

Physics Beyond - Motion

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1 Procedure

1. Define your problem
2. Model your problem mathematically. If the problem is about understanding the physics of something then:
 - Model the description of that something (e.g. model what motion is)
 - Or investigate / find the underlying cause of that something (e.g. what causes motion)
3. Solve the maths
4. Interpret the results
5. (If needed) revise your model

2 Definition:

Definition - change in position (space) over time (time).

1. Put the physics of space time aside
2. Simple models - based on direct experience
 - Space: Euclidean geometry
 - Time: Real numbers \mathbb{R}

3 Modelling Mathematically:

3.1 Q: What is the convenient mathematical model?

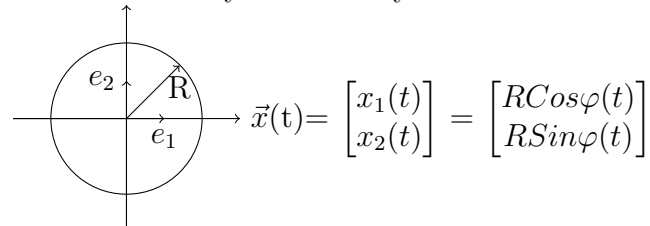
$$\begin{array}{ll} \text{Function} & \text{Time} \rightarrow \text{space} \\ \text{x:} & \mathbb{R} \rightarrow \text{space} \\ & t \mapsto \text{position } x(t) \end{array}$$

3.2 How to model space conveniently?

\leadsto need a reference frame.

$$\begin{array}{l} (0, \vec{e}_1, \vec{e}_2, \vec{e}_3) \\ \vec{x} : \mathbb{R} \rightarrow \text{Euclidean vector-space (after introducing 0)} \\ \vec{x} : \mathbb{R} \rightarrow \mathbb{R}^3 \\ t \mapsto \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \\ \text{(after introducing } [\vec{e}_1, \vec{e}_2, \vec{e}_3] \end{array}$$

Each $x_i : \mathbb{R} \rightarrow \mathbb{R}$ is ordinary an ordinary function we are familiar with

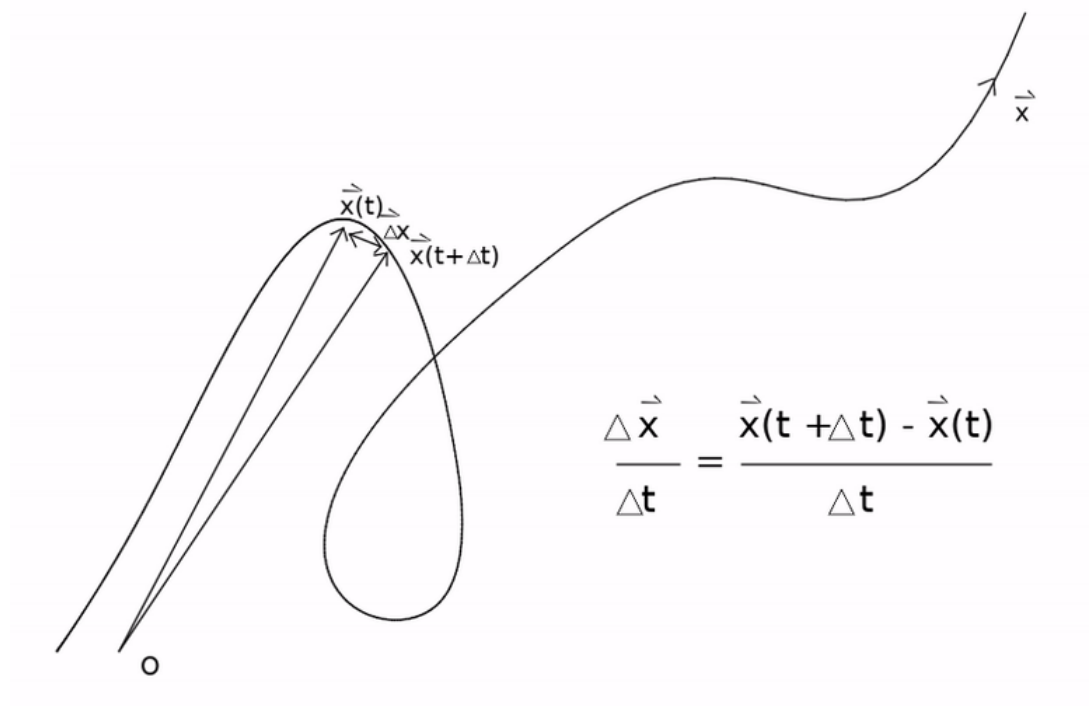


$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} R \cos \varphi(t) \\ R \sin \varphi(t) \end{bmatrix}$$

Observation: interactions change motion. (This is how we detect interactions after all)

3.3 Q: How to model changes in motion?

*****Particle Motion Simulation*****



Problem: I need to find the velocity at one instant in time.

4 Q: How to model this mathematically

→ taking a limit:

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t + \Delta t) - \vec{x}(t)}{\Delta t}$$

→ or using infinitesimals

5 The theory of infinitesimals

Problem: Define rates of change at one instant in time.

The problem is that \vec{v} is the average velocity in the time interval $[t_0, t_0 + \Delta t]$

6 Q: How can we define instantaneous velocity?

1. The idea of a limit:

- make Δt smaller and smaller
- study $\frac{\Delta \vec{x}}{\Delta t}$ changes when Δt approaches 0

\leadsto it becomes a tangent to the curve at $\vec{x}(t_0)$

we write:

$$\vec{v} = \frac{d\vec{x}}{dt} = \dot{\vec{x}} = \lim_{\Delta t \rightarrow 0} \frac{\vec{x}(t_0 + \Delta t) - \vec{x}(t_0)}{\Delta t}$$

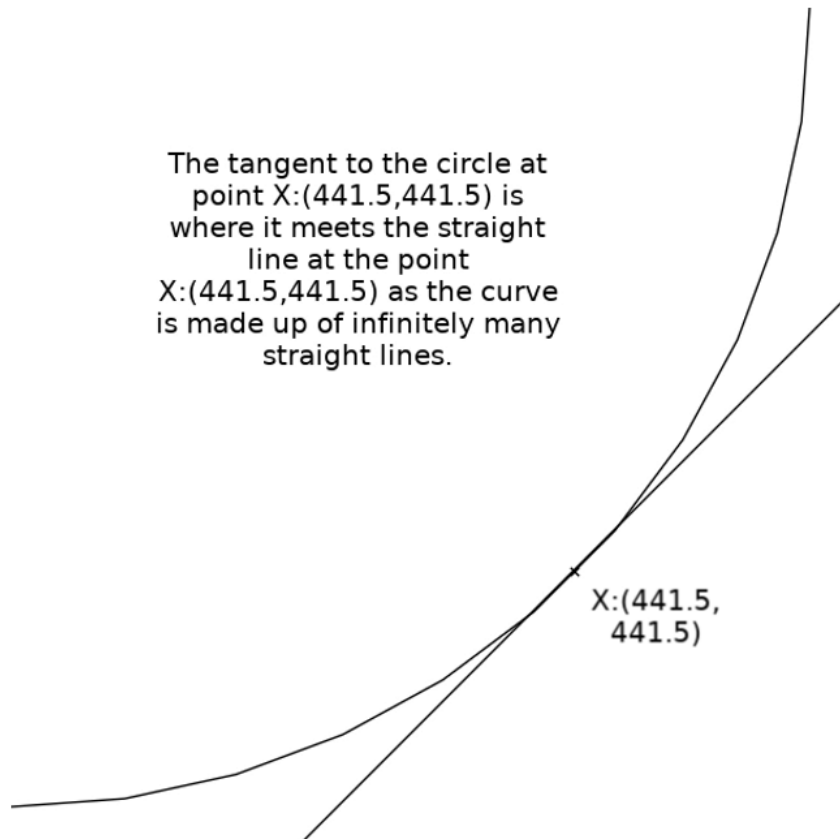
Q: How is $\lim_{\Delta t \rightarrow 0}$ defined mathematically?

\rightarrow it is technical and complicated

2. Idea of infinitesimals:

Intuition: If I zoom in onto a curve infinitely close, it becomes a straight line.

*****Infinitesimals Simulation*****

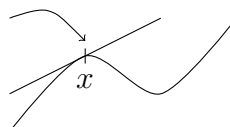


Q: How to define infinitesimals?

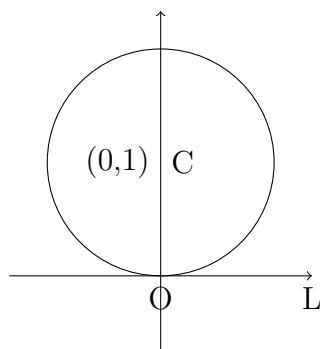
→ Use idea of a tangent touching a curve at a point.

→ We interpret "touching" to mean that the curve and the tangent line intersect in an infinitesimal piece of a curve.

infinitesimal piece of intersection: where the curve agrees with the line



Problem: In general we cannot define what a tangent is except for a circle.



when a line intersects a circle at exactly one point, then it's a tangent to the circle

↪ calculate the intersection of the circle, C, and a line, L.

→ we use a coordinate system

$$C : x^2 + (y - 1)^2 - 1 = 0$$

$$L : y = 0$$

$$(nL = \{(x, 0) | x^2 = 0\})$$

→ to have more than the point (0,0), we do not allow to conclude $x^2 = 0 \rightarrow x = 0$!

Definition: (First-order) infinitesimals are $d \leftarrow R$ such that $d^2 = 0$

$$D := \{d \in R | d^2 = 0\}$$

Q: What is R?

→ It is not \mathbb{R} (the real numbers) because $x^2 = 0 \rightarrow x = 0$ for $x \leftarrow \mathbb{R}$

→ We want R to have the properties similar to \mathbb{R} :

(a) I have addition and subtraction

(b) I have multiplication

(c) $\mathbb{Q}cR$ (in fact $\mathbb{R}cR$) → rational numbers: numbers that can be written as fractions.

Intuition: \mathbb{R} is an extension of the reals by adding all the missing infinitesimals.

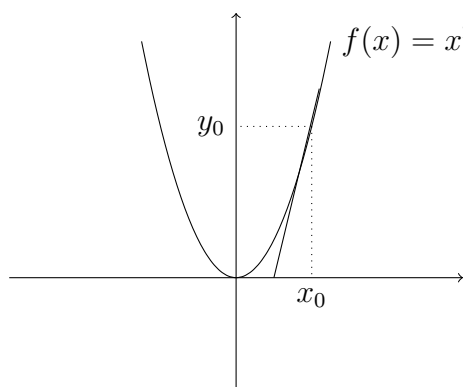
Caution you cannot divide freely by numbers in \mathbb{R}

Suppose $d \leftarrow \mathbb{D}$ has an inverse, d^{-1}

$$d^{-1}d = 1 \text{ but } d^{-1} \underbrace{d \cdot d}_{\substack{d^2=0 \\ d=0}} = d$$

7 Q: Can infinitesimals solve the tangent problem?

Definition (tangent \leftarrow due to Leibniz) - a tangent is a line that intersects a curve in two infinitesimally close points.



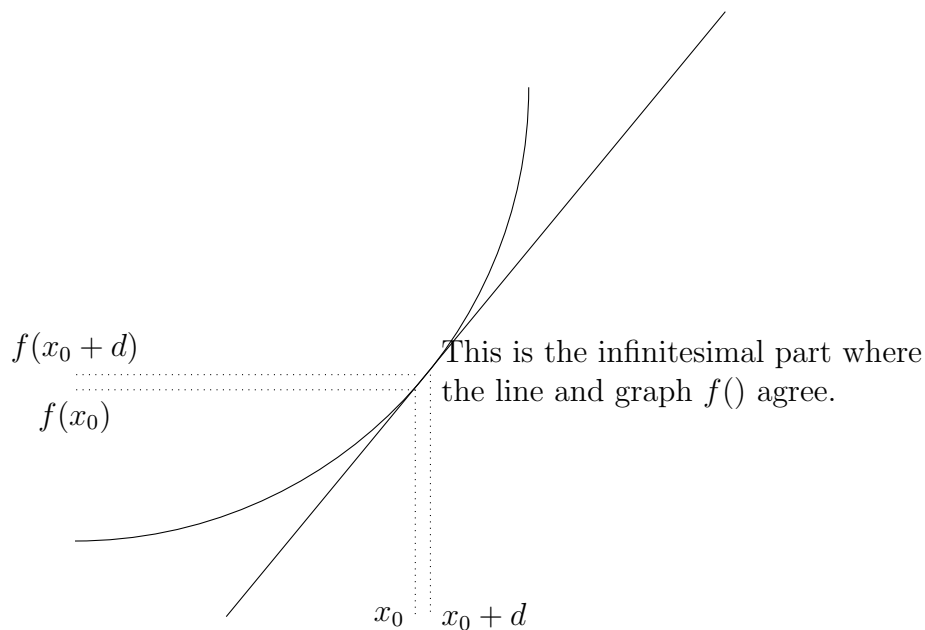
Using definition of Leibniz, we are looking at the secant line through $(x_0, f(x_0)), (x_0 + d, f(x_0 + d))$

1.

$$\begin{aligned} f(x_0 + d) &= (x_0 + d)^2 \\ &= x_0^2 + 2x_0d + d^2 \\ &= x_0^2 + 2x_0d \\ &= f(x_0) + 2x_0d \\ f(x_0 + d) - f(x_0) &= 2x_0d \\ f(x_0 + d) - f(x_0) &= 2x_0 \cdot ((x_0 + d) - x_0) \\ (y_1 - y_0 &= m(x_1 - x_0)) \end{aligned}$$

$\leadsto y = f(x_0) + 2x_0d$ is the infinitesimal piece of the tangent at $(x_0, f(x_0))$

\leadsto for any $h \leftarrow \mathbb{R}$ we get the equation $y = f(x_0) + 2x_0h$



$$2. f : \mathbb{R} \mapsto \mathbb{R} \quad x \mapsto x^3$$

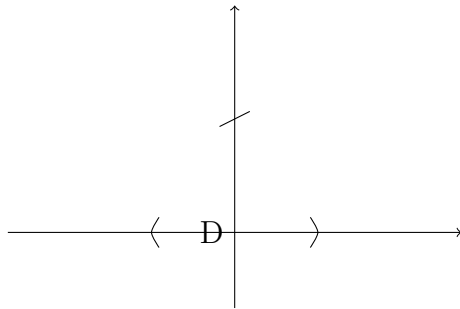
$$\begin{aligned}
 f(x_0 + d) = (x_0 + d)^3 &= (x_0 + d)^2(x_0 + d) \\
 &= (x_0^2 + 2x_0d)(x_0 + d) = x_0^3 + 2x_0^2d + x_0^2d + \underbrace{2x_0d^2}_0 \\
 &= x_0^3 + 3x_0^2d \\
 &= \underbrace{f(x_0)}_{f(x_0)} + 3x_0^2d
 \end{aligned}$$

3.

$$\begin{aligned}
 s(t) &= ut + \frac{1}{2}gt^2 \\
 s(t + dt) &= u(t + dt) + \frac{1}{2}g(t + dt)^2 \\
 &= ut + udt + \frac{1}{2}g(t^2 + 2tdt + dt^2) \\
 &= \underbrace{ut + \frac{1}{2}gt^2}_{s(t)} + (u + gt)dt
 \end{aligned}$$

$$\begin{aligned}
v(t) &= \dot{s}(t) = u + gt \\
v(t + dt) &= u + g(t + dt) = \underbrace{u + gt}_{v(t)} + gdt \\
a(t) &= \dot{v}(t) = \ddot{s}(t) = g
\end{aligned}$$

Axiom: (Kock-Lawvere) For each $f : D \mapsto R$ there are unique $a, b \leftarrow R$ such that $f(d) = a + b \cdot d \ \forall d \leftarrow D$



(This means we model the assumption made that every curve is made of infinitesimal line segments at each point).

$f(0) = a + b \cdot 0 = a$
 \rightsquigarrow b is the gradient of the line and hence the gradient of f at 0. (Because it is simply determined by f).

Consequence:

Consider $f : R \rightarrow R$ at $x_0 \leftarrow R$

Define $f_{x_0} : D \rightarrow R \ d \mapsto f(x_0 + d)$

By K-L we have unique $a_{x_0}, b_{x_0} \leftarrow R$ such that

$$f_{x_0}(d) = a_{x_0} + b_{x_0} \cdot d = f_{x_0}(0) + b_{x_0} \cdot d$$

i.e.

$$f(x_0 + d) = f(x_0) + b_{x_0} \cdot d$$

Def $f'(x_0) := b_{x_0}$ is the derivative of f at x_0

\rightsquigarrow $f'(x_0)$ is the gradient of the infinitesimal line of the graph of f at $(x_0, f(x_0))$,
 "the gradient of f at x_0 ".

8 Rules for Calculating Derivatives:

8.1 Linearity

$$\begin{aligned} f, g : R &\rightarrow R, \lambda \leftarrow R \\ f + \lambda : R &\rightarrow R, x \mapsto f(x) + \lambda g(x) \\ (f + \lambda)' &= f' + \lambda(g') \end{aligned}$$

Example:

$$\begin{aligned} f(x) &= x^2, g(x) = x^3, \lambda = 5 \\ (f + \lambda g)(x) &= x^2 + 5x^3 \\ f'(x) &= 2x, g'(x) = 3x^2 \\ (f + \lambda g)'(x) &= 2x + 15x^2 \end{aligned}$$

8.2 Product Rule

$$\begin{aligned} f.g : R &\rightarrow R, x \mapsto f(x).g(x) \\ (f.g)' &= f'.g + f.g' \end{aligned}$$

Example:

$$\begin{aligned} f(x) &= x, g(x) = x^2 \\ (f.g)(x) &= x^3 \\ f'(x) &= 1, g'(x) = 2x \\ (f.g)'(x) &= 1.x^2 + x.2x = 3x^2 \end{aligned}$$

8.3 Chain Rule

$$\begin{aligned} f, g : R &\rightarrow R \\ f \circ g : R &\rightarrow R, x \mapsto f(g(x)) \text{ (composition)} \\ (f \circ g)' &= (f' \circ g).g' \end{aligned}$$

Examples:

$$\begin{aligned} 1) \quad f(x) &= x^2, g(x) = x^3, f'(x) = 2x, g'(x) = 3x^2 \\ (f \circ g)(x) &= f(g(x)) = f(x^3) = (x^3)^2 = x^6 \\ (f \circ g)'(x) &= f'(g(x)).g'(x) = 2x(x^3).3x^2 = 2x^3.3x^2 = 6x^5 \end{aligned}$$

$$\begin{aligned} 2) \quad f(x) &= x^2, g(x) = 3x^5 + 2x^2 + 5 \\ (f \circ g)(x) &= f(g(x)) = (3x^5 + 2x^2 + 5)^2 \\ f'(x) &= 2x, g'(x) = 3.5x^4 + 2.2x + 0 = 15x^4 + 4x \\ (f \circ g)'(x) &= 2(3x^5 + 2x^2 + 5).(15x^4 + 4x) \end{aligned}$$

Applications:

$$1) \quad f(x) = x, g(x) = \frac{1}{x}$$

take the multiplicative inverse

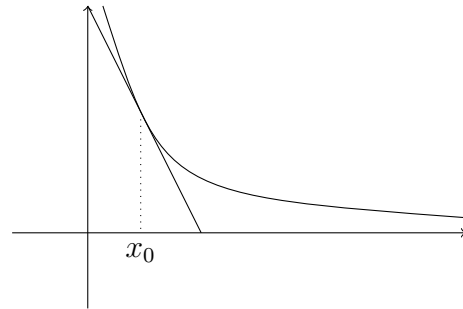
$$g : \begin{matrix} R^x \\ \text{the set of all } x \in R \text{ that are invertible} \end{matrix} \rightarrow R$$

we want to know $g'(x)$!

$$\begin{aligned} (f \cdot g)(x) &= 1 = x \cdot \frac{1}{x} = 1 \\ (f \cdot g)'(x) &= 0 \\ \text{but} \\ (f \cdot g)'(x) &= f'(x) \cdot g(x) + g'(x) \cdot f(x) \\ f'(x) &= 1 \end{aligned}$$

we find:

$$\begin{aligned} 0 &= 1 \cdot g(x) + g'(x) \cdot x \\ &= \frac{1}{x} + g'(x) \cdot x \\ \rightsquigarrow g'(x) &= -\frac{1}{x^2} \end{aligned}$$



$$2) \quad f : R^x \rightarrow R^x, g : R^x \rightarrow R^x, x \mapsto \frac{1}{x}$$

$$\frac{1}{f} = g \circ f$$

$$\left(\frac{1}{f}\right)' = (g \circ f)' \rightsquigarrow \left(\frac{1}{f}\right)'(x) = (g \circ f)'(x) = g'(f(x)) \cdot f'(x) = -\frac{1}{(f(x))^2} \cdot f'(x)$$

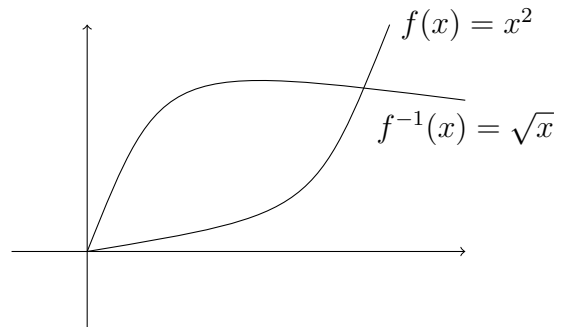
3) Deriving inverse functions:

given a function $f : R \rightarrow R$ such that f is one to one and onto then it has an inverse (for composition).

$$f^{-1} : f^{-1} \circ f = R_{id} \text{ (identity)}$$

$$\text{map } f^{-1}(f(x)) = x$$

$$f \circ f^{-1} = R_{id} \quad f(f^{-1}(y)) = y$$



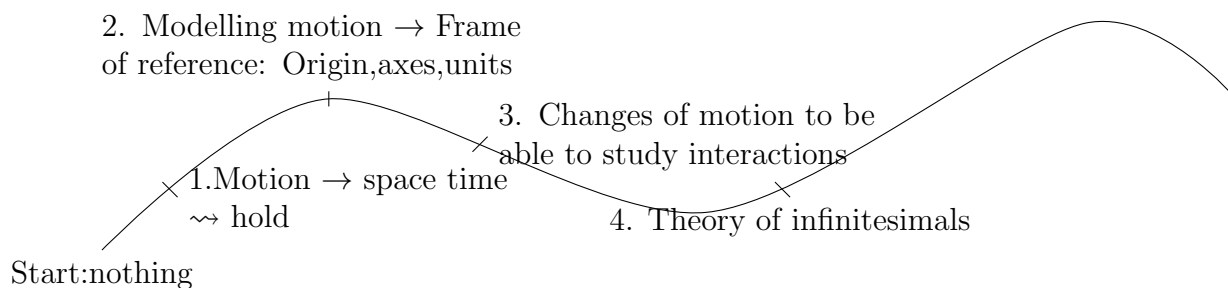
Apply chain rule to

$$\begin{aligned} f^{-1} : f^{-1} \circ f &= R_{id} \\ (f^{-1})'(f(x)) \cdot f'(x) &= 1 \\ (f^{-1})'f(x) &= \frac{1}{f'(x)} \left(\frac{1}{m} = \textit{perpendicular} \right) \end{aligned}$$

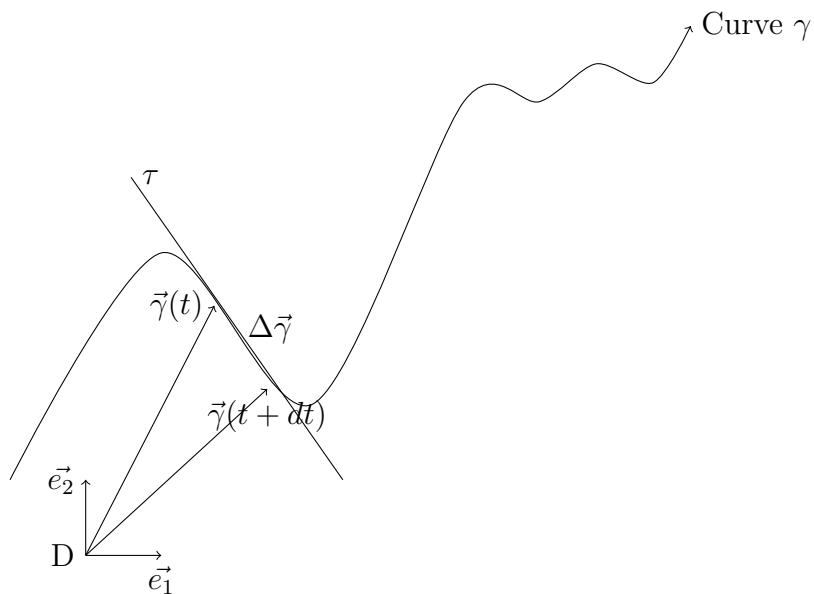
or rewritten

$$\begin{aligned} (f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\ y &= f(x) \\ x &= f^{-1}(f(x)) \\ &= f^{-1}(y) \end{aligned}$$

9 The journey so far:



10 Application: Derivatives of curves



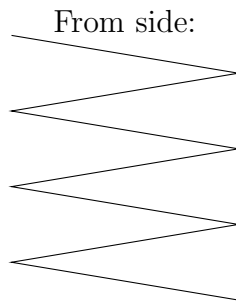
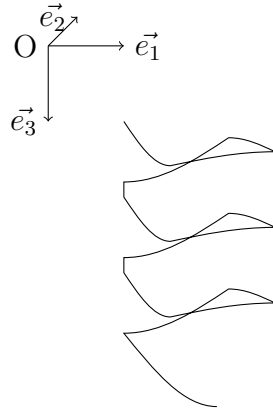
$$dt \leftarrow D$$

intuition says:

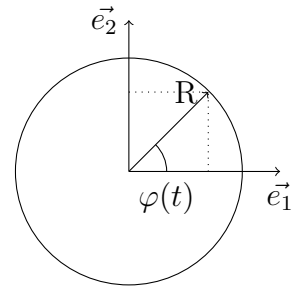
- $\Delta \vec{\gamma}$ agrees with the curve γ .
- a line through $\vec{\gamma}(t)$ in the direction $\vec{\gamma}$ is a tangent at the curve γ .

Theory: $y : R \rightarrow R^2, t \mapsto \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}, \gamma_j : R \rightarrow R, j = 1, 2, \dot{\vec{\gamma}} = \begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \end{bmatrix}$

$$\begin{aligned} \vec{\gamma}(t+dt) &= \begin{bmatrix} \gamma_1(t+dt) \\ \gamma_2(t+dt) \end{bmatrix} \\ &\stackrel{K=L}{=} \begin{bmatrix} \gamma_1(t) + \dot{\gamma}_1(t)dt \\ \gamma_2(t) + \dot{\gamma}_2(t)dt \end{bmatrix} \\ &= \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} + \begin{bmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \end{bmatrix} dt \\ &= \vec{\gamma}(t) + \dot{\vec{\gamma}}(t)dt \\ \Delta \vec{\gamma} &= \vec{\gamma}(t+dt) - \vec{\gamma}(t) = \dot{\vec{\gamma}}(t).dt \leftarrow \text{Velocity} \\ \tau : h &\mapsto \gamma(t) + \dot{\gamma}(t)h \end{aligned}$$



From top: looks like circular motion



$$\begin{aligned}
 \vec{x}(t) &= \begin{bmatrix} R \cos \varphi(t) \\ R \sin \varphi(t) \\ vt \end{bmatrix} \\
 \vec{x}(t) &= \begin{bmatrix} R \cos \varphi(t) \\ R \sin \varphi(t) \\ vt \end{bmatrix} \\
 \dot{\vec{x}}(t) &= \begin{bmatrix} \overbrace{-R \sin \varphi(t) \cdot \dot{\varphi}(t)}^{ut} \\ R \cos \varphi(t) \cdot \dot{\varphi}(t) \\ v \end{bmatrix} \\
 \ddot{\vec{x}}(t) &= \begin{bmatrix} \dot{u}(t) \cdot \dot{\varphi}(t) + \dot{u}(t) \cdot \ddot{\varphi}(t) \\ -R \cos \varphi(t) \cdot (\dot{\varphi}(t))^2 - R \sin \varphi(t) \cdot \ddot{\varphi}(t) \\ -R \sin \varphi(t) \cdot (\dot{\varphi}(t))^2 + R \cos \varphi(t) \cdot \ddot{\varphi}(t) \\ 0 \end{bmatrix}
 \end{aligned}$$

11 Physics of Motion

11.1 Q: What changes motion?

→ Physical interactions.

11.2 Q: Why so general?

↪ any form of physical interaction lead to a changes in motion.

→ we use changes of motion to study interactions.

Introduce force as a measure of the interaction on one object. (Reminder:

- interaction needs at least two objects
- but we need to relate it to the change of motion of one object.)

11.3 Q: How to model force mathematically?

First observation:

1. Has magnitude (measure of the strength of the interaction)
2. Has a direction

↪ model force with a vector.

But! (problems)

→ It matters where the force acts on an object. 3 effects of forces:

1. linear motion
2. rotation
3. deformation

→ This is not captured by a vector.

$$\rightsquigarrow \vec{a}(\vec{F}) = ?$$

Simplest relationship:

$$\vec{a} \propto \vec{F} \checkmark \text{ (Experiment)}$$

Define Inertial mass of an object as the constant of proportionality.

- Inertia measures how much an object resists motion

- Mass > 0

$$\vec{F} = \overset{\text{inertia matrix}}{\widehat{M}} \vec{a} ?$$

Or more simply

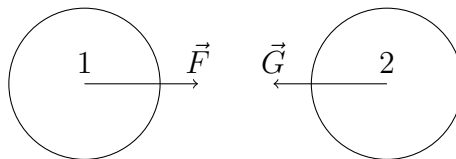
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} M_1 a_1 \\ M_2 a_2 \\ M_3 a_3 \end{bmatrix}$$

so $M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

WE DO NOT OBSERVE THIS

Back to the interaction:

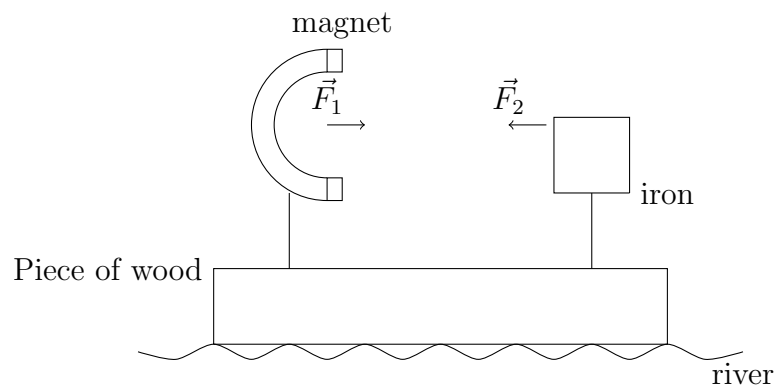
11.4 Q: What is the force acting on the other object



We postulate $\vec{F} = -\vec{G}$

→ simplest assumption consistent with observation that the same objects move symmetrically after an interaction when subjected to the same initial conditions.

→ Experiment:



\rightsquigarrow it does not move $\rightsquigarrow \vec{F}_1 = -\vec{F}_2$

11.5 Q: What is the natural state of motion?

\rightsquigarrow to be compatible with

$$\vec{F} = M\vec{a}$$

We need a natural state if and only if no interaction, if and only if $\vec{F} = 0$

$$\Rightarrow \vec{a} = 0 \quad \vec{v} = \text{constant}$$

a natural state has to be uniform motion with constant velocity.

Problems:

1. Velocity is relative(dependent on frame of reference) but now acceleration is absolute(independent(not true for all reference frames)of frame of reference).
2. I observe objects being naturally at rest! so $\vec{v} = 0$!

Galileo's objection to 2:

→ Ship argument: you cannot tell (with physical experiment) whether you are at rest or moving with a constant velocity. (Galileo relativity of velocity).

Consequence:

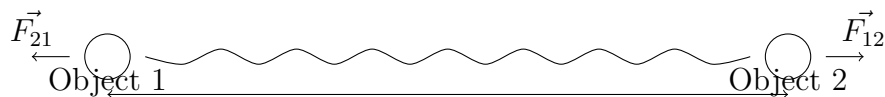
→ We have to explain why things are at rest despite force acting \rightsquigarrow (balanced forces).

→ this makes our explanations more complicated.

But: can explain celestial and terrestrial motion in one theory.

12 Physics of Motion

1. Interaction \rightsquigarrow changes of motion(state:velocity,change:acceleration) \rightsquigarrow acceleration.
2. Interaction



break down interaction into 2 parts, individual to each object.

3. Mathematical model of force?

Observe:

- magnitude(strength of interaction)
- directions

First guess - vector model \rightsquigarrow needs more justification!

Experiments with springs to test vector character ✓

4. Guess physical laws of forces

- Newton's 3rd law: $\vec{F}_{12} = -\vec{F}_{21}$ Experiment: Magnet on a boat ✓.
- Link force and acceleration:

$$\begin{aligned} &\rightarrow \vec{F} \parallel \vec{a} \\ &\rightarrow |\vec{F}| \uparrow \Rightarrow |\vec{a}| \uparrow \\ &\text{(constraints from observations)} \end{aligned}$$

- Simplest link:

$$\vec{a} = k\vec{F} \quad k > 0$$

(but also others that need to be refuted by experiments).

Experiment: ✓ - e.g. ball rolling down a ramp

define $m = \frac{1}{k}$ (inertial mass)

(Physics of inertia?!)

$$\vec{F} = m\vec{a}$$

Problem!

- \vec{F} (interaction) is absolute.
- \vec{a} (acceleration) is relative to frame of reference.

Q: Is there accelerated motion without a force?

\rightarrow No for point-particles? \leftarrow justify!

Newton's 1st Law: $\vec{F} = 0 \Rightarrow \vec{a} = 0 \Rightarrow \vec{v} = \text{constant}$ (law of inertia).

For Newton's first law to hold you need a special frame of reference (inertial frame of reference where the law of inertia holds).

Experiment: Foucault has shown that the Earth is not inertial.

Problem

Absolutist interpretation:
(Newton)

- there is absolute space and absolute time.
- acceleration has to be taken as relative to absolute space.

Relativist interpretation:
(dominating today due to Einstein)

- there is at least one (and infinitely many) inertial forms of reference.
- Newton's laws are only valid in an inertial reference frame.