供应链/物流网络设计与优化

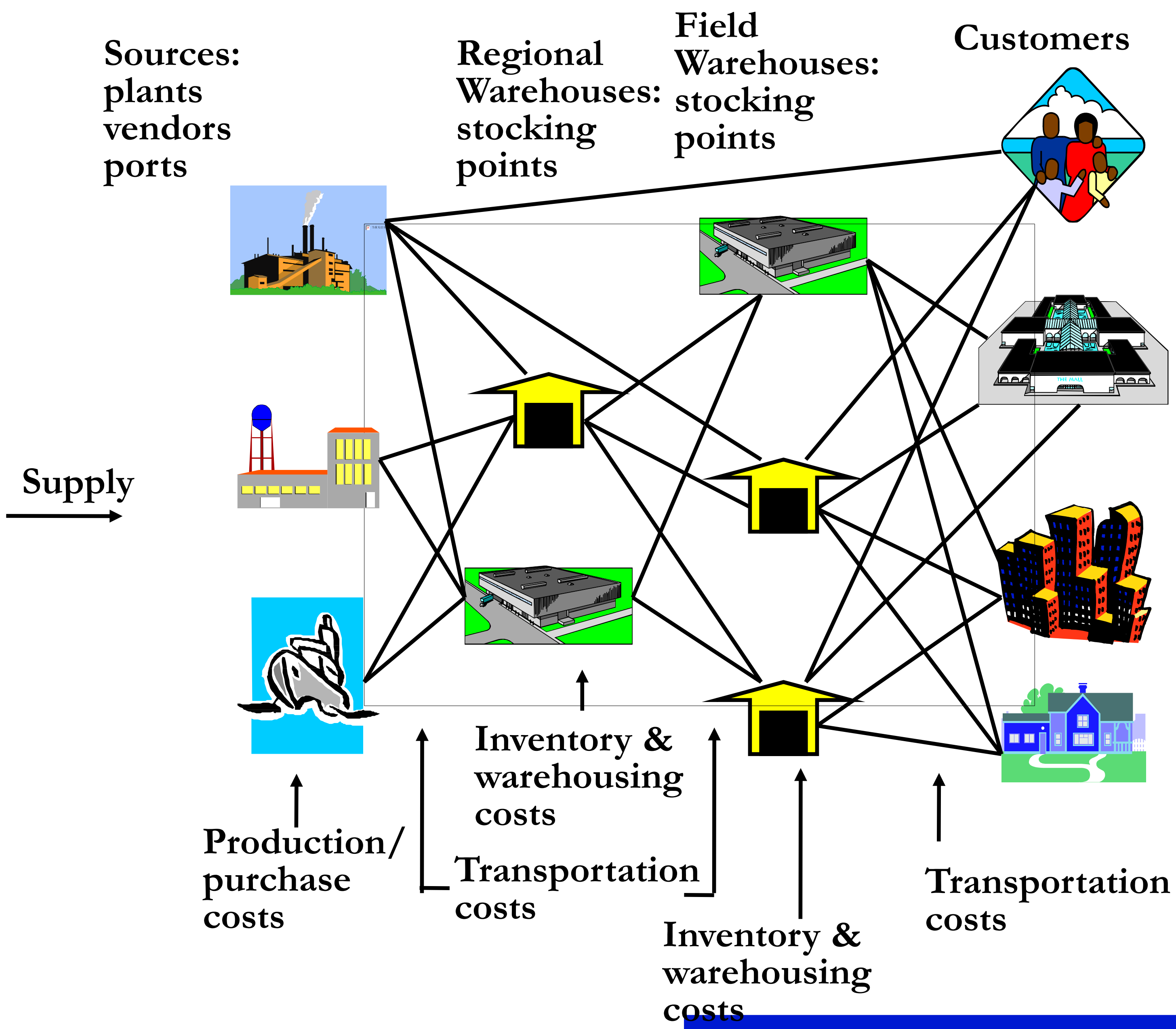
Supply Chain / Logistics Network Configuration

主要内容

- 简介
- 基本模型
- 模型求解方法

供应链/物流网络

- 物流/供应链网络由供应商、制造商、仓库及配送中心、零售网点和客户组成；
- 原材料、在制品和成品库存在各环节之间流动。



Sources:
plants
vendors
ports
Regional
Warehouses:
stocking
points
Field
Warehouses:
stocking
points
Customers
Supply
Production/
purchase
costs
Transportation
costs
Inventory &
warehousing
costs
Transportation
costs
Inventory &
warehousing
costs
Transportation
costs

供应链/物流管理的计划决策层次

- 供应链/物流管理的战略层计划
  - 供应链/物流网络的设计（SC Network Design）
  - 每一个节点企业的工艺布局及工作设计
- 供应链/物流管理的战术层计划
  - 生产计划
  - 运输计划
  - 库存计划/策略
- 供应链/物流管理运作层计划
  - 订单及作业计划 (Scheduling Problem)
  - 车辆送货路线 （Routing Problem）及运输计划（Transportation Scheduling）
  - 同步制造（生产）、准时物流(Just-in-Time)

物流网络配置决策－属战略层决策

- 对企业有长期的效应。
- 包括仓库、工厂和其他物流设施的容量、位置和数量的决策。
- 为每个客户分配为其服务的设施。
- 每个设施内的布局设计和设施投入管理。
- 沟通工具，数据处理方式的设计。

战术层决策

- 在一个周期内（一年或几个月）对制造和配送资源的合理分配，包括：
- 采购与生产决策
- 库存决策
- 运输决策

运作层决策

- 以天为单位的运作决策，包括：
- 生产/仓库作业调度
- 制定运输路线、运输调度计划
- 订单处理
- 车辆装载

物流网络配置是关键战略决策问题

主要决策：（3—5年，Location & Allocation）

- 确定合适的物流设施数量、位置及规模（能力）。
  — **Facility Location**
- 确定客户将从哪个物流设施收到何种产品（包括数量）。—**Flow Allocation**
- 依赖于较长时间跨度上（3—5年）的预测数据
- （客户、产品）数据汇集，基本成本参数的估计
- 永久性资源配置

物流网络配置的目标

- 一般考虑如下**成本**：
  - 生产/采购成本
  - 设施固定成本
  - 库存持有成本
  - 设施运营成本：如搬运、保管、维护等
  - 运输成本
- **目标**：找出一个具有最小总成本的物流网络配置，可以满足客户对产品的需求，并达到预期服务水平。

物流网络中所考虑的物流设施：

- **Industrial facilities :**
  - production units : production, assembly
  - storage facilities : material, intermediate, finished product warehouses
- **Distribution facilities :**
  - distribution warehouses and platforms
  - stores
- **Transport facilities:**
  - vehicles
  - transshipment facilities
  - vehicle garages

数据汇集

- 采用网格或其它聚类技术将距离较近的顾客集合起来。
- 根据以下原则将产品汇集为合理数量的产品组：
  - 产品类型：仅仅在产品型号、款式，或包装形式上有所不同，这些产品就可以汇集成一类。
  - 物流模式：要求汇集成一类中的产品/单品的运输模式与费率、单位存储费用、服务水平一致。

一般可以属于同一产品类型的产品汇集为一类，而在同一类中物料/单品的运输费率、运输模式、单位存储费用、服务水平一致。

物流网络配置中的基本权衡 增加 or 减少？

如果**增加**物流设施数量，一般会：

- 到客户的运输时间， 客户服务水平
- 为保证应对客户需求的不确定性而设置安全库存 ，系统的平均库存
- 设施建设成本和管理费用
- 物流设施的运出成本，即从仓库至客户的运输成本
- 物流设施的运入成本，即从供应商/制造商到仓库的运输成本

关于服务水平需求

- 定义服务水平有多种方法，例如规定客户到其服务的仓库的最大距离，这样可以保证仓库可以在合理的时间内为顾客服务。
- 因此定义服务水平为：**到为其服务的仓库距离不超过规定值的客户数占所有仓库服务客户数的比例。**
- 例如：需要95％的顾客离为其服务的仓库的距离在200公里以内。

关于需求

- 决策所考虑的物流设施数量、位置和规模对公司的影响一般来说至少有3－5年，这意味着**顾客需求在以后数年中的变化**需要在网络设计时予以考虑。

典型的网络设计模型－Facility Location Problem

Dicisions on:

- the number, locations, and activity level of production, storage and distribution facilities

- all product flows and stocks in order to :
    - satisfy customer demand in planning horizon,
    - meet all contraints,
    - minimize all *fixed* and *variable* costs
- a plan to adjust from current to future (multi-period dynamic modelling)

解决技术

- 数学优化技术：
  - 精确算法：找到最优方案。
  - 启发式算法：找到较好的方案，而不是最优方案。
- 仿真技术：供设计者对不同方案进行比较选择。

文献：

Mirchandani P. & Francis R., *Discrete Location theory*, Wiley (1990)

- **Daskin, M.**, *Network & Discrete Location : Models, Algorithms & Applications*, Wiley (1995)
- **Drezner Z., editor (1995),** Facility location, Springer-Verlag
- **Labbé M. & Louveaux F.**, *Location problems*, in Annoted Bibliographies in Combinatorial Optimization, Wiley (1997)
- **Dejax P.,** *Stratégie, planification et implantation du système logistique*, in Maîrise et organisation des flux industriels, J.P. Campagne et P.Burlat, Hermes, Lavoisier, 2001

文献：

- **Aikens C.** (1985), *Facility Location Models for Distribution Planning*
- **Brandeau M. & Chiu S.** (1989), *An Overview of Representative Problems in Location Research*
- **Efroymson M. & Ray T. (1966)，** *A **Branch & Bound** Algorithm for Plant Location*, Op.Res.
- **Erlenkotter D. (1978) ,** *A **dual-based procedure** for uncapacitated facility location*
- **Geoffrion A., G. Graves (1974),** *Multicommodity distribution system design by **Benders decomposition***
- **Geoffrion A., R. McBride (1978),** ***Lagrangean relaxation** applied to capacitated facility location problems*
- **Kuehn A. & Hamburger M.**, *A Heuristic Program for Locating Warehouses*, Mgt.Sc. (1963)

Basic models

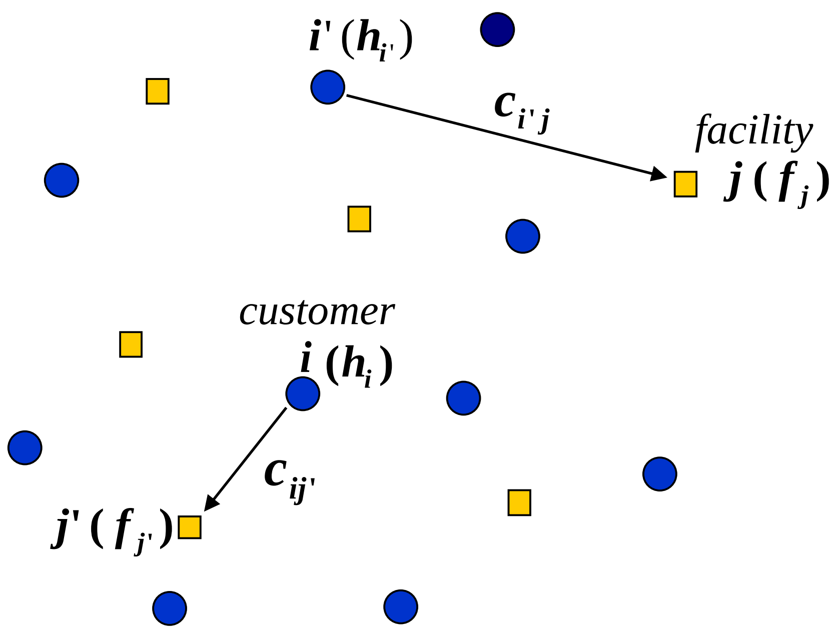
Facility location problem
P-center location problem
Covering problem

Facility Location Problem

Two aspects of decisions: choice of the candidate sites to locate facilities and assignments of customer demands to set-up facilities.

Explicit fixed costs; variable costs;

The objective is for example to minimize the total cost of the system.



$i \in I$

$j \in J$

Facility Location Problem

- ***Uncapacitated Facility Location Problem (UFLP): Mixed Integer Programming model (MIP)***

$$Min \sum_{j} f_j Y_j + \sum_{i} \sum_{j} c_{ij} h_i X_{ij}$$

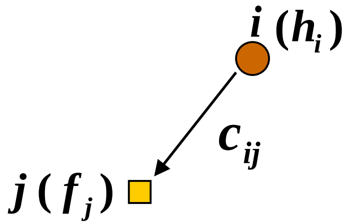
$$s.t.$$

$$\sum_{j} X_{ij} = 1 \qquad \forall i$$

$$X_{ij} \le Y_j \qquad \forall i, j$$

$$X_{ij} \ge 0 \qquad \forall i, j$$

$$Y_j = 0,1 \qquad \forall j$$



Facility Location Problem

- ***Capacitated Facility Location Problem (CFLP). The exogenous specified capacity limit at each candidate facility (k_j) is given:***

- ***With Single Source Constraints:***

$$X_{ij} = 0,1 \quad \forall i, j$$

- ***Many other extended models are proposed:***

Planar Location Problem, Network Location Model， Dynamic Location Problem(multi-period), Competitive Location Problem (in a competitive environment), Combined Location-Routing Problem, Network design model includes technology selection and economies of scales.

P-median Problem

- ***The number (P) of facilities is given instead of an explicit fixed cost***

Covering Location

- 任何客户都能在规定的最大距离（运输时间或成本）内得到一个设施的供应（成品或服务）
- 选择设施的数量和地址
- 最小化设施的固定投入

参数:

I : customers $i = 1, \ldots n$;

t_i : maximal covering distance;

J : candidate facilities $j = 1, \ldots m$;

t_{ij} : distance from i to j;

f_j : fixed cost at j.

Covering Location

I : customers $i = 1, \ldots\ n$;

t_i : maximal covering distance;

J : candidate facilities $j = 1, \ldots\ m$;

t_{ij} : distance from i to j;

f_j : fixed cost at j.

$$Min z = \sum_{j \in J} f_j y_j$$

$$\sum_{j \in N_i} y_j \geq 1 \qquad \forall i \in I$$

$$y_j \in \{0,1\}$$

$$N_i = \{ j : t_{ij} \leq t_i \}$$

思考题

考虑一个由工厂、仓库和客户区域组成的单产品三级供应链网络，工厂向客户区域供货通过仓库进行，现需优化决策仓库的数量、位置、流通量、以及工厂与仓库、仓库与客户区域之间的产品供应关系。网络中已知有P个工厂（$j=1,2,\ldots P$）, 工厂j的最大年生产能力为CP_j，单件生产成本为c_j；有N个客户区域（$i=1,2,\ldots N$），客户区域i的年需求为D_i；仓库的候选地址为K个（$k=1,2,\ldots,K$）, 候选仓库k的年固定成本为F_k, 单件周转作业成本为H_k, 仓库的能力没有限制；工厂j与仓库k之间的单件运输成本为a_{jk}；仓库k与客户区域i之间的单件运输成本为b_{ki}。

（1）请以系统总成本最小化为目标，建立这一问题的线性规划模型 。

（2）用决策变量表示仓库的数量、位置、物料流通量、以及工厂与仓库、仓库与客户区域之间的产品供应数量关系。

思考题

某企业需配送它的产品到8个顾客需求区域，配送通过地区仓库进行。为了适应市场需求的变化和节约成本，企业决定建立一个仓库网络。经过分析有5个候选的仓库地址，在任何一个地址建立仓库的年固定成本投入已知 f_j。不考虑仓库能力的限制。

每个顾客需求点可以接受从任何建立的仓库提供的产品（除下表中给出的个别情况除外）。一个顾客需求点接受一个仓库提供产品的年可变总成本用 C_{ij} 表示，可变成本包括企业工厂提供产品给仓库的成本、搬运成本、运输成本等。

		客户 ***i***			C_{ij}				
仓库 ***j***	f_j	1	2	3	4	5	6	7	8
1	100	120	180	100	-	60	-	180	-
2	70	210	-	150	240	55	210	110	165
3	60	180	190	110	195	50	-	-	195
4	110	210	190	150	180	65	120	160	120
5	80	170	150	110	150	70	195	200	-

思考题

(1) 如果决定只建立一个仓库，那么它应该是哪一个（地址）？计算总成本（固定投入、运作费用）

(2) 如果保持在第一步中建立的仓库，现在要建立第二个仓库，它应该是哪一个？总成本是多少？

(3) 在以上建立的两个仓库基础上，从成本角度出发，是否还需要建立第三个仓库？

(4) 如果在候选地址都建立仓库，计算总成本？

(5) 如果要关闭5个仓库中的一个，应该是哪一个？在这种情况下，总的成本是多少？

(6) 画出总成本、固定成本、可变成本随建立仓库数目的变化曲线。

Solutions to Location Problem

- ***Heuristics***

  (1) basic heuristics [Kuehn and Hamburger (1963), Jacobsen (1983), Feldman et al. (1966), Daskin (1995), Avella et al. (1998)] :

  Construction (Greedy) Heuristic, Interchange Heuristic (Improvement), Hybrid Heuristic

  (2) Meta-heuristics [Metropolis et al. (1953), Kirkpatrick et al. (1983), Glover et al (1993), Holland (1975)]:

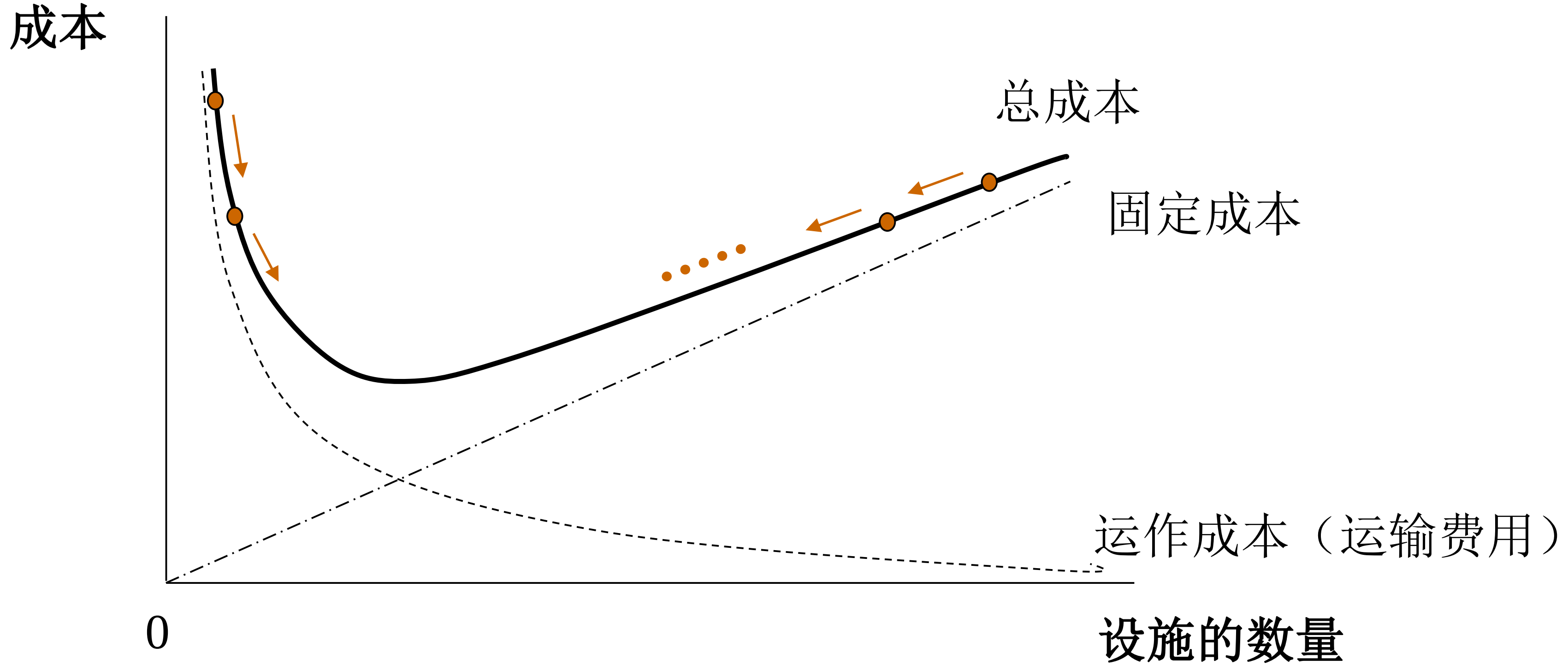
  Simulated annealing, Tabu search and Genetic algorithms

- ***Optimization-based algorithms*** [Parker et al. (1988), Daskin (1995), Minoux (1983); Tragantalerngsak et al. (1997), Sridharan (1995); Erlenkotter (1978); Sa (1969), Ellwein and Gray (1971), Akinc and Khumawala (1977), Davis and Ray (1969); Benders (1962), Geoffrion et al. (1974)]

  Langrangian heuristics, Benders decomposition, Dual ascent, Linear-relaxation based heuristic (rounding solutions) and etc.

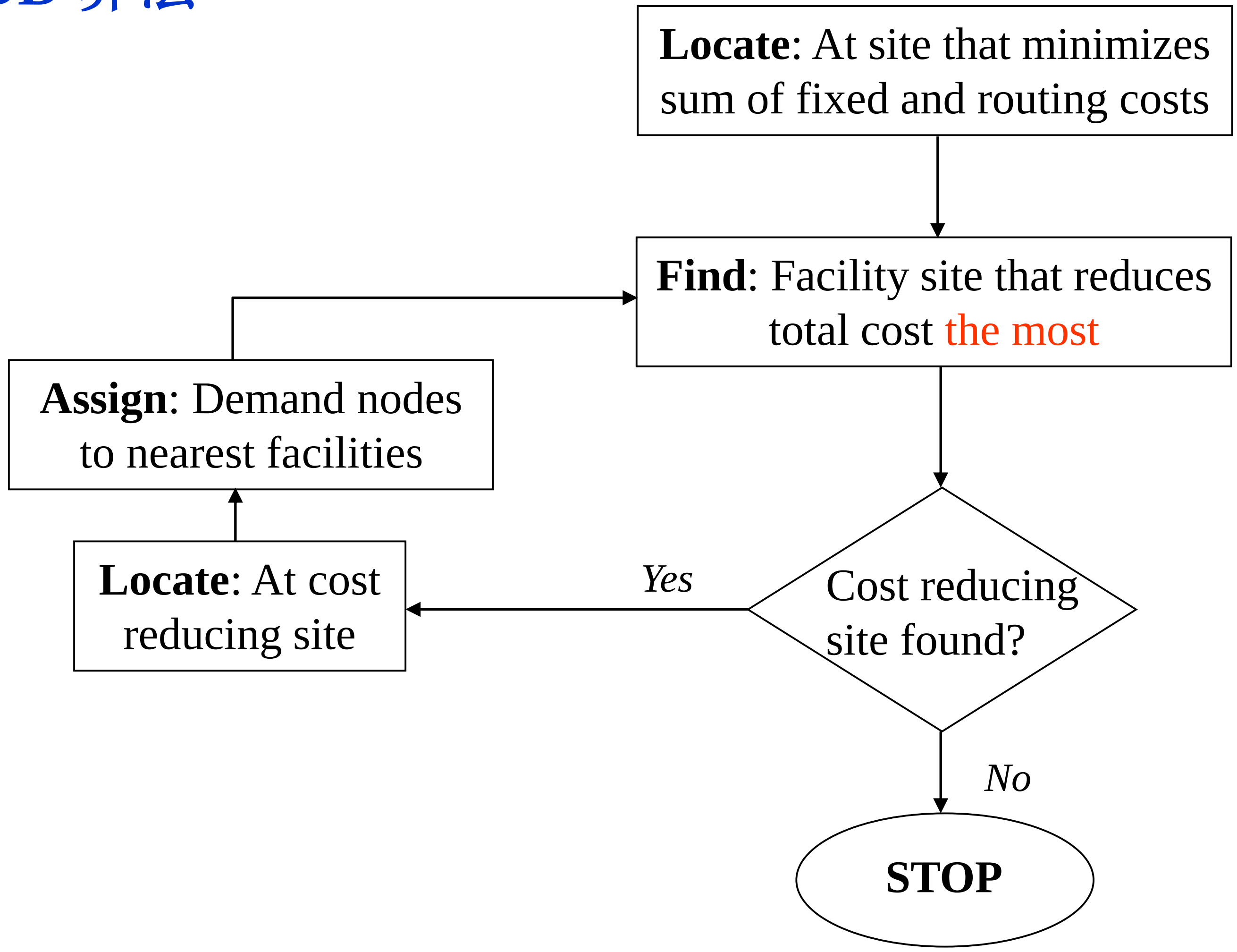
Basic Heuristics

启发式方法

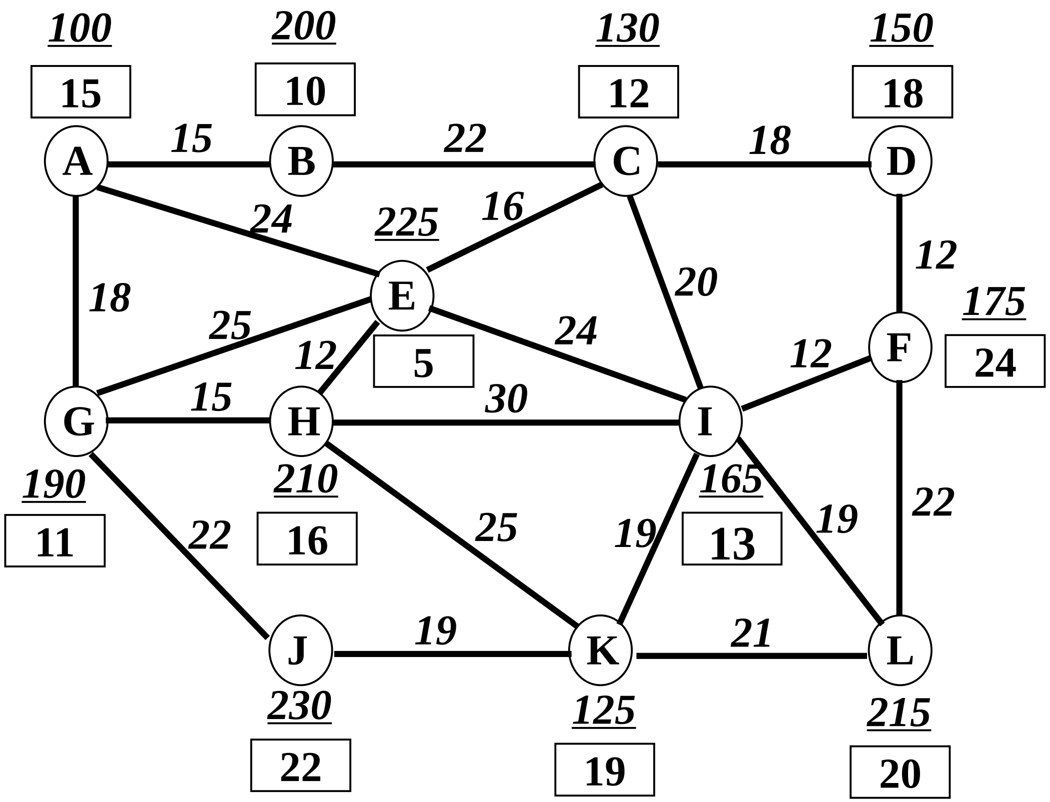


构造（贪婪）算法

－ ADD 算法



ADD算法



$$I = J$$

$$|I| = |J| = |12|$$

A: $f_A = 100$

A: $h_A = 15$

Unit cost per distance per demand **:** $u = 0.35$

Distance **:** d_{ij}

Demand-weighted distance **:** $h_i d_{ij}$

ADD算法

（1）选择第一个设施

候选（设施）地址 *j*

$h_i d_{ij}$

需求点 *i*

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	225	555	825	360	900	270	495	720	600	870	1005
B	150	0	220	400	380	520	330	480	420	550	610	610
C	444	264	0	216	192	360	492	336	240	696	468	468
D	990	720	324	0	612	216	1062	828	432	1116	774	612
E	120	190	80	170	0	180	125	60	120	235	185	215
F	1440	1248	720	288	864	0	1368	1008	288	1200	744	528
		⋮				⋮				⋮		
L	1340	1220	780	680	860	440	1220	920	380	800	420	0
总成本	2836	2970	2179	2410	2249	2191	2618	2300	1835	2640	2077	2445

$$\sum_j f_j + \sum_i \sum_j u h_i d_{ij} = 1835$$

I 被选择

ADD算法

（2）选择下一个设施

候选（设施）地址 *j*

需求点 *i*

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
A	0	225	555	720	360	720	270	495	720	600	720	720
B	150	0	220	400	380	420	330	420	420	420	420	420
⋮												
I	0	0	0	0	0	0	0	0	0	0	0	0
⋮												
L	380	380	380	380	380	380	380	380	380	380	380	0
总成本	1485	1669	1675	1819	1684	1834	1456	1654	1835	1642	1640	1917

G、*I* 被选择

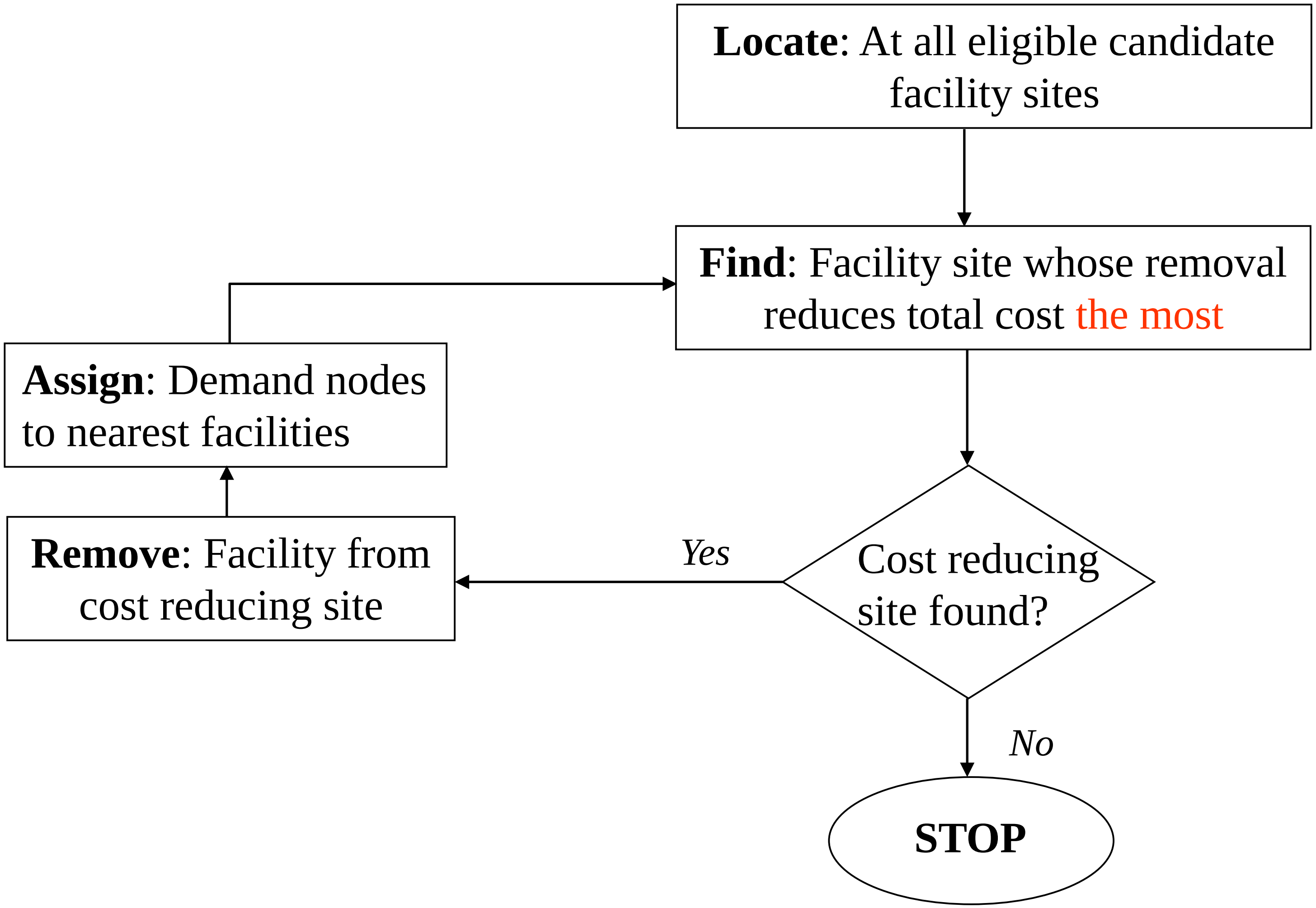
ADD算法

迭代次数	设置设施	需求距离总和	固定投入	总成本
1	*I*	4772	165	1835
2	*G*	3145	355	1456
3	*A*	2695	455	1398
4	*K*	2268	580	1374
5	*D*	1812	730	1364
6	*C*	1556	860	1405

结果: 设置的设施 A、D、G、I、K

总成本（固定投资及运作费用）＝ 1364

DROP算法


Locate: At all eligible candidate facility sites
Find: Facility site whose removal reduces total cost the most
Cost reducing site found?
Yes
Remove: Facility from cost reducing site
Assign: Demand nodes to nearest facilities
No
STOP

DROP算法

(1) 初始化：在每一候选地址都设置设施

总成本: $\sum_{j} f_j = 2115$

设施: ***A, B, C, D, E, F, G, H, I, J, L***

DROP算法

（2）选择第一个去除的设施

候选去除设施 *j*

$h_i d_{ij}$

需求点 *i*

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
A	225	0	0	0	0	0	0	0	0	0	0	0
B	0	150	0	0	0	0	0	0	0	0	0	0
C	0	0	192	0	0	0	0	0	0	0	0	0
D	0	0	0	216	0	0	0	0	0	0	0	0
E	0	0	0	0	60	0	0	0	0	0	0	0
F	0	0	0	0	0	288	0	0	0	0	0	0
		⋮				⋮				⋮		
L	0	0	0	0	0	0	0	0	0	0	0	380
总成本	2094	1968	2052	2041	1911	2041	1983	1972	2005	2031	2116	2033

$$\sum_j f_j + \sum_i \sum_j u h_i d_{ij} = 1911$$

E 被去除

DROP算法

（3）计算下一个去除的设施

候选去除设施 *j*

需求点 *i*

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
A	225	0	0	0	0	0	0	0	0	0	0	0
B	0	150	0	0	0	0	0	0	0	0	0	0
⋮	⋮				⋮				⋮			
E	60	60	60	60	60	60	60	80	60	60	60	60
⋮	⋮				⋮				⋮			
L	0	0	0	0	0	0	0	0	0	0	0	380
总成本	1890	1764	1857	1837	1911	1837	1779	1792	1801	1827	1912	1829

$$\sum_j f_j + \sum_i \sum_j u h_i d_{ij} = 1764$$

***B*、*E* 被去除**

DROP算法

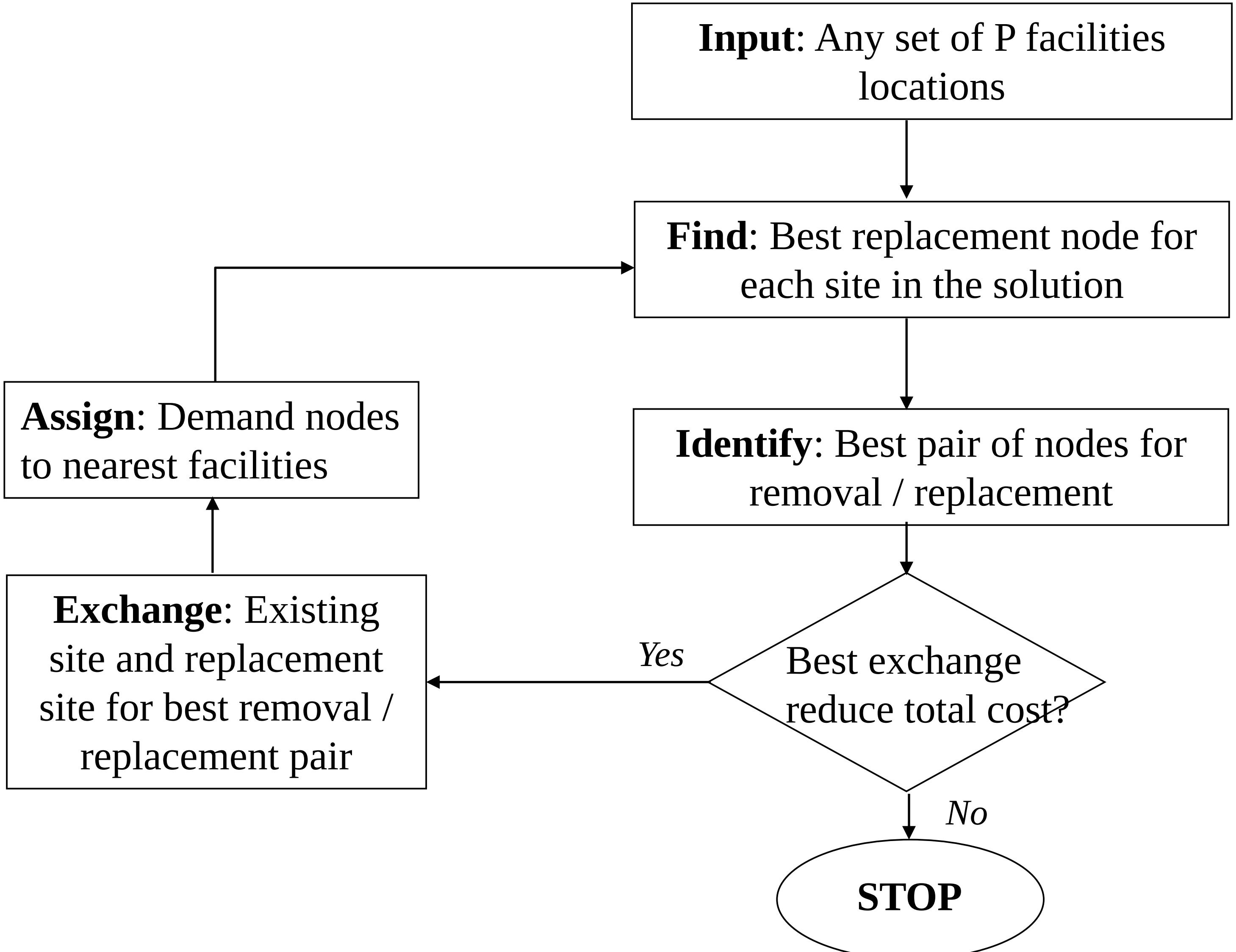
迭代次数	去除的设施	需求距离总和	总固定投入	总成本
1	-	0	2115	2115
2	*E*	60	1890	1911
3	*B*	210	1690	1764
4	*G*	375	1500	1631
5	*I*	531	1335	1521
6	*J*	949	1105	1437
7	*D*	1165	955	1362
8	*L*	1585	740	1295
9	*H*	2038	530	1243
10	*C*	2438	400	1253

去除: *B, D, E, G, H, I, J, L;*

设置设施: *A, C,F, K;*

总成本: 1243

交换（提高）算法



Input: Any set of P facilities locations
Find: Best replacement node for each site in the solution
Identify: Best pair of nodes for removal / replacement
Best exchange reduce total cost?
Yes
No
Exchange: Existing site and replacement site for best removal / replacement pair
Assign: Demand nodes to nearest facilities
STOP

交换（提高）算法

基于ADD算法结果的交换算法的解

迭代过程

Solution	*Original locations*	*Remove node*	*Insert node*	*Total cost*
ADD	*A, D, G, I, K*			**1364.20**
First Ite.	*A, D, G, I, K*	**G**	**H**	**1336.95**
Second Ite.	*A, D, H, I, K*	**I**	**F**	**1314.75**
Third Ite.	*A, D, F, H, K*	**D**	**C**	**1294.75**

解的结果

设施: *A, C, F, H, K*

固定投入: 740.00

需求距离总和: 1585.00

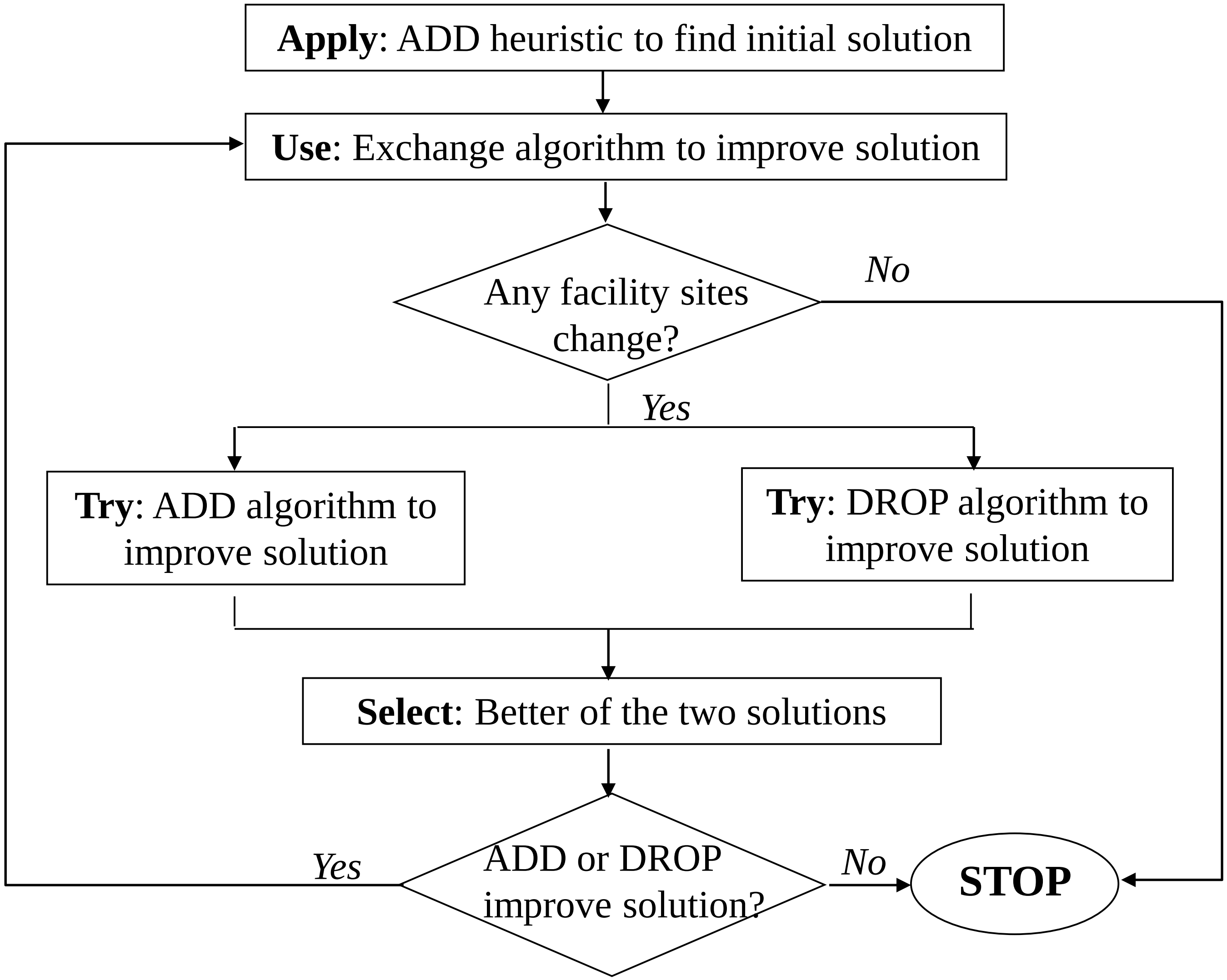
总成本: 1294.75 = 740 + 0.35 * 1585

算法的比较

Algorithm	*Facility locations*	*Total cost*	*Percentage deviation from Optimum*
ADD	*A,D,G,I,K*	1364.20	10.47%
DROP	*A,C,F,K*	1243.00	0.65%
ADD/Improvement	*A,C,F,H,K*	1294.75	4.84%
Lagrangian	*A,D,K*	1234.95	0.00%

注意：在这个例子中，ADD/Improvement 解的精度甚至比单独使用 DROP 算法的精度还要低。

混合方法算法



Apply: ADD heuristic to find initial solution
Use: Exchange algorithm to improve solution
Any facility sites change?
No
Yes
Try: ADD algorithm to improve solution
Try: DROP algorithm to improve solution
Select: Better of the two solutions
Yes
ADD or DROP improve solution?
No
STOP

Optimization-Based Algorithm

Lagrangian Relaxation Approach (1/6)

- *An optimization problem*

$$Z = \boldsymbol{Min}\ \boldsymbol{f}(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{Hx} \le 0 \qquad (OP)$$

- *Lagrangian relaxed problem* $L(x, \lambda)$

$$Z_L(\boldsymbol{x}, \lambda) = \underset{\boldsymbol{x}}{\boldsymbol{Min}}\ \boldsymbol{f}(\boldsymbol{x}) + \varphi(\boldsymbol{x}) \qquad (L(x,\lambda))$$

Pernalized term $\varphi(\boldsymbol{x}) = \lambda \boldsymbol{Hx} = \begin{cases} 0 & \textit{if } x \textit{ is feasible for problem OP} \\ +\infty & \textit{infeasible} \end{cases}$

- *Lagrangian dual DP*

$$Z_D = \underset{x}{Min}[f(x) + \underset{\lambda}{Max}\, \lambda Hx] \qquad (DP)$$

$$= \underset{x}{Min}[\underset{\lambda}{Max}\, L(x, \lambda)] \qquad \lambda \ge 0$$

$$\Rightarrow \underset{\lambda}{Max}[\underset{x}{Min}\, L(x, \lambda)]$$

Lagrangian Relaxation Approach (2/6)

- *Weak lagrangian duality* $\boxed{v(Z_L) \leq v(Z)}$
- *Strong lagrangian duality* $\boxed{v(Z) = v(Z_D)}$

If x solves $L(\hat{x}, \hat{\lambda})$ *for some* $\hat{\lambda}$*, and in addition*

$$H\hat{x} \leq 0; \qquad \hat{\lambda}H\hat{x} = 0$$

then $\hat{x}$ *solves OP.*

Lagrangian Relaxation Approach (3/6)

- *An integer or mixed integer programming problem IP*

$$
\begin{aligned}
& Z = Min\ cx && \textit{(IP)}\\
s.t.\quad & \boxed{Ax \le b} \\
& Dx \le e \\
& x \ge 0 \quad \textit{and integral}
\end{aligned}
$$

where x is $n\times 1$, b is $m\times 1$, e is $k\times 1$, all other matrices have conformable dimensions.

- *Lagrangian relaxed problem LR(u)*

$$
\begin{aligned}
& Z_L(u) = \underset{x}{Min}\, cx + u(Ax - b) && \textit{(LR(u))}\\
s.t.\quad & Dx \le e \\
& x \ge 0 \quad \textit{and integral} \\
& u \ge 0
\end{aligned}
$$

where $u = (u_1, \cdots, u_m)$ is a row vector of lagrangian multipliers.

- *Lagrangian dual DP*

$$Z_D = \underset{u}{Max}\, Z_L(u) \qquad \textit{(DP)}$$

Lagrangian relaxation approach (4/6)

- *Any* $u \geq 0$, $Z_L(u)$ *provides a lower bound for the original problem, that is, for all* $u \geq 0, Z_L(u) = v(LR(u)) \leq Z = v(IP)$.

Proof.
$$Z_L(u) = cx^*_{LR} + u(Ax^*_{LR} - b)$$
$$\leq cx^*_{IP} + u(Ax^*_{IP} - b)$$
$$\leq cx^*_{IP} = v(IP)$$

- *Subgradient optimization to update the value of u. Given an initial value* u^0*, and iterative value of u is obtained by,*

$$u_i^{k+1} = \max(0, u_i^k + t_k(A_i x^k - b_i))$$

where x^k *is an optimal solution to* $LR(u^k)$ *and* t_k *is a positive step size in k th iteration. The step size is in practice determined by,*

$$t_k = \alpha_k (UB(IP) - Z_L(u^k)) / \sum_i (A_i x^k - b_i)^2$$

Where UB(IP) is the best upper bound of problem (IP) found so far, and $0 \leq \alpha \leq 2$.

Lagrangian relaxation approach (5/6)

- ***General procedure***

Step 1. Relax one or more constraints by multiplying the constraints by Lagrange multipliers and bringing the constraints into the objective function.

Step 2. Solve the resulting relaxed problem to find the **optimal solution** to this relaxed problem. This solution provides us a lower bound of the original problem.

Step 3. Use the resulting solution of the relaxed problem found in step 2 to find a feasible solution to the original problem. Update the upper bound if possible.

Step 4*. Update the lagrangian multipliers using the subgradient method (for example).

Step 5. If any terminating condition has been met, stop; otherwise go back to Step 2.

*: In step 4, the value of parameter α is generally decreased (initially from 2 for example) if the lower bound has not increased in a given number of consecutive iterations.

Lagrangian relaxation approach (6/6)

- ***Criteria of Termination***

1. A specified number of iterations.

2. The gap between lower and upper bound is close enough,

$$(UB - LB) / LB < \varepsilon$$

where *UB* and *LB* are respectively the best upper and lower bounds found, ε is a specified parameter.

3. The value of parameter α becomes too small to allow any improvement of the lower bound.

Example on Lagrangian relaxation approach (1/6)

- *An Uncapacitated Facility Location Problem*

$$Min \sum_j f_j Y_j + \sum_i \sum_j c_{ij} h_i X_{ij}$$

$$s.t.$$

$$\boxed{\sum_j X_{ij} = 1 \qquad \forall i} \qquad \lambda = (\lambda_1, \cdots, \lambda_n)$$

$$X_{ij} \le Y_j \qquad \forall i, j$$

$$X_{ij} \ge 0 \qquad \forall i, j$$

$$Y_j = 0,1 \qquad \forall j$$

Example LR (2/6)

- ***Relaxed problem L***

$$\underset{X,Y}{Min}\sum_j f_j Y_j + \sum_i \sum_j (c_{ij} h_i + \lambda_i) X_{ij} - \sum_i \lambda_i$$

$$s.t.$$

$$X_{ij} \le Y_j \qquad \forall i, j$$

$$X_{ij} \ge 0 \qquad \forall i, j$$

$$Y_j = 0,1 \qquad \forall j$$

$$\lambda_i \ unrestricted \qquad \forall i$$

Given a set of values of multipliers λ_i , the problem can actually be decomposed into *j* sub-problems, one for each facility *j*.

Example LR (3/6)

- *Define the sub-problem for any j*

$$LX_j : Min \sum_i (c_{ij} h_i + \lambda_i) X_{ij} + f_j$$

$$s.t.$$

$$0 \le X_{ij} \le 1 \qquad \forall i$$

Solution: $X_{ij} = 1 \quad if \ c_{ij} h_i + \lambda_i < 0$

$X_{ij} = 0 \quad otherwise$

Example LR (4/6)

- *The problem L can be reformulated as (for given λ):*

$$L: Min \sum_j v(LX_j)Y_j - \sum_i \lambda_i$$

$$s.t.$$

$$Y_j = 0,1 \qquad \forall j$$

Solution: $Y_j = 1 \quad if \ v(LX_j) < 0$

$$Y_j = 0 \quad otherwise$$

$$X_{ij} = X^*_{ij}(LX_j) \quad if \ Y_j = 1$$

$$X_{ij} = 0 \qquad if \ Y_j = 0$$

- *The lower bound:* $v(L)$

Example LR (5/6)

- ***The upper bound:***

Solve a transport problem using the value of *Y* determined by the relaxed problem

- ***Update the multipliers***

$$\lambda_i^{k+1} = \lambda_i^k + \frac{\alpha^k (UB - lb^k)(\sum_j X_{ij}^k - 1)}{\sum_i (\sum_j X_{ij}^k - 1)^2}$$

Summary of solutions (6/6)

Algorithm	*Facility locations*	*Total cost*	*Percentage deviation from Optimum*
ADD	*A,D,G,I,K*	1364.20	10.47%
DROP	*A,C,F,K*	1243.00	0.65%
ADD/Improvement	*A,C,F,H,K*	1294.75	4.84%
Lagrangian	*A,D,K*	1234.95	0.00%

Benders Decomposition (1/7)

- *A **Capacitated** Facility Location Problem*

$$Min \sum_j f_j Y_j + \sum_i \sum_j c_{ij} X_{ij}$$

$$s.t.$$

$$\sum_j X_{ij} = h_i \qquad \forall i$$

$$\sum_i X_{ij} \le k_j Y_j \qquad \forall j$$

$$X_{ij} \ge 0 \qquad \forall i, j$$

$$Y_j = 0,1 \qquad \forall j$$

Benders Decomposition (2/7)

- *Sub-problem (SP) — Transportation problem, its objective is conditional on the choice of facility sites as given by the variables Y.*

$$T(X \mid \hat{Y}) = Min \sum_i \sum_j c_{ij} X_{ij}$$

$$s.t.$$

$$\sum_j X_{ij} = h_i \qquad \forall i \qquad U_i$$

$$-\sum_i X_{ij} \geq -k_j \hat{Y}_j \qquad \forall j \qquad W_j$$

$$X_{ij} \geq 0 \qquad \forall i, j$$

Benders Decomposition (3/7)

- *Restate the original problem using sub-problem SP*

$$\underset{Y}{Min}\sum_{j} f_j Y_j + \left\{\underset{X}{Min}[T(X|Y)]\right\}$$

$$s.t.$$

$$\sum_{j} k_j Y_j \geq \sum_{i} h_i$$

$$Y_j = 0,1 \qquad \forall j$$

Benders Decomposition (4/7)

- *The dual problem of T*

$$Max\ D(U,W \mid \hat{Y}) = \sum_i h_i U_i - \sum_j k_j \hat{Y}_j W_j$$

$$s.t.$$

$$U_i - W_j \le c_{ij} \qquad \forall i, j$$

$$U_i \quad unrestricted \qquad \forall i$$

$$W_j \ge 0 \qquad \forall j$$

Benders Decomposition (5/7)

- *Restate the original problem using the dual of sub-problem SP*

$$\underset{Y}{Min}\sum_{j} f_j Y_j + \left\{\underset{U,W}{Max}\left[D(U,W|Y)\right]\right\}$$

$$s.t.$$

$$\sum_{j} k_j Y_j \geq \sum_{i} h_i$$

$$Y_j = 0,1 \qquad \forall j$$

Benders Decomposition (6/7)

- *The master problem MP*

$$Min \sum_j f_j Y_j + D$$

$$s.t.$$

$$\sum_j k_j Y_j \geq \sum_i h_i$$

$$D \geq \sum_i U_i^t - \sum_j k_j Y_j W_j^t \qquad \forall t$$

$$Y_j = 0,1 \qquad \forall j$$

$$D \geq 0$$

Exercise on Benders Decomposition

Master problem

$$Min \sum_j f_j Y_j + D$$

$s.t.$

$$\sum_j k_j Y_j \geq \sum_i h_i$$

$$D \geq \sum_i h_i U_i^t - \sum_j k_j Y_j W_j^t \qquad \forall t$$

$$Y_j = 0,1 \qquad \forall j$$

$$D \geq 0$$

Subproblem

$$T(X \mid \hat{Y}) = Min \sum_i \sum_j c_{ij} X_{ij}$$

$s.t.$

$$\sum_j X_{ij} = h_i \qquad \forall i \qquad U_i$$

$$-\sum_i X_{ij} \geq -k_j \hat{Y}_j \qquad \forall j \qquad W_j$$

$$X_{ij} \geq 0 \qquad \forall i, j$$

$$Max\ D(U,W \mid \hat{Y}) = \sum_i h_i U_i - \sum_j k_j \hat{Y}_j W_j$$

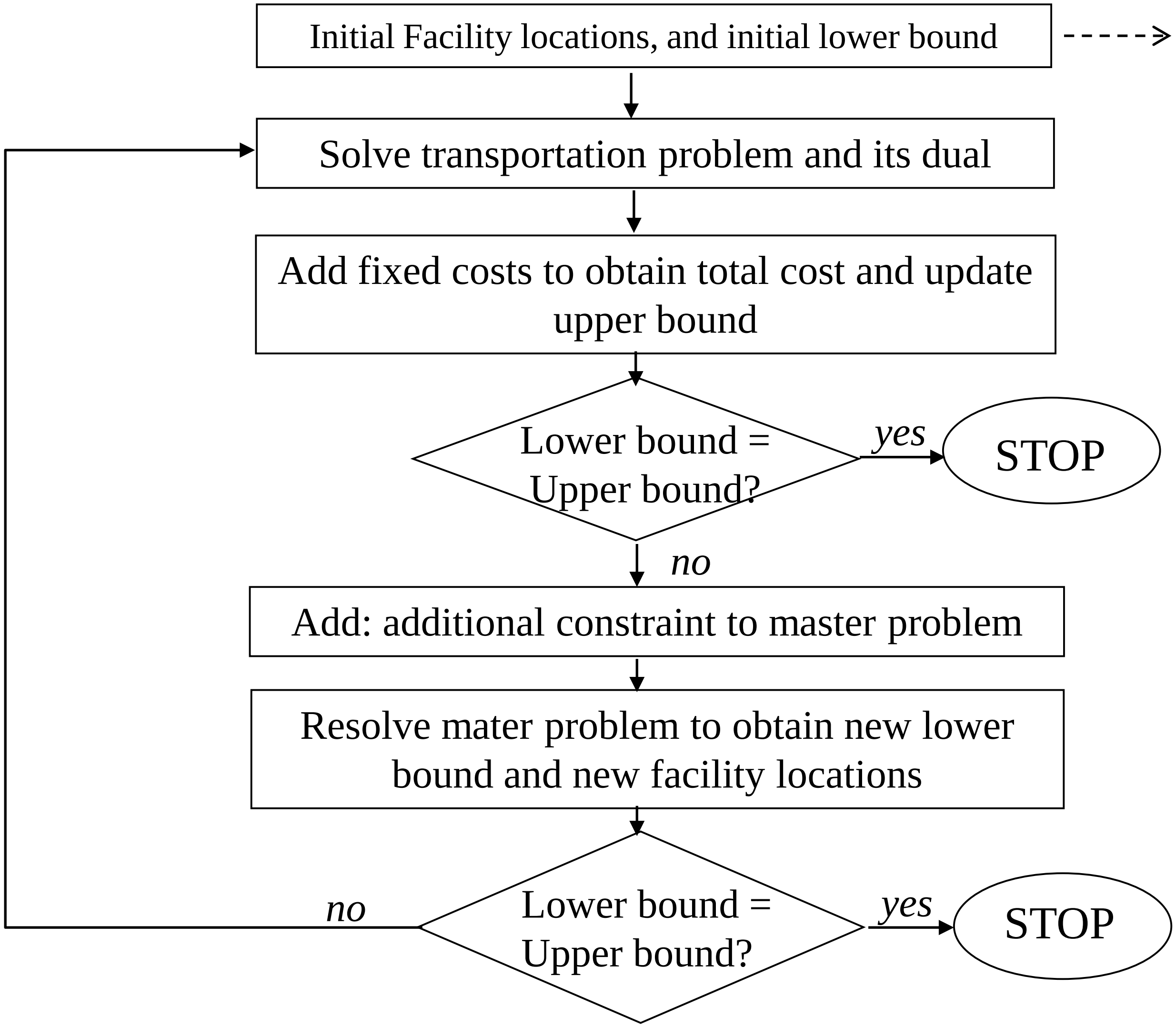
$s.t.$

$$U_i - W_j \leq c_{ij} \qquad \forall i, j$$

$$U_i \quad unrestricted \qquad \forall i$$

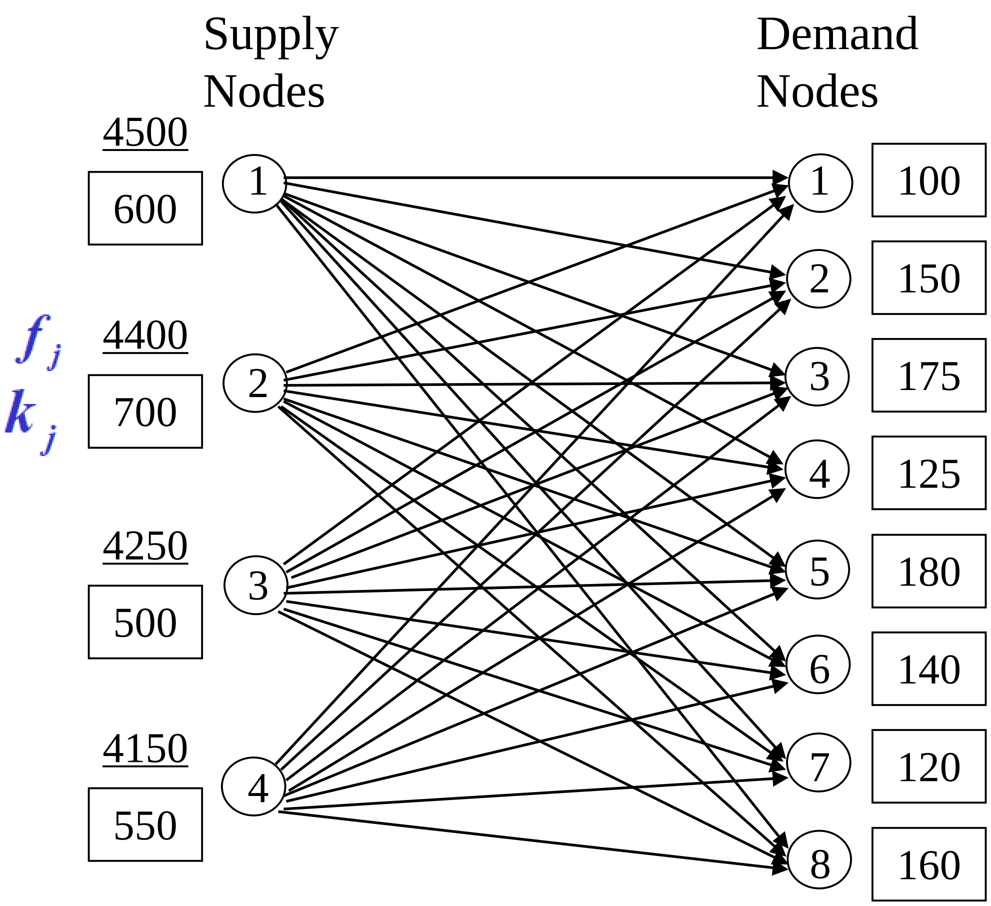
$$W_j \geq 0 \qquad \forall j$$

Benders Decomposition (7/7)



For initial location solution, we can solve the master problem without any “dual constraint”. -- lower bound

Exercise on Benders Decomposition



Unit shipping cost: $\boldsymbol{c}_{ij} = \alpha \boldsymbol{d}_{ij}$

S\D	1	2	3	4	5	6	7	8
1	26	13	19	11	17	25	20	21
2	27	19	22	22	15	26	19	16
3	28	21	12	22	13	27	23	23
4	17	19	26	24	15	23	26	26

Exercise on Benders Decomposition

Initial Master problem

$$Min\, 4500Y_1 + 4400Y_2 + 4250Y_3 + 4150Y_4 + D$$
$$s.t.$$
$$600Y_1 + 700Y_2 + 500Y_3 + 550Y_4 \geq 1150$$
$$Y_1, Y_2, Y_3, Y_4 = 0,1$$
$$D \geq 0$$

Solution: Y_2=Y_4=1; Y_1=Y_3=0; v = 8550 (***lower bound***)

Initial Subproblem

Solution to transportation problem with Y_2=Y_4=1; Y_1=Y_3=0; *Upper bound* = 21910 (Transport cost) + 4400 +4150 (fixed costs) = 30460;

Solution to its dual problem: U^1 = [17, 19, 22, 22, 15, 23, 19, 16]

W^1 = [11, 0, 10, 0]

Adding constraint: $D \geq 21910 - 6600Y_1 - 5000Y_2$

Exercise on Benders Decomposition

The Next Master problem (B&B, linear relaxation(lower bound), rounded solution(upper bound))

$$Min\ 4500Y_1 + 4400Y_2 + 4250Y_3 + 4150Y_4 + D$$

$$s.t.$$

$$600Y_1 + 700Y_2 + 500Y_3 + 550Y_4 \geq 1150$$

$$D \geq 21910 - 6600Y_1 - 5000Y_2$$

$$Y_1, Y_2, Y_3, Y_4 = 0,1$$

$$D \geq 0$$

Solution: $Y_1 = Y_3 = Y_4 = 1$; $Y_2 = 0$; $v = 23210$ (***lower bound***)

Initial Subproblem

Solution to transportation problem with $Y_1 = Y_3 = Y_4 = 1$; $Y_2 = 0$; *Upper bound* = 18445 (Transport cost) + 4500 + 4250 + 4150 (fixed costs) = 31345;

Solution to its dual problem: $U^2 = [17, 13, 12, 11, 13, 23, 29, 21]$

$W^2 = [0, 5, 0, 0]$

Adding constraint: $D \geq 18445 - 3500Y_2$

Exercise on Benders Decomposition

Results of Using Benders Decomposition

Ite.	*Lower Bound*	*Locations*	*Fixed Costs*	*Transport Costs*	*Total Cost*	*Best Upper Bound*
1	8550	2,4	8550	21910	30460	30460
2	23210	1,3,4	12900	18445	31345	30460
3	24210	1,2	8900	20380	**29280**	**29280**
4	25630	2,3	8650	21220	29870	29280
5	27095	1,4	8650	20680	29330	29280
6	28360	1,2,4	13050	19110	32160	29280
7	**29280**					

Some key factors for SC modeling

- **Types of models**
  - deterministic models
  - stochastic models
  - mixed integer programming models
  - simulation models

Some key factors for SC modeling

- single period / multi-periodic models
- number of products, product aggregation
- number of stages : suppliers, production, distribution, customer
- international aspects
- choice of technologies; economies of scale
- stochastic features (uncertain demand)
- costs determination (fixed / variable)
- environment conscious SCM