Physics Beyond - Motion

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June 3, 2020

Contents

1 Procedure

1. Define your problem

2. Model your problem mathematically. If the problem is about understanding the physics of something then:

    - Model the description of that something (e.g. model what motion is)
    - Or investigate / find the underlying cause of that something (e.g. what causes motion)

3. Solve the maths

4. Interpret the results

5. (If needed) revise your model

2 Definition:

Definition - change in position (space) over time (time).

1. Put the physics of space time aside
2. Simple models - based on direct experience
   - Space: Euclidean geometry
   - Time: Real numbers $\mathbb{R}$

3 Modelling Mathematically:

3.1 Q: What is the <u>convenient</u> mathematical model?

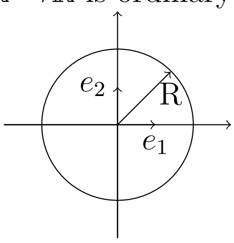
$$\begin{array}{cc} \text{Function} & \text{Time} \to \text{space} \\ \text{x:} & \mathbb{R} \to \text{space} \\ & \text{t} \mapsto \text{position x(t)} \end{array}$$

3.2 How to model space conveniently?

$\rightsquigarrow$ need a reference frame.

$$(0,\ \vec{e_1},\ \vec{e_2},\ \vec{e_3})$$
$$\vec{x} : \mathbb{R} \to \text{Euclidean vector-space (after introducing 0)}$$
$$\vec{x} : \mathbb{R} \to \mathbb{R}^3$$
$$\text{t} \mapsto \vec{x}(\text{t}) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$
$$\text{(after introducing } \begin{bmatrix} \vec{e_1}, \vec{e_2}, \vec{e_3} \end{bmatrix}$$

Each $x_i : \mathbb{R} \to \mathbb{R}$ is ordinary an ordinary function we are familiar with

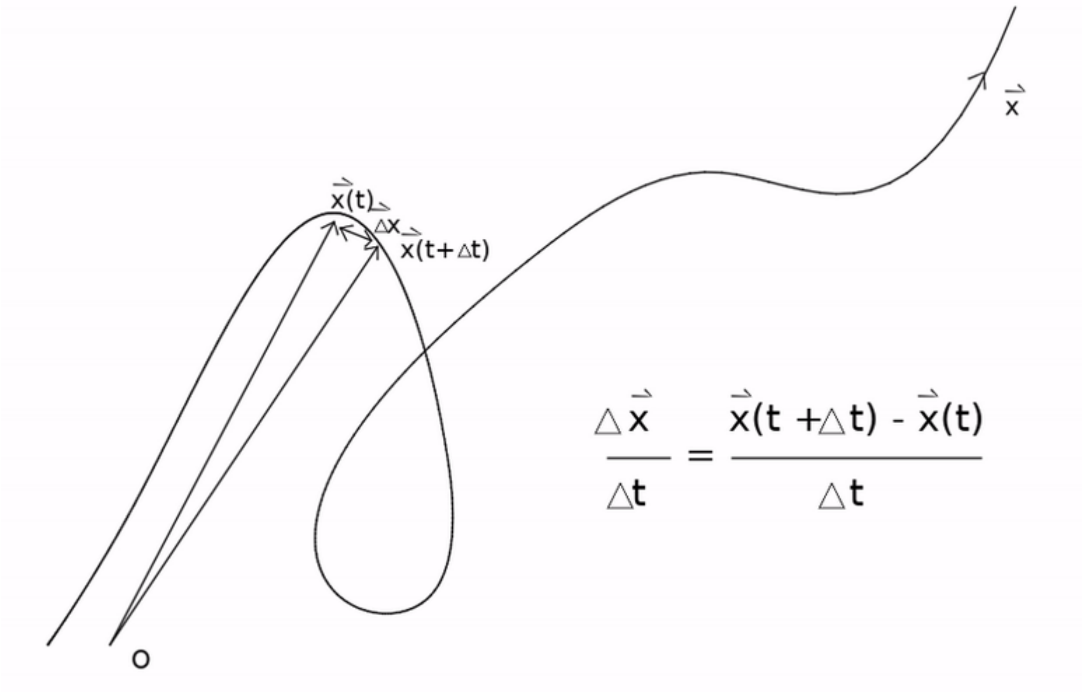


$$\vec{x}(\text{t}) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} RCos\varphi(t) \\ RSin\varphi(t) \end{bmatrix}$$

Observation: interactions change motion.(This is how we detect interactions after all)

3.3 Q: How to model changes in motion?

***********************Particle Motion Simulation*************************



Problem: I need to find the velocity at one instant in time.

4 Q: How to model this mathematically

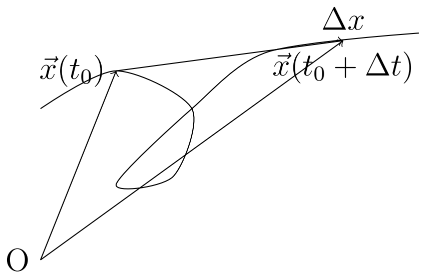
→ taking a limit:

$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t}$

→ or using infinitesimals

5 The theory of infinitesimals

Problem: Define rates of change at one instant in time.



$$\vec{v} = \frac{\Delta \vec{x}}{\Delta t} = \frac{\vec{x}(t_0+\Delta t) - \vec{x}(t_0)}{\Delta t}$$

The problem is that $\vec{v}$ is the average velocity in the time interval $[t_0, t_0 + \Delta t]$

6 Q: How can we define instantaneous velocity?

1. The idea of a limit:

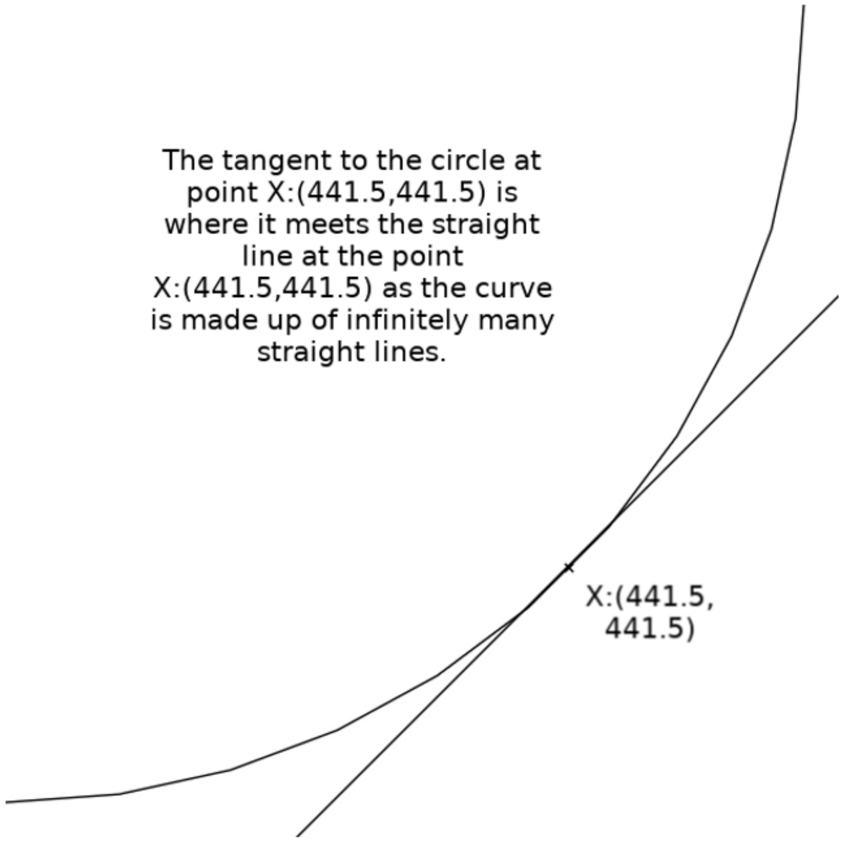
    - make Δt smaller and smaller
    - study $\frac{\Delta \vec{x}}{\Delta t}$ changes when Δtapproaches 0

    $\leadsto$ it becomes a tangent to the curve at $\vec{x}(t_0)$
    we write:

    $$\vec{v} = \tfrac{d\vec{x}}{dt} = \dot{\vec{x}} = \lim_{\Delta t \to 0} \frac{\vec{x}(t_0+\Delta t) - \vec{x}(t_0)}{\Delta t}$$

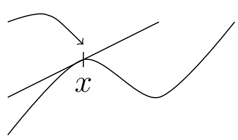
    Q: How is $\lim\limits_{\Delta t \to 0}$ defined mathematically?
    $\rightarrow$ it is technical and complicated
    Remark: derivatives are defined via limits in ordinary maths.

2. Idea of infinitesimals:
    Intuition: If I zoom in onto a curve infinitely close, it becomes a straight line.
    ***********************Infinitesimals Simulation*************************

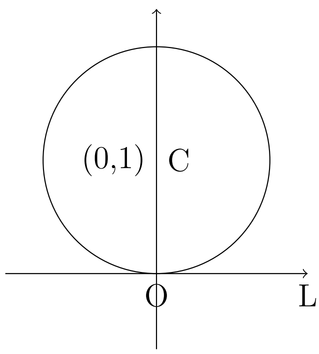


Q: How to define <u>infinitesimals</u>?
→ Use idea of a tangent touching a curve at a point.
→ We interpret "touching" to mean that the curve and the tangent line <u>intersect</u> in an infinitesimal piece of a curve.

infinitesimal piece of intersection: where the curve <u>agrees</u> with the line



<u>Problem</u>: In general we cannot define what a tangent is except for a circle.



when a line intersects a circle at

exactly <u>one</u> point, then it's a tangent to the circle
$\rightsquigarrow$ calculate the intersection of the circle, C, and a line, L.

$\rightarrow$ we use a coordinate system
$C : x^2 + (y-1)^2 - 1 = 0$
$L : y = 0$

$$(nL = \{(x,0)|x^2 = 0\})$$

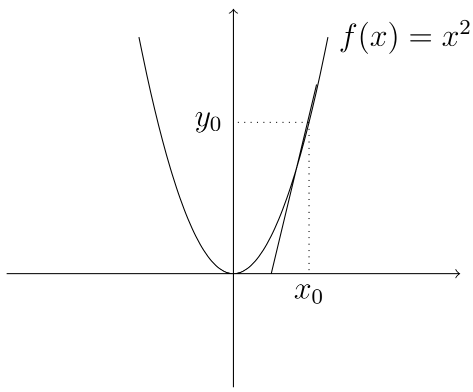
$\rightarrow$ to have more than the point (0,0), we do not allow to conclude $x^2 = 0 \rightarrow x = 0$!
<u>Definition</u>: (First-order) infinitesimals are $d \leftarrow R$ such that $d^2 = 0$
$D := \{\text{do } R | d^2 = 0\}$
Q: What is R?

- $\rightarrow$ It is <u>not</u> $\mathbb{R}$ (the real numbers) because $x^2 = 0 \rightarrow x = 0$ for $x \leftarrow \mathbb{R}$
- $\rightarrow$ We want R to have the properties similar to $\mathbb{R}$:
  - (a) I have addition and subtraction
  - (b) I have multiplication
  - (c) $\mathbb{Q}cR$ (in fact $\mathbb{R}cR$) $\rightarrow$ rational numbers: numbers that can be written as fractions.

<u>Intuition:</u> R is an extension of the reals by adding <u>all</u> the missing infinitesimals.
<u>Caution</u> you cannot divide freely by numbers in R
Suppose d $\leftarrow$ D has an inverse, d^{-1}
$d^{-1}d = 1$ but $d^{-}1 \underbrace{d.d}_{\substack{d^2=0 \\ d=0}} = d$

7 Q: Can infinitesimals solve the <u>tangent problem?</u>

Definition (tangent$\leftarrow$ due to Leibniz) - a tangent is a line that intersects a curve in two infinitesimally close points.

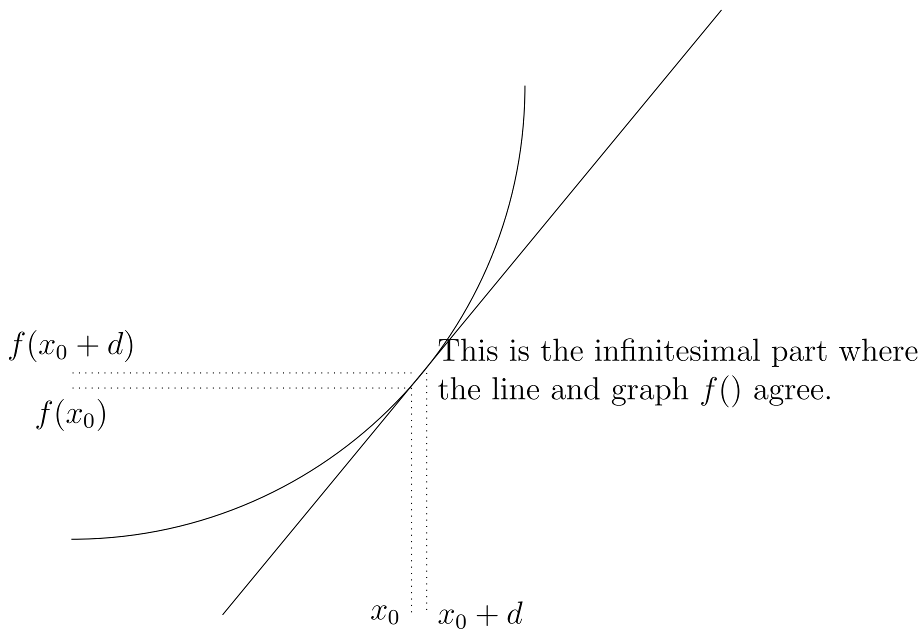


Using definition of Leibniz, we are looking at the secant line through $(x_0, f(x_0)), (x_0 + d, f(x_0 + d))$

1.

$$\begin{aligned}
f(x_0+d) &= (x_0+d)^2 \\
&= {X_0}^2 + 2x_0 d + d^2 \\
&= {X_0}^2 + 2x_0 d \\
&= f(x_0) + 2x_0 d \\
f(x_0+d) - f(x_0) &= 2x_0 d \\
f(x_0+d) - f(x_0) &= 2x_0 \cdot ((x_0+d) - x_0) \\
(y_1 - y_0 &= m(x_1 - x_0))
\end{aligned}$$

$\rightsquigarrow y = f(x_0)+2x_0 d$ is the infinitesimal piece of the tangent at $(x_0, f(x_0))$
$\rightsquigarrow$ for any $h \leftarrow R$ we get the equation $y = f(x_0) + 2x_0 \cdot h$



2. $f : R \to R \; x \to x^3$

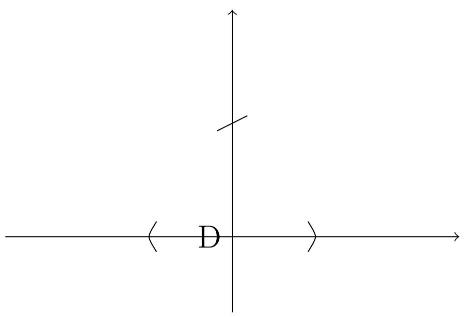
$$\begin{aligned}
f(x_0+d) = (x_0+d)^3 &= (x_0+d)^2(x_0+d) \\
&= ({x_0}^2 + 2x_0 d)(x_0+d) = {x_0}^3 + 2{x_0}^2 d + {x_0}^2 d + \underbrace{2x_0 d^2}_{0} \\
&= {x_0}^3 + 3{x_0}^2 d \\
&= f(x_0) + \underbrace{3{x_0}^2}_{f(x_0)} d
\end{aligned}$$

3.

$$
\begin{aligned}
s(t) &= ut + \frac{1}{2}gt^2 \\
s(t+dt) &= u(t+dt) + \frac{1}{2}g(t+dt)^2 \\
&= ut + udt + \frac{1}{2}g(t^2 2tdt) \\
&= \underbrace{ut + \frac{1}{2}gt^2}_{s(t)} + (u+gt)dt
\end{aligned}
$$

$$
\begin{gathered}
v(t) = \dot{s}(t) = u + gt \\
v(t+dt) = u + g(t+dt) = \underbrace{u+gt}_{v(t)} + gdt \\
a(t) = \dot{v}(t) = \ddot{s}(t) = g
\end{gathered}
$$

Axiom: (Kock-Lawvere) For each $f : D \mapsto R$ there are unique $a, b \leftarrow R$ such that $f(d) = a + b \cdot d \; V d \leftarrow D$



(This means we model the assumption made that every curve is made of infinitesimal line segments at each point).

$f(0) = a + b \cdot 0 = a$

$\rightsquigarrow$ b is the gradient of the line and hence the gradient of f at 0. (Because it is simply determined by f).

Consequence:

Consider $f : R \to R$ at $x_0 \leftarrow R$

Define $f_{x_0} : D \to R \; d \mapsto f(x_0 + d)$

By K-L we have unique $a_{x_0}, b_{x_0} \leftarrow R$ such that

$$f_{x_0}(d) = a_{x_0} + b_{x_0} \cdot d = f_{x_0}(0) + b_{x_0} \cdot d$$

i.e.

$$f(x_0 + d) = f(x_0) + b_{x_0} \cdot d$$

Def $f'(x_0) := b_{x_0}$ is the derivative of f at x_0
$\rightsquigarrow f'(x_0)$ is the gradient of the infinitesimal line of the graph of f at $(x_0, f(x_0))$, "the gradient of f at x_0".

8 Rules for Calculating Derivatives:

8.1 Linearity

$$f, g : R \to R,\ \lambda \leftarrow R$$
$$f + \lambda : R \to R,\ x \mapsto f(x) + \lambda g(x)$$
$$(f + \lambda)' = f' + \lambda(g')$$

Example:

$$f(x) = x^2,\ g(x) = x^3,\ \lambda = 5$$
$$(f + \lambda g)(x) = x^2 + 5x^3$$
$$f'(x) = 2x,\ g'(x) = 3x^2$$
$$(f + \lambda g)'(x) = 2x + 15x^2$$

8.2 Product Rule

$$f.g : R \to R,\ x \mapsto f(x) \cdot g(x)$$
$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

Example:

$$f(x) = x,\ g(x) = x^2$$
$$(f \cdot g)(x) = x^3$$
$$f'(x) = 1,\ g'(x) = 2x$$
$$(f \cdot g)'(x) = 1 \cdot x^2 + x \cdot 2x = 3x^2$$

8.3 Chain Rule

$$f, g : R \to R$$
$$f \circ g : R \to R\ x \mapsto f(g(x))\ \text{(composition)}$$
$$(f \circ g)' = (f' \circ g) \cdot g'$$

Examples:

$$1)\ f(x) = x^2, g(x) = x^3, f'(x) = 2x, g'(x) = 3x^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^3) = (x^3)^2 = x^6$$
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2x(x^3) \cdot 3x^2 = 2x^3 \cdot 3x^2 = 6x^5$$

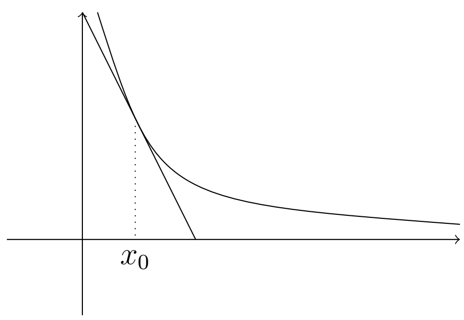
$$2)\ f(x) = x^2, g(x) = 3x^5 + 2x^2 + 5$$
$$(f \circ g)(x) = f(g(x)) = (3x^5 + 2x^2 + 5)^2$$
$$f'(x) = 2x, g'(x) = 3 \cdot 5x^4 + 2 \cdot 2x + 0 = 15x^4 + 4x$$
$$(f \circ g)'(x) = 2(3x^5 + 2x^2 + 5) \cdot (15x^4 + 4x)$$

Applications:

$$1)\ f(x) = x, g(x) = \underbrace{\tfrac{1}{x}}_{\text{take the multiplicative inverse}}$$
$$g : \underbrace{R^x}_{\text{the set of all} x \leftarrow R \text{that are invertible} (y \leftarrow R, xy=1)} \to R$$

we want to know $g'(x)$!

$$(f \cdot g)(x) = 1 = x \cdot \tfrac{1}{x} = 1$$
$$(f \cdot g)'(x) = 0$$
but
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$
$$f'(x) = 1$$



we find:

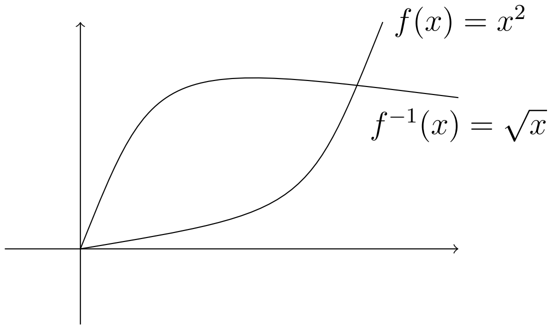
$$\begin{aligned} 0 &= 1 \cdot g(x) + g'(x) \cdot x \\ &= \frac{1}{x} + g'(x) \cdot x \\ \rightsquigarrow g'(x) &= -\frac{1}{x^2} \end{aligned}$$

$$2)\ f : R^x \to R^x, g : R^x \to R^x, x \mapsto \tfrac{1}{x}$$
$$\tfrac{1}{f} = g \circ f$$
$$(\tfrac{1}{f})' = (g \circ f)' \rightsquigarrow (\tfrac{1}{f})'(x) = (g \circ f)'(x) = g'(f(x)) \cdot f'(x) = -\tfrac{1}{(f(x))^2} \cdot f'(x)$$

3) Deriving inverse functions:

given a function $f : R \to R$ such that f is one to one and onto then it has an inverse(for composition). $f^{-1} : f^{-1} \circ f = R_{id}$(identity

$$\text{map})f^{-1}(f(x)) = x$$
$$f \circ f^{-1} = R_{id} \; f(f^{-1}(y)) = y$$



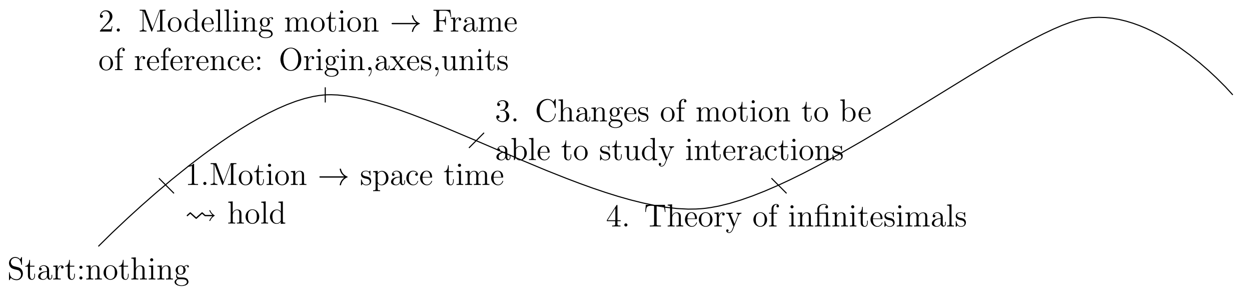
Apply chain rule to

$$\begin{aligned}
f^{-1} : f^{-1} \circ f &= R_{id} \\
(f^{-1})'(f(x)) \cdot f'(x) &= 1 \\
(f^{-1})' f(x) &= \frac{1}{f'(x)} \left(\frac{1}{m} = perpendicular\right)
\end{aligned}$$

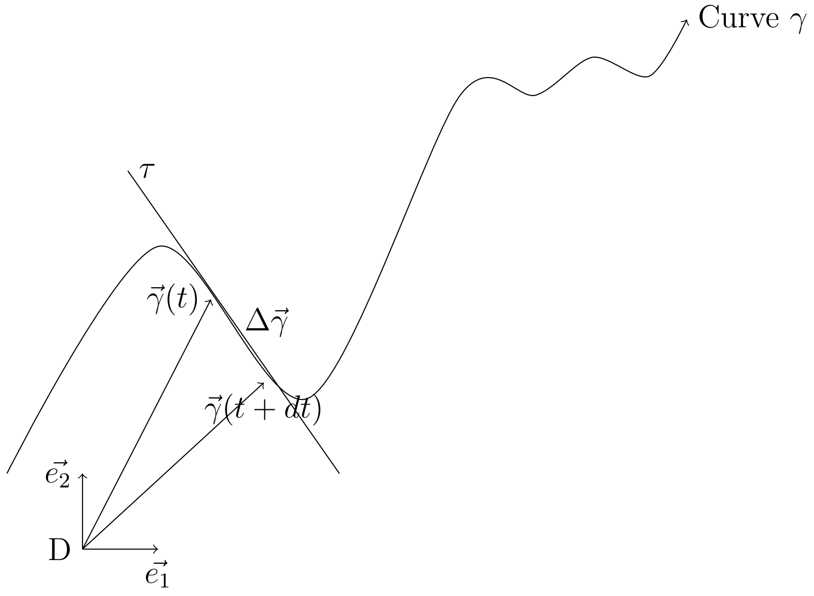
or rewritten

$$\begin{aligned}
(f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\
y &= f(x) \\
x &= f^{-1}(f(x)) \\
&= f^{-1}(y)
\end{aligned}$$

9 The journey so far:



10 Application: Derivatives of curves



$$dt \leftarrow D$$

intuition says:

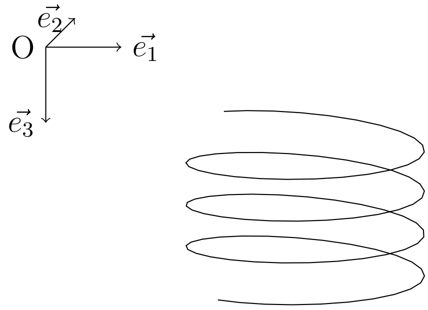
- $\Delta\vec{\gamma}$ agrees with the curve γ.
- a line through $\vec{\gamma}(t)$ in the direction $\vec{\gamma}$ is a <u>tangent</u> at the curve γ.

<u>Theory:</u> $y : R \to R^2$, $t \mapsto \begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix}$, $\gamma_j : R \to R$, j = 1,2 ,$\dot{\vec{\gamma}} = \begin{bmatrix}\dot{\gamma_1}(t)\\ \dot{\gamma_2}(t)\end{bmatrix}$

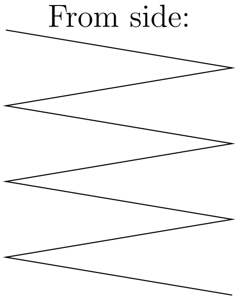
$$\begin{aligned}\vec{\gamma}(t+dt) &= \begin{bmatrix}\gamma_1(t+dt)\\ \gamma_2(t+dt)\end{bmatrix}\\ &\overset{K-L}{=} \begin{bmatrix}\gamma_1(t)+\dot{\gamma_1}(t)dt\\ \gamma_2(t)+\dot{\gamma_2}(t)dt\end{bmatrix}\\ &= \begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix} + \begin{bmatrix}\dot{\gamma_1}(t)\\ \dot{\gamma_2}(t)\end{bmatrix} dt\\ &= \vec{\gamma}(t) + \dot{\vec{\gamma}}(t)dt\end{aligned}$$

$$\Delta\vec{\gamma} = \vec{\gamma}(t+dt) - \vec{\gamma}(t) = \dot{\vec{\gamma}}(t).dt \leftarrow \underline{\text{Velocity}}$$

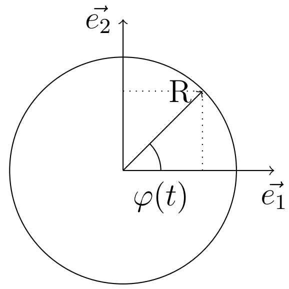
$$\tau : h \mapsto \gamma(t) + \dot{\gamma}(t)h$$



From side:



From top: looks like circular motion



$$\vec{x}(t) = \begin{bmatrix} Rcos\varphi(t) \\ Rsin\varphi(t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} Rcos\varphi(t) \\ Rsin\varphi(t) \\ vt \end{bmatrix}$$

$$\dot{\vec{x}}(t) = \begin{bmatrix} \overbrace{-Rsin\varphi(t)}^{ut}.\dot{\varphi}(t) \\ Rcos\varphi(t).\dot{\varphi}(t) \\ v \end{bmatrix}$$

$$\ddot{\vec{x}}(t) = \begin{bmatrix} \dot{u}(t).\dot{\varphi}(t) + \dot{u}t.\ddot{\varphi}(t) \\ \\ \\ \end{bmatrix}$$

$$= \begin{bmatrix} -Rcos\varphi(t).(\dot{\varphi}(t))^2 - Rsin\varphi(t).\ddot{\varphi}(t) \\ -Rsin\varphi(t).(\dot{\varphi}(t))^2 + Rcos\varphi(t).\ddot{\varphi}(t) \\ 0 \end{bmatrix}$$

11 Physics of Motion

11.1 Q:What changes motion?

$\rightarrow$ Physical interactions.

11.2 Q: Why so general?

$\rightsquigarrow$ any form of physical interaction lead to a changes in motion.
$\rightarrow$ we use changes of motion to study interactions.
Introduce force as a measure of the interaction on one object.(Reminder:

- interaction needs at least two objects
- but we need to relate it to the change of motion of one object.)

11.3 Q: How to model force mathematically?

First observation:

1. Has magnitude (measure of the strength of the interaction)
2. Has a direction

$\rightsquigarrow$ model force with a vector.
But! (problems)
$\rightarrow$ It matters where the force acts on an object. 3 effects of forces:

1. linear motion
2. rotation
3. deformation

$\rightarrow$ This is not captured by a vector.

$$\rightsquigarrow \vec{a}(\vec{F}) = ?$$

Simplest relationship:

$$\vec{a} \propto \vec{F} \checkmark \text{ (Experiment)}$$

Define Inertial mass of an object as the constant of proportionality.

- Inertia measures how much an object resists motion

- Mass > 0

$$\vec{F} = \overbrace{M}^{\text{inertia matrix}} \vec{a} \;?$$

Or more simply

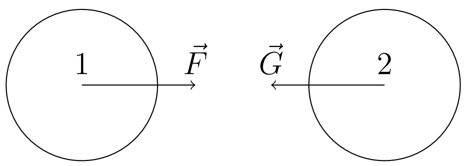
$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} M_1 a_1 \\ M_2 a_2 \\ M_3 a_3 \end{bmatrix}$$

$$\text{so } M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

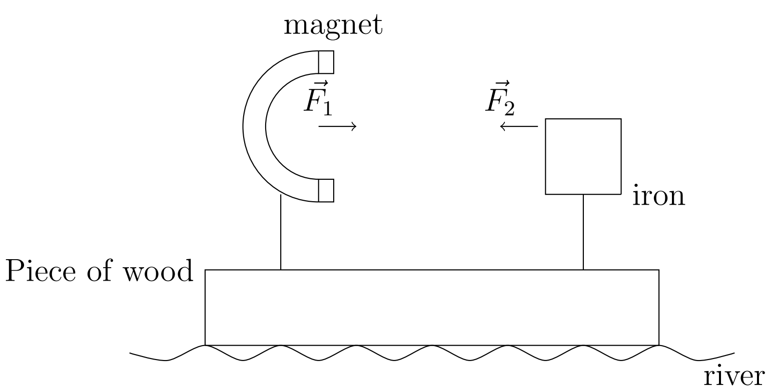
<u>WE DO NOT OBSERVE THIS</u>

Back to the interaction:

11.4 <u>Q:</u> What is the force acting on the other object



We postulate $\vec{F} = -\vec{G}$
$\rightarrow$ simplest assumption consistent with observation that the same objects move symmetrically after an interaction when subjected to the same initial conditions.
$\rightarrow$ Experiment:



$\rightsquigarrow$ it does not move $\rightsquigarrow$ $\vec{F}_1 = -\vec{F}_2$

11.5 Q: What is the natural state of motion?

$\rightsquigarrow$ to be compatible with

$$\vec{F} = M\vec{a}$$

We need a natural state if and only if no interaction, if and only if $\vec{F} = 0$

$$\Rightarrow \vec{a} = 0 \ \vec{v} = \text{constant}$$

a natural state has to be uniform motion with constant velocity.
Problems:

1. Velocity is relative(dependent on frame of reference) but now acceleration is absolute(independent(not true for all reference frames)of frame of reference).

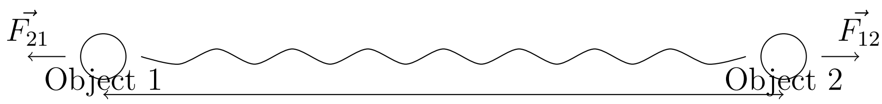
2. I observe objects being naturally at rest! so $\vec{v} = 0$!

Galileo's objection to 2:
$\rightarrow$ Ship argument: you cannot tell (with physical experiment) whether you are at rest or moving with a constant velocity. (Galileo relativity of velocity).
Consequence:
$\rightarrow$ We have to explain why things are at rest despite force acting $\rightsquigarrow$(balanced forces).
$\rightarrow$ this makes our explanations more complicated.
But: can explain celestial and terrestrial motion in one theory.

12 Physics of Motion

1. Interaction $\rightsquigarrow$ changes of motion(state:velocity,change:acceleration) $\rightsquigarrow$ acceleration.

2. Interaction



break down interaction into 2 parts, individual to each object.

3. Mathematical model of force?
Observe:

- magnitude(strength of interaction)
- directions

First guess - vector model $\rightsquigarrow$ needs more justification!
Experiments with springs to test vector character ✓

4. Guess physical laws of forces

   - Newton's 3rd law: $\vec{F_{12}} = -\vec{F_{21}}$ Experiment: Magnet on a boat ✓.
   - Link force and acceleration:

   $$\rightarrow \vec{F} \parallel \vec{a}$$
   $$\rightarrow |\vec{F}| \uparrow \Rightarrow |\vec{a}| \uparrow$$
   (constraints from observations)

   - Simplest link:

   $$\vec{a} = k\vec{F} \; k > 0$$

   (but also others that need to be refuted by experiments).
   Experiment:✓ - e.g. ball rolling down a ramp
   define $m = \frac{1}{k}$ (inertial mass)
   (Physics of inertia?!)

   $$\vec{F} = m\vec{a}$$

   Problem!

   - $\vec{F}$ (interaction) is absolute.
   - $\vec{a}$ (acceleration) is relative to frame of reference.

   Q: Is there accelerated motion without a force?
   $\rightarrow$ No for point-particles?$\leftarrow$ justify!
   Newton's 1st Law: $\vec{F} = 0 \Rightarrow \vec{a} = 0 \Rightarrow \vec{v} =$ constant (law of inertia).
   For Newton's first law to hold you need a special frame of reference (inertial frame of reference where the law of inertia holds).
   Experiment: Foucault has shown that the Earth is not inertial.

Problem

Absolutist interpretation:
(Newton)

- → there is absolute space and absolute time.
- → acceleration has to be taken as relative to absolute space.

Relativist interpretation:
(dominating today due to Einstein)

- → there is at least one(and infinitely many) inertial forms of reference.
- → Newton's laws are only valid in an inertial reference frame.