Physics Beyond - Modelling Problems in Mechanics

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1 Modelling Problems in Mechanics

Problem: Knowing the interactions modelled by forces find the motion of a particle.
Analyse:

$$\vec{a} = \frac{\vec{F}}{m}$$

1.1 Q: Link acceleration and position?

$$\begin{aligned}
\vec{a} &= \ddot{\vec{x}} \\
\ddot{\vec{x}} &= \frac{\vec{F}}{m}\vec{x}(t) \leftarrow \text{but } \vec{F} \text{ needs to depend on position!} \\
\vec{a}(t) &= \ddot{\vec{x}}(t)
\end{aligned}$$

$\rightsquigarrow$ Vector model fails!
replace with a vector field $\vec{F}$

$$\vec{F} : \text{positions} \to \text{vectors}$$
$$x \to \vec{F}(x)$$

Use reference frame

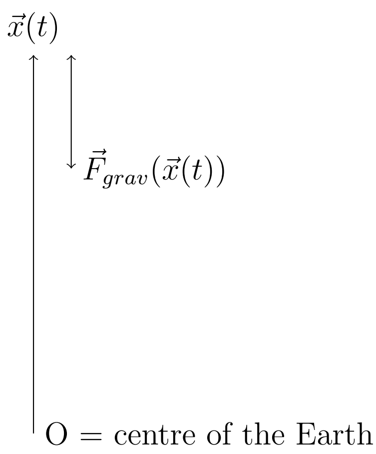
$$\vec{F} : R^3 \to R^3$$
$$\vec{x} \mapsto \vec{F}(\vec{x})$$

Example: Force of gravity (Close to Earth surface)

1)
$$\ddot{\vec{x}} = \frac{m\vec{g}}{m} = \vec{g}$$

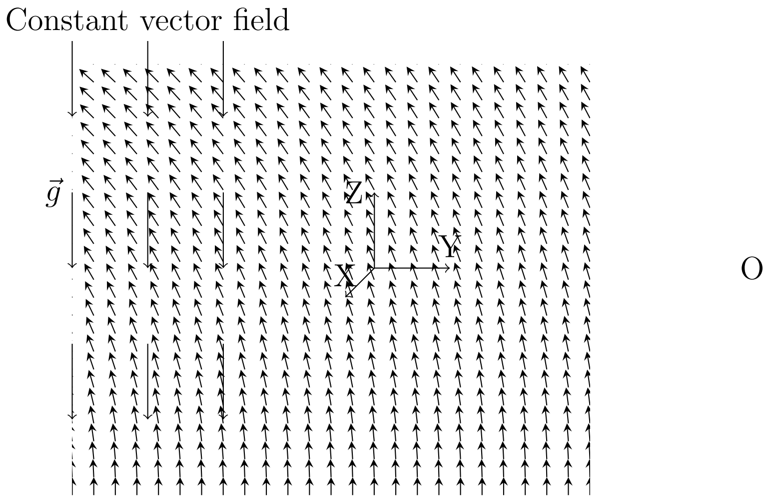
2)
$$\ddot{\vec{x}} = \frac{M_{Earth}}{|\vec{x}(t) - \vec{x}_0|^3}$$

x_0 = centre of the earth



in case of 1)

$\vec{g} : R^3 \to R^3, \vec{x} \mapsto \vec{g}$(Constant vector field)



1.2 Q: What are we dealing with mathematically?

$\rightarrow$ 2nd order differential equation

$$\begin{aligned} \vec{v}(t+dt) &= \dot{\vec{x}}(t+dt) \\ &= \dot{\vec{x}}(t) + \ddot{\vec{x}}(t)dt \\ &= \vec{v}(t) + \dot{\vec{v}}(t)dt \end{aligned}$$

$$(=)\vec{v}(t+dt) - \vec{v}(t) = \dot{\vec{v}}(t)dt$$

$$d\vec{v} = \dot{\vec{v}}(t)dt = \frac{\vec{F}(\vec{x}(t))}{m}dt$$

1.3 Q: How do we go about this?

$\rightarrow$ Suppose we are given an initial position: $\vec{x}(t=0) = \vec{x_0}$

$$\dot{x}(t=0) = \vec{v_0}$$

I)

$$\begin{aligned} \vec{v}(0+dt) &= \vec{v}(0) + \dot{\vec{v}}(0)dt \\ &= \vec{v_0} + \frac{\vec{F}(\vec{x}(0))}{m}dt \\ &= \vec{v_0} + \frac{\vec{F}(\vec{x_0})}{m}dt \end{aligned}$$

II)

$$\begin{aligned}\vec{x}(0+dt) &= \vec{x_0} + \dot{\vec{x}}(0)dt \\ &= \vec{x_0} + \vec{v_0}dt\end{aligned}$$

III)

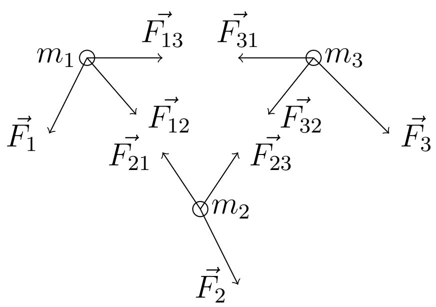
$$\begin{aligned}\vec{x}(0+dt+dt_2) &= \vec{x}(0+dt) + \dot{\vec{x}}(dt)dt_2 \\ &= \vec{x_0} + \vec{v_0}dt + \vec{v_0}dt_2 + \frac{\vec{F}(\vec{x_0})}{m}dtdt_2 \\ &= \vec{x_0} + \vec{v_0}(dt+dt_2) + \frac{\vec{F}(\vec{x_0})}{m}dtdt_2\end{aligned}$$

2 Solving Equations of motion

2.1 Method: (for N particles)

1) Free body diagram of each particle
2) Find the resultant forces
3) Set up equations of motion
4) Solve equations of motion

1)



2) Resultant force:

$$\vec{F_i} = \sum_{f=1, i\neq j}^{N} \vec{F_{ij}} + \vec{F_{ext,i}} \text{ for all particles i}$$

3)

$$m_i\ddot{\vec{x_i}} = \vec{F_i}$$

4) Task: find particles as a function of time $\vec{x_i}(t)$ for all particles based on the knowledge of forces.
$\rightarrow$ last time: force $\vec{F_{ij}}$ are modelled as vector fields.

$$\vec{F_{ij}} : R^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$(\underbrace{\vec{x_i}}_{\text{Position of i-th particle}}, \overbrace{\vec{x_j}}^{\text{Position of j-th particle}}) \mapsto \vec{F_{ij}}(\vec{x_i}, \vec{x_j}) \leftarrow$$ adding force on i-th particle due to j-th particles

$$m_i\ddot{\vec{x_i}} = \vec{F_i}(\vec{x_1}, \cdots, No.\vec{x_i}, \cdots, \vec{x_N})$$

Example:

2.2 Newton's law of gravity

$$\vec{F_{EM}}(\vec{x_E}, \vec{x_M})$$

$$\begin{aligned} F &= G\frac{M_E M_M}{R^2} \\ &= \frac{GM_E M_M}{|\vec{x_M} - \vec{x_E}|^2} \cdot \frac{(\vec{x_M} - \vec{x_E})}{|\vec{x_M} - \vec{x_E}|} \\ &= \frac{GM_E M_M}{|\vec{x_M} - \vec{x_E}|^3} \cdot (\vec{x_M} - \vec{x_E}) \end{aligned}$$

Mathematically, the problem we are facing is to solve a system of 2nd order ordinary differential equation that is coupled.

3 Ordinary Differential Equations(ODE):

3.1 First Order Differential Equations

Idea: Flow of a river ***************River simulation.gif*****************
$\rightarrow$ Introduce a frame of reference

$$\vec{v} : R^2 \rightarrow \mathbb{R}^2$$

$$\overbrace{\vec{x}}^{\text{point on the river}} \quad \mapsto \quad \overbrace{v(\vec{x})}^{\text{velocity at that point}}$$

$$\vec{x} : \overbrace{\mathbb{R}}^{\text{time}} \to \overbrace{\mathbb{R}^2}^{\text{position on the river}}$$

$$t \mapsto \vec{x}(t)$$

$$\dot{\vec{x}}(t) \text{velocity of leaf}$$

Since the leaf follows the flow of the river

$$\dot{\vec{x}}(t) = \vec{v}(\vec{x}(t))$$

First order differential equation

3.2 ODE from an infinitesimal viewpoint

$$\vec{x_0} = \vec{x}(t_0) \xrightarrow{\vec{v}(\vec{x_0})} \vec{x}(t + dt)$$

$$\vec{x}(t_0 + dt) \overset{\text{kock - lawvere}}{=} \vec{x}(t_0) + \dot{\vec{x}}(t_0)dt$$

$$dt = D$$

according to ODE

$$\begin{aligned} \dot{\vec{x}}(t_0) &= \vec{v}(\vec{x}(t_0)) \\ &= v(\vec{x_0}) \end{aligned}$$

$$\rightsquigarrow \vec{x}(t_0 + dt) = \vec{x_0} + \vec{v}(\vec{x_0})dt$$

$dt_1, dt_2 \leftarrow D$

$$\vec{x}((t_0 + dt_1) + dt_2) \overset{\text{K-L}}{=} \vec{x}(t_0 + dt_1) + \dot{\vec{x}}(t_0 + dt_1)dt_2$$

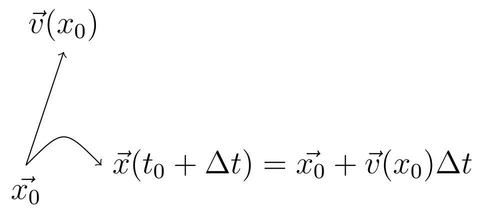
$$\text{ODE } \dot{\vec{x}}(t_0 + dt_1) = \vec{v}(\vec{x}(t_0 + dt_1))$$

$$\vec{x}(t_0 + dt_1 + dt_2) = \vec{x}(t_0 + dt_1) + \vec{v}(\vec{x}(t_0 + dt_1))dt_2$$

$\to$ Iterating this method leads to what is called the infinitesimal Euler method.

3.3 Numerical Euler Method

Instead of infinitesimal time steps $dt \leftarrow D$, use finite but small time steps,$\Delta t > 0$.



This allows you to find an approximate solution $\vec{x}(t)$ of the 1st order ODE $\dot{\vec{x}} = \vec{v}(x)$

3.3.1 Method:

1)
$$\vec{x}(t_0) = \vec{x_0}(\text{initial conditions})$$

2)
$$t_1 = t_0 + \Delta t$$
$$\vec{x}(t_1) = \vec{x}(t_0 + \Delta t) = \vec{x}(t_0) + \vec{v}(\vec{x}(t_0))\Delta t$$

3) after time
$$t_n$$
$$t_{n+1} = t_n + \Delta t$$
$$\vec{x}(t_{n+1}) = \vec{x}(t_n) + \vec{v}(\vec{x}(t_n))\Delta t$$

4 1st Order ODEs from the differential point of view

4.1 Consider ODE

$$\text{for } v : \overbrace{R^n}^{\text{point}} \to \overbrace{R^n}^{\text{Velocity vector at point}}$$
$$\dot{x} = v(x)$$

i.e we are looking for a curve/trajectory $x : R \to R^n, t \mapsto x(t)$

$$\dot{x}(t) = v(x(t))$$

for a solution to be uniquely determined we need an initial value $x(t_0) = x_0$

4.2 Infinitesimal point of view

for an infinitesimal time: dt and dx are <u>both</u> infinitesimal differences.
for $dt \leftarrow D$

$$x(t+dt) \overset{\text{K-L}}{=} x(t) + \dot{x}(t)dt$$
$$\Rightarrow \underbrace{x(t+dt) - x(t)}_{dx} = \dot{x}(t)dt$$

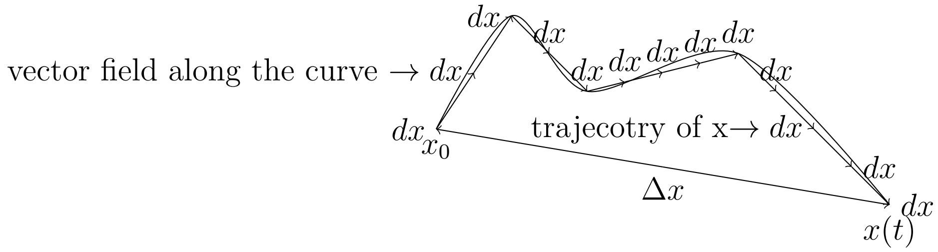
This is the differential of x (dx), it is the infinitesimal displacement along $x(t)$.
(Reminder from multivariable calculus $dx \leftarrow D(n)$)
Using this notation and substituting ODE we find:

$$dx = v(x)dt$$

$\rightsquigarrow$ an <u>equation between differentials.</u>
$\rightsquigarrow$ strategy to find the <u>finite</u> difference in displacement:

$$\Delta x = x(t) - x(t_0)$$

from the differential equation we have to sum up all the infinitesimal displacements $dx = v(x)dt$ along the curve $x(t)$.



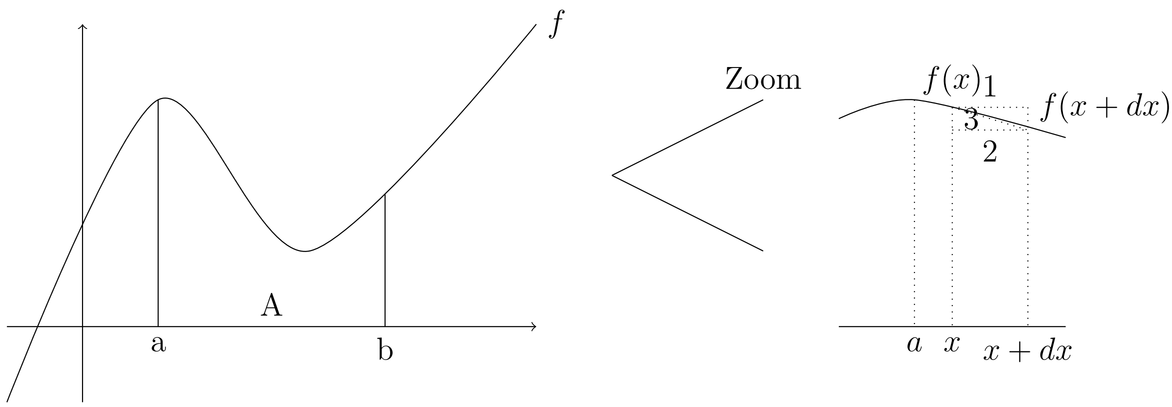
<u>Integration:</u>(Here along the curve!)
Is the idea to sum up all the infinitely many infinitesimal contributions (here along a curve) to get a finite result.

$$\text{stands for s like } \underline{\text{sum}} \rightarrow \int_{x_0}^{x(t)} dx = \int_{t_0}^{t} v(\tau)d\tau$$

need to develop a theory of integration sufficiently strong to integrate vector-valued infinitesimal contributions.

4.3 Theory of integration from the infinitesimal viewpoint

Basic Probelm:



Area of rectangle 1: $dA = f(x)dx$
Area of rectangle 2: $dA = f(x+dx)dx = (f(x) + f'(x)dx)dx = f(x)dx$
Area of trapezium 3: $dA = \frac{1}{2}(f(x) + f(x+dx))dx = \frac{1}{2}(f(x) + f(x) + f'(x)dx)dx = f(x)dx$
Q: How to define

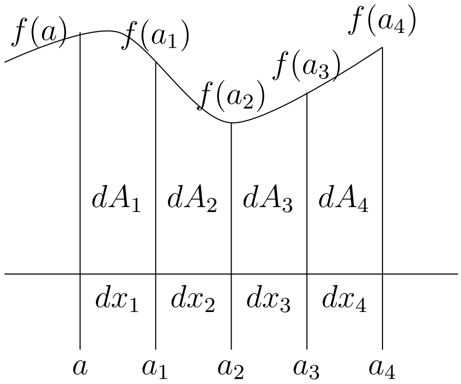
$$A = \int_a^b dA = \int_a^b f(x)dx?$$

$\rightsquigarrow$ problem: are not able to give a direct intuitive definition, as the theory has not been developed so far!
Q: Can you find a way to evaluate $\int_a^b f(x)dx$ and use that as an effective definition?
Observation:

$$F(x+dx) \overset{\text{K-L}}{=} F(x) + F'(x)dx$$

$$F(x+dx) - F(x) = F'(x)dx$$

Suppose I can find $F : R \to R$ such that $F'(x) = f(x), vx \leftarrow R$ then I get $F(x+dx) - F(x) = F'(x)dx = f(x)dx$.



$$
\begin{aligned}
\text{Total Area} &= dA_1 + dA_2 + dA_3 + dA_4 + ... \\
&= f(a)dx_1 + f(a_1)dx_2 + f(a_2)dx_3 + f(a_3)dx_4 + ... \\
&= F(a_1) - F(a) + F(a_2) - F(a_1) + F(a_3) - F(a_2) + F(a_4) - F(a_3) \\
&= F(a_4) - F(a)
\end{aligned}
$$

Idea: No matter how you would define an infinite sum of infinitesimals this cancellation process when summing up

$$f(x)dx = dF = F(x + dx) - F(x)$$

should only depend on the boundary values.
(Fudge)definition:

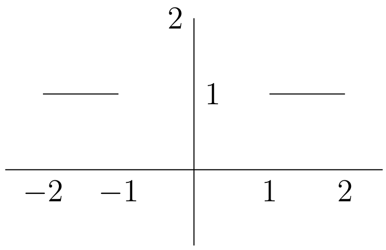
$$\int_a^b f(x)dx = F(b) - F(a) \text{ for an antiderivative } F : R \to R \text{ of } f$$

Q: How do we know an antiderivative exists?
$\to$ we don't, so we postulate it in the theory.

4.4 Integration axiom

- For every $f : R \to R$ there is an antiderivative $F : R \to R$, i.e $F' = f$
- If F and F are antiderivatives of f then $F - F$ is a constant function

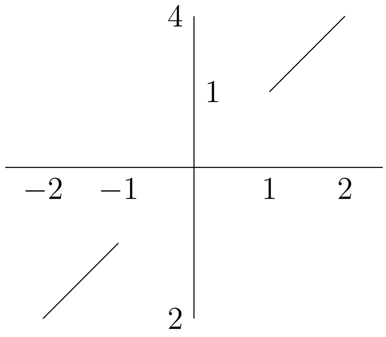
Why difference constant?



$$f : [-2, 1] \cup [1, 2] \to R$$

$$f(x) = \begin{cases} 1, -2 \leq x \leq 1 \\ 2, 1 \leq x \leq 2 \end{cases}$$

$$F_1 : [-2, 1] \cup [1, 2] \to R$$



$$x \mapsto \begin{cases} x, -2 \leq x \leq 1 \\ 2x, 1 \leq x \leq 2 \end{cases}$$

$$F_2 : [-2, 1] \cup [1, 2] \to R$$

$$x \mapsto \begin{cases} x + 4, -2 \leq x \leq 1 \\ 2x - 10, 1 \leq x \leq 2 \end{cases}$$

We have $F_2' = f$ but $F_2(x) - F_1(x) = \begin{cases} 4, -2 \leq x \leq 1 \\ -10, 1 \leq x \leq 2 \end{cases}$ is not constant.

intuition: (has gaps)
On a domain that is 'connected' the antiderivatives do not have to differ by a constant!

4.5 Differentiation rules

4.5.1 Linearity

$$(f + g)' = f' + g' (\text{pointwise sum})$$
$$(f + g)'(x) = f'(x) + g'(x)$$
$$(\lambda f)' = \lambda f'$$
$$(\lambda f)'(x) = \lambda f'(x)$$

4.5.2 Product Rule

$$(f \cdot g)' = f' \cdot g + f \cdot g' (\text{pointwise product})$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

4.5.3 Chain Rule

$$(f \circ g)' = (f' \circ g)'$$
$$(f \circ g)(x) = f(g(x))$$
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

4.6 Integration rules

4.6.1 additivity

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

Proof:

$$\begin{aligned}
F(x) &= \int f(x)dx \\
&= [F(x)]_a^c + [F(x)]_c^b \\
&= F(c) - F(a) + F(b) - F(c) \\
&= F(b) - F(a) \\
&= \int_a^b f(x)dx
\end{aligned}$$

4.6.2 linearity

$$\int_a^b \lambda f(x) + g(x)dx = \lambda \int_a^b f(x)dx + \int_a^b g(x)dx$$

Proof:

$$\begin{aligned}
\int_a^b \lambda f(x) + g(x)dx &= \\
&= [\int \lambda f(x) + g(x)dx]_a^b \\
&= [\int \lambda f(x) + g(x)dx]^b - [\int \lambda f(x) + g(x)dx]_a \\
&= [\int \lambda f(x)dx]^b - [\int \lambda f(x)dx]_a + [\int g(x)dx]^b - [\int g(x)dx]_a \\
&= [\int \lambda f(x)dx]_a^b + [\int g(x)dx]_a^b \\
&= \lambda[\int f(x)dx]_a^b + \int_a^b g(x)dx \\
&= \lambda \int_a^b f(x)dx + \int_a^b g(x)dx
\end{aligned}$$

4.6.3 partial differentiation

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx$$

Proof:

$$\begin{aligned}
\text{Intergrate product rule: } (f \cdot g)'(x) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\
\int_a^b (f \cdot g)'(x)dx &= \int_a^b f'(x) \cdot g(x)dx + \int_a^b f(x) \cdot g'(x)dx \\
\big[(f \cdot g)(x)\big]_a^b &= \int_a^b f'(x) \cdot g(x)dx + \int_a^b f(x) \cdot g'(x)dx \\
\rightsquigarrow \int_a^b f'(x) \cdot g(x)dx &= [(f \cdot g)(x)]_a^b - \int_a^b f(x) \cdot g'(x)dx
\end{aligned}$$

Example:

$$\begin{aligned}
\int_0^1 x sin(x)dx &= \int_0^1 g(x)f'(x)dx \\
&= [-cos(x)\cdot x]_0^1 - \int_0^1 -cos(x)\cdot 1dx \\
&= -cos(1)+0+\int_0^1 cos(x)dx \\
&= -cos(1)+sin(1)-sin(0) \\
&= sin(1)-cos(1)
\end{aligned}$$

4.6.4 substitution rule

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

I do the substitution $u = g(x)$

$$du = g(x+dx) - g(x) = g(x) + g'(x)dx - g(x) = g'(x)dx$$

Example:

$$\int_0^1 sin(x^2)2xdx, u = x^2, (x^2)^1 = 2x$$

$$\begin{aligned}
&= \int_0^1 sin(u)du \\
&= [-cos(u)]_0^1 = cos(0) - cos(1) = 1 - cos(1)
\end{aligned}$$

Example: Let F be an antiderivative of f ($F' = f$) (Exists due to integration axiom)
Consider $(F\circ g)' = F'\circ g\cdot g' = f\circ g\cdot g'$
$\rightsquigarrow F\circ g$ is the antiderivative of $f\circ g$

$$\rightsquigarrow \int_a^b (f(g(x))g'(x)dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du \text{ (by definition)}$$

Remark: Although the proof of integration by substitution is straight forward with the definitions we made, geometrically it is not straight forward.
Q: What if $g(x) = g(x+dx)$? (i.e. x is a stationary point)
$\rightsquigarrow du = 0$, but dx is not probably equal to 0.
This is not a problem as:

$$du = g'(x)dx \text{ and } g'(x) = 0$$

Note:

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

$$\int_a^{a+d} f(x)dx = F(a+d) - F(a) = f(a)d \text{ (K-L)}$$

5 Vector Valued Integration

Reminder: For ODEs we had to consider

$$dx = v(x)dt$$

5.1 Q: How do we link this back to the integral we just discussed?

$\rightsquigarrow$ Introduce coordinates
Assume $x : R \to R^n$ is the solution to our ODE $dx = v(x)dt$

$$dx = x(t + dt) - x(t) = \dot{x}(t)dt = v(x(t))dt$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$dx = \begin{pmatrix} x_1(t+dt) - x_1(t) \\ \vdots \\ x_n(t+dt) - x_n(t) \end{pmatrix} = \begin{pmatrix} v_1(x_1(t) & \cdots & x_n(t))dt \\ \vdots & \ddots & \vdots \\ v_n(x_1(t) & \cdots & x_n(t))dt \end{pmatrix}$$

$\rightsquigarrow$ we can sum up the infinitely many infinitesimal vectors $v(x(t))dt$
in 1D summing over $f(x)dx$, $v(x(t))dt$

5.2 Definition: (vector valued integral)

$$\gamma : R \to R^n, t \mapsto \gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix} \text{ (a curve)}$$

$$\int_{t_0}^{t_n} \gamma(t)dt = \begin{pmatrix} \int_{t_0}^{t_n} \gamma_1(t)dt \\ \vdots \\ \int_{t_0}^{t_n} \gamma_n(t)dt \end{pmatrix}$$

5.3 Q: Is that well defined?

Let $T_j : R \to R$ be the antiderivative of γ_j for all $1 <= j <= n$

Consider $T : R \to R^n$, $t \mapsto \begin{pmatrix} T_1(t) \\ \vdots \\ T_n(t) \end{pmatrix}$ (a curve)

$$\dot{T}(t) = \begin{pmatrix} \dot{T}_1(t) \\ \vdots \\ \dot{T}_n(t) \end{pmatrix} = \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix} = \gamma(t)$$

$\rightsquigarrow$ T is an antiderivative for γ.

$$\int_{t_0}^{t_1} \gamma(t)dt = T(t_1) - T(t_0) \begin{pmatrix} T_1(t_1) - T_1(t_0) \\ \vdots \\ T_n(t_1) - T_n(t_0) \end{pmatrix} = \begin{pmatrix} \int_{t_0}^{t_1} \gamma_1(t)dt \\ \vdots \\ \int_{t_0}^{t_1} \gamma_2(t)dt \end{pmatrix}$$

Can we apply vector valued integrals to ODEs?
Kind of.

5.4 Q: What is the problem?

$\to$ We need to integrate $v(x(t))dt$ to get $x(t)$ $\to$ but we need to know the curve x in advance to do this. Like in 1D what we really get is an equation involving a vector valued integral. But now we have a definition of the vector valued integral or the RHS.

6 Separation of variables

6.1 Example: 1D

$v : R \to R$, $x \mapsto x$, $\dot{x} = x$, $x(0) = x_0$ (Initial Value Problem - IVP)
rewrite this as a differential equation:

$$\dot{x} = \frac{dx}{dt}$$

$$dx = x(t + dt) - x(t) \underbrace{=}_{\text{K-L}} \dot{x}dt \underbrace{=}_{\text{ODE}} xdt \to dx = xdt$$

$$\frac{1}{x}dx = dt \underbrace{\rightsquigarrow}_{\text{Integrate}} \int_{x_0}^{x(t)} \frac{1}{x}dx \quad = \quad \int_0^t d\tau$$

$$\begin{aligned} &(=) \quad [lnx]_{x_0}^{x(t)} = [\tau]_0^t \\ &(=) \quad ln\frac{x(t)}{x_0} = t \\ &(=) \quad \frac{x(t)}{x_0} = e^t \\ &(=) \quad x(t) = x_0 e^t \end{aligned}$$

<u>Check:</u>

$$\dot{x}(t) = x_0 e^t = x(t)\checkmark$$

$$x(0) = x_0 e^0 = x_0 \cdot t = x_0\checkmark$$

If $x(0) = 0$ initially, $x(t) = 0$

6.2 Method 1:

(separation of variables $v : R \to R$, $x \mapsto f(x)$, $f(x) = x$
<u>Assume:</u> For each $x \leftarrow R$ $f(x)$ has a multiplicative inverse.

$$dx = f(x)dt \text{ (differential form of the ODE x = f(x))}$$

1. Separate the variables x and t

$$\frac{1}{f(x)}dx = dt$$

2. Integrate both sides (IVP $\dot{x} = f(x)$, $x(t_0) = x_0$)

$$\int_{x_0}^{x_t} \frac{1}{f(x)}dx = \int_{t_0}^{t} dt = t - t_0$$

3. Solve this equation for $x(t)$.

6.3 <u>Q:</u> Does the equation have a solution? Is this solution unique?

$$G : R \to R, y \mapsto \int_{x_0}^{y} \frac{1}{f(x)}dx$$

$\rightsquigarrow$ the equation becomes $G(x(t)) = t - t_0$
We notice
$$G(x_0) = \int_{x_0}^{x_0} \frac{1}{f(x)} dx = 0$$
$$t_0 - t_0 = 0$$
We have a solution for $t_0 - t_0 = 0$:
$$G'(y) = \frac{1}{f(y)}$$
This is different for zero $\rightsquigarrow$ No stationary points.
$\rightarrow$ It is either increasing or decreasing (won't change monotonicity).
$\rightarrow$ Will always be able to find solution and solution is unique as G is one to one (locally, for t close to t_0).

6.4 Method 2:

$v : R \times R \to R$(time dependent vector field)$(x,t) \mapsto v(x,t) = f(x)g(t)(g(t) = 1$ in method 1)

Assumption: $f(x)$ has a multiplicative inverse in R
Consider the IVP $\dot{x} = v(x,t), x(0) = x_0, \dot{x}(t) = v(x(t), t)$

1. $dx = v(x,t)dt = f(x)g(t)dt$
2. Separate the variables: $\frac{1}{f(x)}dx = g(t)dt$
3. Integrate $\int_{x_0}^{x(t)} \frac{1}{f(x)}dx = \int_{t_0}^{t} g(\tau)d\tau$

Solve for x(t)

7 2nd Order ODEs

$$\ddot{x} = F(x, \dot{x}, t)$$
$$F : R^n x R^n \times R \to R^n$$

7.1 Q: How to make it relate to 1st order ODEs?

is a system of two 1st order ODEs $\begin{cases} \dot{x} = v \leftarrow \text{introduce a 'velocity'.} \\ \dot{v} = \ddot{x} = F(x, v, t) \end{cases}$

7.2 Q: How to turn this into one 1st order ODE?

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix}}_{\dot{z}} = \underbrace{\begin{pmatrix} v \\ F(x,v,t) \end{pmatrix}}_{\tilde{F}(z,t)}$$

$$z = \begin{pmatrix} x \\ v \end{pmatrix}, \tilde{F} : R^{2n} \times R \to R^{2n}, (z,t) \mapsto \begin{pmatrix} v \\ F(z,t) \end{pmatrix}$$

A solution: $z : R \to R^{2n}, t \mapsto z(t)$ of $\dot{z} = \tilde{F}(z,t)$
i.e.

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(x(t), v(t), t) \end{pmatrix}$$

$$\rightsquigarrow \dot{x}(t) = v(t)$$

$$\dot{v}(t) = F(x(t), v(t), t)$$

$$\underbrace{\rightsquigarrow}_{\text{substitute } \dot{x}=v} \quad \dot{v}(t) = \ddot{x}(t) = F(x(t), \dot{x}(t), t)\checkmark$$

7.3 Example: (Free Fall)

$$\ddot{x} = -g, F : R \to R, x \mapsto -g(\text{constant})$$

$$\text{Systen if 1st order ODEs} \begin{pmatrix} \dot{x} \\ v \end{pmatrix} = \begin{pmatrix} v \\ -g \end{pmatrix} \begin{cases} \dot{x} = v(I) \\ \dot{v} = -g(II) \end{cases}$$

Integrate:

$$(I)v(t) - v(0) = -gt \rightsquigarrow v(t) = v(0) - gt$$

Substitute in (I):

$$\dot{x} = v(t) = v(0) - gt(w : R \times R \to R, (x,t) \mapsto v(0) - gt, \dot{x} = w(x,t))$$

Integrate:

$$x(t) - x(0) = v(0)t - \frac{1}{2}gt^2$$

$$x(t) = x(0) + v(0)t - \frac{1}{2}gt^2$$

7.4 Example: Pendulum

(We ignore centripetal force due to constraint)
T = tension
F_c = centrifugal force
W = weight
contraint: rod is rigid
$\rightsquigarrow$ bob of mass m is going to move on a circle of radius l.
$\rightsquigarrow$ resulting force:

$$W_{\|} = -mgsin\varphi$$

$$m\ddot{x} = W_{\|} = -mgsin\varphi$$

(use $x = \varphi l$)

$$\ddot{x} = \ddot{\varphi}l = -gsin\varphi$$

$$\rightsquigarrow \ddot{\varphi} = -\frac{g}{l}sin\varphi$$

Step 1 $\rightarrow$ rewrite this as a system of 1st order ODEs:

$$\phi = \nu$$

$$\dot{\nu} = -\frac{g}{l}sin\varphi$$

Try separation of variables:

$$d\nu = \frac{-g}{l}sin\varphi dt, d\varphi = \nu dt$$

Doesn't work as per method. Try further:

$$dt = \frac{d\varphi}{\nu}(\text{ careful } \nu = 0 \text{ is possible})$$

$$\rightsquigarrow d\nu = \frac{-g}{l}sin\varphi\frac{d\varphi}{\nu}$$

Now we can separate:

$$\nu d\nu = -\frac{g}{l}sin\varphi d\varphi$$

(We have lost the time variable) Integrate:

$$\int_{\nu(\varphi_0)}^{\nu(\varphi)} \nu d\nu = -\frac{g}{l}\int_{\nu(\varphi_0)}^{\nu(\varphi)} sin\varphi d\varphi$$

(φ_0 - pulled up the bob an angle of φ_0 and release from rest $\nu(\varphi_0) = 0$)

$$\rightsquigarrow \frac{1}{2}\nu(\varphi)^2 = \frac{g}{l}(cos\varphi - cos\varphi_0)$$

$$(=)\nu(\varphi) = \sqrt{\frac{2g}{l}(cos\varphi - cos\varphi_0)}$$

7.5 Q: What have we figured out?

We found the angular velocity as a function of the angle not the time.
This gives us an ODE:

$$\dot{\varphi} = \nu(\varphi) = \sqrt{\frac{2g}{l}(cos\varphi - cos\varphi_0)}$$

(Apply separation of variables:)

$$\frac{1}{\sqrt{\frac{2g}{l}(cos\varphi - cos\varphi_0)}}d\varphi = dt$$

$$\sqrt{\frac{1}{2g}}\int_{\varphi_0}^{\varphi(t)} \frac{1}{\sqrt{cos\varphi - cos\varphi_0}}d\varphi = t$$

Solve for t: (problem: this is an integral that has not got an elementary function as an antiderivative)

$$E(z) = \int_{\varphi_0}^{z} \frac{1}{\sqrt{cos\varphi - cos\varphi_0}}d\varphi \text{(eliptic integral)}$$