Supplementary Material for: Spatio-Temporal Cross-Covariance Functions under the Lagrangian Framework with Multiple Advections

1 Proof of Theorem 2

The derivation below follows the derivation in Schlather (2010). In order to easily follow the derivation, we color coded the terms that change from the left hand side of the equation to the right hand side. Let $\mathbf{T} = (t_1\mathbf{I}_d \quad -t_2\mathbf{I}_d)$ and $\mathbf{v}_{ij} = (\mathbf{v}_{ii}^\top \quad \mathbf{v}_{jj}^\top)^\top$. Operating on the exponentials, we have

$$
\begin{aligned}
& (\mathbf{s}_1 - \mathbf{s}_2 - t_1\mathbf{v}_{ii} + t_2\mathbf{v}_{jj})^\top(\mathbf{s}_1 - \mathbf{s}_2 - t_1\mathbf{v}_{ii} + t_2\mathbf{v}_{jj}) + (\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}(\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
= \; & (\mathbf{h} - \mathbf{T}\mathbf{v}_{ij})^\top(\mathbf{h} - \mathbf{T}\mathbf{v}_{ij}) + (\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}(\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
\end{aligned}
$$

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\begin{aligned}
\Rightarrow \; & (\mathbf{h} - \mathbf{T}\mathbf{v}_{ij})^\top(\mathbf{h} - \mathbf{T}\mathbf{v}_{ij}) + (\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}(\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
= \; & \mathbf{h}^\top\mathbf{h} - \mathbf{h}^\top\mathbf{T}\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{h} + \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{T}\mathbf{v}_{ij} + (\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}(\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
\end{aligned}
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\begin{aligned}
\Rightarrow \; & \mathbf{h}^\top\mathbf{h} - \mathbf{h}^\top\mathbf{T}\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{h} + \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{T}\mathbf{v}_{ij} + (\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}(\mathbf{v}_{ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
= \; & \mathbf{h}^\top\mathbf{h} - \mathbf{h}^\top\mathbf{T}\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{h} + \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{T}\mathbf{v}_{ij} + \mathbf{v}_{ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\mathbf{v}_{ij} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} - \mathbf{v}_{ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\mathbf{v}_{ij}
\end{aligned}
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\begin{aligned}
\Rightarrow \; & \mathbf{h}^\top\mathbf{h} - \mathbf{h}^\top\mathbf{T}\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{h} + \mathbf{v}_{ij}^\top\mathbf{T}^\top\mathbf{T}\mathbf{v}_{ij} + \mathbf{v}_{ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\mathbf{v}_{ij} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} - \mathbf{v}_{ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} - \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\mathbf{v}_{ij} \\
= \; & \mathbf{h}^\top\mathbf{h} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} + \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{h} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) - (\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij}
\end{aligned}
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\begin{aligned}
\Rightarrow \; & \mathbf{h}^\top\mathbf{h} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} + \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{h} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) - (\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} \\
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& -(\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-1}(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij}
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\begin{aligned}
\Rightarrow \; & \mathbf{h}^\top\mathbf{h} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} + \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-1}(\mathbf{T}^\top\mathbf{h} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
& -(\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-1}(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} \\
= \; & \mathbf{h}^\top\mathbf{h} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} + \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-\top}(\mathbf{T}^\top\mathbf{h} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
& -(\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-\top}(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij}
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\Rightarrow \; & \mathbf{h}^\top\mathbf{h} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij} + \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij} - \mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-\top}(\mathbf{T}^\top\mathbf{h} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}) \\
& -(\mathbf{h}^\top\mathbf{T} + \boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}^\top\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})^{-\top}(\mathbf{T}^\top\mathbf{T} + \boldsymbol{\Sigma}_{\boldsymbol{\mathcal{V}},ij}^{-1})\mathbf{v}_{ij}
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\begin{aligned}
=\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}+\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}-\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}\}^\top\\
&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}
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\Rightarrow\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}+\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}-\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}\}^\top\\
&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
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&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
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&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
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\Rightarrow\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}+\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}-\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}\}^\top\\
&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
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=\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}+\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}-\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}\}^\top\\
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&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
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\Rightarrow\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}+\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}-\mathbf{v}_{ij}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}\}^\top\\
&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
&+\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}^\top(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\\
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&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
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&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
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&-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{h}^\top\mathbf{T}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^\top\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\mathbf{v}_{ij}\\
&+\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\\
&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\\
=\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}\\
&+\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}\\
&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})
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&-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\\
=\;& \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij}\\
&+\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\mathcal{V}},ij}\boldsymbol{\mu}_{\boldsymbol{\mathcal{V}},ij})\}
\end{aligned}
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$$\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}\\
& +\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}\\
& -\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\\
=\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}\\
& +\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}\\
& -\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})
\end{aligned}$$

Let $\boldsymbol{\xi}=\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}$.

$$\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}\\
& +\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\{\mathbf{v}_{ij}-(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}\\
& -\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\\
=\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})-\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}
\end{aligned}$$

Taking the exponentials

$$\begin{aligned}
\Rightarrow\quad & \int_{\mathbb{R}^d}\exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\mathrm{d}\mathbf{v}_{ij}\\
=\quad & \exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\int_{\mathbb{R}^d}\exp\{-(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})\}\mathrm{d}\mathbf{v}_{ij}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\int_{\mathbb{R}^d}\exp\{-(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})\}\mathrm{d}\mathbf{v}_{ij}\\
=\quad & \exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\frac{(2\pi)^{d/2}}{(2\pi)^{d/2}}\frac{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})|^{1/2}}{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})|^{1/2}}\int_{\mathbb{R}^d}\exp\{-\frac{1}{2}(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top 2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})\}\mathrm{d}\mathbf{v}_{ij}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\frac{(2\pi)^{d/2}}{(2\pi)^{d/2}}\frac{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})|^{1/2}}{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})|^{1/2}}\int_{\mathbb{R}^d}\exp\{-\frac{1}{2}(\mathbf{v}_{ij}-\boldsymbol{\xi})^\top 2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{v}_{ij}-\boldsymbol{\xi})\}\mathrm{d}\mathbf{v}_{ij}\\
=\quad & \exp\{-\mathbf{h}^\top\mathbf{h}-\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}+\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\}\frac{(2\pi)^{d/2}}{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})|^{1/2}}\times 1 \qquad (***)
\end{aligned}$$

Working on the exponents of the exponential,

$$\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\xi}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})\boldsymbol{\xi}\\
=\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\}^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\\
=\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-1}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})\\
=\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}^\top_{\mathcal{V},ij}\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij}-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}^{-1}_{\mathcal{V},ij}\boldsymbol{\mu}_{\mathcal{V},ij})
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij})^\top(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}(\mathbf{T}^\top\mathbf{h}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij})\\
= \quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\\
= \quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-\top}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\\
= \quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\}^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\boldsymbol{\Sigma}_{\mathcal{V},ij}\}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\}^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\}^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}\{(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\boldsymbol{\Sigma}_{\mathcal{V},ij}\}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})\}^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\\
= \quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})\}^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}\{(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})\}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})\}^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-\top}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})\}^{-\top}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}\{(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})\}^{-\top}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})\}^{-\top}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\\
= \quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})\}^{-1}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})\}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})\}^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\mathbf{h}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\mathbf{h}\\
& -\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\mathbf{T}^\top\mathbf{h}-\mathbf{h}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\\
= \quad & \mathbf{h}^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\mathbf{T}^\top\mathbf{h}\\
& -\mathbf{h}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mathbf{h}^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}\mathbf{h}+\boldsymbol{\mu}_{\mathcal{V},ij}^\top\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\mathbf{T}^\top\mathbf{h}\\
& -\mathbf{h}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}-\boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\boldsymbol{\mu}_{\mathcal{V},ij}\quad (****)\\
= \quad & (\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})\quad (**)
\end{aligned}
$$

Combining (**), (***), and other terms in $\mathrm{E}_{\mathcal{V}}\left\{C_{ij}^S(\mathbf{h}-\mathbf{V}_{ii}t_1+\mathbf{V}_{ii}t_2)\right\}$, i.e.,

$$
\begin{aligned}
& \mathrm{E}_{\mathcal{V}}\left\{C_{ij}^S(\mathbf{h}-\mathbf{V}_{ii}t_1+\mathbf{V}_{ii}t_2)\right\}\\
= \quad & \frac{1}{(2\pi)^d|\boldsymbol{\Sigma}_{\mathcal{V},ij}|^{1/2}}(***)\exp\{-(**)\}\\
= \quad & \frac{1}{(2\pi)^d|\boldsymbol{\Sigma}_{\mathcal{V},ij}|^{1/2}}\frac{(2\pi)^{d/2}}{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})|^{1/2}}\exp\Big[-(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \frac{1}{(2\pi)^d|\boldsymbol{\Sigma}_{\mathcal{V},ij}|^{1/2}} \frac{(2\pi)^{d/2}}{|2(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})|^{1/2}} \exp\Big[-(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})\Big] \\
= \quad & \frac{1}{|\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d}|^{1/2}} \exp\Big[-(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top\{\mathbf{I}_d-\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}(\mathbf{h}-\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})\Big]
\end{aligned}
$$

Note that the expression

$$
\begin{aligned}
(\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top[\mathbf{I}_d-\mathbf{T}\{\mathbf{T}^\top\mathbf{T}+(\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\}^{-1}\mathbf{T}^\top](\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij}) &= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\mathbf{T}\}\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}\{(\mathbf{T}^\top\mathbf{T})^{-1}\mathbf{T}^\top\mathbf{T}+(\mathbf{T}^\top\mathbf{T})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\}^{-1}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}\{\mathbf{I}_{2d}+(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}\}^{-1}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}\}\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}\{(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}-(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\} \\
&\quad \boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{T}^\top\mathbf{T}-\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T} \\
&\quad +\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}\}\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}\{\mathbf{I}_{2d}+(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T})^{-1}\}^{-1}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\}\boldsymbol{\mu}_{\mathcal{V},ij},
\end{aligned}
$$

where we used the Sherman Woodbury matrix inverse formula, is equal to the 2nd and 5th term in (****), i.e.,

$$
\begin{aligned}
\boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}-(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\}\boldsymbol{\mu}_{\mathcal{V},ij} &= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}-\{\mathbf{I}_{2d}-\mathbf{I}_{2d}\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}]\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1}\}\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\}\boldsymbol{\mu}_{\mathcal{V},ij} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{T}^\top\mathbf{T}(\boldsymbol{\Sigma}_{\mathcal{V},ij}\mathbf{T}^\top\mathbf{T}+\mathbf{I}_{2d})^{-1}\}\boldsymbol{\mu}_{\mathcal{V},ij}.
\end{aligned}
$$

Also

$$
\begin{aligned}
(\mathbf{T}\boldsymbol{\mu}_{\mathcal{V},ij})^\top\{\mathbf{I}_d+\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}^{-1})^{-1}\mathbf{T}^\top\}\mathbf{h} &= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\mathbf{T}^\top\{\mathbf{I}_d+\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\}\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\mathbf{T}^\top\mathbf{h}-\boldsymbol{\mu}^\top\mathbf{T}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\mathbf{T}^\top\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top\{\mathbf{I}_{2d}-\mathbf{T}^\top\mathbf{T}(\mathbf{T}^\top\mathbf{T}+\boldsymbol{\Sigma}_{\mathcal{V},ij}^{-1})^{-1}\}\mathbf{T}^\top\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{I}_{2d}-\{\mathbf{I}_{2d}+(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\}^{-1}]\mathbf{T}^\top\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{I}_{2d}-[\mathbf{I}_{2d}-(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\{\mathbf{I}_{2d}+(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\}^{-1}])\mathbf{T}^\top\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top[\mathbf{I}_{2d}-\mathbf{I}_{2d}+(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\{\mathbf{I}_{2d}+(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij})^{-1}\}^{-1}]\mathbf{T}^\top\mathbf{h} \\
&= \boldsymbol{\mu}_{\mathcal{V},ij}^\top(\mathbf{T}^\top\mathbf{T}\boldsymbol{\Sigma}_{\mathcal{V},ij}+\mathbf{I}_{2d})^{-1}\mathbf{T}^\top\mathbf{h},
\end{aligned}
$$

which is the 3rd and 4th term in (****).

Sherman Woodbury formula,

$$
\begin{aligned}
(A+B)^{-1} &= A^{-1}-A^{-1}B(A+B)^{-1} \\
(u^2D+\mathbf{1}_{d\times d})^{-1} &= (u^2D)^{-1}-(u^2D)^{-1}\mathbf{1}_{d\times d}(u^2D+\mathbf{1}_{d\times d}) = (u^2D)^{-1}-(u^2D)^{-1}(u^2D+\mathbf{1}_{d\times d}) \\
(\mathbf{1}_{d\times d}+u^2D)^{-1} &= \mathbf{1}_{d\times d}-\mathbf{1}_{d\times d}(u^2D)(\mathbf{1}_{d\times d}+u^2D)^{-1} = \mathbf{1}_{d\times d}-\left\{(u^2D)^{-1}+\mathbf{1}_{d\times d}\right\}^{-1} \\
(\mathbf{1}_{d\times d}+u^2D)^{-T} &= \mathbf{1}_{d\times d}-\left\{(u^2D)^{-1}+\mathbf{1}_{d\times d}\right\}^{-T}
\end{aligned}
$$

2 Estimation

2.1 Universal Kriging

Throughout this work, we assume the following model for our spatio-temporal processes:

$$\mathbf{Y}(\mathbf{s},t) = \boldsymbol{\mu}(\mathbf{s},t) + \mathbf{Z}(\mathbf{s},t), \quad (\mathbf{s},t) \in \mathbb{R}^d \times \mathbb{R}, \tag{1}$$

where $\boldsymbol{\mu}(\mathbf{s},t)$ is the vector-valued spatio-temporal mean of multivariate random field that may be spatially varying and $\mathbf{Z}(\mathbf{s},t)$ is a zero mean multivariate second-order stationary spatio-temporal random field. Assume that the mean function in (1) can be characterized as a linear combination of some covariates $X_1, X_2, \ldots, X_M$. Denote by $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \ldots, \boldsymbol{\beta}_p^\top)^\top \in \mathbb{R}^{Mp}$ the vector of mean parameters, where $\boldsymbol{\beta}_i = (\beta_{1,i}, \ldots, \beta_{M,i})^\top \in \mathbb{R}^M$, for $i = 1, \ldots, p$, and $\mathbf{X} = \left\{\mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_1,t_1)^\top, \mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_2,t_2)^\top, \ldots, \mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_n,t_n)^\top\right\}^\top \in \mathbb{R}^{np\times Mp}$, where $\mathbf{X}(\mathbf{s},t) = \{X_1(\mathbf{s},t), \ldots, X_M(\mathbf{s},t)\} \in \mathbb{R}^M$. The model in (1) becomes $\mathbf{Y}(\mathbf{s},t) = \left\{\mathbf{I}_p \otimes \mathbf{X}(\mathbf{s},t)^\top\right\}\boldsymbol{\beta} + \mathbf{Z}(\mathbf{s},t)$. The universal cokriging predictor $\hat{\mathbf{Z}}(\mathbf{s}_0,t_0)$ of $\mathbf{Z}(\mathbf{s},t)$ at unsampled spatio-temporal location $(\mathbf{s}_0,t_0)$ is

$$\hat{\mathbf{Y}}(\mathbf{s}_0,t_0) = \left[\mathbf{c} + \mathbf{X}\{\mathbf{X}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{X}\}^{-1}\{\mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_0,t_0)^\top - \mathbf{X}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{c}\}\right]^\top \boldsymbol{\Sigma}^{-1}\mathbf{Y},$$

where $\mathbf{c} = (\left[\left\{C_{ij}(\mathbf{s}_l - \mathbf{s}_0, t_l, t_0; \hat{\boldsymbol{\Theta}})\right\}_{i,j=1}^2\right]_{l=1}^n)^\top \in \mathbb{R}^{np\times p}$. The corresponding cokriging variance is

$$\begin{aligned}\mathrm{var}\{\hat{\mathbf{Y}}(\mathbf{s}_0,t_0)\} = \mathrm{trace}[&\mathbf{C}(\mathbf{0},0;\hat{\boldsymbol{\Theta}}) - \mathbf{c}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{c} \\ &+ \{\mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_0,t_0)^\top - \mathbf{X}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{c}\}^\top\{\mathbf{X}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{X}\}^{-1}\{\mathbf{I}_p \otimes \mathbf{X}(\mathbf{s}_0,t_0)^\top - \mathbf{X}^\top\boldsymbol{\Sigma}(\hat{\boldsymbol{\Theta}})^{-1}\mathbf{c}\}],\end{aligned}$$

3 Simulation Study

Figures S2 and S3 present the boxplots when fitting M2 and M3 to data generated from M3 at other values of k, the parameter that controls the dependence in time, such that the dependence in time decreases as k increases. The main manuscript shows the results for $k = 0.001$, representing

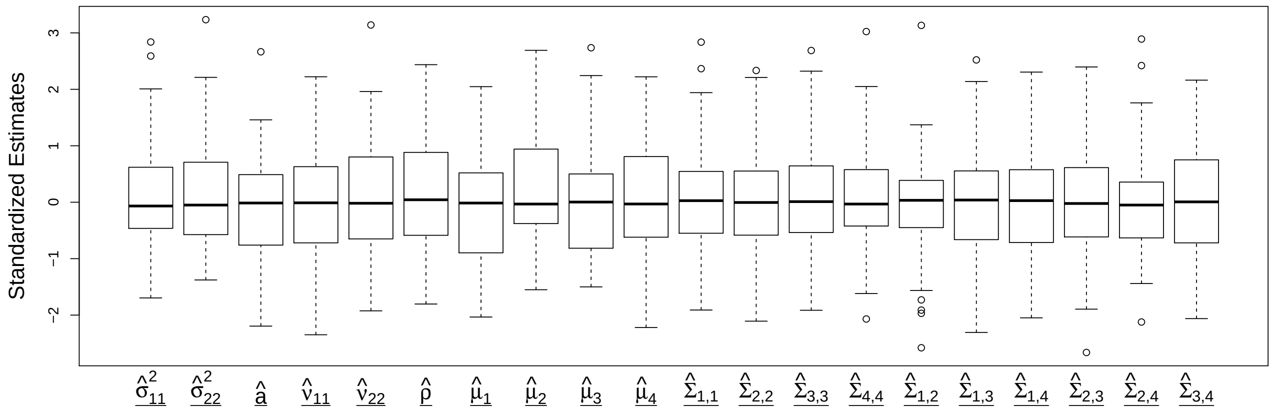

Figure S1: Boxplots of standardized estimated parameters of via multi-step MLE.

a long range / strong temporal dependence. The models with $k = 0.1$ and $k = 1$ represent the moderate and weak temporal dependence, respectively. It can be seen that the disparity between models M2 and M3 disappears as the dependence in time diminishes. However, it remains the case that when the random fields have long range / strong temporal dependence, it is crucial to recognize and model the presence of multiple advections.

4 Application

4.1 Timeseries Plots

4.2 Testing for Stationarity of Spatio-Temporal Data

Since the data has a very short timeseries, we only test for stationarity in the spatial domain. Using the testing method in Jun and Genton (2012), we test the second-order stationarity of the residuals obtained using the generalized least squares regression with covariates, namely, relative humidity and temperature. The testing method requires the computation of the empirical spatial marginal and cross-covariances of two disjoint spatial domains. We perform the test two times. First, we divide Saudi Arabia vertically, i.e. $[35, 45] \times [17, 32]$ and $[45, 55] \times [17, 32]$, and test whether there is enough evidence to reject the assumption that the empirical spatial marginal and cross-covariances in the West and East of Saudi Arabia are the same. Next, we divide Saudi

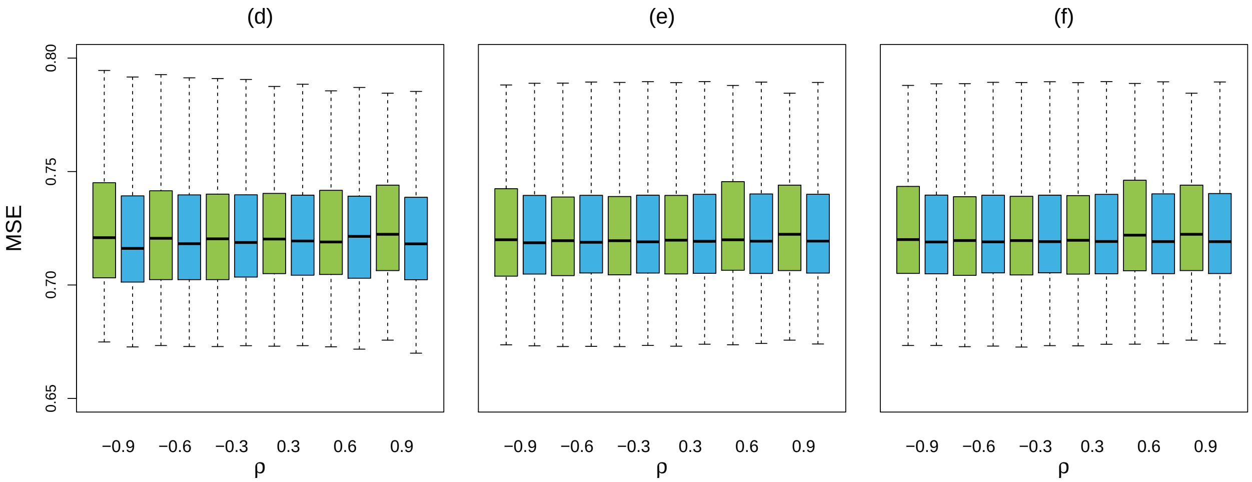

Figure S2: Boxplots of the MSEs under scenarios (d)-(f) when M2 (green) and M3 (blue) are fitted to data generated from M3 at different values of ρ at $k = 1$.

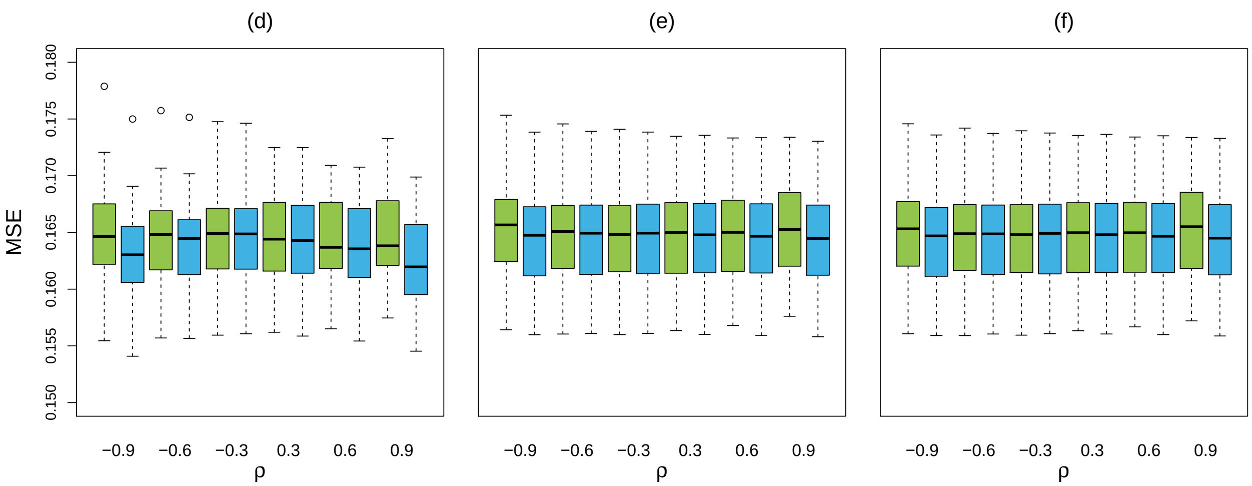

Figure S3: Boxplots of the MSEs under scenarios (d)-(f) when M2 (green) and M3 (blue) are fitted to data generated from M3 at different values of ρ at $k = 0.1$.

Arabia horizontally, i.e. $[35, 55] \times [17, 24.5]$ and $[35, 55] \times [24.5, 32]$, and perform the test.

We choose the following spatial lags at which we compute the p-values:

$$
\begin{aligned}
\Lambda_1 &= \{\mathbf{h} = (0, 2), (2, 0), (2, 2)\}, \\
\Lambda_2 &= \{\mathbf{h} = (0, 2), (2, 0), (2, 2), (0, 4), (4, 0), (4, 4)\}, \\
\Lambda_3 &= \{\mathbf{h} = (0, 2), (2, 0), (2, 2), (0, 4), (4, 0), (4, 4), (0, 6), (6, 0), (6, 6)\}
\end{aligned}
$$

in $\times 10^2$ km. Tables 1 and 2 present the p-values from the test for stationarity with the empirical

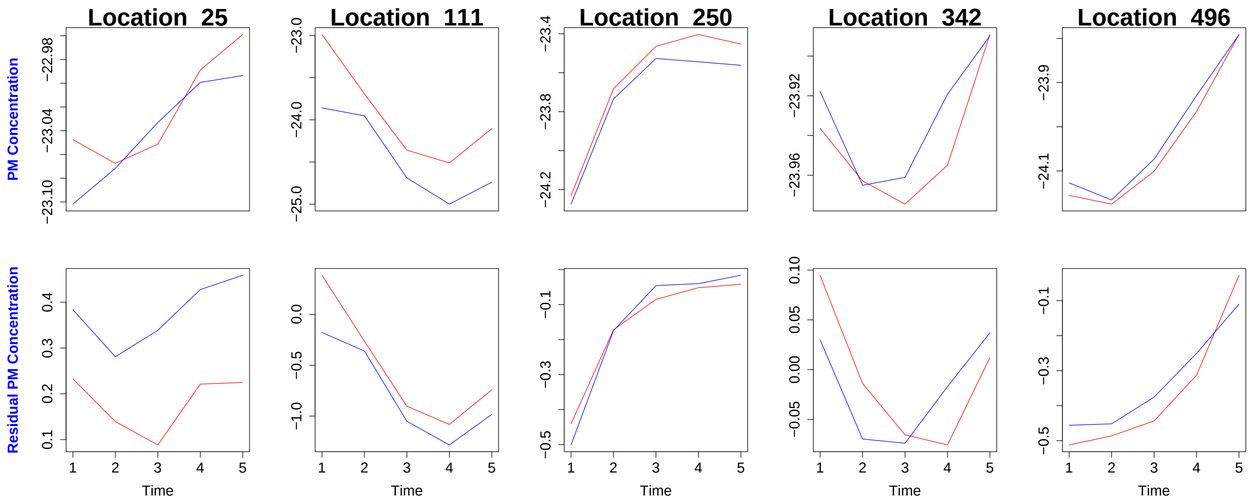

Figure S4: Timeseries plots of PM concentrations on January 18, 2019, 9:00 to 21:00, at 880 hPa (red) and 925 hPa (blue), in log scale of $\mu g/m^3$.

marginal and cross-covariances calculated using spatial lag $\mathbf{h}$ or anisotropy-corrected spatial lag $\mathbf{Rh}$. From the p-values, it can be seen that we need to reject the stationarity assumption using a significance level of $\alpha = 0.05$. Using the anisotropy corrected spatial lags, however, generally results to a failure in rejecting the stationarity assumption using a significance level of $\alpha = 0.05$

Table 1: P-values from the test for several combinations of spatial lags when dividing Saudi Arabia vertically and computing for the empirical C_{11}, C_{22}, and C_{12}.

	C_{11}			C_{22}			C_{12}		
	Λ_1	Λ_2	Λ_3	Λ_1	Λ_2	Λ_3	Λ_1	Λ_2	Λ_3
$\mathbf{h}$	0.6081	0.1339	0.0286	0.6204	0.0451	0.0013	0.2636	0.0863	0.0136
$\mathbf{Rh}$	0.9671	0.0881	0.0782	0.8838	0.0292	0.0348	0.9233	0.0928	0.0538

Table 2: P-values from the test for several combinations of spatial lags when dividing Saudi Arabia horizontally and computing for the empirical C_{11}, C_{22}, and C_{12}. Note that Λ_3 cannot be properly computed when dividing Saudi Arabia horizontally.

	C_{11}			C_{22}			C_{12}		
	Λ_1	Λ_2	Λ_3	Λ_1	Λ_2	Λ_3	Λ_1	Λ_2	Λ_3
$\mathbf{h}$	0.1102	0.2939	-	0.0147	0.0362	-	0.0481	0.1266	-
$\mathbf{Rh}$	0.1433	0.2349	-	0.0364	0.0517	-	0.0840	0.1801	-

References

Jun, M. and Genton, M. G. (2012). A test for stationarity of spatio-temporal random fields on planar and spherical domains. *Statistica Sinica*, 22:1737–1764.

Schlather, M. (2010). Some covariance models based on normal scale mixtures. *Bernoulli*, 16(3):780–797.