Synthesis of Feedback Controls Using Optimization Theory—An Example*

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Summary—**This paper illustrates the use of optimization theory in the synthesis of a linear time-varying feedback control by carrying out the design of an aircraft landing system. The optimization method employed is the Parametric Expansion Method.**

A number of different controls are synthesized by selecting different functional forms for the weighting factors appearing in the error index formulated from the performance requirements. These controls are compared by presenting the landing trajectories of the aircraft.

I. Introduction

Many CONTROL problems are characterized by the necessity for satisfying multiple performance requirements and constraints. In such cases, the synthesis of the control by conventional trial-and-error methods may prove exceedingly difficult. If, in addition, the performance requirements are such that the need for a time-varying control is indicated, the control engineer is faced with the difficult task of selecting not only the configuration of the control but also the values of its time-varying parameters or gains. For such problems, synthesis methods based on optimization theory offer distinct advantages over trial-and-error methods. In particular, the selection of the time-varying parameters or gains is replaced essentially by the selection of constant parameters which are the magnitudes of the weighting factors. These weighting factors appear in an error index formulated from the performance requirements. In addition, the absolute stability of the system is not a consideration in the selection of the weighting factors, whereas, in the selection of time-varying gains by conventional methods, consideration of absolute stability would assume primary importance.

This paper is intended to be tutorial in nature and its purpose is to illustrate the use of optimization theory in the synthesis of a linear time-varying feedback control for a process which is characterized by difficult dynamics and is subjected to multiple performance requirements and constraints. This is accomplished by carrying out the design of an aircraft landing system using the Parametric Expansion Method.[1–3] Included in this paper are a definition of the aircraft landing problem, a development of a suitable linear aircraft model, a specification of the performance requirements and constraints, the formulation of a mathematical error index, a brief review of the basic concepts of the Parametric Expansion Method, the synthesis of a number of different aircraft landing systems, and finally a brief description of one possible mechanization of the most suitable system.

II. Aircraft Landing Problem Definition

The landing of an aircraft consists of several phases. First, the aircraft is guided toward the airport with approximately the correct heading by RDF equipment. Within a few miles of the airport, radio contact is made with the radio beam of the instrument landing system (ILS). In following this beam, the pilot guides the aircraft along a glide path angle of approximately $-3°$ toward the runway. Finally, at an altitude of approximately 100 feet, the flare-out phase of the landing begins. During this final phase of the landing, the ILS radio beam is no longer effective for guiding the aircraft due to electromagnetic disturbances. Nor is the $-3°$ glide path angle particularly desirable from the viewpoint of safety and comfort. Hence, the pilot must guide the aircraft along the desired flare-path by making visual contact with the ground.

The landing problem described here is concerned only with the final phase of the landing, that is, with the last 100 feet of the aircraft's descent. It is assumed that the aircraft is guided to the proper location by air traffic control, and that its altitude and rate of ascent at the beginning of this flare-out phase may range from 80 to 120 feet and -16 feet/sec to -24 feet/sec, respectively. For values outside of this range, it is assumed that the aircraft is waved off. Finally, it is assumed that only the longitudinal motion (motion in a vertical plane) need be considered in this final phase of the landing. Lateral motion of the aircraft is required primarily to point the aircraft down the runway. For the most part, this lateral motion is accomplished prior to the final flare-out phase of the landing.

During the flare-out, the aircraft may be subjected to both steady winds and wind gusts. Wind gusts are of primary importance since they tend to be random. On the other hand, steady winds parallel to the ground can be counteracted by merely a steady-state change in the heading of the aircraft. In this problem it is assumed that the wind gusts are zero. The wind disturbances could be treated, but this would require a discussion of the statistical formulation of the Parametric Expansion Method which is beyond the scope of this paper.

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[1] C. W. Merriam, III, "Use of a mathematical error criterion in the design of adaptive control systems," *Trans. AIEE*, vol. 78 (*Appl. and Ind., pt. II*), pp. 506–512; January, 1960.

[2] C. W. Merriam, III, "An optimization theory for feedback control system design," *Information and Control*, vol. 3, pp. 32–59; March, 1960.

[3] C. W. Merriam, III, "Synthesis of Adaptive Controls," Sc.D. dissertation, Massachusetts Institute of Technology, Cambridge; May, 1958.

III. Aircraft Equations of Motion

A first step in the synthesis of a control system is the development of a suitable mathematical description of the process or plant to be controlled. This description may take the form of a differential equation of some order, a set of first-order differential equations, or a transfer function. In this landing problem, a description of the aircraft relating the response (output) variables, measurable state variables, and control (input) variables is necessary.

In the past, a great deal of attention has been directed toward the development of the equations of motion of an aircraft.[4] This development proceeds from a consideration of the aerodynamic forces and moments, and from the application of the fundamental laws of mechanics. The resulting equations are then linearized based on the assumption that the deviation from the equilibrium flight condition is small. In addition, the assumption often is made that the glide path angle γ is sufficiently small so that the small angle approximations, $\sin\gamma=\gamma$ and $\cos\gamma=1$ can be made. This assumption is valid in this case due to the landing geometry. Finally, it is assumed for this problem that the velocity V of the aircraft is maintained essentially constant during the landing by utilizing throttle control. Thus, the longitudinal motion of the aircraft is governed entirely by the elevator deflection $\delta_e(t)$ and this becomes the only control signal. The use of these assumptions leads to the so-called short period equations of motion of the aircraft. These equations can be rewritten in terms of the following transfer function relating elevator deflection, δ_e, and pitch rate, θ'.

$$\theta'(s) = \frac{K_s(T_s s + 1)}{\left(\dfrac{s^2}{\omega_s^2} + \dfrac{2\zeta s}{\omega_s} + 1\right)} \delta_e(s). \tag{1}$$

Throughout this paper, the prime (′) is used to denote the time derivative of the variable to the left of the prime. Thus, in (1) the symbol $\theta'(s)$ denotes the Laplace Transform of the pitch rate. The pitch angle θ and the stability axes x and z of the aircraft are defined in Fig. 1.

The aircraft parameters K_s, T_s, ω_s, and ζ often are referred to as follows:

K_s = short period gain,
T_s = path time constant,
ω_s = short period resonant frequency,
ζ = short period damping factor.

These parameters are assumed to be time invariant and the numerical values used in this study are:

$K_s = -0.95\ \text{sec}^{-1}$,
$T_s = 2.5$ sec,
$\omega_s = 1$ radian/sec,
$\zeta = 0.5$.

[4] C. D. Perkins and R. E. Hage, "Airplane Performance Stability and Control," John Wiley and Sons, Inc., New York, N. Y., pp. 374–407; 1949.

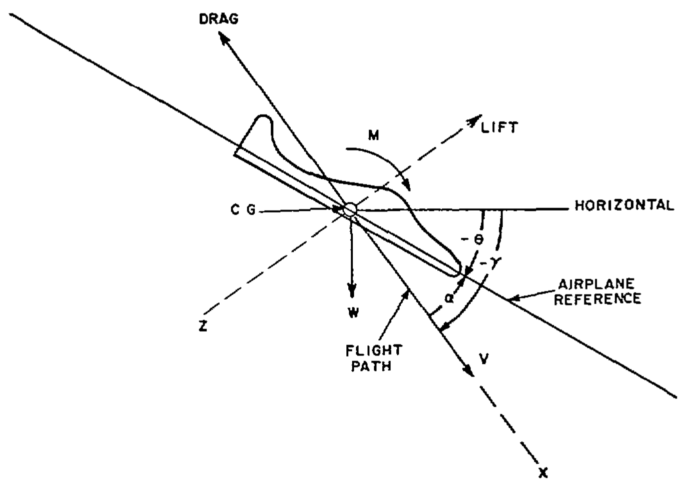


Fig. 1—Definition of aircraft coordinates and angles.

To complete the description of the aircraft, an equation relating pitch rate θ' and altitude, or a time derivative of altitude, is required. Such a relationship exists between vertical acceleration and pitch rate and is given by

$$h''(s) = \frac{V}{(T_s s + 1)} \theta'(s). \tag{2}$$

For this problem, the velocity V of the aircraft is assumed to be constant during the flare-out at a value of 256 ft/sec. Noting that

$$h(s) = \frac{1}{s^2} h''(s) \tag{3}$$

the over-all aircraft transfer function relating altitude and elevator deflection is given by

$$h(s) = \frac{K_s V}{s^2\left(\dfrac{s^2}{\omega_s^2} + \dfrac{2\zeta s}{\omega_s} + 1\right)} \delta_e(s). \tag{4}$$

This relationship also can be written in terms of a fourth-order linear differential equation as follows

$$\frac{d^4h(t)}{dt^4} + 2\zeta\omega_s\frac{d^3h(t)}{dt^3} + \omega_s^2\frac{d^2h(t)}{dt^2} = K_s V \omega_s^2 \delta_e(t). \tag{5}$$

Eq. 5 describes the behavior of the aircraft in terms of the four state signals $h(t)$, $h'(t)$, $h''(t)$, $h'''(t)$. This fact can be demonstrated by drawing from this equation a block diagram of the aircraft containing four integrators, with $\delta_e(t)$ as the input to the diagram and $h(t)$ as the output. The outputs of the four integrators are the state signals. The derivative $h''''(t)$ is not a state signal; it appears as the input to the first integrator in the block diagram.

Because the state signals ultimately appear as measured variables in the feedback portion of the optimum control system, the state signals selected to describe the process must be measurable with available instrumentation in order to construct the system. For instance, as indicated above, a complete set of state signals for the aircraft is $h(t)$, $h'(t)$, $h''(t)$ and $h'''(t)$. The altitude $h(t)$ can be measured wih a radar altimeter, and the rate-of-ascent $h'(t)$ can be measured with a barometric-rate-meter. However, the derivative of acceleration $h'''(t)$ is not readily measurable. Another complete set of state signals is $h(t)$, $h'(t)$, $\theta(t)$, $\theta'(t)$. Pitch angle $\theta(t)$ and pitch rate $\theta'(t)$ are readily measurable with gyros. Hence this second set of state signals is preferable. To use this set, the relationships between $\theta'(t)$, $\theta(t)$, $h'''(t)$, and $h''(t)$ must be utilized in order to obtain from (5) an equation in terms of the new state signals. The equations relating $h''(t)$, $h'''(t)$, $\theta'(t)$, and $\theta(t)$ are found from (2) to be

$$h''(t) = \frac{V}{T_s}\theta(t) - \frac{1}{T_s}h'(t) \tag{6}$$

$$h'''(t) = \frac{V}{T_s}\theta'(t) - \frac{1}{T_s}h''(t) \tag{7}$$

$$h''''(t) = \frac{V}{T_s}\theta''(t) - \frac{1}{T_s}h'''(t). \tag{8}$$

Substituting (6), (7), and (8) into (5) gives

$$\begin{aligned}\frac{V}{T_s}\theta''(t) &- \left(\frac{V}{T_s^2} - 2\zeta\omega_s\frac{V}{T_s}\right)\theta'(t) \\ &- \left(2\zeta\omega_s\frac{V}{T_s^2} - \omega_s^2\frac{V}{T_s} - \frac{V}{T_s^3}\right)\theta(t) \\ &- \left(\frac{1}{T_s^3} - 2\zeta\omega_s\frac{1}{T_s^2} + \frac{\omega_s^2}{T_s}\right)h'(t) \\ &= \omega_s^2K_sV\delta_e(t).\end{aligned} \tag{9}$$

Eq. 9 also is a valid description of the behavior of the aircraft.

A block diagram for the aircraft may be drawn from (6) and (9). This is shown in Fig. 2. Since the aircraft description is of fourth order, this diagram contains four integrators, and the output of each integrator is a state signal.

In carrying out the synthesis of the system using optimization theory, it is convenient to describe the behavior of the aircraft in terms of four first-order differential equations rather than a single higher order differential equation. This is accomplished using (6) and (9) by assigning new symbols to the four state signals and the control signal. Thus, letting

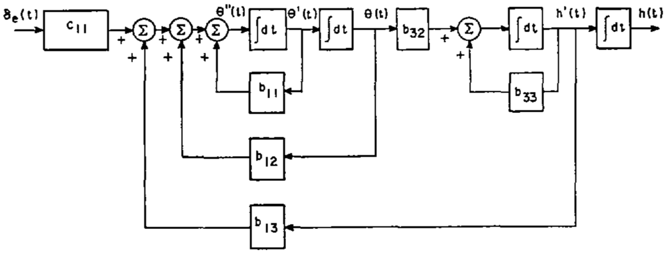
$$x_1(t) = \theta'(t),\ \ x_2(t) = \theta(t),\ \ x_3(t) = h'(t),\ \ x_4(t) = h(t) \tag{10}$$

and

$$m_1(t) = \delta_e(t) \tag{11}$$

the motion of the aircraft is described by

$$\begin{aligned}x_1'(t) &= b_{11}x_1(t) + b_{12}x_2(t) + b_{13}x_3(t) + c_{11}m_1(t) \\ x_2'(t) &= x_1(t) \\ x_3'(t) &= b_{32}x_2(t) + b_{33}x_3(t) \\ x_4'(t) &= x_3(t).\end{aligned} \tag{12}$$



$b_{11} = \frac{1}{T_s} - 2\zeta\omega_s$ $\qquad b_{32} = \frac{V}{T_s}$

$b_{12} = \frac{2\zeta\omega_s}{T_s} - \omega_s^2 - \frac{1}{T_s^2}$ $\qquad b_{33} = -\frac{1}{T_s}$

$b_{13} = \frac{1}{VT_s^2} - \frac{2\zeta\omega_s}{VT_s} + \frac{\omega_s^2}{V}$ $\qquad c_{11} = \omega_s^2K_sT_s$

Fig. 2—Aircraft block diagram in terms of measurable state signals.

Eq. 12 may be written as a single equation using matrix notation by letting

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \equiv \text{state signal vector} \tag{13}$$

$$\boldsymbol{m}(t) = [m_1(t)] \equiv \text{control signal vector} \tag{14}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \equiv \text{“}B\text{” matrix} \tag{15}$$

$$C = \begin{bmatrix} c_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix} \equiv \text{“}C\text{” matrix.} \tag{16}$$

Thus, (12) becomes

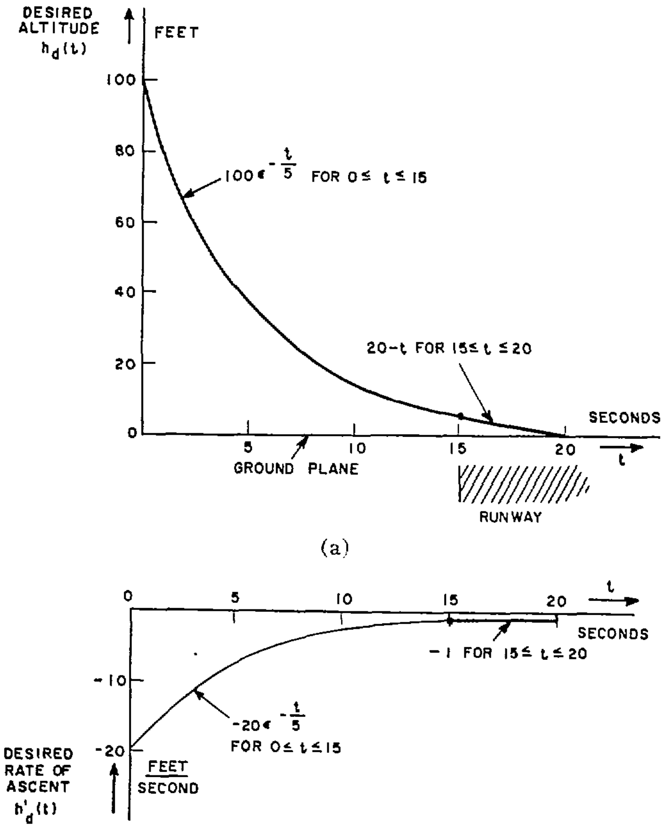
$$\mathbf{x}'(t) = B\mathbf{x}(t) + C\boldsymbol{m}(t). \tag{17}$$

Some authors refer to this as the "equation of state" of the dynamic process. For this aircraft landing problem, it completely describes the behavior of the aircraft in response to the control signal. In the more general case, an additional term usually appears in this equation to include the effects of disturbances on the response of the process.

IV. Performance Requirements and Constraints

An aircraft landing system is satisfactory only if certain prescribed performance requirements and constraints are satisfied. Often these are described in terms of desired response signals, desired control signals, and in terms of limits on these signals. The following requirements and constraints are considered to be of primary importance and are treated in this problem.

1) The desired altitude $h_d(t)$ of the aircraft at each instant of time during the landing is described by the flare-path shown in Fig. 3(a). It consists of an exponen-



(a)

(b)

(c) Pitch angle at touchdown: $0° \leq \theta(T) \leq 10°$.
(d) Angle of attack: $\alpha(t) \leq 18°$.
(e) Elevator deflection: $-35° \leq \delta_e(t) \leq +15°$.

Fig. 3—Performance requirements and constraints.

tial function of time followed by a linear function. This desired path is described by

$$h_d(t) \atop \text{(feet)} = \begin{cases} 100\epsilon^{-t/5}, & 0 \leqq t \leqq 15. \\ 20 - t, & 15 \leqq t \leqq 20. \end{cases} \tag{18}$$

An exponential-linear path of this form ensures a safe and comfortable landing. The desired duration of the flare-out is 20 sec, including 5 sec over the runway. This value is appropriate for an aircraft flying at 175 mph and beginning to flare-out at an altitude of 100 feet.

2) The desired rate of ascent $h_d'(t)$ of the aircraft is given by the time derivative of (18) and is shown in Fig. 3(b). That is,

$$h_d'(t) \atop \text{(feet/sec)} = \begin{cases} -20\epsilon^{-t/5}, & 0 \leqq t \leqq 15 \\ -1, & 15 \leqq t \leqq 20. \end{cases} \tag{19}$$

The magnitude of the rate of ascent of the aircraft is important primarily at touchdown. A value other than zero is desirable to prevent the aircraft from floating down the runway and, hence, perhaps overshooting it. A very large negative value for the rate of ascent may overstress the landing gear and, thus, is equally undesirable. The value of -1 foot/sec at touchdown given in (19) is equal to -60 feet/minute which is well below the maximum permissible value for a modern aircraft.

3) The pitch angle $\theta(t)$ of the aircraft at the desired touchdown time $t = T$ must lie between 0° and +10°. That is

$$0° \leqq \theta(T) \leqq +10°. \tag{20}$$

The lower limit is necessary to prevent the nose wheel of an aircraft with a tricycle landing gear from touching down first. The upper limit on the pitch angle is required to prevent the tail gear from touching down first.

4) During the landing, the angle of attack $\alpha(t)$ must remain below the stall value. For this problem, the stall value was assumed to be +18°. The aircraft enters the flare-out phase in equilibrium with an angle of attack of approximately 80 per cent of the stall value. Hence, the permissible positive increment in the angle of attack during the flare-out is equal to 20 per cent of the stall value. Thus, the restrictions on the angle of attack are

$$\alpha(t) < 18° \tag{21}$$

$$\Delta\alpha(t) < 3.6°. \tag{22}$$

5) The elevator, which controls the longitudinal behavior of the aircraft, is restricted to motion between mechanical stops. For this problem it is assumed that these stops exist at $-35°$ and $+15°$ (elevator trailing edge down). Hence, during the flare-out, the elevator deflection $\delta_e(t)$ is constrained between these two physically limiting values. That is,

$$-35° \leqq \delta_e(t) \leqq +15°. \tag{23}$$

For a linear controller, the elevator is permitted to attain these limiting values instantaneously but not to exceed them. That is, for linear operation, the elevator is not permitted to operate against the mechanical stops for nonzero periods of time during the landing. In the synthesis of this aircraft landing system, saturation effects are not treated.

V. Mathematical Error Index

An important step in the synthesis of a control system by using optimization theory is the formulation of a mathematical error index. The error index is important because it, to a large degree, determines the nature of the resulting optimum control. That is, the resulting control may be linear, nonlinear, stationary, or time-varying depending on the form of the error index. Thus, the control engineer is able to influence the nature of the resulting system by the manner in which he formulates this index based on the requirements of the problem at hand. In general, these requirements may include not only the performance requirements but also any restrictions on the form of the optimum control to ensure physical realizability.

A first step in the formulation of an error index is the establishment of an instantaneous error measure which takes into account all of the important performance requirements and constraints. In general, the instantaneous error measure consists of the sum of the weighted response signal and control signal errors. In this landing problem, the important response signal errors are: the deviation of the aircraft altitude from the desired altitude given in (18), the deviation of the aircraft rate of ascent from the desired rate of ascent given in (19), the deviation of the pitch angle at touchdown from the desired value of $+2°$ which is within the prescribed range given in (20), and the deviation of the angle of attack from its initial value at $t=0$. In addition, the pitch rate of the aircraft may be important. The aircraft equations of motion indicate that the pitch rate of the aircraft governs the magnitude of the angle of attack since the pitch rate term enters the angle of attack equation as the driving function. Hence, it is possible to control the magnitudes of both the angle of attack and the pitch rate by introducing a weighted pitch-rate-error term into the instantaneous error measure. Of course, it also is possible to include both pitch rate and angle of attack error terms in this measure. However, this requires the selection of two weighting factors instead of one. In this problem, the use of only the weighted pitch-rate-error term was found to be completely satisfactory for controlling the magnitudes of both the angle of attack and the pitch rate. The desired pitch rate $\theta_d'(t)$ was selected to be zero. The final term in the instantaneous error measure is the control signal error, or, in this case, the elevator deflection error. To maintain the elevator deflection within the prescribed limits it is convenient to select the value of the desired control signal to be zero. Thus, for the landing problem an appropriate instantaneous error measure is

$$\begin{aligned} e_m(t) = {} & \phi_h(t)[h_d(t) - h(t)]^2 + \phi_{h'}(t)[h_d'(t) - h'(t)]^2 \\ & + \phi_\theta(t)[\theta_d(t) - \theta(t)]^2 + \phi_{\theta'}(t)[\theta_d'(t) - \theta'(t)]^2 \\ & + [\delta_e(t)]^2 \end{aligned} \tag{24}$$

where $\phi_h(t)$, $\phi_{h'}(t)$, $\phi_\theta(t)$ and $\phi_{\theta'}(t)$ are the time-varying weighting factors which indicate the relative importance of the various terms in the error measure. In carrying out the design of the optimum control, these weighting factors must be selected such that the performance requirements and constraints are satisfied.

As already mentioned, the error measure may assume a variety of forms depending on the nature of the problem to be solved and the requirements to be met. For this landing problem, a weighted quadratic error measure was selected because it was felt that this form would insure satisfactory system behavior. In addition, the use of a weighted quadratic error measure in conjunction with the linear aircraft description results in a linear control system with feedback and feed-forward gains which are not a function of the state of the aircraft but merely functions of the time-to-go before touchdown. Such a system is relatively simple to implement physically and is a practical airborne landing system.

The control engineer is interested not so much in an instantaneous value of the error measure as he is in the cumulative effect of this instantaneous measure throughout an interval of time. Hence, the error index $e(t)$ often is expressed as the time integral of the error measure over a suitable future interval of time, t to T, throughout which the system performance is of interest. That is

$$e(t) = \int_t^T e_m(\sigma)d\sigma \tag{25}$$

where σ is a dummy time variable. The lower limit of integration $\sigma=t$ is present or real time. For the landing problem, the future interval of time which is of importance is the time remaining before touchdown occurs. Hence, the upper limit of integration $\sigma=T$ is set equal to 20 sec. The error index then can be written,

$$\begin{aligned} e(t) = \int_t^{20} \{ & \phi_h(\sigma)[h_d(\sigma) - h(\sigma)]^2 + \phi_{h'}(\sigma)[h_d'(\sigma) - h'(\sigma)]^2 \\ & + \phi_\theta(\sigma)[\theta_d(\sigma) - \theta(\sigma)]^2 + \phi_{\theta'}(\sigma)[\theta_d'(\sigma) - \theta'(\sigma)]^2 \\ & + [\delta_e(\sigma)]^2 \} d\sigma. \end{aligned} \tag{26}$$

The performance requirements indicate that the altitude and rate of ascent errors should be small at the desired touchdown point $\sigma=20$ to insure actual touchdown very close to this point. Large altitude and rate of ascent errors at $\sigma=20$ may result in touchdown prior to the start of the runway, or so far down the runway that the aircraft cannot be brought to a stop before the end of the runway. Hence, impulse weighting of these errors at $\sigma=20$ in addition to weighting of the errors during the flare-out appears to be desirable. Thus, $\phi_h(\sigma)$ and $\phi_{h'}(\sigma)$ can be written

$$\phi_h(\sigma) = \phi_4(\sigma) + \phi_{4,T}u_0(20 - \sigma) \tag{27}$$

and

$$\phi_{h'}(\sigma) = \phi_3(\sigma) + \phi_{3,T}u_0(20 - \sigma). \tag{28}$$

Here $u_0(20-\sigma)$ is an impulse of unit area occurring at $\sigma=20$ and $\phi_{4,T}$ and $\phi_{3,T}$ are the areas of the impulse functions. The selection of the time functions $\phi_4(\sigma)$ and $\phi_3(\sigma)$ is treated in a later paragraph.

The performance requirements indicate that the pitch angle error is important only at the touchdown point. Hence, the functional form of the weighting factor $\phi_\theta(\sigma)$ should be selected such that only the pitch angle error at the desired touchdown point contributes to the value of the error index. An impulse function exhibits the required form. Thus, $\phi_\theta(\sigma)$ can be written

$$\phi_\theta(\sigma) = \phi_{2,T}u_0(20 - \sigma). \tag{29}$$

Since the angle of attack and the pitch rate are important throughout the entire landing interval, $\phi_{\theta'}(\sigma)$ is written

$$\phi_{\theta'}(\sigma) = \phi_1(\sigma). \tag{30}$$

The performance requirements suggest the functional forms which $\phi_4(\sigma)$, $\phi_3(\sigma)$, and $\phi_1(\sigma)$ should assume. Since altitude and pitch rate errors are important throughout the flare-out, it is reasonable to make these weighting factors constants. On the other hand the rate of ascent errors are not of primary interest prior to the start of the runway. Thus, $\phi_3(\sigma)$ should be zero prior to the start of the runway and a constant over the runway. The resulting expressions for these weighting factors are

$$\phi_4(\sigma) = \phi_4 \tag{31}$$

$$\phi_3(\sigma) = \begin{cases} 0, & 0 \leqq \sigma < 15 \\ \phi_3, & 15 \leqq \sigma \leqq 20 \end{cases} \tag{32}$$

$$\phi_1(\sigma) = \phi_1. \tag{33}$$

The selection of the numerical values of ϕ_4, $\phi_{4,T}$, ϕ_3, $\phi_{3,T}$, $\phi_{2,T}$ and ϕ_1, and the effects of these values on the resulting aircraft performance are discussed in Section VII.

VI. System Synthesis

Having formulated the error index, the control engineer endeavors to minimize the value of this index in such a manner that he obtains information about the configuration and parameters of the control system being synthesized. In the Parametric Expansion Method, the error index is minimized with respect to the components of the control signal vector $\boldsymbol{m}(t)$. In general, this leads directly to an expression for each control signal in terms of the measurable state signals and time-varying coefficients. These coefficients can be considered time-varying gains in the optimum control system. This expression for each control signal is referred to as the control law. In the case of the Parametric Expansion Method, this law defines a feedback configuration for the system. As a result, having derived the control law, the block diagram of the optimum system immediately can be drawn. The application of the Parametric Expansion Method to the aircraft landing problem is described in the Appendix. A more general treatment of this subject including the rigorous development of the principles involved may be found in the technical literature.[1–3]

As indicated in the Appendix, the use of the Parametric Expansion Method results in the following expression for the optimum elevator deflection as a function of time

$$\delta_e(t) = \omega_s^2 K_s T_s[k_1(t) - k_{11}(t)\theta'(t) - k_{12}(t)\theta(t) - k_{13}(t)h'(t) - k_{14}(t)h(t)]. \tag{34}$$

This expression defines the configuration of the landing system. The block diagram of the system is drawn with the aid of (34) and is shown in Fig. 4. The optimum system contains four feedback loops with a measured state signal fed back in each loop. The gain in each feedback loop is time-varying. The input to the system also is time-varying.

The use of the Parametric Expansion Method also results in a set of K-equations which, when solved, provide the values of the time-varying input signal and feedback gains of the system. These equations are derived in the Appendix and are given in (57) through (71).

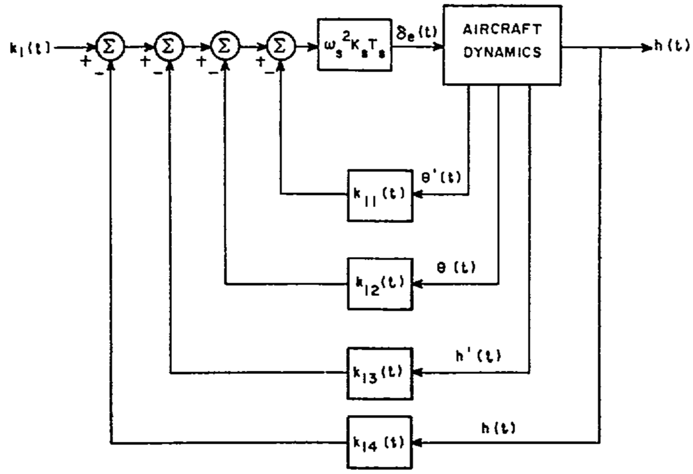


Fig. 4—Block diagram of optimum aircraft landing system.

Thus, this synthesis procedure provides directly the configuration of the optimum system and a set of equations which may be solved to obtain the system parameter values.

VII. Aircraft Landing Systems

As a result of the use of the Parametric Expansion Method, the synthesis procedure is reduced to the solution of the K-equations. In general, the solution of these equations requires the use of an analog or digital computer. The systems described in this paper were synthesized using an IBM 704 digital computer. The computer program consists of two major parts. In the first part, the K-equations are solved. In the second part, the aircraft equations of motion are solved to obtain the actual landing trajectories. This two part program is illustrated in Fig. 5. The integration algorithm used in this study is the Gill modification of the Runge-Kutta fourth-order extrapolation algorithm.[5] However, a number of algorithms possessing similar properties also could be utilized.

As indicated in Fig. 5, the landing trajectories are computed for three sets of four initial conditions. These conditions are tabulated in Fig. 6. It is assumed that the initial vertical acceleration $h''(0)$ is zero and the initial altitude, vertical velocity, and pitch rate are as given in the table. Using these values, the initial pitch angle is computed from the aircraft equations of motion.

Sets 1 and 3 of the initial conditions correspond to the worst possible combinations of the largest permissible altitude and rate of ascent errors at the beginning of the flare-out. That is, in set 1 the aircraft is 20 per cent above the desired landing trajectory and it is descend-

[5] S. Gill, "A process for the step-by-step integration of differential equations in an automatic digital computing machine," *Proc. Camb. Phil. Soc.*, vol. 47, pt. 1, pp. 96–108; 1951.

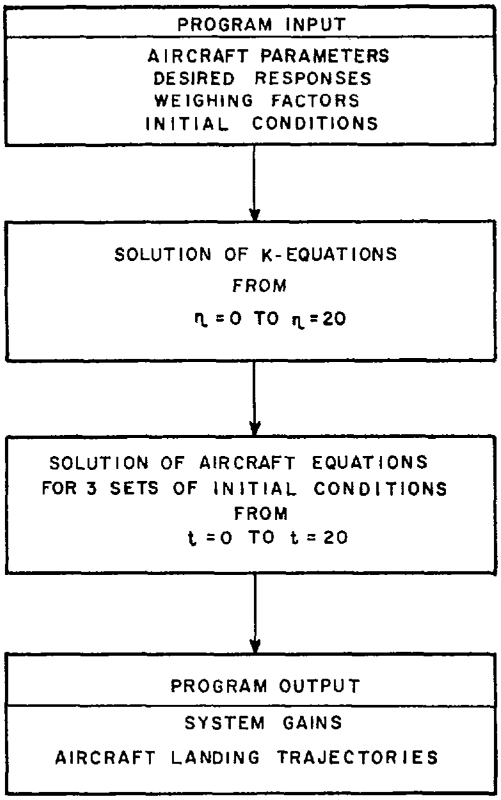


Fig. 5—Simplified flow diagram of digital computer program.

ing 20 per cent slower than desired. Hence, the aircraft is diverging from the desired trajectory at the largest permissible rate. A similar situation occurs in set 3 where the aircraft is below the desired trajectory and is descending at the maximum permissible rate. In set 2 the aircraft is on the desired trajectory and is descending at the rate prescribed by the performance requirements. This wide range in initial conditions serves as a severe test for the aircraft landing system.

As described in the Appendix, the solution of the K-equations cannot be carried out until the desired responses, aircraft parameters, and weighting factors are specified. In the preceding sections, the desired responses and the aircraft parameters have already been specified. Hence, the design procedure reduces to merely the selection of a set of weighting factors which will result in satisfactory system performance for the three sets of initial conditions.

There are two important aspects associated with each weighting factor $\phi(\sigma)$: its functional form $f(\sigma)$ and its magnitude Φ. That is, $\phi(\sigma)$ may be written

$$\phi(\sigma) = \Phi f(\sigma). \tag{35}$$

The functional form $f(\sigma)$ usually is suggested by the performance requirements for the system and, hence, only the magnitude Φ remains to be selected. For example, the pitch angle requirement at touchdown has led to the selection of an impulse function as the functional form of $\phi_\theta(\sigma)$. However, its magnitude $\phi_{2,T}$ is yet to be specified. Thus, the synthesis procedure ultimately is

SET	ALTITUDE h (0) (FT)	VERTICAL VELOCITY h'(0) (FT/SEC)	PITCH ANGLE θ (0) (RAD)	PITCH RATE θ'(0) (RAD/SEC)
1	120	-16	-0.0625	0
2	100	-20	-0.0781	0
3	80	-24	-0.0938	0

Fig. 6—Aircraft initial conditions.

reduced to merely the selection of the magnitudes of a set of weighting factors.

In this paper, three different sets of weighting factors are selected to illustrate the effects on the nature of the resulting system and on the landing trajectories. These three cases are described in the paragraphs that follow.

Case I—Constant Weighting of Altitude Error and Elevator Deflection

In this case, altitude error and elevator deflection are weighted by a constant over the entire 20-sec landing interval, and all other weighting factors are zero. That is,

$$\phi_h(\sigma) = \phi_4 = \text{constant} \tag{36}$$

$$\phi_1 = \phi_{2,T} = \phi_3 = \phi_{3,T} = \phi_{4,T} = 0. \tag{37}$$

The error index becomes

$$e(t) = \int_t^{20} \{\phi_4[h_d(\sigma) - h(\sigma)]^2 + [\delta_e(\sigma)]^2\} d\sigma. \tag{38}$$

The value of ϕ_4 is determined by a consideration of the relative importance of altitude errors and elevator deflection. At the beginning of the flare-out, the largest positive altitude error is 20 feet due to the initial conditions. The landing system will attempt to correct this error by calling for a large positive elevator deflection. The largest permissible positive elevator deflection is 15°, or 0.262 radian. For this case, altitude errors and elevator deflection are assumed of equal importance in the performance of the landing system. Therefore, the two terms in the integrand of the error index of (38) should contribute equally to the value of this index. Hence, the value of ϕ_4 may be calculated by equating these two terms and inserting the above altitude error and elevator deflection values. That is,

$$\phi_4[20]^2 = [0.262]^2 \tag{39}$$

or

$$\phi_4 = 0.000171. \tag{40}$$

The performance of the aircraft for $\phi_4 = 0.0001$ is shown in Fig. 7. This figure contains four graphs. From

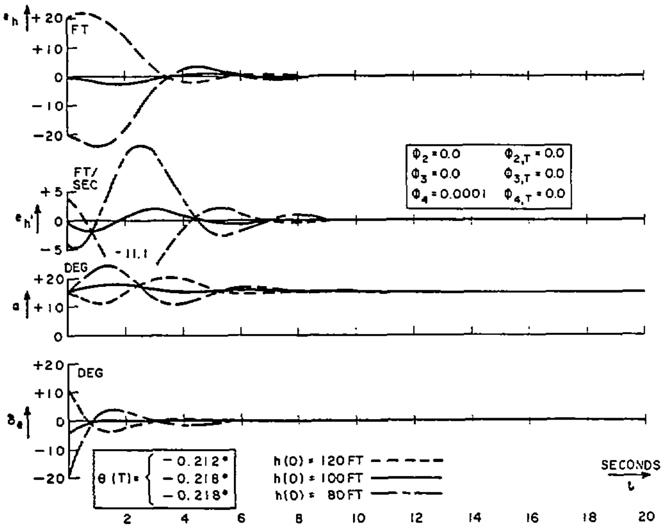


Fig. 7—System performance for Case I.

top to bottom these are:

1) altitude error vs time;
2) rate of ascent error vs time;
3) angle of attack vs time;
4) elevator deflection vs time.

Each graph, in turn, contains three curves, one for each of the three sets of initial conditions. In addition, the values of the pitch angle $\theta(T)$ at $t=20$ sec are tabulated at the bottom of the figure for the three sets of initial conditions. The $\theta(T)$ values correspond, from top to bottom, to the three sets of initial conditions given in Fig. 6. The values of the weighting factors also are listed on Fig. 7.

For this figure, e_h and $e_{h'}$ are given by

$$e_h(t) = [h(t) - h_d(t)](\text{feet}) \tag{41}$$

and

$$e_{h'}(t) = [h'(t) - h'_d(t)](\text{feet/sec}). \tag{42}$$

When the aircraft is at an altitude of 120 feet at the beginning of the flare-out, $e_h = 20$ feet. The rate of ascent of the aircraft, for this same condition, is -16 feet/sec giving $e_{h'} = 4$ feet/sec. Also at $t=0$, the angle of attack α is at its equilibrium value of $+14.4°$ (80 per cent of the stall value) for all three sets of initial conditions.

The set of weighting factors listed in Fig. 7 results in a system which meets the performance requirements on altitude, rate of ascent, and elevator deflection. However, the constraints on angle of attack and pitch angle at touchdown are violated.

Another aspect of the performance is not evident from these figures due to the ordinate scale selected for $e_{h'}$. The rate-of-ascent error at $t=20$ varies substantially with the initial conditions of the aircraft. Hence, the point at which touchdown actually occurs will vary with the initial conditions. This is undesirable as already mentioned. Therefore, for many reasons the simple error index of (38) is not suitable for this problem.

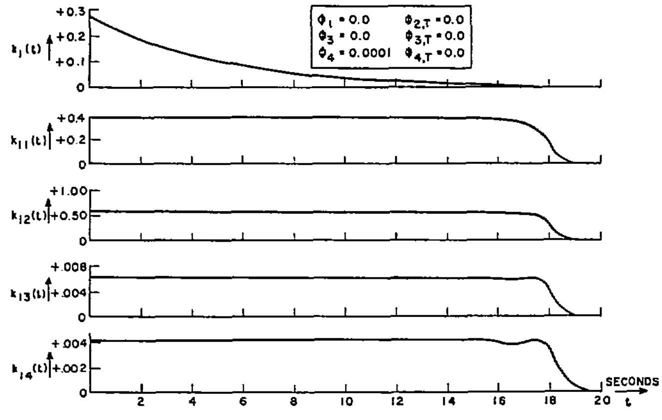


Fig. 8—System gains for Case I.

Before proceeding to another error index, it is worthwhile to consider the feedback gains and the input signal associated with the weighting factors and performance curves of Fig. 7. These gains are shown in Fig. 8. The input signal $k_1(t)$ appears to be similar in shape to the desired altitude $h_d(t)$ although substantially reduced in amplitude. The feedback gains $k_{11}(t)$, $k_{12}(t)$, $k_{13}(t)$ and $k_{14}(t)$ are essentially constant until the last three secs of the landing when they gradually decrease to zero. Hence, just prior to touchdown, the landing system is operating essentially open loop. This should cause no concern since it is not possible to alter significantly the trajectory of the aircraft during this short interval with the finite control effort available due to the lags in the aircraft.

Case II—Impulse Weighting of Altitude Error at Touchdown

A great deal has been written in the literature about so-called terminal control systems.[6–8] Such a system attempts to reduce an error to zero at a future point in time called the terminal point. A system with this property also can be synthesized using optimization theory by selecting the weighting factor associated with this undesirable future error to be an impulse. Hence, if the only performance requirement for this landing system were to reduce the altitude error at the desired touchdown point to the smallest possible value without exceeding the elevator deflection limits, a suitable error index might be

$$e(t) = \int_t^{20} \{\phi_{4,T}u_0(20-\sigma)[h_d(\sigma) - h(\sigma)]^2 + [\delta_e(\sigma)]^2\}d\sigma. \tag{43}$$

[6] M. V. Mathews and C. W. Steeg, "Terminal Controller Synthesis," presented at the Princeton Symp. on Non-Linear Control Systems, Princeton, N. J.; March 26–27, 1956.

[7] R. C. Booton, Jr., "Optimum design of final-value control systems," *Proc. Polytechnic Institute of Brooklyn Symp. on Non-Linear Circuit Analysis*, Brooklyn, N. Y., April 25–27, 1956, Polytechnic Institute of Brooklyn, N. Y.; 1957.

[8] E. A. O'Hearn and R. K. Smyth, "Terminal control system applications," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-6, pp. 142–153; May, 1961.

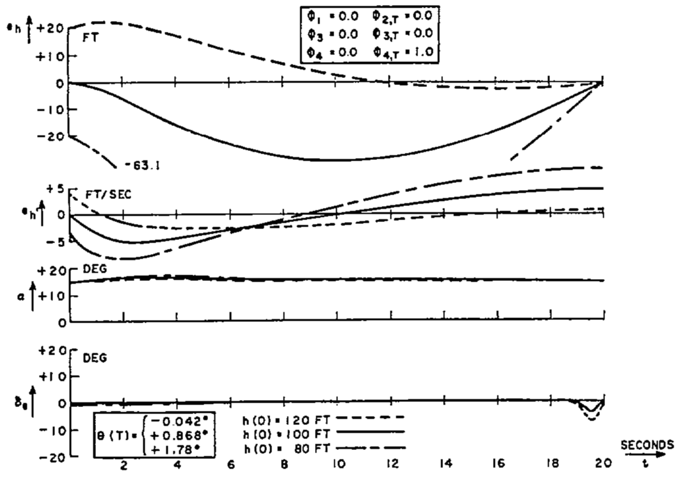


Fig. 9—System performance for Case II.

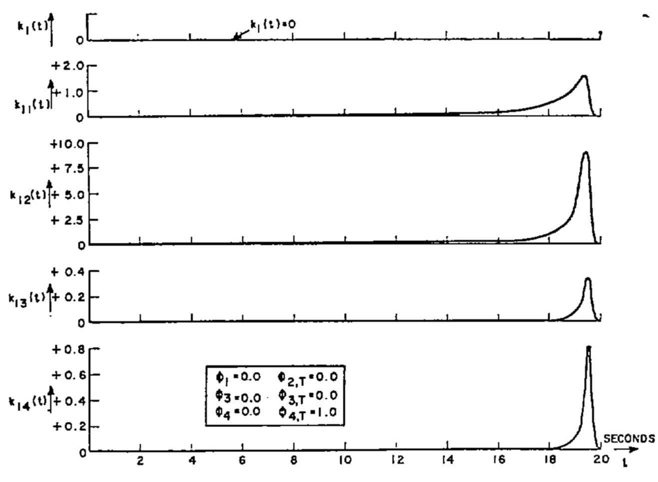


Fig. 10—System gains for Case II.

The resulting performance for $\phi_{4,T}=1.0$ is shown in Fig. 9.

The altitude error at $t=20$ is quite small, but, unfortunately, the aircraft crashes for two sets of initial conditions due to the large altitude errors occurring during the earlier part of the landing interval. In addition, the rate of ascent error is undesirably large at touchdown.

The system input signal and the feedback gains for this case are shown in Fig. 10. The input signal $k_1(t)$ is zero during the entire landing interval as might be expected. The feedback gains are negligibly small until the aircraft begins to approach the desired touchdown point. These gains then become quite large in order to reduce the altitude error to a small value at $t=20$. Since $k_{14}(t)$ essentially is zero for a large part of the landing interval, the altitude feedback loop in essence is open during this time and large altitude errors arise, as already demonstrated.

Case III—Constant Weighting of Altitude Error, Rate of Ascent Error over the Runway, Pitch Rate Error, and Elevator Deflection; Impulse Weighting at Touchdown of Pitch Angle Error, Altitude Error, and Rate of Ascent Error

The error index for this case is

$$\begin{aligned} e(t) = \int_t^{20} \{ &[\phi_4 + \phi_{4,T}u_0(20-\sigma)][h_d(\sigma) - h(\sigma)]^2 \\ &+ [\phi_3(\sigma) + \phi_{3,T}u_0(20-\sigma)][h_d'(\sigma) - h'(\sigma)]^2 \\ &+ \phi_{2,T}u_0(20-\sigma)[\theta_d(\sigma) - \theta(\sigma)]^2 \\ &+ \phi_1[\theta_d'(\sigma) - \theta'(\sigma)]^2 + [\delta_e(\sigma)]^2 \} d\sigma. \end{aligned} \tag{44}$$

Here, some trial-and-error is required to arrive at a suitable set of weighting factors. The values of the weighting factors and the resulting performance are shown in Fig. 11. This system essentially meets all of the performance requirements. The extremely small negative pitch angle at touchdown existing for one set of initial conditions could be avoided by a slight increase in the value of $\phi_{2,T}$.

Fig. 11 indicates that there is very little margin between the performance obtained and the performance requirements for the system. This suggests that the performance requirements are very difficult to satisfy with the given aircraft dynamics and the assumed initial conditions. Unfortunately, a great many control problems are characterized by the need to satisfy a set of very restrictive performance requirements. It is in the solution of such problems that optimization theory is of particular value.

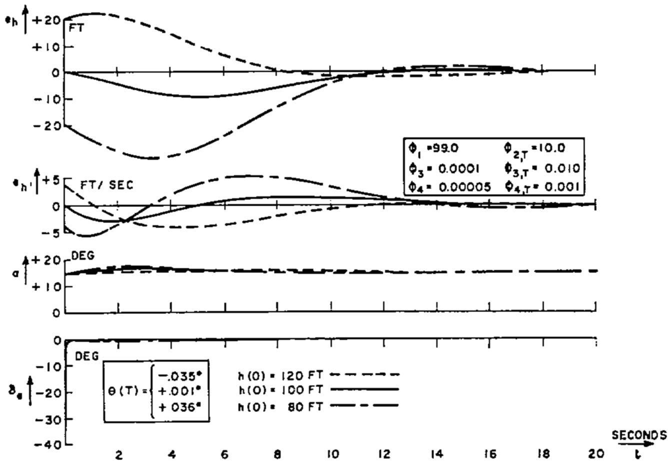


Fig. 11—System performance for Case III.

The input signal and the feedback gains associated with the performance of Fig. 11 are shown in Fig. 12. The peaks in the gains $k_{12}(t)$, $k_{13}(t)$, $k_{14}(t)$ near the touchdown point are due to the use of the impulse weighting factors $\phi_{2,T}$, $\phi_{3,T}$ and $\phi_{4,T}$. It is important to note that in this case the input signal $k_1(t)$ goes negative during the final phase of the landing and becomes positive again just prior to $t=20$. The negative and then positive excursions evidently are required to satisfy the pitch angle constraint at touchdown. It appears that in problems where multiple performance requirements exist, the system input signal, in essence, is a composite function derived from all of the performance requirements.

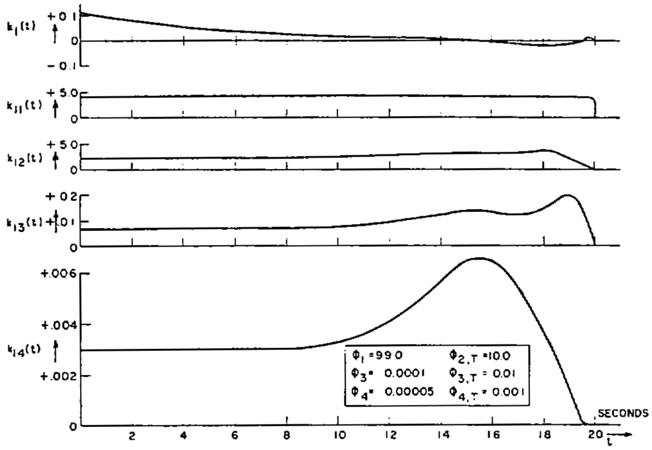


Fig. 12—System gains for Case III.

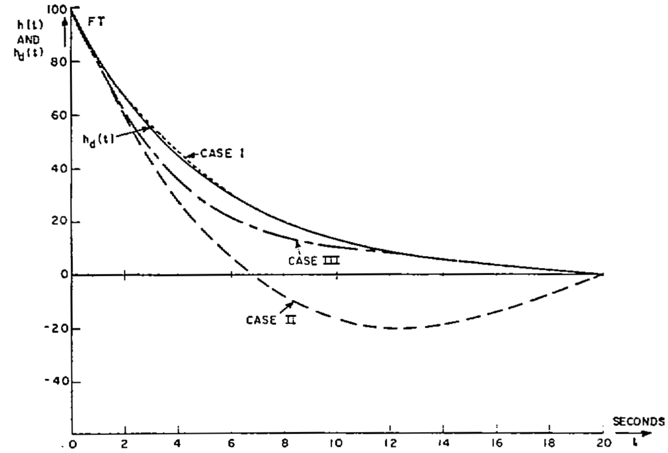


Fig. 14—Aircraft landing trajectories for $h(0)=100$ feet.

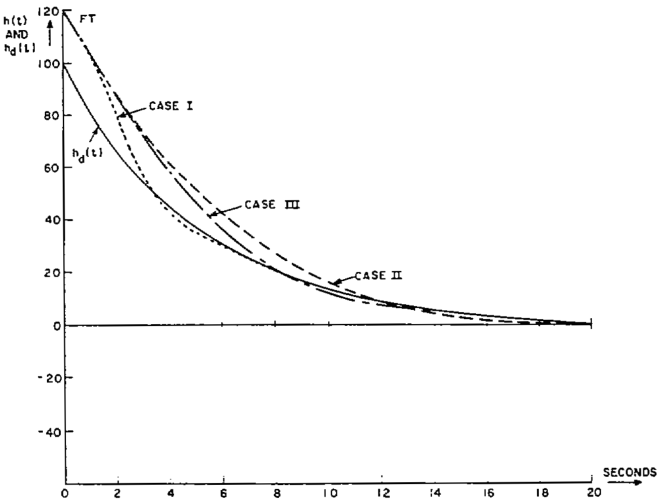


Fig. 13—Aircraft landing trajectories for $h(0)=120$ feet.

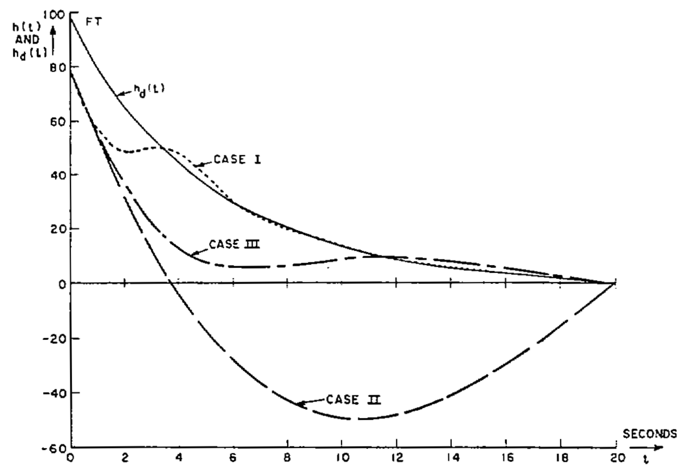


Fig. 15—Aircraft landing trajectories for $h(0)=80$ feet.

The behavior of the systems of Cases I, II, III also may be compared by displaying the aircraft landing trajectories for these cases on three separate graphs, one for each set of initial conditions. These trajectories are shown in Figs. 13, 14 and 15 along with the desired altitude $h_d(t)$.

The system of Case I follows the desired flare-path more closely than do the other systems. This is due primarily to the greater weighting of altitude errors throughout the landing interval in the design of the system of Case I. Of course, it is not evident from these curves alone that other performance requirements are violated by the system of Case I.

As already mentioned, the system of Case II causes the aircraft to crash for two sets of initial conditions and, therefore, is completely unsatisfactory.

The system of Case III meets all of the performance requirements but it takes substantially longer than the system of Case I to correct the initial condition errors. The persistence of the initial altitude errors for a longer period of time is the price one has to pay to satisfy the other performance requirements.

In order to mechanize the aircraft landing system, it is necessary to program the feedback gains and the input signal as functions of time-to-go (20-t). In other words, time-to-go is a quantity that must be available throughout the landing phase in order to construct the control law specified by the theory. Undoubtedly, the measurement of time-to-go is the most difficult measurement associated with the physical construction of the proposed landing system. First of all, the aircraft velocity will not be absolutely constant as has been assumed. However, appropriate thrust control should keep perturbations in the aircraft velocity small enough in order to be considered second order. On the other hand, the measurement of the total aircraft velocity itself is difficult. This aircraft velocity is considered to be the velocity of the aircraft with respect to an inertial frame or rather with respect to the ground. In other words, the aircraft velocity in question is ground speed. Because landing situations with nonzero winds will occur, ground speed cannot be measured from the aircraft itself. Therefore, the success of this landing system requires the successful measurement of ground speed from radar. A number of possible radar systems exist. For instance, a system of ground-based radar beacons could be used in order to estimate the time required to pass over two fixed points on the earth's surface. Another possibility is the use of Doppler radar which is ground based. In this case, the velocity information would be relayed to the aircraft. Due to these practical considerations, the measurement of time-to-go could be obtained from

$$\tau \approx \frac{d(t)}{V(t)} \tag{45}$$

where the time-to-go τ is defined by

$$\tau = 20 - t. \tag{46}$$

In the estimation of time-to-go, the quantity $d(t)$ is defined as the distance from the aircraft to the desired point of touchdown on the runway, and $V(t)$ is the ground speed of the aircraft. Both this distance $d(t)$ and this velocity $V(t)$ could be measured from a system of ground-based radar.

It may be desirable to construct the engineering version of the control system in terms of analog components. Therefore some brief comments concerning the analog construction of the control system are presented here. The block diagram shown in Fig. 16 indicates one possible implementation of the time-varying gains appearing in the control system. If it is assumed that time-to-go τ is available as an angular shaft position, the time-varying multiplications can be realized from a set of loaded potentiometers. Such potentiometers have multiple taps and the appropriate k-parameter functions of time-to-go can be constructed by applying appropriate voltages to the taps on the potentiometer. The signal on the wiper of such a potentiometer would then be the appropriate function of the shaft angle τ times the excitation voltage applied to the loaded pot. The summing and gain functions indicated in Fig. 16 are achieved from standard analog computer components. The shaft angle τ is achieved by placing an input signal τ to a position servo. The input signal τ could be generated from the expression for time-to-go given in (45) by the appropriate division of the distance to the touchdown point by the indicated ground speed of the aircraft.

Another possibility for the generation of the signal τ arises under ideal situations where the ground speed is a known constant and the initial distance to the touchdown point is known. Under these ideal conditions, a simple clock consisting of an integrator with a constant input can be used for the generation of time-to-go. Both of these possibilities for the generation of the signal τ are indicated in Fig. 16. The system indicated in Fig. 16 is highly suited for airborne applications because this system is light, cheap, and reliable. Therefore, we can conclude that if the measurements of the state variables and also time-to-go are readily available and reasonably accurate, the theoretical results of this example form the basis for a practical landing system that exhibits high performance.

If the time-to-go measurement is in error during the initial portion of the flare-out interval but becomes

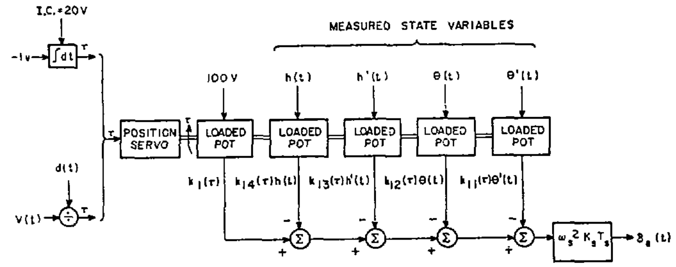


Fig. 16—Mechanization of aircraft landing system.

accurate as the desired touchdown point is approached, the control system described in this paper still is an optimum system. This system treats these measurement errors over a period of time as initial condition errors occurring at the instant the time-to-go measurement becomes accurate. Since this system is optimum for all initial conditions, it is also optimum for this situation.

VIII. Conclusions

A number of important advantages to be gained by the use of optimization theory as the basis for the design of feedback control systems have been pointed out by the example of the aircraft landing system. In particular, multiple design considerations associated with difficult dynamic situations readily can be included in the system design. The functional forms of the weighting factors appearing in the error index are chosen directly from considerations of the time domain performance requirements. Except for certain mathematical restrictions, the functional forms of these weighting factors can be chosen without consideration of the system stability *per se*. However, one of the disadvantages of using optimization theory is that, generally speaking, a unique set of weighting factors does not exist when a set of design considerations is given. In fact, the simpler the design problem from dynamic considerations, the larger the number of sets of acceptable weighting factors.

The design problem chosen for this paper is made difficult because a large number of design specifications must be met in order to consider the results to be successful. It is difficult to meet all of these design considerations because the trajectory requirements of flare-out are somewhat incompatible with the dynamic characteristics of the aircraft. In particular, the trajectory corrections required for passing from the glide phase to the flare-out phase are large in comparison to the amount of control effort allowable during the flare-out portion of the trajectory. Roughly speaking, the amount of control effort is considered to be the magnitude of the control signal times the amount of time during which this control signal can be applied to the aircraft. Optimization theory assumes its most useful purpose under the conditions where the potential response errors are large relative to the amount of control effort which can be applied to the system.

The results of this particular example indicate that the design specifications for the aircraft landing system can be met. The digital computer simulation of the aircraft trajectories shows that under ideal conditions successful automatic aircraft landings can be achieved. However, because a number of engineering questions have been neglected in the study presented here, additional work is required in order to build a practical landing system from the theoretical results presented thus far. In particular, the effects of wind gusts and measurement noise should be included in trajectory studies in order to ascertain their effect on the performance of the system. These statistical disturbances could be included in a digital computer simulation. However, it may be more efficient to perform these additional studies on an analog computer.

Another area which should be investigated before the proposed landing system is considered to be checked out in an engineering sense is the area of sensitivity analysis. In particular, the aircraft parameters assumed in the previous results are only approximations, and therefore the sensitivity of the system trajectories to perturbations around these assumed parameters should be investigated. Also the effects of approximations to the k-parameters should be investigated because the engineering system may only be able to generate these time-varying gains in an approximate sense. This work is being done and will be the subject of a later paper.

Appendix

The Parametric Expansion Method

In this section, the Parametric Expansion Method is applied to the aircraft landing problem. No attempt is made to justify the procedure followed. Rather, the reader is referred to papers in the available technical literature which develop, in a rigorous fashion, the principles employed here.

The first consideration in the minimization of the error index is the derivation of the condition under which the minimum exists. This has been carried out by Bellman in his development of the principles of Dynamic Programming.[9] The condition under which the error index of (26) is a minimum is given by

$$\min_{\delta_e(\mu)} \Big\{ \phi_h(\mu)[h_d(\mu) - h(\mu)]^2 + \phi_{h'}(\mu)[h_d'(\mu) - h'(\mu)]^2 + \phi_\theta(\mu)[\theta_d(\mu) - \theta(\mu)]^2 + \phi_{\theta'}(\mu)[\theta_d'(\mu) - \theta'(\mu)]^2 + [\delta_e(\mu)]^2 + \frac{dE}{d\mu} \Big\} = 0 \tag{47}$$

where $\sigma = \mu$ is an arbitrary instant of time in the interval $\sigma = t$ to $\sigma = T$. Also, E is the minimum value of the error index and is called the minimum error function. Note that the first five terms of (47) are the terms appearing in the integrand of the error index given in (26).

[9] R. E. Bellman, "Dynamic Programming," Princeton University Press, Princeton, N. J.; 1957.

It has been shown in the literature that, for state determined processes, the minimum value of the error index is a function only of the measured state $\mathbf{x}(\mu)$ of the dynamic process (the aircraft) and the time μ at which the measurement is made.[1] Thus, functionally $E = E[\mathbf{x}(\mu), \mu]$. As a result of this functional relationship, the total derivative of E with respect to μ appearing in (47) may be written as follows:

$$\frac{dE}{d\mu} = \frac{dE[\mathbf{x}(\mu), \mu]}{d\mu} = \frac{\partial E[\mathbf{x}(\mu), \mu]}{\partial \mu} + \sum_{n=1}^{N} \frac{dx_n}{d\mu} \frac{\partial E[\mathbf{x}(\mu), \mu]}{\partial x_n} \tag{48}$$

where $N = 4$ because the aircraft description is of fourth order. Also, $dx_n/d\mu$ is a time derivative of a state signal which appears in the dynamic process equations given by (12). Thus, (47) becomes

$$\min_{\delta_e(\mu)} \Big\{ \phi_h(\mu)[h_d(\mu) - h(\mu)]^2 + \phi_{h'}(\mu)[h_d'(\mu) - h'(\mu)]^2 + \phi_\theta(\mu)[\theta_d(\mu) - \theta(\mu)]^2 + \phi_{\theta'}(\mu)[\theta_d'(\mu) - \theta'(\mu)]^2 + [\delta_e(\mu)]^2 + \frac{\partial E}{\partial \mu} + x_1'(\mu)\frac{\partial E}{\partial x_1} + x_2'(\mu)\frac{\partial E}{\partial x_2} + x_3'(\mu)\frac{\partial E}{\partial x_3} + x_4'(\mu)\frac{\partial E}{\partial x_4} \Big\} = 0. \tag{49}$$

Eq. (49) states that when the quantity within the braces is minimized with respect to $\delta_e(\mu)$, its minimum value is zero. In other words, when the optimum control signal $\delta_e^*(\mu)$ is substituted into the terms within the braces, these terms add up to zero. But, the quantity within the braces is a minimum when the partial derivative of this quantity with respect to $\delta_e(\mu)$ is equal to zero. Performing this last operation gives the expression for the optimum control signal $\delta_e^*(\mu)$. This expression is

$$\delta_e^*(\mu) = -\frac{c_{11}}{2}\frac{\partial E}{\partial x_1} \tag{50}$$

since terms 1–4, 6, and 8–10 in (49) are not functions of $\delta_e(\mu)$, and $x_1'(\mu)$ is a function of $\delta_e(\mu)$ as indicated in (11) and (12).

Substituting (12) and then (50) into (49) gives

$$\phi_h(\mu)[h_d(\mu) - h(\mu)]^2 + \phi_{h'}(\mu)[h_d'(\mu) - h'(\mu)]^2 + \phi_\theta(\mu)[\theta_d(\mu) - \theta(\mu)]^2 + \phi_{\theta'}(\mu)[\theta_d'(\mu) - \theta'(\mu)]^2 + \left[-\frac{c_{11}}{2}\frac{\partial E}{\partial x_1}\right]^2 + \frac{\partial E}{\partial \mu} + \left[b_{11}x_1(\mu) + b_{12}x_2(\mu) + b_{13}x_3(\mu) - \frac{c_{11}^2}{2}\frac{\partial E}{\partial x_1}\right]\frac{\partial E}{\partial x_1} + x_1(\mu)\frac{\partial E}{\partial x_2} + [b_{32}x_2(\mu) + b_{33}x_3(\mu)]\frac{\partial E}{\partial x_3} + x_3(\mu)\frac{\partial E}{\partial x_4} = 0. \tag{51}$$

Eq. (51) is a partial differential equation defining the minimum error function E. If this equation can be solved for E, then the optimum control signal $\delta_e^*(\mu)$ can be found using (50). This is the goal of the control engineer.

In the Parametric Expansion Method, the solution of (51) is carried out in a linear case (linear process and weighted quadratic error index) by assuming the following expression for E:

$$E = k(\mu) - 2 \sum_{m=1}^{N} k_m(\mu) x_m(\mu) + \sum_{m=1}^{N} \sum_{p=1}^{N} k_{mp}(\mu) x_m(\mu) x_p(\mu) \tag{52}$$

where $N=4$ due to the fourth-order representation of the aircraft. Also, $k_{mp}(\mu) = k_{pm}(\mu)$. This expansion is called a Parametric Expansion because the coefficients $k_m(\mu)$ and $k_{mp}(\mu)$ turn out to be parameters of the optimum control system. This is demonstrated in the succeeding paragraphs.

The expression for the optimum control signal given by (50) can now be expressed in terms of the state signals and the k-parameters using (52). Taking the partial derivative of E with respect to x_1 and substituting into (50) gives

$$\delta_e^*(\mu) = c_{11}[k_1(\mu) - k_{11}(\mu)x_1(\mu) - k_{12}(\mu)x_2(\mu) - k_{13}(\mu)x_3(\mu) - k_{14}(\mu)x_4(\mu)]. \tag{53}$$

Since μ may take on the value t, which is present or real time, (53) defines the control law for the optimum system to be

$$\delta_e(t) = \omega_s^2 K_s T_s[k_1(t) - k_{11}(t)\theta'(t) - k_{12}(t)\theta(t) - k_{13}(t)h'(t) - k_{14}(t)h(t)] \tag{54}$$

where the expressions for c_{11} and the state signals have been inserted. This control law defines the configuration of the landing system. The block diagram of the system is drawn with the aid of (54) and is shown in Fig. 4. There are a number of important observations to be made from this block diagram:

1) The number of feedback loops is equal to the order of the process (aircraft).
2) The quantities fed back are the measurable state signals.
3) The feedback gains are double subscripted k-parameters.
4) The feedback gains, for this linear case (linear aircraft description and weighted quadratic error index), are merely time-varying and not functions of the state signals.
5) The system input signal is a single subscripted k-parameter.
6) The system input signal, for this linear case, is merely time-varying and not a function of the state signals.

The only remaining information required by the control engineer in order to carry out the mechanization of the system is the values of the time-varying gains. These are found by differentiating (52) to obtain the derivatives of E appearing in (51); then, these derivatives are substituted into (51) and the terms are collected such that the resulting expression is of the following form

$$\begin{aligned} P_0 &+ P_1x_1(\mu) + P_2x_2(\mu) + P_3x_3(\mu) + P_4x_4(\mu) \\ &+ P_{11}x_1^2(\mu) + P_{12}x_1(\mu)x_2(\mu) + P_{13}x_1(\mu)x_3(\mu) \\ &+ P_{14}x_1(\mu)x_4(\mu) + P_{22}x_2^2(\mu) + P_{23}x_2(\mu)x_3(\mu) \\ &+ P_{24}x_2(\mu)x_4(\mu) + P_{33}x_3^2(\mu) + P_{34}x_3(\mu)x_4(\mu) \\ &+ P_{44}x_4^2(\mu) = 0. \end{aligned} \tag{55}$$

The P-coefficients consist of terms containing the k-parameters, the desired response signals, the elements of the B and C matrices which describe the aircraft, and the weighting factors.

If (55) is to be valid for all possible values of the state signals, each P-coefficient independently must be equal to zero. Hence,

$$P_0 = P_m = P_{mp} = 0 \qquad m = 1, 2, \cdots, p;\ p = 1, 2, \cdots, N. \tag{56}$$

Eq. (56) yields a set of $1+N+(N/2)(N+1)$ independent ordinary first-order differential equations which define the k-parameters. These are called the k-equations. For this aircraft landing problem, the fifteen k-equations are

$$\begin{aligned} -k'(\mu) &= \phi_{\theta'}(\mu)[\theta_d'(\mu)]^2 + \phi_\theta(\mu)[\theta_d(\mu)]^2 \\ &\quad + \phi_{h'}(\mu)[h_{d'}(\mu)]^2 + \phi_h(\mu)[h_d(\mu)]^2 \\ &\quad - c_{11}^2k_1^2(\mu) \end{aligned} \tag{57}$$

$$-k_1'(\mu) = \phi_{\theta'}(\mu)\theta_d'(\mu) + b_{11}k_1(\mu) + k_2(\mu) - c_{11}^2k_1(\mu)k_{11}(\mu) \tag{58}$$

$$-k_2'(\mu) = \phi_\theta(\mu)\theta_d(\mu) + b_{12}k_1(\mu) + b_{32}k_3(\mu) - c_{11}^2k_1(\mu)k_{12}(\mu) \tag{59}$$

$$-k_3'(\mu) = \phi_{h'}(\mu)h_d'(\mu) + b_{13}k_1(\mu) + b_{33}k_3(\mu) + k_4(\mu) - c_{11}^2k_1(\mu)k_{13}(\mu) \tag{60}$$

$$-k_4'(\mu) = \phi_h(\mu)h_d(\mu) - c_{11}^2k_1(\mu)k_{14}(\mu) \tag{61}$$

$$-k_{11}'(\mu) = \phi_{\theta'}(\mu) + 2b_{11}k_{11}(\mu) + 2k_{12}(\mu) - c_{11}^2k_{11}^2(\mu) \tag{62}$$

$$-k_{12}'(\mu) = b_{11}k_{12}(\mu) + b_{12}k_{11}(\mu) + k_{22}(\mu) + b_{32}k_{13}(\mu) - c_{11}^2k_{11}(\mu)k_{12}(\mu) \tag{63}$$

$$\begin{aligned} -k_{13}'(\mu) &= b_{11}k_{13}(\mu) + b_{13}k_{11}(\mu) + k_{23}(\mu) \\ &\quad + b_{33}k_{13}(\mu) + k_{14}(\mu) \\ &\quad - c_{11}^2k_{11}(\mu)k_{13}(\mu) \end{aligned} \tag{64}$$

$$-k_{14}'(\mu) = b_{11}k_{14}(\mu) + k_{24}(\mu) - c_{11}^2k_{11}(\mu)k_{14}(\mu) \tag{65}$$

$$-k_{22}'(\mu) = \phi_\theta(\mu) + 2b_{12}k_{12}(\mu) + 2b_{32}k_{23}(\mu) - c_{11}^2k_{12}^2(\mu) \tag{66}$$

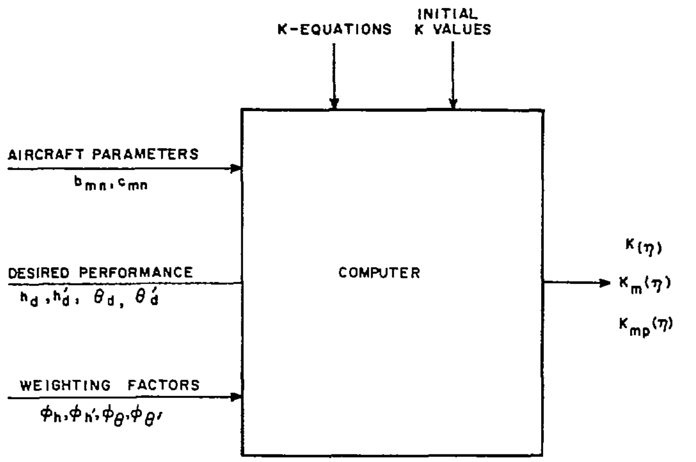


Fig. 17—Solution of the K-equations.

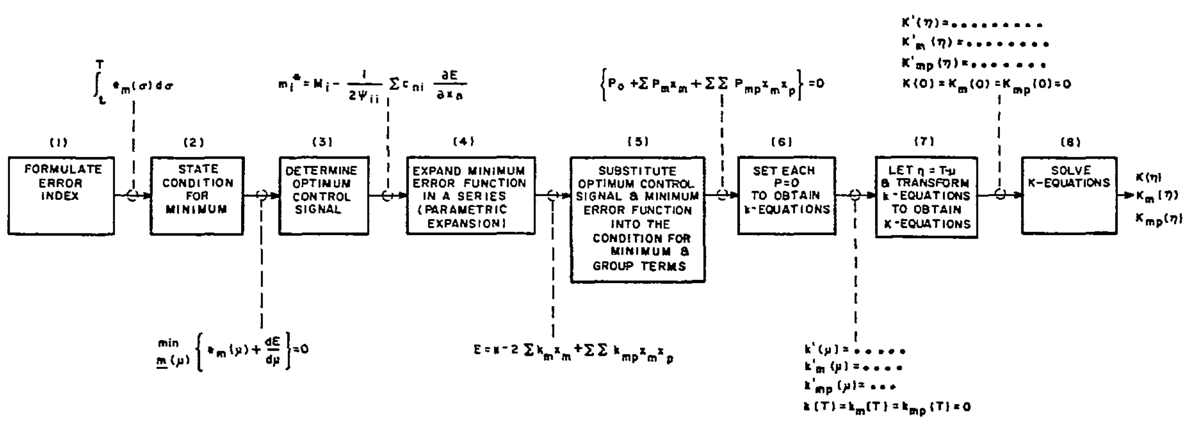


Fig. 18—Block diagram of the steps involved in the use of the Parametric Expansion Method.

$$
\begin{aligned}
-k_{23}'(\mu) &= b_{33}k_{23}(\mu) + b_{12}k_{13}(\mu) + b_{13}k_{12}(\mu) \\
&\quad + b_{32}k_{33}(\mu) + k_{24}(\mu) \\
&\quad - c_{11}^2k_{12}(\mu)k_{13}(\mu) \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
-k_{24}'(\mu) &= b_{12}k_{14}(\mu) + b_{32}k_{34}(\mu) \\
&\quad - c_{11}^2k_{12}(\mu)k_{14}(\mu) \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
-k_{33}'(\mu) &= \phi_{h'}(\mu) + 2b_{33}k_{33}(\mu) + 2b_{13}k_{13}(\mu) \\
&\quad + 2k_{34}(\mu) - c_{11}^2k_{13}^2(\mu) \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
-k_{34}'(\mu) &= b_{33}k_{34}(\mu) + b_{13}k_{14}(\mu) + k_{44}(\mu) \\
&\quad - c_{11}^2k_{13}(\mu)k_{14}(\mu) \qquad (70)
\end{aligned}
$$

$$
-k_{44}'(\mu) = \phi_h(\mu) - c_{11}^2k_{14}^2(\mu) \qquad (71)
$$

The boundary conditions on these k-parameters can be shown to be

$$
k(20) = k_m(20) = k_{mp}(20) = 0; \quad m, p = 1, 2, 3, 4. \qquad (72)
$$

That is, the values of the k-parameters are known at the terminal point $\mu = T = 20$ rather than at the initial point.

Hence, it is necessary to solve these equations backward in time. As a result, it is convenient to introduce a new dummy time variable η, where

$$
\eta = 20 - \mu. \qquad (73)
$$

Then, after having transformed the $k(\mu)$-equations into $K(\eta)$-equations using (73), the solution of these new equations can be carried out from $\eta = 0$ to $\eta = 20$, as indicated in Fig. 5.

The solution of the K-equations is illustrated in Fig. 17. Fig. 17 indicates the quantities which must be specified before the solution can be carried out. These quantities appear in the k-equations, (57)–(71). Thus, included in these equations is all of the information pertinent to the problem.

As has been described in this section, the Parametric Expansion Method provides directly the configuration of the optimum system and a set of equations which may be solved to obtain the system parameter values. The procedure described here is summarized in Fig. 18.

Nomenclature

B matrix relating $x'(t)$ and $x(t)$

b_{mn} an element of the B matrix

C matrix relating $x'(t)$ and $\boldsymbol{m}(t)$

c_{mn} an element of the C matrix

E minimum value of the error index $e(t)$, also called the minimum error function

$e(t)$	value of the error index
$e_m(t)$	value of the instantaneous error measure
$h(t)$	aircraft altitude
$h'(t)$, $h''(t)$, $h'''(t)$	time derivatives of aircraft altitude
$h_d(t)$	desired aircraft altitude
$h_d'(t)$	desired aircraft rate of ascent
K_s	aircraft short period gain
$K(\eta)$, $K_m(\eta)$, $K_{mp}(\eta)$	transformed k-parameters using $\eta = T - \mu$
$k(\mu)$, $k_m(\mu)$, $k_{mp}(\mu)$	k-parameters in the expansion of the minimum error function
m	summation index
$m(t)$, $m_1(t)$, $m_2(t)$, $\cdots$, $m_n(t)$	process control signals
$\mathbf{m}(t)$	control signal vector
$\mathbf{m}^*(t)$	optimum control signal vector
N	order of the dynamic process
n	summation index
p	summation index
s	complex variable
T	a future instant of time
T_s	aircraft path time constant
t	real time
$u_0(20-\sigma)$	unit impulse occurring at $\sigma = 20$
V	aircraft total velocity
W	weight of the aircraft
$x(t)$, $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$	process state signals
$\mathbf{x}(t)$	state signal vector
$\mathbf{x}'(t)$	time derivative of state signal vector
$x_1'(t)$, $x_2'(t)$, $\cdots$, $x_n'(t)$	time derivatives of process state signals
$\alpha(t)$	aircraft angle of attack
$\gamma(t)$	aircraft glide path angle
$\Delta\alpha(t)$	change in angle of attack from the equilibrium value
$\delta_e(t)$	aircraft elevator deflection
ζ	aircraft short period damping factor
η	dummy time variable
$\theta(T)$	aircraft pitch angle at $\sigma = T$
$\theta(t)$	aircraft pitch angle
$\theta'(t)$	aircraft pitch rate
$\theta_d(t)$	desired aircraft pitch angle
$\theta_d'(t)$	desired aircraft pitch rate
σ	dummy time variable
τ	time-to-go before touchdown
ϕ_1	weighting factor for pitch rate errors
ϕ_3	weighting factor for rate of ascent errors
ϕ_4	weighting factor for altitude errors
$\phi_{2.T}$	magnitude of the impulse weighting factor for pitch angle errors
$\phi_{3.T}$	magnitude of the impulse weighting factor for rate of ascent errors
$\phi_{4.T}$	magnitude of the impulse weighting factor for altitude errors
$\phi_h(t)$	weighting factor for altitude errors
$\phi_{h'}(t)$	weighting factor for rate of ascent errors
$\phi_\theta(t)$	weighting factor for pitch angle errors
$\phi_{\theta'}(t)$	weighting factor for pitch rate errors
ω_s	aircraft short period resonant frequency

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