\int f(t) dt\right]_{t=0} \qquad 9.12$$

where $\left[\int f(t) dt\right]_{t=0}$ is the value of the integral at $t = 0$.

- Find the Laplace transform of $f(t) = 1$.

- ❑

$$L\{f(t)\} = L\{1\} = \int_0^\infty e^{-pt} \cdot 1 dt = -\frac{e^{-pt}}{p}\bigg|_0^\infty = -\frac{e^{-\infty p}}{p} - \left(-\frac{e^0}{p}\right) = \frac{1}{p} \qquad 9.13$$

Therefore, the inverse Laplace transform of $\frac{1}{p}$, i.e., $L^{-1}\left\{\frac{1}{p}\right\} = 1$

- Find the Laplace transform of e^{-at}, where a is a constant.

- ❑

$$L\{e^{-at}\} = \int_0^\infty e^{-pt} e^{-at} dt = \int_0^\infty e^{-(p+a)t} dt = \frac{1}{p+a} e^{-(p+a)t}\Big|_0^\infty = \frac{1}{p+a} \qquad 9.14$$

- Find the Laplace transform of $\cos\omega t$.

- ❑

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$L\{\cos\omega t\} = L\left\{\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right\} = \frac{1}{2}\left[L\{e^{i\omega t}\} + L\{e^{-i\omega t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p - i\omega} + \frac{1}{p + i\omega}\right] = \frac{p}{p^2 + \omega^2} \qquad 9.15$$

We have made use of the linearity property.

- Show that $L\{\sin\omega t\} = \dfrac{\omega}{p^2 + \omega^2}$ 9.16

- ❑

$$\sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$L\{\sin\omega t\} = L\left\{\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\} = \frac{1}{2i}\left[L\{e^{i\omega t}\} - L\{e^{-i\omega t}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{p-i\omega}-\frac{1}{p+i\omega}\right]=\frac{\omega}{p^2+\omega^2}$$

We have used the linearity property here.

- Find the Laplace transform of t^n.

❑

$$L\{t^n\}=\int_0^\infty t^n e^{-pt}dt$$

where n is a constant.

Let $I_n=\int_0^\infty t^n e^{-pt}dt=\frac{-1t^n}{p}e^{-pt}+\frac{n}{p}\int_0^\infty t^{n-1}e^{-pt}dt=\frac{n}{p}I_{n-1}$

where we have made use of $u=t^n$, $du=nt^{n-1}dt$, $dv=e^{-pt}dt$ and $v=-\frac{1}{p}e^{-pt}$

and $\int udv=uv-\int vdu$

$$I_{n-1}=\int_0^\infty t^{n-1}e^{-pt}dt$$

$$= \frac{n}{p}\cdot\frac{(n-1)}{p}I_{n-2}$$

$$= \frac{n}{p}.\frac{(n-1)}{p}\cdot\frac{(n-2)}{p}\cdots I_0$$

$$I_0=\int_0^\infty t^{1-1}e^{-pt}dt=\frac{1}{p}$$

Hence, $\int_0^\infty t^n e^{-pt}dt=\frac{n!}{p^{n+1}}$ 9.17

- Find the Laplace tranform of $f(t-t_0)$.

❑

$$L\{f(t-t_0)\}=\int_0^\infty e^{-pt}f(t-t_0)dt= I$$

Let $t-t_0=\tau$

Then, $t=t_0+\tau$

$dt=d\tau$

Therefore, $I = \int_{-t_0}^{\infty} e^{-p(t_0+\tau)} f(\tau) d\tau$

Check that the limits are indeed as indicated in the last equation, given the transformation from t to τ .

Therefore, $I = e^{-pt_0} \int_{-t_0}^{\infty} e^{-p\tau} f(\tau) d\tau$

$= e^{-pt_0} F(p)$

Therefore,

$$L\{f(t-t_0)\} = e^{-pt_0} F(p) \qquad 9.18$$

This is the t-shift rule.

▪ Find the Laplace transform of $\frac{d^2}{dt^2} t^3$

❑

From equation (9.17), $L\{t^3\} = \frac{3!}{p^4}$

$$L\left\{\frac{d^2}{dt^2} t^3\right\} = p^2 L\{t^3\} - p^1 \cdot 0 - p^0 \cdot 0$$

Here, we have applied the differentiation rule.

Check: $L\{6t\} = 6\left(\frac{1}{p^2}\right)$

9.3 Rules of Partial Fraction Decomposition

You may find the following rules of partial fraction decomposition useful.

(i) Every linear factor $ax+b$ in the denominator of the expression has a partial fraction of the form,

$\frac{A}{ax+b}$, where A is a real number.

(ii) For any linear factor $ax+b$ that occurs *r* times in the denominator, there is a sum of *r* partial fractions,

$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_r}{(ax+b)^r}$, where A_1, A_2, … A_r are all real numbers.

(iii) For every factor ax^2+bx+c in the denominator, there is a partial fraction of the form,

$$\frac{Ax+B}{x^2+bx+c} \text{ for real } A \text{ and } B.$$

9.4 Application of Laplace Transform in solving single Differential, Integral and Integro-Differential Equations

Laplace transform can be applied to solve differential, integral and integro-differential equations. First, we take the Laplace transform of the equation of interest, and solve for the Laplace transform of the variable of interest. We can then find the inverse Laplace transform, which is the variable we set out to find in the first place.

As an example of a single ordinary differential equation with constant coefficients, we solve the ordinary differential equation $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=e^{-3t}$; $x(0)=2$; $x'(0)=3$

Taking the Laplace transform of both sides.

$$L\left\{\frac{d^2x}{dt^2}\right\}=p^2L\{x\}-px(0)-p^0x'(0)$$

$$= p^2L\{x\}-2p-3$$

$$3L\left\{\frac{dx}{dt}\right\}=3\big(pL\{x\}-2\big)$$

$$L\{e^{-3t}\}=\frac{1}{p+3}$$

Let $L\{x\}=X$

$$(p^2X-2p-3)+3(pX-2)+2X=\frac{1}{p+3}$$

$$(p^2+3p+2)X=\frac{1}{p+3}+2p+3+6$$

$$=\frac{1+(p+3)(2p+9)}{p+3}$$

$$=\frac{1+2p^2+15p+27}{p+3}$$

$$X=\frac{2p^2+15p+28}{(p+3)(p+2)(p+1)}$$

$$X=\frac{1/2}{p+3}-\frac{6}{p+2}+\frac{15/2}{p+1}=L\{x\}$$

The Inverse Laplace transform

$$x = L^{-1}\{X\} = L^{-1}\left\{\frac{1/2}{p+3}\right\} - L^{-1}\left\{\frac{6}{p+2}\right\} + L^{-1}\left\{\frac{15/2}{p+1}\right\}$$

$$= \frac{1}{2}e^{-3t} - 6e^{-2t} + \frac{15}{2}e^{-t}$$

- Given that $i(0) = 0$, solve for **i(t)** in the circuit in Fig. 9.1, given that $V(t)$ = sin5t , R = 2 Ω and L = 2 H.

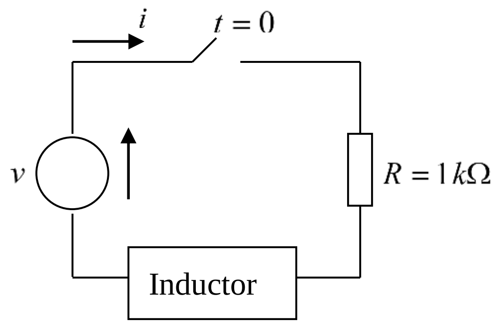


Fig. 9.1

❑

$$Ri + L\frac{di}{dt} = v$$

$$2i + 2\frac{di}{dt} = \sin 5t$$

$$2I + 2(pI - i(0)) = \frac{5}{p^2+25}$$

$$i_0 = i(0) = 0$$

$$2I + 2pI = \frac{5}{p^2+25}$$

$$(1+p)I = \frac{5}{p^2+25}$$

$$I = \frac{5}{(p+1)(p^2+25)} = \frac{A}{p+1} + \frac{Bp+C}{p^2+25}$$

So

5 = A(p^2 + 25) + (Bp + C)(p + 1)
We need to solve for A, B and C.
Let $p = -1$:
5 = 26A

Thus A = $\frac{5}{26}$

Equating the coefficients of p^2:

$0 = A + B$ gives $B = -A = -\frac{5}{26}$

Equating coefficients of s:

$0 = B + C$ gives $C = -B = \frac{5}{26}$

So,

$$I = \frac{5}{26}\left(\frac{1}{p+1} - \frac{p}{p^2+25} + \frac{1}{p^2+25}\right)$$

or

$$i = \frac{5}{26}\left(e^{-t} - \cos 5t + \sin 5t\right)$$

Summary of Study Session 9

In Study Session 9, you learnt to do the following:

1. Write the expression for the Laplace transform of a given function of a variable.
2. Find the Laplace transform of a given function.
3. State the properties of the Laplace transform.
4. Apply the properties of the Laplace transform.
5. Find the inverse Laplace transform of some functions.
6. Find the Laplace transform of a periodic function.
7. Laplace transform a differential equation.
8. Apply Laplace transform in solving an ordinary differential equation with constant coefficients.
9. Apply Laplace transform in solving an integral equation.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self-Assessment Questions for Study Session 9

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this Module.

SAQ 9.1 (tests Learning Outcome 9.3)

List the properties of the Laplace transform.

SAQ 9.2 (test Learning Outcomes 9.2, 9.3 and 9.4)

Find the Laplace transform of $e^{2t}\sin(t+2)$.

SAQ 9.3 (test Learning Outcomes 9.1, 9.2, 9.3 and 9.4)

Obtain the Laplace transform of the following:

(i) $\sin(t-4)$

(ii) $f(at)$; given that $L\{f(t)\}=F(p)$

SAQ 9.4 (test Learning Outcomes 9.2, 9.3 and 9.4)

Find the Laplace transform of the following:

(i) $(t-4)^2$ (ii) $e^{-3t}\cos\omega(t-5)$ (iii) te^{at}

SAQ 9.5 (tests Learning Outcomes 9.3 and 9.7)

Find the Laplace transform of the following ordinary differential equations:

(i) $y'-x'+y+2x=e^t$; $x(0)$, $y(0)$ = 1.

(ii) $2\dot{x}-y=12e^{2t}+2e^{-2t}$; $x(0)=2; y(0)=2$

SAQ 9.6 (tests Learning Outcome 9.6)

Find the Laplace transform of a periodic function of period a.

SAQ 9.7 (tests Learning Outcomes 9.2, 9.5, 9.7 and 9.8)

Solve the following ordinary differential equation with the aid of the Laplace transform.

$y''-y'-2y=e^{2t}$; $y(0)=0$, $y'(0)=1$

SAQ 9.8 (tests Learning Outcomes 9.2, 9.3, 9.4, 9.5 and 9.9)

Consider the circuit in Fig. 9.2 when the switch is closed at t = 0 with $V_C(0)$ = 1.0 V. By applying Laplace transform, solve for the current $i(t)$ in the circuit.

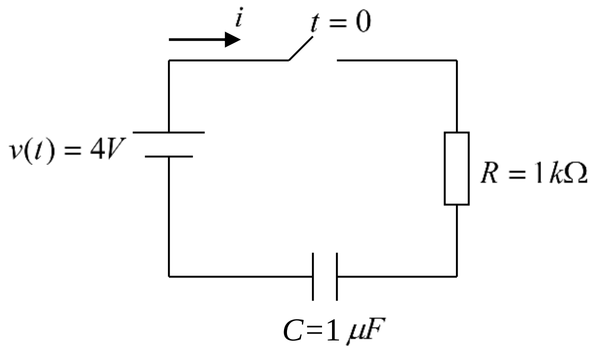


Fig. 9.2

Solutions of SAQs

SAQ 9.1

Properties of the Laplace Transform:

(i) Linearity of Laplace transform and inverse Laplace transform:

$L(af(t)+bg(t)) = aL(f(t))+bL(g(t))$

where a and b are constants

(ii) The time-shift or t-shift rule

$L\{f(t-t_0)\} = e^{-pt_0}F(p)$

(iii) Frequency shift

$L(e^{at}f(t)) = F(p-a)$

(iv) Time scaling

$$L\{f(at)\} = \frac{1}{|a|}F\left(\frac{p}{a}\right)$$

(v) Differentiation Rule

$L\{f^n(t)\} = p^n L\{f(t)\} - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - p^0 f^{(n-1)}(0)$

SAQ 9.2

The Laplace transform of $\sin t$ is $\dfrac{1}{p^2+1}$

Laplace transform of $\sin(t+2)$ is $e^{2p}\dfrac{1}{p^2+1}$ (t-shift rule)

Laplace transform of $e^{2t}\sin(t+2)$ is $e^{2(p-2)}\dfrac{1}{(p-2)^2+1}$ (frequency shift)

SAQ 9.3

(i) $L\{\sin t\} = \int_0^\infty (\sin t)e^{-pt}dt = \dfrac{\omega}{p^2+\omega^2}$

Hence, using the t-shift rule,

$$L\{\sin(t-4)\} = \frac{e^{-4p}\omega}{p^2+\omega^2}$$

(ii) $L(f(at)) = \int_0^\infty f(at)e^{-pt}dt = \dfrac{1}{a}\int_0^\infty f(\tau)e^{-p\tau/a}d\tau$, that is, if we set $\tau = at$, so that $d\tau = adt$, and $t = \tau/a$.

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$$

SAQ 9.4

(i) Find the Laplace transform of $(t-4)^2$

$$L\{t^2\} = \frac{2!}{p^3} = F(p)$$

$$t_0 = 4$$

$$L\{(t-4)^2\} = e^{-4p}F(p) = \frac{2!}{p^3}e^{-4p}$$

using the t-shift rule.

(ii) $$L\{\cos\omega t\} = \frac{p}{p^2+\omega^2}$$

$$L\{\cos\omega(t-5)\} = \frac{p}{p^2+\omega^2}e^{-5p}$$

Therefore, $L\{e^{-3t}\cos\omega(t-5)\} = \frac{e^{-5(p+3)}(p+3)}{(p+3)^2+\omega^2}$

using both the t-shift rule and the frequency shift.

(iii) $$L\{te^{at}\} = \int_0^\infty te^{at}e^{-pt}dt = \int_0^\infty te^{-(p-a)t}dt$$

Let $t = u$, then $dt = du$

$$e^{-(p-a)t}dt = dv \quad v = -\frac{1}{p-a}e^{-(p-a)t}$$

$$I = -\frac{t}{p-a}e^{-(p-a)t}\bigg|_0^\infty + \frac{1}{p-a}\int_0^\infty e^{-(p-a)t}dt = \frac{1}{(p-a)^2}e^{-(p-a)t}\bigg|_0^\infty = \frac{1}{(p-a)^2}$$

SAQ 9.5

Let the Laplace transform of *x* be *X*, and that of *y*, *Y*.

$$L(\dot{x}) = pL(x) - x(0) = pX - 2$$

$$L(\dot{y}) = pL(Y) - y(0) = pY - 2$$

(i) $$(pY - y_0) - (pX - x_0) + Y + 2X = \frac{1}{p-1}$$

(ii) $$2pX - 4 - Y = \frac{12}{p-2} + \frac{2}{p+2}$$

SAQ 9.6

Let $f(t) = f(t+na)$

$$L(f(t)) = \int_0^\infty f(t)e^{-st} = \int_0^a f(t)e^{-st}dt + \int_a^{2a} f(t)e^{-st}dt + \int_{2a}^{3a} f(t)e^{-st}dt + \ldots$$

$$= \int_0^a f(t)e^{-st}dt + \int_0^a f(t+a)e^{-s(t+a)}dt + \int_0^a f(t+2a)e^{-s(t+2a)}dt + \ldots$$

$$= \int_0^a f(t)e^{-st}dt + \int_0^a f(t)e^{-s(t+a)}dt + \int_0^a f(t)e^{-s(t+2a)}dt + \ldots$$

$$= (1 + e^{-sa} + e^{-2sa} + \ldots)\int_0^a f(t)e^{-st}dt = \frac{1}{1-e^{-sa}}\int_0^a f(t)e^{-st}dt$$

SAQ 9.7

$L\{y''\} = p^2Y - py(0) - y'(0) = p^2Y - 1$

$L\{y'\} = pY - y(0) = pY$

$L\{2y\} = 2Y$

$L\{e^{2t}\} = \frac{1}{p-2}$

Hence,

$$p^2Y - 1 - pY - 2Y = \frac{1}{p-2}$$

$$Y = \frac{\frac{1}{p-2} + 1}{p^2 - p - 2} = \frac{p-2+1}{(p-2)(p^2-p-2)} = \frac{p-1}{(p-2)(p^2-p-2)}$$

$$= \frac{p-1}{(p-2)(p-2)(p+1)} = \frac{p-1}{(p+1)(p-2)^2}$$

Resolving into partial fractions,

$$\frac{p-1}{(p+1)(p-2)^2} = \frac{A}{p+1} + \frac{B}{p-2} + \frac{C}{(p-2)^2}$$

Multiplying through by $(p+1)(p-2)^2$,

$$p - 1 = A(p-2)^2 + B(p+1)(p-2) + C(p+1)$$

Let $p = 2$, then,

$$2 - 1 = A(0) + B(0) + 3C$$

Hence,

$$C = \frac{1}{3}$$

Let $p = -1$, then,

$$-2 = 9A$$

or

$$A = -\frac{2}{9}$$

Let $p = 0$,

$$-1 = 4A - 2B + C = 4\left(-\frac{2}{9}\right) - 2B + \frac{1}{3}$$

Hence,

$$B = \frac{-1 + \frac{8}{9} - \frac{1}{3}}{-2} = \frac{-9 + 8 - 3}{-18} = \frac{4}{18} = \frac{2}{9}$$

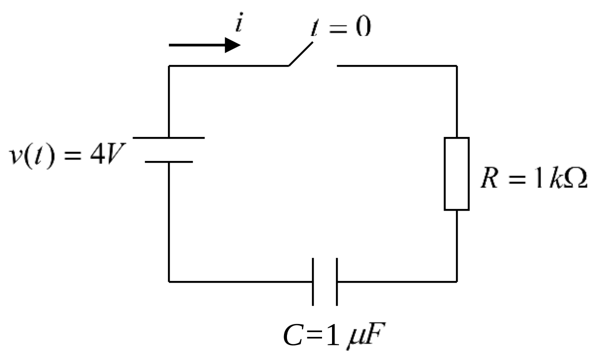
Therefore,

$$Y = -\frac{2}{9}\frac{1}{p+1} + \frac{2}{9}\frac{1}{p-2} + \frac{1}{3}\frac{1}{(p-2)^2}$$

Hence,

$$y = L^{-1}\{Y\} = -\frac{2}{9}e^{-t} + \frac{2}{9}e^{-2t} + \frac{1}{3}te^{2t}$$

SAQ 9.8



$$\frac{1}{C}\int idt + Ri = v(t)$$

$$\frac{1}{10^{-6}}\int idt + 10^3 i = 4$$

Multiplying throughout by 10^{-6} :

$$\int idt + 10^{-3} i = 4 \times 10^{-6}$$

Taking Laplace transform:

$$\frac{I}{p}+\frac{1}{p}\left[\int idt\right]_{t=0}+10^{-3}I=\frac{4\times10^{-6}}{p}$$

$$V_c=2.0\ .$$

So

$$V_c(0)=\frac{1}{C}\left[\int idt\right]_{t=0}=2$$

That is,

$$\frac{1}{10^{-6}}\left[\int idt\right]_{t=0}=2$$

or $\left[\int idt\right]_{t=0}=2\times10^{-6}=q_0$

So,

$$\frac{I}{p}+\frac{2\times10^{-6}}{p}+10^{-3}I=\frac{4\times10^{-6}}{p}$$

Rearranging,

$$\left(\frac{1}{p}+10^{-3}\right)I=\frac{4\times10^{-6}}{p}-\frac{2\times10^{-6}}{p}=\frac{2\times10^{-6}}{p}$$

$$(1+10^{-3}s)I=2\times10^{-6}\ (1+10^{-3}p)I=2\times10^{-6}$$

$$I=\frac{2\times10^{-6}}{1+10^{-3}p}=2\times10^{-6}\times\frac{1}{1000+p}$$

Finding the inverse Laplace transform gives us

$i=2\times10^{-3}e^{-1000t}$

Study Session 10 Laplace Transform and Application II

In Study Session 9 we examined Laplace transforms, and with the method, solved single ordinary differential equations with constant coefficients as well as an integral equation. We explore the powerful tool of Laplace transform further in this study session, by applying it to a system of equations.

Learning Outcomes for Study Session 10

At the end of this study session, you should be able to do the following:

10.1 Apply Laplace transform in converting a system of differential, integral and integro-differential equations into a system of algebraic equations (SAQ 10.1-10.4).

10.2 Solve the system of algebraic equations (SAQ 10.1-10.4).

10.3 Recover the solution to the original system via inverse Laplace transform (SAQ 10.1-10.4).

10.1 Simultaneous Linear ODE with constant coefficients

❑ Solving a set of simultaneous linear ordinary differential equations follows the same procedure as in the case of a single equation of the same classification. We illustrate with the following problem.

❑

▪ Solve the system:

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3x = 3t + 3 \tag{10.1}$$

$$2\frac{dx}{dt} - \frac{dy}{dt} + 3y = 6t - 3 \tag{10.2}$$

subject to $x(0) = y(0) = 0$

❑ Let $L\{x\} = X$, $L\{y\} = Y$

From equation (10.1),

$$pX + 4pY + 3X = \frac{3}{p^2} + \frac{3}{p} \tag{10.3}$$

From equation (10.2),

$$2pX - pY + 3Y = \frac{6}{p^2} - \frac{3}{p} \tag{10.4}$$

From (10.3), $(p+3)X + 4pY = \dfrac{3}{p^2} + \dfrac{3}{p}$

which implies

$$X + \frac{4pY}{p+3} = \frac{\frac{3}{p^2} + \frac{3}{p}}{p+3} = \frac{3+3p}{p^2(p+3)}$$

$$X = \frac{3+3p}{p^2(p+3)} - \frac{4pY}{p+3} \qquad 10.5$$

Similarly,

$$X + \frac{(3-p)Y}{2p} = \frac{\frac{6}{p^2} - \frac{3}{p}}{2p} = \frac{6-3p}{2p^3}$$

$$X = \frac{6-3p}{2p^3} - \frac{(3-p)Y}{2p} \qquad 10.6$$

Equating equations (10.5) and (10.6),

$$\frac{3+3p}{p^2(p+3)} - \frac{4pY}{p+3} = \frac{6-3p}{2p^3} - \frac{(3-p)Y}{2p}$$

$$\frac{(3-p)Y}{2p} - \frac{4pY}{p+3} = \frac{6-3p}{2p^3} - \frac{3+3p}{p^2(p+3)}$$

$$\frac{(3-p)(p+3)Y - 8p^2Y}{2p(p+3)} = \frac{(p+3)(6-3p) - 2p(3+3p)}{2p^3(p+3)}$$

$$[(3-p)(p+3) - 8p^2]Y = \frac{(p+3)(6-3p) - 2p(3+3p)}{p^2}$$

$$[9 - p^2 - 8p^2]Y = \frac{-3p^2 + 6p - 9p + 18 - 6p - 6p^2}{p^2} = \frac{-9p^2 - 9p + 18}{p^2}$$

$$[9 - 9p^2]Y = \frac{-9p^2 - 9p + 18}{p^2}$$

$$[1 - p^2]Y = \frac{-p^2 - p + 2}{p^2}$$

$$Y = \frac{-p^2 - p + 2}{p^2(1-p^2)} = \frac{-(p-1)(p+2)}{p^2(1-p)(1+p)} = \frac{(p+2)}{p^2(p+1)} \qquad 10.7$$

Resolving into partial fractions,

$$Y = \frac{(p+2)}{p^2(p+1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+1}$$

Hence,

$$Ap(p+1) + B(p+1) + Cp^2 = p+2$$

Setting $p=0$, $B=2$

Setting $p=-1$, $C=1$

Setting $p=1$, $2A+2B+C=3 \quad \Rightarrow \quad 2A+2(2)+1=3$

Hence, $A=\dfrac{3-5}{2}=-1$

Therefore,

$$Y=\frac{2}{p^2}-\frac{1}{p}+\frac{1}{p+1} \tag{10.8}$$

Substituting into equation (10.6),

$$X=\frac{6-3p}{2p^3}-\frac{(3-p)}{2p}\frac{(p+2)}{p^2(p+1)}=\frac{6-3p}{2p^3}-\frac{(3-p)(p+2)}{2p^3(p+1)}$$

$$=\frac{(p+1)(6-3p)-(3-p)(p+2)}{2p^3(p+1)}$$

$$=\frac{6p-3p^2-3p+6+p^2-p-6}{2p^3(p+1)}$$

$$=\frac{-2p^2+2p}{2p^3(p+1)}=\frac{p(1-p)}{p^3(p+1)}=\frac{1-p}{p^2(p+1)}$$

$$=\frac{1}{p^2}-\frac{2}{p}+\frac{2}{p+1} \tag{10.9}$$

Taking the inverse Laplace transform of equations (10.8) and (10.9),

$x=t-2+2\sin t$ and $y=2t-1+\sin t$

10.2 Application to Electric Circuits

- The system in Fig 10.1 is quiescent. Find the loop current $\mathbf{i_2(t)}$.

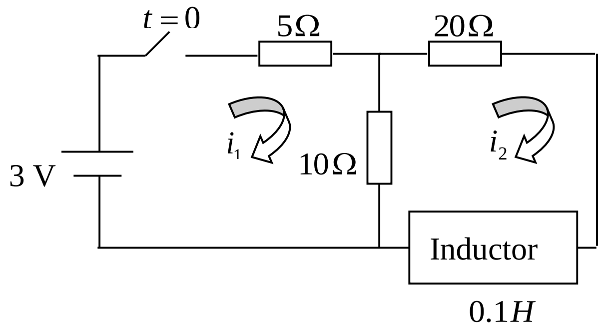


Fig. 10.1

- If the circuit is quiescent, then, both i_1 and i_2 and their derivatives are zero:

$$i_1 = i_2 = i_1' = i_2' = 0$$

For loop 1:

$$5i_1 + 10(i_1 - i_2) = 3$$

$$15i_i - 10i_2 = 3 \qquad 10.10$$

For loop 2:

$$10i_2 + 0.1\frac{di_2}{dt} + 10(i_2 - i_1) = 0$$

$$-10i_1 + 20i_2 + 0.1\frac{di_2}{dt} = 0$$

$$-100i_1 + 200i_2 + \frac{di_2}{dt} = 0 \qquad 10.11$$

From equation (10.10),

$$i_1 = \frac{3 + 10i_2}{15}$$

Putting this in equation (10.11),

$$-100\left(\frac{3 + 10i_2}{15}\right) + 200i_2 + \frac{di_2}{dt} = 0$$

$$-300 + 2000i_2 + 15\frac{di_2}{dt} = 0$$

or

$$15\frac{di_2}{dt} + 2000i_2 = 300$$

Taking Laplace transform:

$$15(pI_2 - i_2(0)) + 2000I_2 = \frac{300}{p}$$

But $i_2(0) = 0$, so,

$$3pI_2 + 400I_2 = \frac{60}{p}$$

$$(3p + 400)I_2 = \frac{60}{p}$$

$$I_2 = \frac{60}{p(3p + 400)}$$

Resolving into partial fractions,

$$\frac{60}{p(3p + 400)} = \frac{A}{p} + \frac{B}{3p + 400}$$

Multiplying through by $p(3p + 400)$,

$$60 = A(3p + 400) + Bp$$

Setting p equal to zero gives,

$$60 = 400A$$

implying

$$A = \frac{6}{40} = \frac{3}{20}$$

Setting $3p = -400$,

$$60 = -\frac{400}{3}B$$

$$B = -\frac{120}{400} = -\frac{3}{10}$$

Hence,

$$I_2 = \frac{3}{20p} - \frac{3}{30(p + 400/3)}$$

Taking inverse Laplace transform of both sides,

$$i_2 = \frac{3}{20} \times 1 - \frac{1}{10}e^{-400t/3} = \frac{1}{10}(\frac{3}{2} - e^{-400t/3})$$

Summary of Study Session 10

In Study Session 10, you learnt how to:

1. Apply Laplace transform in converting a system of differential, integral and integro-differential equations into a system of algebraic equations.
2. Solve the system of algebraic equations.
3. Recover the solution to the original system via inverse Laplace transform.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self-Assessment Questions

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this Module.

SAQ 10.1 (tests Learning Outcomes 10.1, 10.2 and 10.3)

Using the method of Laplace transform, solve the simultaneous ordinary differential equations

$$\dot{x} + \dot{y} = 6e^{2t} - 2e^{-2t}\,, \quad 2\dot{x} - y = 12e^{2t} + 2e^{-2t}\,; \qquad x(0) = 2; y(0) = 2$$

SAQ 10.2 (tests Learning Outcomes 10.1, 10.2 and 10.3)

Subject to the initial conditions $x(0) = y(0) = 0$, solve the system of linear ordinary differential equations:

$$\dot{x} + x + 2y = e^{2t}$$

$$2\dot{x} + \dot{y} - x = 0$$

SAQ 10.3 (tests Learning Outcomes 10.1, 10.2 and 10.3)

Solve the ordinary differential equation:

$$\dot{x} + y = 3e^{3t} + e^{5t}\,, \qquad x + \dot{y} = 5e^{5t} + e^{3t}\,; \qquad x(0) = y(0) = 1$$

SAQ 10.4 (tests Learning Outcomes 10.1, 10.2 and 10.3)

Find the values of *x* and *y* that satisfy the ordinary differential equations,

$$\frac{dx}{dt} + x + 2y = e^{2t}$$

$$2\frac{dx}{dt} + \frac{dy}{dt} - x = 0$$

$$x(0) = y(0) = 0$$

Solutions to SAQs
SAQ 10.1

$$\dot{x} + \dot{y} = 6e^{2t} - 2e^{-2t} \qquad 10.12$$

$$2\dot{x} - y = 12e^{2t} + 2e^{-2t} \qquad 10.13$$

$$x(0) = 2; y(0) = 2$$

Let $L\{x\} = X$, $L\{y\} = Y$

$$L(\dot{x}) = pL(x) - x(0) = pX - 2$$

$$L(\dot{y}) = pL(Y) - y(0) = pY - 2$$

Therefore,

$$pX + pY = \frac{6}{p-2} - \frac{2}{p+2} \qquad 10.14$$

$$2pX - 4 - Y = \frac{12}{p-2} + \frac{2}{p+2} \qquad 10.15$$

$2(9.14) - (9.15)$ gives,

$$Y(2p+1) = -\frac{6}{p+2} + 4 = \frac{-6+4p+8}{p+2} = \frac{2(2p+1)}{p+2}$$

$$Y = \frac{2}{p+2} \qquad 10.16$$

$$y = L^{-1}(Y) = 2e^{-2t} \qquad 10.17$$

Putting $Y = \dfrac{2}{p+2}$ into equation (10.15),

$$2pX - 4 - \frac{2}{p+2} = \frac{12}{p-2} + \frac{2}{p+2}$$

$$2pX - 4 = \frac{12}{p-2} + \frac{4}{p+2}$$

$$2pX = \frac{12}{p-2} + \frac{4}{p+2} + 4 = \frac{4p^2 - 16 + 12p + 24 + 4p - 8}{(p-2)(p+2)} = \frac{4p^2 + 16p}{(p-2)(p+2)}$$

$$= \frac{4p(p+4)}{(p-2)(p+2)}$$

Hence, $X = \dfrac{2(p+4)}{(p-2)(p+2)} = \dfrac{A}{p-2} + \dfrac{B}{p+2}$

or $2(p+4) = A(p+2) + B(p-2)$

When $p = 2$, $12 = 4A$ or $A = 3$

When $p=-2$, $4=-4B$ or $B=-1$

Therefore,

$$X=\frac{3}{p-2}-\frac{1}{p+2}$$

$$x=L^{-1}(X)=3e^{2t}-e^{-2t}$$

10.18

SAQ 10.2

Transforming,

$$(p+1)X(p)+2Y(p) = \frac{1}{p-2} \qquad \text{(i)}$$

$$(2p-1)X(p)+pY(p) = 0 \qquad \text{(ii)}$$

Multiply (i) by p and (ii) by – 2:

$$p(p+1)X+2pY = \frac{p}{p-2}$$

$$-2(2p-1)X(p)+pY(p) = 0$$

Subtracting,

$$(p^2-3p+2)X=\frac{p}{p-2}$$

Hence,

$$X(p) = \frac{p}{(p-1)(p-2)^2}$$

$$Y(p) = \frac{1-2p}{(p-1)(p-2)^2}$$

Expanding into partial fractions,

$$X=\frac{2p}{(p-2)^2}-\frac{1}{p-2}+\frac{1}{p-1}$$

Therefore,

$$x(t)=L^{-1}(X) = (2t-1)e^{2t}+e^{t}$$

$$y(t)=L^{-1}(Y) = (1-3t)e^{2t}-e^{t}$$

SAQ 10.3

$$-pX+1-Y=\frac{-3}{p-3}-\frac{1}{p-5}$$

$$X-1+pY=\frac{5}{p-5}+\frac{1}{p-3}$$

$$pX-p+p^2Y=\frac{5p}{p-5}+\frac{p}{p-3}$$

$$\left(p^2-1\right)Y+1-p=\frac{5p-1}{p-5}+\frac{p-3}{p-3}$$

$$\therefore \quad \left(p^2-1\right)Y=\frac{5p-1}{p-5}+p$$

$$Y=\frac{5p-1+p^2-5p}{(p-5)(p^2-1)}=\frac{1}{p-5}$$

$$X=\frac{5}{p-5}+\frac{1}{p-3}+1-\frac{p}{p-5}=\frac{5-p}{p-5}+\frac{1}{p-3}+1$$

$$=\frac{1}{p-3}$$

Therefore, $x=L^{-1}(X)=e^{3t}$

$$y=L^{-1}(Y)=e^{5t}$$

SAQ 10.4

$$(p+1)X+2Y=\frac{1}{p-2}$$

$$(2p-1)X+pY=0$$

Solve these simultaneously to get

$$X=\frac{p}{(p-1)(p-2)^2}$$

$$Y=\frac{1-2p}{(p-1)(p-2)^2}$$

$$\frac{p}{(p-1)(p-2)^2}=\frac{A}{p-1}+\frac{B}{p-2}+\frac{C}{(p-2)^2}$$

$$p=A(p-2)^2+B(p-1)(p-2)+C(p-1)$$

$p=1$: $A=1$

$p=2:\ C=2$

$p=0: \qquad 4A+2B-C=0$

$\qquad 4+2B-2=0$ or $B=-1$

$$\frac{p}{(p-1)(p-2)^2}=\frac{1}{p-1}-\frac{1}{p-2}+\frac{2}{(p-2)^2}$$

Therefore,

$$x=e^t-e^{2t}+2te^t$$

$$\frac{1-2p}{(p-1)(p-2)^2}=\frac{A}{p-1}+\frac{B}{p-2}+\frac{C}{(p-2)^2}$$

$1-2p=A(p-2)^2+B(p-1)(p-2)+C(p-1)$

$p=1:\ A=-1$

$p=2:\ C=-3$

$p=0: \qquad 4A+2B-C=1$

$\qquad -4+2B+3=1$ or $B=1$

$$\frac{1-2p}{(p-1)(p-2)^2}=-\frac{1}{p-1}+\frac{1}{p-2}-\frac{3}{(p-2)^2}$$

Therefore,

$$x=e^t-e^{2t}+2te^{2t},\ y=-e^t+e^{2t}-3te^{2t}$$

Test: $\quad 2\frac{dx}{dt}+\frac{dy}{dt}-x=0:$

$2(e^t-2e^{2t}+2e^{2t}+4te^{2t})+(-e^t+2e^{2t}-3e^{2t}-6te^{2t})-(e^t-e^{2t}+2te^{2t})$

$=(2e^t+8te^{2t}-e^t-e^{2t}-6te^{2t})-(e^t-e^{2t}+2te^{2t})$

$=(e^t-e^{2t}+2te^{2t})-(e^t-e^{2t}+2te^{2t})=0$

Study Session 11 Complex Analysis

Introduction

The square root of a negative number gives the motivation for complex numbers. The fact that simple harmonic motion involves solutions of the form sine and cosine which are exponentials of complex variables underscores the importance of complex numbers. Wavefunctions in Quantum mechanics are generally complex functions, and that is why the wavefunction on its own might not make any physical sense. But by taking its square, we get the probability distribution function $|\psi(r)|^2$. Moreover, some integrals would prove difficult ordinarily, but such a problem could be solved with the help of complex integration.

Learning Outcomes of Study Session 11:

By the time you are through with this study session, you would be able to do the following:

11.1 Carry out simple operations on complex numbers, such as, the addition, subtraction, multiplication of complex numbers, complex conjugation, etc. (SAQ 11.1)

❑ Write a function of a complex variable and be able to identify the real and imaginary parts of such a function (SAQ 11.3, 11.4).

11.2 Calculate the roots of a complex number (SAQ 11.2, 11.5).

11.1 Complex Numbers

A complex number can be written as (Fig. 11.1)

$$z = x + iy \qquad 11.1$$

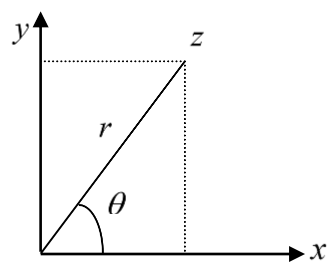


Fig. 11.1

Both x and y are real numbers. Also from the figure, we notice that in polar form, we may also write $re^{i\theta}$, r is a real number. Obviously,

$$r = \sqrt{x^2 + y^2}\ , \qquad 11.2$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad 11.3$$

$$i = \sqrt{-1}\,.$$

$$x = r\cos\theta\,,\ \ y = r\sin\theta \qquad 11.4$$

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta = x + iy$$

To get the complex conjugate of any complex number, we replace i by $-i$. For example, let $z = x + iy$, then, the complex conjugate $\bar{z} = \overline{x + iy} = x - iy$.

Note that $x = \operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ and $y = \operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$.

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

In another form,

$$z\bar{z} = re^{i\theta} re^{-i\theta} = r^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$$

$$|z| = r,\ |z|^2 = r^2$$

$$z\bar{z} = |z|^2$$

Let $z = 3 + 2i$

$$|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

- Show that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \overline{\left(\dfrac{x_1 + iy_1}{x_2 + iy_2}\right)} = \dfrac{x_1 - iy_1}{x_2 - iy_2} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Note that

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

So,

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n e^{i(\theta_1 + \theta_2 + \ldots + \theta_n)}$$

In the particular case where $z_1 = z_2 = \ldots = z_n$,

$$z^n = r^n e^{in\theta}$$

n can be a fraction, say, $\dfrac{1}{m}$. Then,

$$z^{1/m} = r^{1/m}e^{i\theta/m} = r^{1/m}e^{i\frac{\theta+2\pi k}{m}} \qquad 11.5$$

$$= r^{1/m}\left[\cos\frac{\theta+2\pi k}{m} + i\sin\frac{\theta+2\pi k}{m}\right] \qquad 11.6$$

The only unique members of the set are for $k = 0,1,2,\ldots,m-1$.

- ❑ Find all the square roots of –1.

- ❑

$$\sqrt{-1} = (-1)^{1/2} = \left(re^{i\theta}\right)^{1/2}$$
$$-1 = x + iy = -1 + i\times 0$$
$$r = \sqrt{(-1)^2 + 0^2} = 1$$
$$\theta = \tan^{-1}\left(\frac{0}{-1}\right) = \pi$$

The roots are,

$$r^{1/m}\left[\cos\frac{\pi+2\pi k}{m} + i\sin\frac{\pi+2\pi k}{m}\right]$$
$$k = 0,1$$
$$k = 0,\ z^{1/2} = i,\ i^2 = -1$$
$$k = 1,\ z^{1/2} = -i,\ (-i)^2 = -1$$

- ▪ Find the cube roots of $2 - 2i\sqrt{3}$

- ❑

$$r = 4,\ \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

$x = 2$ and $y = -2\sqrt{2}$, meaning that the complex number is in the 4th quadrant. $\theta = 2\pi - \pi/3 = 5\pi/3$. But since it is in the fourth quadrant, you could also leave the angle as it is, but bearing in mind that the angle has been measured in the clockwise sense from the positive *x* direction.

The roots are:

$$4^{1/3}\left[\cos\frac{-\frac{\pi}{3}+2\pi k}{3} + i\sin\frac{-\frac{\pi}{3}+2\pi k}{3}\right]$$

where $k = 0,\ 1,\ 2$

$k = 0 \quad : \qquad 4^{1/3}\left[\cos\left(-\frac{\pi}{9}\right) + i\sin\left(-\frac{\pi}{9}\right)\right] = 0.9397 - 0.342i$

$k = 1 : \qquad 4^{1/3}\left[\cos\left(\frac{5\pi}{9}\right) + i\sin\left(\frac{5\pi}{9}\right)\right] = -0.1736 + 0.9848i$

$k = 2 : \qquad 4^{1/3}\left[\cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)\right] = -0.766 - 0.6428i$

11.2 Functions of a Complex Variable

Let z be a point in the complex plane. Where it exists, we shall denote a function of z by $f(z)$. At times, we shall write $f(z) = w$. We have been writing $z = x + iy$, where x and y are real. We shall also write

$$f(z) = w = u + iv \qquad 11.7$$

where u and v are real. We can see f as a transformation taking a point in the z-plane to the w-plane (Fig. 11.1).

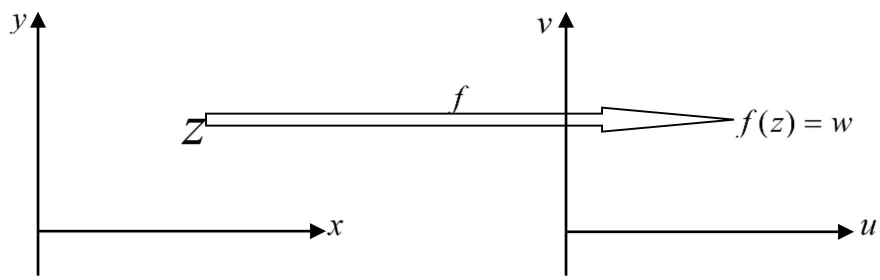


Fig. 11.1: Functions of a Complex Variable

- Find u and v in the following transformations:

(e) $f(z) = z$ (ii) $f(z) = z^2$ (iii) $f(z) = |z|^2$ (iv) $f(z) = z^2 - 3z$

❑

(i) $f(z) = z = x + iy$

but $f(z) = u + iv$

Thus, $u = x$ and $v = y$

(ii) $f(z) = z^2$

$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

Thus, $u = x^2 - y^2$, $v = 2xy$

(iii) $f(z) = |z|^2$

$|z|^2 = x^2 + y^2$

$u = x^2 + y^2$, $v = 0$

(iv) $f(z) = z^2 - 3z$

$$= (x+iy)^2 - 3x - 3iy$$
$$= x^2 + i3xy - y^2 - 3x - i3y$$
$$= x^2 - y^2 - 3x - i3y + i2xy$$

Hence,

$$u = x^2 - y^2 - 3x\,,\ v = y(2x-3)$$

Summary of Study Session 11

In Study Session 11, you learnt to:

1. Carry out simple operations on complex numbers.
2. Write a function of a complex variable and identify the real and imaginary parts of such a function.
3. Calculate the roots of a complex number.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.
4. Hefferson, J. (2012). Linear Algebra, http://joshua.smcvt.edu/linearalgebra/book.pdf
5. Hefferson, J. (2012). Answers to Exercises, http://joshua.smcvt.edu/linearalgebra/jhanswer.pdf

Self-Assessment Questions for Study Session 11

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this Module.

SAQ 11.1 (tests Learning Outcome 11.1)

Show that (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (ii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

SAQ 11.1 (tests Learning Outcome 11.3)

Get all the cube roots of $-2 - 2i\sqrt{3}$

SAQ 11.2 (tests Learning Outcome 11.3)

Find all the cube roots of $z = \sqrt{2} + i\sqrt{2}$.

SAQ 11.3 (tests Learning Outcome 11.2)
Given that $f(z) = z^3 + 2z + 3$, find $u(x, y)$ and $v(x, y)$.

SAQ 11.4 (tests Learning Outcome 11.2)
Given that $f(z) = z + \dfrac{1}{z}$, find $u(x, y)$ and $v(x, y)$, if $f(z) = u(x, y) + iv(x, y)$.

SAQ 11.5 (tests Learning Outcome 11.3)
Find all solutions of $z^3 = -8i$.

Solutions to SAQs

SAQ 11.1

$$\begin{aligned}\overline{z_1+z_2} &= \overline{(x_1+iy_1)+(x_2+iy_2)} = x_1 - iy_1 + x_2 - iy_2 \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= \bar{z}_1 + \bar{z}_2\end{aligned}$$

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(x_1+iy_1)(x_2+iy_2)} \\ &= \overline{x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2} \\ &= x_1x_2 - ix_1y_2 - iy_1x_2 - y_1y_2 \\ &= (x_1 - iy_1)(x_2 - iy_2) \\ &= \bar{z}_1\bar{z}_2\end{aligned}$$

SAQ 11.1

$$r = 4,\ \theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{4\pi}{3}$$

(Third quadrant, since both x and y components are negative.)

The roots are:

$$4^{1/3}\left[\cos\frac{\frac{4\pi}{3}+2\pi k}{3} + i\sin\frac{\frac{4\pi}{3}+2\pi k}{3}\right]$$

where $k = 0, 1, 2$

$k = 0$: $\quad 4^{1/3}\left[\cos\left(4\frac{\pi}{9}\right) + i\sin\left(4\frac{\pi}{9}\right)\right]$

$k = 1$: $\quad 4^{1/3}\left[\cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)\right]$

$k = 2$: $\quad 4^{1/3}\left[\cos\left(\frac{16\pi}{9}\right) + i\sin\left(\frac{16\pi}{9}\right)\right]$

SAQ 11.2

Let $z = x + iy$ and $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$

$$r = |z| = |\sqrt{2} + i\sqrt{2}| = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \tan^{-1}(1) = \pi/4$$

So $z = 2e^{i\frac{\pi}{4}}$.

$$z^{1/m} = |z|^{1/m}\ \exp\left(\frac{i(\theta + 2k\pi)}{m}\right),\ k = 0,\ 1,\ 2,\ \ldots,\ m-1$$

So the cube roots of $2e^{i\frac{\pi}{4}}$ are

$$2^{1/3}\exp\left(\frac{i}{3}\left(\frac{\pi}{4} + 2k\pi\right)\right),\ k = 0,\ 1,\ 2$$

or $2^{1/3}e^{i\frac{\pi}{12}}, 2^{1/3}e^{i\frac{9\pi}{12}}, 2^{1/3}e^{i\frac{17\pi}{12}}$

SAQ 11.3

$$\begin{aligned} z^3 &= (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + 3ixy^2 - 3xy^2 - 3iy^3 \end{aligned}$$

$$2z = 2x + 2iy$$

$$\begin{aligned} f(z) &= z^3 + 2z + 3 = x^3 + 3ixy^2 - 3xy^2 - 3iy^3 + 2y + 2iy + 3 \\ &= x^3 - 3xy^2 + 2y + 3 + i(3xy^2 - 3y^3 + 2y) \end{aligned}$$

Hence,

$$u(x, y) = x^3 - 3xy^2 + 2y + 3$$
$$v(x, y) = 3xy^2 - 3y^3 + 2y$$

SAQ 11.4

$$\begin{aligned} f(z) &= z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= r\cos\theta + ir\sin\theta + \frac{1}{r}(\cos\theta - i\sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta \end{aligned}$$

Hence,

$$u(x, y) = \left(r + \frac{1}{r}\right)\cos\theta\ ;\ v(x, y) = \left(r - \frac{1}{r}\right)\sin\theta$$

SAQ 11.5

$$z^3 = -8i = 8e^{-i\pi/2} \qquad (\text{since } e^{i\pi/2} = \cos(\pi/2) - i\sin(\pi/2)\ )$$

Hence,

$$z = \sqrt[3]{8}\exp i\left(\frac{\pi}{2} + 2\pi k\right)/3 = 2\exp i\left(\frac{\pi}{6} + \frac{2\pi k}{3}\right)$$

$$= 2\exp i\left(\frac{\pi + 12\pi k}{6}\right)$$

$k = 0$:

$$z = 2\exp\frac{i\pi}{6}$$

$k = 1$:

$$z = 2\exp\frac{13i\pi}{6}$$

$k = 2$:

$$z = 2\exp\frac{25i\pi}{6}$$

Study Session 12 Differentiation of a Complex Variable

Introduction

Now that you have taken a look at the algebra of complex numbers, the question that comes readily to mind is whether, just as is the case with real functions, is it possible to differentiate complex functions. The answer is, yes. We differentiate complex functions, but our answers are not as easily obtained as is the case with real functions. You shall have cause to come across **analytic functions** and the attendant **Cauchy-Riemann equations** that have much application in Physics. You shall see that each of the pair of functions involved in the Cauchy-Riemann equations is a **harmonic function**.

Learning Outcomes of Study Session 12

At the end of this study session, you should be able to:

12.1 Understand and correctly use key words printed in **bold** (SAQ 12.1).
12.2 Prove whether a function of a complex variable is analytic (SAQ 12.2, 12.5, 12.6)
12.3 Differentiate a function of complex variables from first principles (SAQ 12.3).
12.4 Derive the Cauchy-Riemann equations (SAQ 12.4)
12.5 Find a harmonic function between two lines (SAQ 12.6, 12.7).
12.6 Distinguish between between the differentiation of a real function and that of a complex function (SAQ 12.8).

12.1 Differentiation of a Complex Function

A function of a complex variable z is said to have a derivative at a point z_0 if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad 12.1$$

exists and it is independent of how z approaches z_0 (Fig. 12.1).

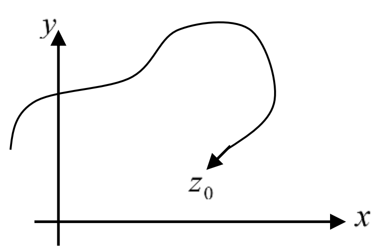


Fig. 12.1: The limit of a complex variable at z_0 is irrespective of the direction of approach

We can also write $z = z_0 + h$

Therefore, $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$

▪ Find the differentials of the following functions of a complex variable:

(f) $f(z) = z^n$ (ii) $f(z) = \operatorname{Re} z$

❑

(i) $f(z) = z^n$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{(z+h)^n - z^n}{h}$$

$$= nz^{n-1}$$

(ii) $f(z) = \operatorname{Re} z$

$\operatorname{Re} z = x$

$$f'(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{x - x_0}{(x - x_0) + i(y - y_0)}$$

Let $z \to z_0$ through the line $x = x_0$, then, $f'(z) = 0$.

Let $z \to z_0$ through the line $y = y_0$, then $f'(z) = 1$

Therefore, $\operatorname{Re} z$ has no derivative. This is because the two derivatives are not the same.

▪ Let $f(z) = \bar{z}$. Does $f'(z)$ exist?

❑

$$f'(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{\bar{z} - \bar{z}_0}{(x - x_0) + i(y - y_0)}$$

$$= \lim_{z\to z_0} \frac{(x - iy) - (x_0 - iy_0)}{(x - x_0) + i(y - y_0)}$$

$$= \lim_{z\to z_0} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) + i(y - y_0)}$$

Let $z \to z_0$ through the line $x = x_0$, then, $f'(z) = -1$.

Let $z \to z_0$ through the line $y = y_0$, then $f'(z) = 1$.

Therefore, $f(z)=\bar{z}$ has no derivative. The two derivatives are not the same.

12.2 Cauchy-Riemann Equations

Suppose $f'(z)$ exists at a point. Then the limit

$f'(z)=\lim_{g\to 0}\dfrac{f(z+g)-f(z)}{g}$ exists.

Let $g=h+ik$

$f(z)=u(x,y)+iv(x,y)$

$f(z+g)=u(x+h,y+k)+iv(x+h,y+k)$

$$f'(z)=\lim_{g\to 0}\frac{[u(x+h,y+k)-u(x,y)]+i[v(x+h,y+k)-v(x,y)]}{h+ik}$$

Let $g\to 0$ along the x - axis, i.e., $k=0$.

$$L_1=\lim_{h\to 0}\frac{[u(x+h,y)-u(x,y)]+i[v(x+h,y)-v(x,y)]}{h}$$

$$=\lim_{h\to 0}\left[\frac{[u(x+h,y)-u(x,y)]}{h}+\frac{i[v(x+h,y)-v(x,y)]}{h}\right]$$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \qquad 12.2$$

Let $g\to 0$ along the y - axis, i.e., $h=0$.

$$L_2=\lim_{k\to 0}\frac{[u(x,y+k)-u(x,y)]+i[v(x,y+k)-v(x,y)]}{ik}$$

$$=\lim_{k\to 0}\left[\frac{[u(x,y+k)-u(x,y)]}{ik}+\frac{i[v(x,y+k)-v(x,y)]}{ik}\right]$$

$$=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} \qquad 12.3$$

Since $L_1=L_2$, we equate (12.10) and (12.11) to arrive at

$f'(z)=u_x+iv_x=v_y-iu_y$

This implies that (equating real and imaginary parts)

$$u_x=v_y \qquad 12.4a$$

and

$$v_x=-u_y \qquad 12.4b$$

where we have written $\dfrac{\partial u}{\partial x}=u_x$, etc.

The equations (12.4) are called the Cauchy-Riemann equations.

12.3 Analyticity

A function is said to be analytic at a point if the Cauchy-Riemann equations are satisfied.

▪ $f(z) = z^2 = x^2 - y^2 + 2ixy$. Show that it satisfies the Cauchy-Riemann equations.

❑

$u_x = 2x$; $u_y = -2y$

$v_x = 2y$; and $v_y = 2x$

Therefore, the function $f(z) = z^2$ is analytic on the $x - y$ plane.

12.4 Harmonic Functions

Let the Cauchy-Riemann equations hold.
Differentiate equation (12.4a) with respect to x

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 12.5$$

which is Laplace equation in 2-dimensions

Similarly, it can be proved that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad 12.6$$

u and v that satisfy these relations are said to be harmonic functions. Indeed, they are orthogonal. In electrostatics, for example, u may be equipotential function and v the field (Fig. 12.2).

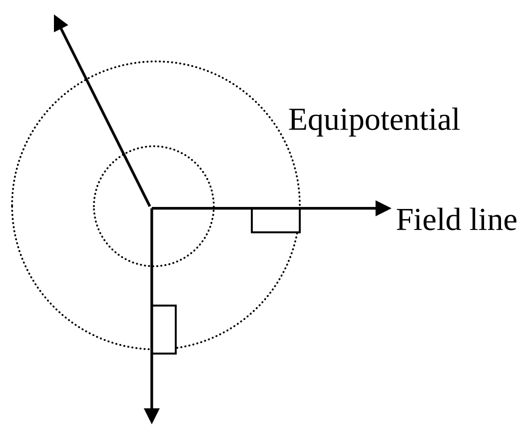


Fig. 12.2: Equipotentials and the electric field lines of a point charge

Note

1. A function is analytic at a point if it is differentiable at that point and every point in a neighbourhood of the point.

2. $f'(z)$ exists does not imply $f(z)$ is analytic
   $f(z)$ analytic implies $f'(z)$ exists

Analyticity is not a "point" idea, but that of a "neighbourhood". Thus, for a function to be analytic at z_0, $f'(z_0)$ must exist, and in addition, $f'(z)$ must exist in a certain neighbourhood of the point.

12.5 Laws of Algebra for Differentiating Complex Functions

The laws of algebra for complex numbers are the same as the laws of algebra for real numbers. Indeed, the rules for differentiating complex algebraic expressions are the same as the rules for differentiating real algebraic expressions:

Given that $f_1(z)$ and $f_2(z)$, then

$$\frac{d}{dz}(f_1(z) \pm f_2(z)) = \frac{d}{dz} f_1(z) \pm \frac{d}{dz} f_2(z) \qquad 12.7$$

$$\frac{d}{dz}(f_1(z) f_2(z)) = \left(\frac{d}{dz} f_1(z)\right) f_2(z) + f_1(z)\frac{d}{dz} f_2(z) \qquad 12.8$$

$$\frac{d}{dz}\left(\frac{f_1(z)}{f_2(z)}\right) = \frac{\left(\frac{d}{dz} f_1(z)\right) f_2(z) - f_1(z)\frac{d}{dz} f_2(z)}{(f_2(z))^2} \qquad 12.9$$

Just as is the case in real polynomial functions and rational functions hold.

$$\frac{d}{dz}(a_0 + a_1 z + a_2 z^2 + ... + a_n z^n) = a_1 + 2a_2 z + ... + na_n z^{n-1} \qquad 12.10$$

Rule 12.9 applies to a rational function.

▪ Find the differential with respect to *z* of $f(z) = z^{-3}$, $z \neq 0$.

❑

$$\frac{d}{dz} z^{-3} = -3z^{-3-1} = -3z^{-4}$$

12.5.1 Differentiation of Exponential, trigonometric and logarithmic functions

Given $f(z) = e^{az}$, where *a* is a constant, then

$$\frac{d}{dz} e^{az} = ae^{az}$$

If $f(z) = \cos az$, where a is a constant, then,

$$\frac{d}{dz}\cos az = -a\sin az$$

Also, for $f(z) = \sin az$,

$$\frac{d}{dz} = a\sin az$$

For $f(z) = \ln z$, then

$$\frac{d}{dz}\ln z = \frac{1}{z}$$

Summary of Study Session 12

In Study Session 12, you learnt how to:

1. Prove whether a function of a complex variable is analytic.
2. Differentiate a function of complex variables from first principles.
3. Derive the Cauchy-Riemann equations.
4. Find a harmonic function between two lines.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self-Assessment Questions for Study Session 12

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SAQ 12.1 (tests Learning Outcome 12.1)

What are analytic functions?

When do we say a function is harmonic?

SAQ 12.2 (tests Learning Outcome 12.2)

Prove that $f(z) = |\bar{z}|^2$ is not analytic.

SAQ 12.3 (tests Learning Outcome 12.3)

Find $\frac{d}{dz}\left[\frac{1+z}{1-z}\right]$.

SAQ 12.4 (tests Learning Outcome 12.4)
Suppose $f'(z)$ exists at a point, prove that the Cauchy-Riemann equations are satisfied.

SAQ 12.5 ((tests Learning Outcome 12.2)
Investigate the analyticity of the function of complex variable, $f(z) = x^2 - iy^2$.

SAQ 12.6 (tests Learning Outcome 12.2)
Is the function $f(z) = \bar{z}$ analytic? Hence, show whether or not $f(z) = \cos y - i\sin y$ is analytic.

SAQ 12.7 (tests Learning Outcome 12.5)
Find a function that is harmonic on the vertical strip from $x = 1$ to 3 and equals 20 and 35 at $x = 1$ and 3.

SAQ 12.8 (tests Learning Outcome 12.5)
Find a function that is harmonic on the strip between the lines $y = -x + 3$, $y = -x - 3$ that takes the values -40 and 10 on the lower and upper lines.

SAQ 12.9 (tests Learning Outcome 12.6)
In what way is the differentiation of a function of a real variable different from that of a function of a complex variable?

Solutions to SAQs

SAQ 12.1

A function is said to be analytic at a point if the Cauchy-Riemann equations are satisfied.

$u_x = v_y$ and $v_x = -u_y$

Each of u and v which satisfy the Cauchy-Riemann equations also satisfies

$$\frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} = 0$$

and as such, is said to be a harmonic function.

SAQ 12.2

$$f(z) = |\bar{z}|^2 = x^2$$
$$u = x^2, v = 0$$
$$u_x = 2x, \; v_y = 0$$
$$f(z) = |z+1|^2$$
$$u_y = 0, \; v_x = 0$$

We conclude that $f(z) = |\bar{z}|^2$ is not analytic.

SAQ 12.3

$$\frac{d}{dz}\left[\frac{1+z}{1-z}\right] = \lim_{\Delta z \to 0} \frac{\left(\frac{1+z+\Delta z}{1-z-\Delta z}\right) - \left(\frac{1+z}{1-z}\right)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(1-z)(1+z+\Delta z) - (1+z)(1-z)}{\Delta z(1-z-\Delta z)(1-z)}$$

$$= = \lim_{\Delta z \to 0} \frac{2}{(1-z)(1-z-\Delta z)} = \frac{2}{(1-z)^2}$$

SAQ 12.4

Suppose $f'(z)$ exists at a point. Then the limit

$f'(z) = \lim_{g \to 0} \frac{f(z+g) - f(z)}{g}$ exists.

Let $g = h + ik$

$f(z) = u(x, y) + iv(x, y)$

$f(z+g) = u(x+h, y+k) + iv(x+h, y+k)$

$$f'(z) = \lim_{g\to 0} \frac{[u(x+h, y+k) - u(x,y)] + i[v(x+h, y+k) - v(x,y)]}{h+ik}$$

Let $g \to 0$ along the x - axis, i.e., $k = 0$.

$$L_1 = \lim_{h\to 0} \frac{[u(x+h, y) - u(x,y)] + i[v(x+h, y) - v(x,y)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{[u(x+h, y) - u(x,y)]}{h} + \frac{i[v(x+h, y) - v(x,y)]}{h} \right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Let $g \to 0$ along the y - axis, i.e., $h = 0$.

$$L_2 = \lim_{k\to 0} \frac{[u(x, y+k) - u(x,y)] + i[v(x, y+k) - v(x,y)]}{ik}$$

$$= \lim_{k\to 0} \left[\frac{[u(x, y+k) - u(x,y)]}{ik} + \frac{i[v(x, y+k) - v(x,y)]}{ik} \right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since $L_1 = L_2$, we equate (12.10) and (12.11) to arrive at

$f'(z) = u_x + iv_x = v_y - iu_y$

This implies that (equating real and imaginary parts)

$$u_x = v_y$$

and

$$v_x = -u_y$$

SAQ 12.5

$f(z) = x^2 - iy^2$

$u(x,y) = x^2$; $v(x,y) = -y^2$

$u_x = 2x$, $v_y = -2y$

$u_y = 0$, $v_x = 0$

$u_x = v_y$ implies $x = -y$

Thus, the function is analytic only on the line $y = -x$.

SAQ 12.6

$f(z) = \bar{z} = x - iy$

$u(x,y) = x$, $v(x,y) = -y$

$u_x = 1$, $u_y = -1$

$f(z) = \bar{z}$ is not analytic.

It follows, therefore, that if any complex function should depend on $\bar{z}$, it cannot be analytic.

$f(z) = \cos y - i \sin y = \cos\frac{z-\bar{z}}{2i} - i\sin\frac{z-\bar{z}}{2i}$ is not analytic because it depends on $\bar{z}$.

SAQ 12.7

We guess $\phi(x, y) = Ax + B$, as the function is to be harmonic on a vertical strip, it could be independent of *y*.

Then

$\phi(1,0) = A + B = 20$

$\phi(3,0) = 3A + B = 35$

So, $2A = 15$ or $A = 7.5$, and $B = 20 - 7.5 = 12.5$

$\phi(x, y) = 7.5x + 12.5$

SAQ 12.8

Guess $\phi(x, y) = Ax + By + C$

Find the values of *A*, *B* and *C*.

$\phi(3,0) = 3A + C = 10$

$\phi(-3,0) = -3A + C = -40$

$2C = -30$

$C = -15$

$A = \frac{10 - C}{3} = \frac{10 + 30}{3} = \frac{40}{3}$

$A = \frac{10 - C}{3} = \frac{10 + 15}{3} = \frac{25}{3}$

$\phi(0,3) = 3B + C = 10$

$B = \frac{10 - C}{3} = \frac{10 + 15}{3} = \frac{25}{3}$

The solution is

$\theta(x, y) = \frac{25}{3}(x + y) - 15$

SAQ 12.9

A function of a complex variable is said to have a derivative at a point z_0 if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is independent of how *z* approaches z_0 .

For the differentiation of a real variable, it is sufficient that the limit exists.

Study Session 13 Complex Integration

Introduction

Just as you may integrate a function of a real variable, you can also integrate a function of a complex variable. We shall find that it might even be easier to convert a real integral into a complex integral in a way that would make the integration easier. We shall be learning some powerful tools that would assist us in doing this.

Learning Outcomes of Study Session 13

At the end of this study session, you should be able to do the following:

13.1 Define and correctly use all the key words printed in **bold** (SAQ 13.1).

13.2 Develop a Laurent series for a given function of a complex variable (SAQ 13.2)

13.3 Calculate the residues of a function of a complex variable (SAQ 13.3-13.8).

13.4 Evaluate the integral of a complex function with the residue theorem (SAQ 13.3,13.4, 13.6-13.8).

13.5Evaluate real integrals with the aid of the residue theorem (SAQ 13.4).

13.1 Laurent's Theorem

This states: If $f(z)$ is analytic on C_1 and C_2 and throughout the annulus between C_1 and C_2 (Fig. 13.1) , then at each point in the annulus, $f(z)$ can be written as

$$f(z)=\sum_{m=1}^{\infty}a_m(z-z_0)^{-m}+\sum_{n=0}^{\infty}a_n(z-z_0)^n \qquad 13.1$$

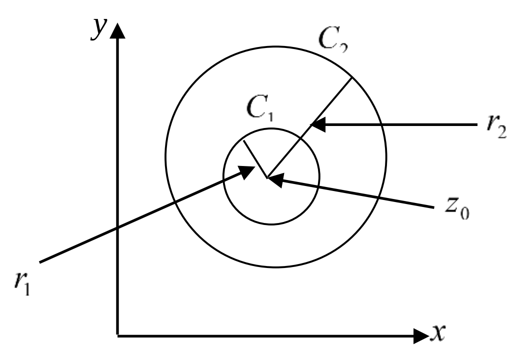

Fig. 13.1: Figure depicting nomenclature for Laurent's Theorem

Where $a_m=\dfrac{1}{2\pi i}\oint\dfrac{f(\xi)d\xi}{(\xi-z_0)^{m+1}}$ 13.2

$$b_m=\frac{1}{2\pi i}\oint\frac{f(\xi)d\xi}{(\xi-z_0)^{-m+1}} \qquad 13.3$$

Definition

A point z_0 is a **singular point** of the function $f(z)$ if and only if it fails to be analytic at z_0 and every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.

Definition

Let z_0 be a singular point of an analytic function. If the neighbourhood of z_0 contains no other singular points, then, $z = z_0$ is an **isolated singularity**.

Definition

In the Laurent series if all but a finite number of $b_m's$ are zero, then z_0 is a pole of $f(z)$. If k is the highest integer such that $b_k \neq 0$, then z_0 is said to be a **pole** of order k.

Definition

The coefficient of b_1 in the Laurent expansion is called the **residue** of $f(z)$ at z_0, denoted by $Res(f, z_0)$.

$Res(f, z_0) = b_1$

Let the singularity at $z = z_0$ be a pole of order m. Since

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + b_2 (z - z_0)^{-2} + \ldots + b_m (z - z_0)^{-m} + \ldots$$

The coefficient of b_1 is

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m f(z) \right\} \qquad 13.4$$

Proof:

$$(z - z_0)^m f(z) = (z - z_0)^m \left(\sum_{n=1}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + b_2 (z - z_0)^{-2} + \ldots + b_m (z - z_0)^{-m} + \ldots \right)$$

$$= (z - z_0)^m \sum_{n=1}^{\infty} a_n (z - z_0)^n + (z - z_0)^m \left(\frac{b_1}{z - z_0} + b_2 (z - z_0)^{-2} + \ldots + b_m (z - z_0)^{-m} + \ldots \right)$$

$$= (z - z_0)^m \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

$$+\left(\frac{b_1(z-z_0)^m}{z-z_0}+b_2(z-z_0)^{-2}(z-z_0)^m+\ldots+b_m(z-z_0)^{-m}(z-z_0)^m+\ldots\right)$$

$$=(z-z_0)^m\sum_{n=1}^{\infty}a_n(z-z_0)^n+\left(b_1(z-z_0)^{m-1}+b_2(z-z_0)^{m-2}+\ldots+b_m+\ldots\right)$$

Differentiating the term involving b_1 with respect to z, we get

$$\frac{d}{dz}b_1(z-z_0)^{m-1}=(m-1)b_1(z-z_0)^{m-2}$$

Taking the differential with respect to z once more,

$$\frac{d^2}{dz^2}b_1(z-z_0)^{m-1}=(m-1)\times(m-2)b_1(z-z_0)^{m-2}$$

Therefore, differentiating $m-1$ times,

$$\frac{d^{m-1}}{dz^{m-1}}b_1(z-z_0)^{m-1}=(m-1)\times(m-2)\times\cdots\times 3\times 2\times 1\times b_1$$

Dividing through by $(m-1)!$,

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}b_1(z-z_0)^{m-1}=b_1$$

13.2 Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except at $z=z_0$. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the residue of $f(z)$ at $z=z_0$.

$$\oint_C f(z)dz=2\pi i Res(f,z_0) \qquad 13.5$$

- With the aid of the residue theorem, evaluate the integral $\int_{|z|=2}\frac{1}{z^2-1}dz$.

❑

$$\int_{|z|=2}\frac{1}{z^2-1}dz=\int_{|z|=2}\frac{dz}{(z+1)(z-1)}$$

$$f(z)=\frac{1}{(z+1)(z-1)}$$

There are two poles, $z=1$ and $z=-1$.

$$Res(f(z),1)=\frac{d^0}{dz^0}(z-1)\frac{1}{(z+1)(z-1)}\bigg|_1$$

$$Res(f(z),-1) = \frac{d^0}{dz^0}(z+1)\frac{1}{(z+1)(z-1)}\bigg|_{-1}$$

Therefore,

$$\int_{|z|=2} \frac{1}{z^2-1}dz = 2\pi i\left(\frac{1}{2}-\frac{1}{2}\right) = 0$$

13.3 Evaluation of Real Integrals by the Residue Theorem

Let us consider the integral $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$, F being the quotient of two polynomials in $\sin\theta$ and $\cos\theta$. We can reduce the evaluation of such an integral to the calculation of the integral of a rational function of z along the circle $|z| = 1$, since rational functions have no singularities other than poles, we can make use of the residue theorems.

Let $z = e^{i\theta}$ $\quad$ 13.6

Then,

$$dz = e^{i\theta}d\theta \qquad 13.7$$

or

$$d\theta = \frac{dz}{iz} \qquad 13.8$$

Therefore,

$$\cos\theta = \frac{z+z^{-1}}{2} \qquad 13.9$$

and

$$\sin z = \frac{z-z^{-1}}{2i} \qquad 13.10$$

Putting equations (13.9) and (13.10) in $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$, we get $\int_C R(z)dz$ where $R(z)$ is a rational function of z and the integral is over the circular path $|z| = 1$. The residue theorem thus yields $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta = 2\pi i \sum Res(R(z))$ at the poles within the unit circle.

- Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1+\alpha\sin\theta};\ 0 < \alpha < 1$$

- Substituting equations (13.9) to (13.10) in the last equation yields

$$I = \frac{2}{\alpha}\int_C \frac{dz}{z^2 + \frac{2iz}{\alpha} - 1}$$

$$= \frac{2}{\alpha}\int_C \frac{dz}{(z - z_1)(z - z_2)},$$

where $z_1 = -\frac{i}{\alpha}\left(1 + \sqrt{1-\alpha^2}\right)$ and $z_2 = -\frac{i}{\alpha}\left(1 - \sqrt{1-\alpha^2}\right)$

Summary of Study Session 13

In Study Session 13, you learnt the following:

1. Develop a Laurent series for a given function of a complex variable.
2. Calculate the residues of a function of a complex variable.
3. Evaluate the integral of a complex function with the residue theorem.
4. Evaluate real integrals with the aid of the residue theorem.

References

1. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
2. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self-Assessment Questions (SAQs) for Study Session 13

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 13.1 (tests Learning Outcome 13.1)

Define the Following:

(i) singular point (ii) isolated singularity (iii) a pole of order k

(iv) residue of a function of a complex variable at a point in the complex plane.

SAQ 13.2 (tests Learning Outcome 13.2)

Develop a Laurent expansion for the function

$$f(z) = \frac{1}{(1-z)^m}\frac{1}{z^n}$$

around the point $z=0$, where m and n are positive integers. Hence, show that

$$\oint_C \frac{dz}{(1-z)^n z^m} = 2\pi i \frac{(n+m-2)!}{(n-1)!(m-1)!}$$

where C is a circle with radius less than unity, centred on the origin.

SAQ 13.3 (tests Learning Outcome 13.3)

Evaluate the integral: $\int_{|z|=2} \frac{1}{z^2+1} dz$

SAQ 13.4 (tests Learning Outcome 13.3)

Evaluate the integral:

$$I = \int_0^{2\pi} \frac{d\theta}{\sin\theta + 5/4}$$

SAQ 13.5 (tests Learning Outcome 13.3)

Find the residues of $f(z)$ at all its isolated singular points and at infinity, where $f(z)$ is given as

$$f(z) = \frac{z^2+z-1}{z^2(z-1)}$$

Assume infinity is not a limit point of singular points.

SAQ 13.6 (tests Learning Outcomes 13.3 and 13.4)

Evaluate the integral

$$\int_{|z|=3} \frac{dz}{(z^2+1)^2(z^2+4)}$$

SAQ 13.6 (tests Learning Outcomes 13.3 and 13.4)

Evaluate $\oint_{|z|=1} \frac{e^{kz}}{z} dz$, where k is a real constant. Hence, deduce the following integrals:

$$\int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta) d\theta = 2\pi \quad \int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta) d\theta = 0$$

SAQ 13.7 (tests Learning Outcomes 13.3 and 13.4)

Evaluate the integral $\oint\limits_{|z|=4} \frac{dz\cos z}{z^2-6z+5}$.

SAQ 13.8 (tests Learning Outcomes 13.3 and 13.4)

Evaluate $\oint_{|z|=5} \frac{\cos z}{z^2(z-\pi)^3} dz$.

Solutions to SAQs

SAQ 13.1

(i) A point z_0 is a singular point of the function $f(z)$ if and only if it fails to be analytic at z_0 and every neighbourhood of z_0 contains at least one point at which $f(z)$ is analytic.

(ii) Let z_0 be a singular point of an analytic function. If the neighbourhood of z_0 contains no other singular points, then, $z = z_0$ is an **isolated singularity**.

(iii) In the Laurent series if all but a finite number of $b_m{}'s$ are zero, then z_0 is a pole of $f(z)$. If k is the highest integer such that $b_k \neq 0$, then z_0 is said to be a **pole** of order k .

(iv) The coefficient of b_1 in the Laurent expansion is called the **residue** of $f(z)$ at z_0 , denoted by $Res(f, z_0)$.

$Res(f, z_0) = b_1$

SAQ 13.2

$$f(z) = \frac{1}{(1-z)^n}\frac{1}{z^m}$$

$$= \frac{1}{z^m}(1-z)^{-n} = \frac{1}{z^m}\left[1 + nz + \frac{(-n)(-n-1)z^2}{2} + \frac{(-n)(-n-1)(-n-2)z^3}{6} + \ldots\right]$$

$$= \frac{1}{z^m} + \frac{n}{z^{m-1}} + \frac{(-n)(-n-1)}{2!\,z^{m-2}} + \ldots + \frac{(-n)(-n-1)\ldots(-n+m-1-1)}{(m-1)!\,z^{m-(m-1)}} + \ldots$$

The Laurent series is:

$$= \frac{1}{z^m} + \frac{n}{z^{m-1}} + \frac{(-n)(-n-1)}{2!\,z^{m-2}} + \ldots + \frac{(-n)(-n-1)\ldots(-n-m+2)}{(m-1)!\,z^{m-(m-1)}} + \ldots$$

$$= \frac{1}{z^m} + \frac{n}{z^{m-1}} + \frac{(-n)(-n-1)}{2!\,z^{m-2}} + \ldots + \frac{(n+m-2)!}{(n-1)!(m-1)!\,z^{m-(m-1)}} + \ldots$$

since $n! = n(n-1)! \Rightarrow n = \dfrac{n!}{(n-1)!}$.

Therefore, $a_{-1} = \dfrac{(n+m-2)!}{(n-1)!(m-1)!}$

= Residue at $z = 0$.

Therefore, the integral

$$\oint_C \frac{dz}{(1-z)^n z^m} = 2\pi i\,\frac{(n+m-2)!}{(n-1)!(m-1)!}$$

SAQ 13.3

$$\int_{|z|=2} \frac{1}{z^2+1}\, dz = \int_{|z|=2} \frac{dz}{(z+i)(z-i)}$$

$$f(z) = \frac{1}{(z+i)(z-i)}$$

There are two poles, $z = i$ and $z = -i$.

$$Res(f(z), i) = \frac{d^0}{dz^0}(z-i)\frac{1}{(z+i)(z-i)}\bigg|_i$$

$$Res(f(z), -i) = \frac{d^0}{dz^0}(z+i)\frac{1}{(z+i)(z-i)}\bigg|_{-i}$$

Therefore,

$$\int_{|z|=2} \frac{1}{z^2+1}\, dz = 2\pi i\left(\frac{1}{2i} - \frac{1}{2i}\right) = 0$$

SAQ 13.4

Let $z = e^{i\theta}$

Then, $d\theta = \dfrac{dz}{iz}$, $\sin\theta = \dfrac{z - z^{-1}}{2i}$

Therefore,

$$I = \int_C \frac{dz}{iz\left(\frac{5}{4} + \frac{z - z^{-1}}{2}\right)} = 2\int_C \frac{dz}{z^2 + \frac{5}{4}iz - 2}$$

$$= 2\int_C \frac{dz}{\left(z + \frac{i}{2}\right)(z + 2i)}.$$

The poles of the integrand are at $z = -2i$ and $-i/2$. Only the latter lies within the unit circle

$$res(-i/2) = z \xrightarrow{\lim} -i/2 \;\left(z + \frac{i}{2}\right)\frac{1}{\left(z + \frac{i}{2}\right)(z + 2i)}$$

$$= \frac{-2}{3i}$$

Therefore,

$$I = 2 \times 2\pi i \times -\frac{2i}{3} = \frac{8\pi}{3}$$

SAQ 13.5

This has poles at $z = 0$, of multiplicity $k = 2$ and at $z = 1$, of multiplicity $k = 1$.

$$\mathrm{Re}\, s_{z=z_0} f(z) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]$$

$$\mathrm{Re}\, s_{z=1} = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{z^2 + z - 1}{z^2} = 1$$

$$\mathrm{Re}\, s_{z=0} = \lim_{z \to 1} \frac{d}{dz} z^2 f(z) = \lim_{z \to 0} \frac{z^2 + z - 1}{z - 1} = \frac{(2z+1)(z-1) - (z^2 + z - 1)}{(z-1)^2}$$

If $f(z)$ is analytic except at isolated singular points, then the sum of all the residues of $f(z)$ is 0.

Hence, $\mathrm{Re}\, s_{z=0} + \mathrm{Re}\, s_{z=1} + \mathrm{Re}\, s_{z=\infty} = 1 + \mathrm{Re}\, s_{z=\infty} = 0$

Therefore, $\mathrm{Re}\, s_{z \to \infty} = -1$.

SAQ 13.6

The integrand has a pole of order 2 at $z = i$ and a pole of order 1 at $z = 2i$, which lie within the circle $|z| = 3$.

$$\mathrm{Res}\,(z = i) = \lim_{z \to i} \frac{d}{dz} \left[\frac{1}{(z+i)(z+i)(z^2+4)} \right]$$

$$= \lim_{z \to i} \left[\frac{-2}{(z+i)^3 (z^2+4)} - \frac{2z}{(z+i)^2 (z^2+4)^2} \right]$$

$$= \frac{-2}{3(2i)^3} - \frac{2i}{9(2i)^2}$$

$$= -\frac{i}{36}$$

Similarly,

$$\mathrm{Res}\,(z = 2i) = \lim_{z \to 2i} \left[\frac{1}{(z^2+1)(z+2i)} \right]$$

$$= -\frac{i}{36}$$

Therefore,

$$\int_{|z|=3} \frac{dz}{(z^2+1)^2(z^2+4)} = 2\pi i\left(-\frac{i}{36}-\frac{i}{36}\right)$$

$$= \frac{\pi}{9}$$

SAQ 13.6

The only pole is at $z=0$, and the attendant residue is $e^{kz}\Big|_{z=0} = 1$. Hence,

$$\oint_{|z|=1} \frac{e^{kz}}{z} dz = 2\pi i \times 1 = 2\pi i$$

But

$$\oint_{|z|=1} \frac{e^{kz}}{z} dz = \int_0^{2\pi} \frac{e^{k(\cos\theta + i\sin\theta)}}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} e^{k(\cos\theta+i\sin\theta)} id\theta$$

$$= \int_0^{2\pi} e^{k\cos\theta} e^{ik\sin\theta} id\theta$$

$$= \int_0^{2\pi} e^{k\cos\theta} [\cos(k\sin\theta) + i\sin(k\sin\theta)] id\theta$$

$$= \int_0^{2\pi} e^{k\cos\theta} [i\cos(k\sin\theta) - \sin(k\sin\theta)] d\theta$$

which in turn is equal to $2\pi i$

Equating real and parts,

$$\int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta) d\theta = 2\pi$$

$$\int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta) d\theta = 0$$

SAQ 13.7

$$\oint_{|z|=4} \frac{dz\cos z}{z^2-6z+5} = \int_{|z|=4} \frac{\cos dz}{(z-1)(z-5)}$$

$$f(z) = \frac{\cos z}{(z-1)(z-5)}$$

There are two poles, $z=1$ and $z=5$. Only the first pole lies within the contour.

$$Res(f(z),1) = \frac{d^0}{dz^0}(z-1)\frac{\cos z}{(z-1)(z-5)}\Bigg|_1 = -\frac{1}{4}\cos(1)$$

Therefore,

$$\int_{|z|=4} \frac{dz}{(z-1)(z-5)} = -2\pi i \times \frac{1}{4}\cos(1)$$

$$\oint_{|z|=4} \frac{dz\cos z}{(z-1)(z-5)} = -\frac{\pi i \cos(1)}{2}$$

SAQ 13.8

$$\oint_{|z|=5} \frac{\cos z}{z^2(z-\pi)^3} dz$$

$$= 2\pi i(\operatorname{Re} s_{z=0} + \operatorname{Re} s_{z=\pi})$$

$$= 2\pi i\left(\lim_{z\to 0}\frac{d}{dz}\frac{\cos z}{(z-\pi)^3} + \frac{1}{2}\lim_{z\to\pi}\frac{d^2}{dz^2}\frac{\cos z}{z^2}\right)$$

$$= 2\pi i\left(-\frac{3}{\pi^4} + \frac{\pi^2-6}{2\pi^4}\right) = \frac{\pi^2-12}{\pi^3} i$$

Study Session 14 Eigenvalue Problems

Introduction

Eigenvalue problems crop up in every area of Physics, and is perhaps one of the most fundamental ideas behind Quantum mechanics, an area of Physics where a physical system can only take a certain set of values. As an example, the energy of a quantum-mechanical oscillator can only take a discrete set of values; likewise the electron in a hydrogen atom. Indeed, every physical observable in Quantum mechanics has associated with it a Hermitian operator, the eigenvalues of which are the only possible values the physical observable can attain.

Learning Outcomes of Study Session 14

At the end of this study session you would be able to do the following:

14.1 Correctly define and make use of all the key words printed in **bold** (SAQ 14.1).

14.2 Recognise and work with the eigenvalue equation (SAQ 14.2).

14.3 Write the characteristic equation for an eigenvalue problem (SAQ 14.2-14.5).

14.4 Find the eigenvalues and the eigenfunctions of a square matrix (SAQ 14.2-14.5).

14.1 The Eigenvalue Problem

Let an operator A act on a state ψ, then, we can write

$$A\psi = \phi \qquad 14.1$$

This means that the result of the action of the **operator** A on the **eigenstate** (one of the possible states the system described by A can be found) ψ is the state ϕ.

If indeed ϕ is a certain multiple of ψ, ϕ can be written as $\phi = \lambda\psi$, then

$$A\psi = \lambda\psi \qquad 14.2$$

and this is the **eigenvalue equation**.

In Physics, we have cause to deal with eigenvalue problems in several ways: the operator could be a matrix, or a function. Indeed, the eigenvalue problem involves finding values of λ for which equation 14.2 holds. We shall see later that in Quantum mechanics, the eigenvalues of an operator are the possible values the physical quantity represented by the operator takes in some allowable states. Once we have obtained the eigenvalues, we put each of them back in equation 14.2 and then find the eigenstate corresponding to it. The eigenvalues are also called the spectrum of the A.

14.1.1 Matrix Space

Let

$$A\psi = \lambda\psi$$

where A is a square matrix, and I is the unit or identity matrix of the same dimension as *A*.

Then,

$$(A - \lambda I)\psi = 0 \qquad 14.3$$

The identity matrix *I* is necessary as *A* is matrix, and there is no way you would subtract a number from a matrix. What this effectively does is to take λ away from the diagonal element of *A*. For this equation to have a unique solution, the determinant

$$|A - \lambda I| = 0 \qquad 14.4$$

This gives a polynomial equation in λ. The order of the equation is the dimension of the *A*. This polynomial equation is called the indicial or characteristic equation. We solve the equation for the allowable values of λ. We then put these expressions back into equation 14.2 in order to get the corresponding eigenstates. We shall assume initially that all the eigenvalues are different. The case where two or more are equal, we say the states are degenerate. This case will be handled separately.

▪ Find the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$.

❑ The eigenvalue equation is

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 14.5$$

where $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a typical eigenstate.

This we can write equation 14.5 as,

$$\begin{pmatrix} 2-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

For equation … to have a unique solution, we require that,

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \qquad 14.6$$

The indicial equation is,

$$(2-\lambda)(-2-\lambda) + 1 = 0$$

$$-(2-\lambda)(2+\lambda) + 1 = 0$$

$$\lambda^2 - 4 + 1 = 0$$

Hence,

$$\lambda = \pm\sqrt{3}$$

The eigenvalues are $\sqrt{3}$ and $-\sqrt{3}$

Case (i): $\lambda = \sqrt{3}$

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sqrt{3}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$2u_1 - u_2 = \sqrt{3}u_1$$

We can choose u_1 as 1, then,

$$u_2 = 2u_1 - \sqrt{3}u_1 = 2 - \sqrt{3}$$

Hence, the eigenstate corresponding to $\sqrt{3}$ is

$$\begin{pmatrix} 1 \\ 2-\sqrt{3} \end{pmatrix}$$

Case (ii): $\lambda = -\sqrt{3}$

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\sqrt{3}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$2u_1 - u_2 = -\sqrt{3}u_1$$

We can choose u_1 as 1, then,

$$u_2 = 2u_1 - \sqrt{3}u_1 = 2 + \sqrt{3}$$

Hence, the eigenstate corresponding to $-\sqrt{3}$ is

$$\begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}$$

What this means is that the quantity represented by the operator $\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$ can exist in only two eigenstates $\mathbf{u} = \begin{pmatrix} 1 \\ 2-\sqrt{3} \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}$. The value of the property measured in the two possible states are the respective eigenvalues $\sqrt{3}$ and $-\sqrt{3}$.

▪ Let $\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find the eigenvalues of Ω.

❑ The characteristic equation is formed by $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$$-\lambda^3 + \lambda = 0$$

The eigenvalues are 0, 1 and -1.

For $\omega = 0$, eigenvector is given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a_1 = 0$, $a_3 = 0$. We can choose $a_2 = 1$ without loss of generality.

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1: \ \Omega = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -1: \Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

14.1.2 Function Space

In function space, let the possible eigenstates be denoted by ϕ or $|\phi>$, the latter which we refer to as the ket. In standard notation, eigenstates in quantum mechanics are denoted by kets. The expression $(\mathbf{v}_1, \mathbf{v}_2)$ can be written as $<\mathbf{v}, \mathbf{v}>$ or $<\mathbf{v}|\mathbf{v}>$ in bra-ket notation. Just as we have learnt in Study Session 2, the bra $<\mathbf{v}|$ is a row matrix, while $|\mathbf{v}>$ is a column matrix.

- For a particle in an infinite potential well $0 \le x \le L$, the possible eigenstates are,

  $|\phi_n>=\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$; $n = 1, 2, \ldots$

  an infinite number of states that merges into a continuum as n tends to infinity. The usual operators are $\hat{x}$, the position operator, $\hat{p} = -i\hbar\frac{d}{dx}$, etc.

  Let a physical observable A be represented by $\hat{A}$. Show that the expectation value of the operator is the eigenvalue of the operator.

- Then, the expectation value of the physical observable is given as,

  $$<A>=<\phi_n|\hat{A}|\phi_n>=<\phi_n|\lambda_n|\phi_n> \quad 14.7$$

  But λ_n, the eigenvalue of A in state $|\phi_n>$, that is, the average value of the physical quantity A, such that if a measurement is made of A, λ_n is the value we would obtain, albeit with a certain probability (see Section …).

  We then see that we can write equation 14.7 as,

  $<A>=\lambda_n<\phi_n|\phi_n>=\lambda_n$ if the set $\{|\phi_j>\}_{j=1}^{\infty}$ is an orthonormal set.

  You can now see that indeed, the expectation value is the eigenvalue λ_n.

- Given that the Hamiltonian for the Harmonic Oscillator in the ground state is $\frac{1}{2}\hbar\omega_0$, show that the expectation of the energy of the harmonic oscillator in the ground state is $\frac{1}{2}\hbar\omega_0$. Assume that the eigenvectors are normalised.

- The energy expectation value of for the ground state of the simple harmonic oscillator:

  $$<E>=\int_{-\infty}^{\infty}\psi_0^*\hat{H}\psi_0 dx = \int_{-\infty}^{\infty}\psi_0^*\left(\frac{1}{2}\hbar\omega_0\right)\psi_0 = \frac{1}{2}\hbar\omega_0\int_{-\infty}^{\infty}\psi_0^*\psi_0 dx = \frac{1}{2}\hbar\omega_0$$

  since ψ_0 is normalised.

- It is known that the eigenvalue of square of the orbital angular momentum and the z-component of the angular momentum of the electron in the hydrogen atom are, respectively $l(l+1)\hbar^2$ and $m\hbar$ in the states represented by the Spherical Harmonics $Y_{lm}(\theta,\phi)$. Write the eigenvalue equation for each of the physical observables.

- The eigenvectors of the angular momentum L and the *z*-component of the angular momentum, L_z, of an electron in the Hydrogen atom is $Y_{lm}(\theta,\phi)$, in the spherical coordinates (r,θ,ϕ). The eigenvalues are respectively, $\sqrt{l(l+1)}\hbar$ and $m\hbar$, where *l* is the orbital angular momentum and m is the azimuthal angular momentum. We write,

$$LY_{lm}(\theta,\phi) = \sqrt{l(l+1)}\hbar Y_{lm}(\theta,\phi)$$

$$L_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$$

Summary of Study Session 14

In Study Session 14, you learnt to do the following:

1. Recognise and work with the eigenvalue equation.
2. Write the characteristic equation for an eigenvalue problem.
3. Find the eigenvalues and the eigenfunctions of a square matrix.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self Assessment Questions for Study Session 14

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 14.1 (tests Learning Outcome 14.1)

Explain the following:

(i) Eigenvalue (ii) Eigenvector (iii) Eigenvalue equation

(iv) Characteristic or indicial equation

SAQ 14.2 (tests Learning Outcome 14.2)

Let O be a Hermitian operator. If in addition, $O^3 = 2I$, prove that $O = 2^{1/3} I$.

SAQ 14.3 (tests Learning Outcomes 14.3 and 14.4)

Find all the possible values (spectrum) $L_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ in the state $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

SAQ 14.4 (tests Learning Outcomes 14.3 and 14.4)

Find the eigenvalues of S_y. Write the eigenvectors in terms of the spin-1/2 spinors.

$$S_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

SAQ 14.5 (tests Learning Outcomes 14.3 and 14.4)

a. Find the eigenvalues and the corresponding eigenfunctions of the matrix.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b. If this matrix represents a physically observable attribute of a particle, what is the expectation value of the attribute in each of the possible states. Comment on your results.

Solutions to SAQs

SAQ 14.1

(i) The eigenvalues or spectrum of an operator is the possible values a system represented by the operator can assume.

(ii) The eigenvectors of an operator is the possible states (eigenstates) a system represented by the operator can assume.

(iii) The eigenvalue equation is of the form,

$$A\psi = \lambda\psi$$

where ψ is the eigenstate, λ, the corresponding eigenvalue and A is the operator.

(iv) The characteristic or indicial equation is that which results from the determinant,

$$\left|A - \lambda I\right| = 0$$

where all symbols have their usual meanings and I is the appropriate unit matrix.

SAQ 14.2

$$O^3\psi = 2I\psi$$

$$O^3\psi = O^2 O\psi = O^2\lambda\psi = \lambda O\lambda\psi = \lambda^2 O\psi = \lambda^3\psi$$

The indicial equation is,

$$| 2I - \lambda^3 | = 0$$

or

$$\begin{vmatrix} 2-\lambda^3 & 0 & 0 \\ 0 & 2-\lambda^3 & 0 \\ 0 & 0 & 2-\lambda^3 \end{vmatrix} = 0$$

$$\lambda = 2^{1/3}$$

Now, Get the cube roots of 2.

$$r = 2^{1/3},\ \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

The roots are:

$$2^{1/3}\left[\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right]$$

where $k = 0, 1, 2$

$$k = 0 \quad : \qquad 2^{1/3}\left[\cos 0 + i\sin 0\right] \quad = \; 2^{1/3}$$

$$k = 1: \qquad 2^{1/3}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \quad = 2^{1/3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$k = 2: \qquad 2^{1/3}\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] \quad = 2^{1/3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

A Hermitian operator can only have real eigenvalues. Thus, the only permissible eigenvalue of O is $2^{1/3}$.

Thus,

$$O\psi = 2^{1/3}\psi$$

Hence, $O = 2^{1/3} I$.

SAQ 14.3

$$L_x \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix} = \frac{\hbar}{\sqrt{6}\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$L_x \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\begin{vmatrix}0-\lambda & 1 & 0\\1 & -\lambda & 1\\0 & 1 & -\lambda\end{vmatrix} = 0$$

or

$$-\lambda(\lambda^2 - 1) - 1(-\lambda) = -\lambda^3 + \lambda + \lambda = 0$$

$$\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2 - 2) = 0$$

$\lambda = -\sqrt{2}$, 0 and $\sqrt{2}$

SAQ 14.4

The indicial equation is given by,

$$\begin{vmatrix}0-\lambda & -i\hbar/2\\i\hbar/2 & 0-\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & -i\hbar/2\\i\hbar/2 & -\lambda\end{vmatrix} = 0$$

$$\lambda^2 - (\hbar/2)^2 = 0$$

Hence,

$$\lambda = \pm\frac{\hbar}{2}$$

Therefore, the eigenvalues are $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$.

For $\lambda = \hbar/2$,

$$\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \hbar/2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$-i\hbar u_2/2 = \hbar u_1/2$$

or

$$-iu_2 = u_1$$

$$u_2 = iu_1$$

Hence,

$$\mathbf{e}_y = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda = -\hbar/2$,

$$\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\hbar/2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$-i\hbar u_2/2 = -\hbar u_1/2$$

or

$$iu_2 = u_1$$

$$u_2 = -iu_1$$

Hence,

$$\mathbf{e}_y = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvalues of S_y are:

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SAQ 14.5

a. The characteristic equation is formed by $\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$$-\lambda^3 + \lambda = 0$$

Eigenvalues are 0, 1 and -1.

For λ = 0, eigenvector is given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

Or $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The normalised eigenfunction is $\psi_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\lambda = 1: \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The normalised wavefunction is $\psi_2 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\lambda = -1: \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Normalised wavefunction is $\psi_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

b. The expectation value of A in state $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is

$$< \psi_1 \mid A \mid \psi_1 >= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$< \psi_2 \mid A \mid \psi_2 >= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \times 2 = 1$$

$$< \psi_3 \mid A \mid \psi_3 >= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2} \times -2 = -1$$

Comment: The expectation values are the eigenvalues we got earlier. This is another way of getting the eigenvalues of an operator.

Study Session 15 Diagonalisation of a Matrix

Introduction

We have learnt how to get the eigenvalues and the eigenvectors of a given square matrix. In this study session, we shall apply the knowledge gained to diagonalise a given square matrix. We shall see the condition that would need to be satisfied if a square matrix is to be diagonalisable. We shall also dwell more on the properties of some matrices and how they can furnish us with a rich family of orthonormal eigenvectors.

Learning Outcomes of Study Session 15

By the end of this study session you should be able to do the following:

15.1 Understand and correctly use the keywords in **bold** print (SAQ 15.1).

15.2 Determine if a given matrix can be diagonalised or not (SAQ 15.4).

15.3 Diagonalise a given matrix (SAQ 15.2, 15.3, 15.5).

15.4 Calculate the modal matrix for a given diagonalisable matrix (SAQ 15.2, 15.3).

15.5 Determine the kind of matrix that can diagonalise a given type of matrix (SAQ 15.5).

15.1 Diagonalisation of a Matrix

Given a matrix **A**, we can find the corresponding eigenvalues $\{\lambda_i\}_{i=1}^n$ in corresponding eigenstates $(\mathbf{x}_i)_{i=1}^n$, i.e.,

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

The eigenvalue system is given by $\mathbf{Ax} = \lambda \mathbf{x}$, where λ is the eigenvalue corresponding to the eigenvector $\mathbf{x} \neq \mathbf{0}$. In matrix form, this is equivalent to

Or $$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

We can write this system, which is $\mathbf{Ax} = \lambda\mathbf{x}$ as $(\mathbf{A} - \lambda\,\mathbf{I})\mathbf{x} = \mathbf{0}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.

This equation has a unique solution if and only if $|\,\mathbf{A} - \lambda\,\mathbf{I}\,| = 0$, giving the characteristic equation from which the n eigenvalues $\{\lambda_i\}_{i=1}^{n}$ could be found. It follows that all the λ_i could be distinct, or some could be repeated.

Suppose we have obtained the eigenvalues of a given matrix and the corresponding eigenvectors, then we can write,

$$M = \begin{pmatrix} x_{11} & x_{21} & . & . & . & x_{n1} \\ x_{12} & x_{22} & . & . & . & x_{n2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{1n} & x_{2n} & . & . & . & x_{nn} \end{pmatrix}$$

where we have arranged the eigenvectors vertically. M is the **modal matrix**.

$A\mathbf{x} = \lambda\mathbf{x}$, or in component form, $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$, …, $A\mathbf{x}_n = \lambda_n\mathbf{x}_n$, where

$$\mathbf{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ . \\ . \\ . \\ x_{1n} \end{pmatrix}, \; \mathbf{x}_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ . \\ . \\ . \\ x_{2n} \end{pmatrix}, \ldots, \; \mathbf{x}_n = \begin{pmatrix} x_{n1} \\ x_{n2} \\ . \\ . \\ . \\ x_{nn} \end{pmatrix}$$

$$AM = A(\mathbf{x}_1 \quad \mathbf{x}_2 \quad . \quad . \quad . \quad \mathbf{x}_n) = (\lambda_1\mathbf{x}_1 \quad \lambda_2\mathbf{x}_2 \quad . \quad . \quad . \quad \lambda_n\mathbf{x}_n)$$

$$= \begin{pmatrix} x_{11} & x_{21} & . & . & . & x_{n1} \\ x_{12} & x_{22} & . & . & . & x_{n2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{1n} & x_{2n} & . & . & . & x_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{pmatrix}$$

Hence,

$$M^{-1}AM = \begin{pmatrix} x_{11} & x_{21} & . & . & . & x_{n1} \\ x_{12} & x_{22} & . & . & . & x_{n2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{1n} & x_{2n} & . & . & . & x_{nn} \end{pmatrix}^{-1} \begin{pmatrix} x_{11} & x_{21} & . & . & . & x_{n1} \\ x_{12} & x_{22} & . & . & . & x_{n2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ x_{1n} & x_{2n} & . & . & . & x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_n \end{pmatrix} = D$$

$$D = M^{-1}AM$$

This is a **similarity transformation**, and the matrices A and D are said to be **similar matrices**.

M thus diagonalises *A*. *M* is called the modal matrix, and the diagonal matrix is the spectral matrix that has only an eigenvalue in each column (or row). The diagonal matrix is said to be the canonical or diagonal form. Notice that the spectral matrix has the same eigenvalues as the matrix *A*. Indeed, the eigenvalues are arranged in the same way in both matrices. We can diagonalise a matrix *A* if we can get a modal matrix, which of course must be invertible. In other words, *M* should not be a singular matrix. How do we ensure that it is not a singular matrix? If it is *n*-dimensional, then, its columns, and hence rows, must be linearly independent. One way we can be sure of this is that all the eigenvalues are distinct. We could still have a situation where the eigenvalues are repeated and we can still get *n* linearly independent eigenvectors. Such a case we shall visit later in the study session.

The modal matrix is not the only avenue for finding the spectral matrix. Indeed, if we normalise each vector, we would still get the spectral matrix. But now, the matrix could have some other special properties.

▪ Diagonalise the matrix

$$E = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

using both the eigenvectors and the normalised eigenvectors.

- The eigenvectors of the matrix are:

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

respectively for -1, 1, 0.

The modal matrix is (for the eigenvalues as ordered above)

$$M = \begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Then,

$$M^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

The spectral matrix is,

$$D = M^{-1}EM = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using normalised eigenvectors, we normalise each vector,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

The matrix composed of these eigenvectors is,

$$\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{pmatrix}$$

The inverse matrix is,

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & -\sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{pmatrix}$$

$$\begin{pmatrix} -\sqrt{2} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & -\sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

15.2 Adjoint or Hermitian conjugate of a Matrix

The adjoint or Hermitian conjugate of a matrix A is given by

$$Adj(A) = A^{+}$$

15.3 Normal Matrices

A normal matrix is one that commutes with its Hermitian conjugate.

$$AA^{+} = A^{+}A$$

15.4 Unitary Matrices

Recall that for an orthogonal matrix Q, $QQ^{T} = I$, and $Q^{T} = Q^{-1}$. The complex analogue of a real orthogonal matrix is a **unitary** matrix, i.e., $AA^{+} = I$ or, equivalently, $A^{+} = A^{-1}$. Note that the condition on matrices reduces to that for orthogonal matrices if the entries are all real. It follows, therefore, that,

$$UU^{+} = U^{+}U = I$$

or

$$U^{+} = U^{-1}$$

We then expect the columns of a unitary matrix to be orthonormal. Also, the rows should also be orthonormal.

Notice that a unitary matrix satisfies a more stringent condition. Not only is it normal, but in addition, the product of the matrix and its Hermitian adjoint is equal to the unit or identity matrix.

15.5 Hermitian Matrices

A Hermitian matrix is the complex equivalent of a symmetric matrix, satisfying,

$$A^{+} = A$$

Similarly, the corresponding skew-Hermitian or anti-Hermitian matrix is the complex equivalent of a skew-symmetric matrix.

In quantum mechanics, every (real) observable has associated with it a Hermitian operator because the eigenvalues must be real, being the real values the observable can assume under measurement. The following theorem is therefore in order.

Theorem

To every Hermitian matrix A, there exists a unitary matrix U, built out of the eigenfunctions of A, such that $U^{+}AU$ is diagonal, with the eigenvalues on the diagonal.

Proof

Let $\quad A|\mathbf{u}_1 >= b_1 |\mathbf{u}_1 >$

$$A|\mathbf{u}_2 >= b_2 |\mathbf{u}_2 >$$

$$.$$

$$.$$

$$A|\mathbf{u}_n >= b_n |\mathbf{u}_n >$$

where $\mathbf{u}_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ . \\ . \\ u_{1n} \end{pmatrix}$, etc.

Let $U = \begin{pmatrix} u_{11} & u_{21} & . & . & u_{n1} \\ u_{12} & u_{22} & . & . & u_{n2} \\ . & . & . & . & . \\ . & . & . & . & . \\ u_{1n} & u_{2n} & . & . & u_{nn} \end{pmatrix}$ as in the earlier case, where we have represented each **orthonormalised** eigenvector by a column.

U is a unitary matrix, since $U^{+}U = I$, or equivalently, $U^{+} = U^{-1}$. If the matrix we wish to diagonalise is a real symmetric matrix (real analogue of a Hermitian matrix), then the unitary matrix reduces to an orthogonal matrix, Q.

▪ Diagonalise the symmetric matrix $\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ using the appropriate orthogonal matrix.

❑

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is formed by

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda = 0$$

Eigenvalues are 0, 1 and -1.

For λ = 0, the eigenvector is given by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 1$:

$$\Omega = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{(normalised)}$$

For $\lambda = -1$:

$$\Omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{(normalised)}$$

The appropriate orthogonal matrix is,

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and

$$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$D = Q^T \Omega Q \quad = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that you can only diagonalise a matrix the columns (or rows) of which are linearly independent. We learnt earlier that the columns (as well as the rows) of an orthogonal matrix are linearly independent. In addition, a unitary matrix is the complex analogue of an orthogonal matrix. As such, we conclude that every orthogonal matrix and every unitary matrix is diagonalisable. Conversely, we expect that orthogonal and unitary matrices would diagonalise some matrices, being composed of linearly independent (normalised) vectors arranged in order, and hence non-singular.

Real symmetric matrices can be diagonalised by orthogonal matrices. As such, real symmetric matrices with *n* distinct eigenvalues are orthogonally diagonalisable.

▪ Show that the matrix $B = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$ is Hermitian. Hence, find the equivalent diagonal matrix.

❑

$$B^{+} = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} = B$$

Hence, B is Hermitian.

The eigenvectors are, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}$

Arranging the eigenvectors in order in a matrix, $P = \begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. $P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -i & 0 & 1 \\ i & 0 & 1 \end{bmatrix}$

$$P^{-1}BP = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -i & 0 & 1 \\ i & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have successfully diagonalised B. But notice that the matrix P is not a unitary matrix.

$$PP^{+} = \begin{bmatrix} 0 & i & -i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -i & 0 & 1 \\ i & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \neq I_3$$

This is because the vectors composing P are not normalised. They satisfy orthogonality however. Let us normalise each vector constituting P and label the new matrix U, the columns of which will now be orthonormal.

Then,

$$U = \begin{bmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$U^+ = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$UU^+ = \begin{bmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

showing that U is a unitary matrix. The columns are orthonormal, and indeed, $U^+ = U^{-1}$.

It is much easier calculating the Hermitian adjoint of a matrix than finding the inverse of the matrix. The former entails finding the transpose and finding the complex conjugate of every element; much simpler than the rigorous work required to find the inverse of a matrix, especially if the dimension is more than 2.

We could also diagonalise B with the matrix U instead of P. We shall now make use of the matrix U to show that a unitary matrix diagonalises a Hermitian matrix:

$$U^+BU = \begin{bmatrix} 0 & 1 & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \\ i/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A Hermitian matrix is normal, since $A^+ = A$, then $AA^+ = A^+A$. Similarly, a real symmetric (skew-symmetric) matrix, is normal. Indeed, a less stringent condition is satisfied since the complex conjugate of a real number is the number itself. For real symmetric and skew-symmetric matrices, it suffices that $AA^T = A^TA$. A unitary matrix diagonalises a matrix if and only if the matrix is normal.

Summary of Study Session 15

In Study Session 15, you learnt how to do the following:

1. Determine whether a given matrix can be diagonalised or not.
2. Diagonalise a given matrix.
3. Calculate the modal matrix for a given diagonalisable matrix.
4. Determine the kind of matrix that can diagonalise a given type of matrix.

References

1. Hill, K. (1997). Introductory Linear Algebra with Applications, Prentice Hall.
2. Butkov, E. (1968). Mathematical Physics, Addison-Wesley.
3. MacQuarrie, D. A. (2003). Mathematical Methods for Scientists & Engineers, University Science Books.

Self Assessment Questions for Study Session 15

You have now completed this study session. You may now assess how well you have achieved the Learning Outcomes by answering the following questions. Write your answers in your Study Diary and discuss them with your Tutor at the next Study Support Meeting. You can check your answers with the solutions to the Self-Assessment Questions at the end of this study session.

SAQ 15.1 (tests Learning Outcome 15.1)

Define the following matrices:

(i) Modal matrix (ii) Spectral matrix (iii) Unitary matrix

(iv) Normal matrix (v) Hermitian matrix

SAQ 15.2 (tests Learning Outcome 15.3)

Diagonalise the matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ -1 & -6 & -2 \\ 5 & 0 & 1 \end{pmatrix}$.

SAQ 15.3 (tests Learning Outcome 15.3)

Diagonalise the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

SAQ 15.4 (tests Learning Outcome 15.2)

Show that the matrix $H = \begin{pmatrix} 1 & i & -i \\ -i & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ is Hermitian, but not diagonalisable.

SAQ 15.5 (tests Learning Outcomes 15.3, 15.5)

Diagonalise the Hermitian matrix $H = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}$.

Solutions to SAQs

SAQ 15.1

(i) A modal matrix is a matrix composed of the eigenvectors of a square matrix arranged in order, each eigenvector written as a column vector.

(ii) The spectral matrix is the diagonal equivalent of a given diagonalisable matrix. The eigenvalues are the only non-zero elements of the matrix and are arranged along the main diagonal.

(iii) Unitary matrix is a matrix that satisfies the condition $U^{+} = U^{-1}$ or equivalently, $U^{+}U = UU^{+} = I$, that is, the Hermitian conjugate of the matrix is the inverse of the matrix.

(iv) A normal matrix that commutes with its Hermitian conjugate, that is, $NN^{+} = N^{+}N$.

(v) A Hermitian matrix is a complex-valued symmetric matrix that satisfies the condition $H^{+} = H$, that is, it is its own Hermitian conjugate. We say such a matrix is self-adjoint.

SAQ 15.2

We find the eigenvalues of the matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ -1 & -6 & -2 \\ 5 & 0 & 1 \end{pmatrix}$.

The characteristic equation is given by,

$$\begin{vmatrix} 4-\lambda & 0 & 0 \\ -1 & -6-\lambda & -2 \\ 5 & 0 & 1-\lambda \end{vmatrix} = 0$$

Hence,

$$(-6-\lambda)(1-\lambda)(4-\lambda) = 0$$

$\lambda = -6$, 1 and 4.

The eigenvectors are,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -7/2 \end{pmatrix}, \begin{pmatrix} -30/13 \\ 1 \\ -50/13 \end{pmatrix}$$

Let us arrange the eigenvectors in descending order of the eigenvalue (this ensures uniqueness, even though there is no fixed way of arranging the eigenvectors):

$$M = \begin{pmatrix} -30/13 & 0 & 0 \\ 1 & 1 & 1 \\ -50/13 & -7/2 & 0 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -13/30 & 0 & 0 \\ 10/21 & 0 & -2/7 \\ -3/70 & 1 & 2/7 \end{pmatrix}$$

$$\begin{pmatrix} -13/30 & 0 & 0 \\ 10/21 & 0 & -2/7 \\ -3/70 & 1 & 2/7 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ -1 & -6 & -2 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} -30/13 & 0 & 0 \\ 1 & 1 & 1 \\ -50/13 & -7/2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

SAQ 15.3

The eigenvectors are given by,

$$\begin{vmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$$

The indicial or characteristic equation is therefore,

$$\lambda^3 - 3\lambda - 2 = (\lambda - 2)(\lambda + 1)^2 = 0$$

The eigenvalues are 2, -1, -1.

The corresponding eigenvectors are,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 2 \text{, and } \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda = -1.$$

Why do we have two eigenvectors for the same eigenvalue?

Let us set $\lambda = -1$. Then,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u_1 + u_2 + u_3 = 0$$

satisfied by the set $\{u_1, u_2, u_3\} = \{-1, 1, 0\}$ and the set $\{u_1, u_2, u_3\} = \{-1, 0, 1\}$, which are linearly independent.

All the vectors are linearly independent.

Arranging the eigenvectors such that the corresponding eigenvalues are in decreasing order (the order of the corresponding eigenvectors for repeated eigenvalues is immaterial), the matrix

$\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is the modal matrix, *M*,

The inverse is $M^{-1} = \dfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$$M^{-1}AM = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

M is the modal matrix and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is the spectral matrix.

SAQ 15.4

$$H^{+} = \begin{pmatrix} 1 & i & -i \\ -i & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = H$$

Hence, *H* is Hermitian.

It is not diagonalisable because the last two columns are not linearly independent:

$\alpha(i,0,0) + \beta(-i,0,0) = \mathbf{0}$ implies $\alpha = -\beta$. Neither constant has to be zero.

SAQ 15.5

$$H = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}$$

The eigenvalues are, -1 and 1, twice.

The corresponding eigenvalues are,

$\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$ for $\lambda = -1$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = 1$

The normalised eigenvectors are,

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -i/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

The matrix required for obtaining the spectral matrix is the unitary matrix,

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$U^{+} = \begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 1 & 0 \\ i/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \\ -i/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$