Deriving the Distribution of the t Statistic

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Let $X_1, ..., X_n$ be a sample from $N(\mu, \sigma^2)$ where $\sigma^2 > 0$ and $\mu \in \mathbb{R}$ and let μ_0 be a fixed real number. We define the statistic

$$t = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s},$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

Recall that $Z = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ has a standard normal distribution and that $W = \frac{(n-1)s^2}{\sigma^2}$ has a χ^2 distribution with $n-1$ degrees of freedom. We have that

$$\begin{aligned} t &= \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \\ &= \frac{\sqrt{n}(\bar{X} - \mu + \mu - \mu_0)/\sigma}{s/\sigma} \\ &= \frac{Z + \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}}{\sqrt{\frac{W}{n-1}}} \end{aligned}$$

Let $U_1 = Z + \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}$ and let $U_2 = W$. Note that from the properties of the normal distribution, we know that U_1 and U_2 are independent. We will make the following transformation of variables:

$$\begin{aligned} V_1 &= t = (n-1)^{\frac{1}{2}} U_1 U_2^{-\frac{1}{2}} \\ V_2 &= U_2 \end{aligned}$$

The inverse transformation is given by:

$$\begin{aligned} U_1 &= (n-1)^{-\frac{1}{2}} V_1 V_2^{\frac{1}{2}} \\ U_2 &= V_2 \end{aligned}$$

Next we need to compute the Jacobian. Since $\frac{\partial U_2}{\partial V_1} = 0$, the Jacobian is just

$$J = \left| \frac{\partial U_1}{\partial V_1} \frac{\partial U_2}{\partial V_2} \right| = (n-1)^{-\frac{1}{2}} V_2^{\frac{1}{2}}.$$

Since Z is standard normal and U_1 is equal to $Z + \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}$, we have that $U_1 \sim N(\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}, 1)$. Furthermore, $U_2 = W$ has a χ^2_{n-1} distribution and is independent of U_1. Therefore, we can easily give the joint pdf of U_1 and U_2 as follows:

$$\begin{aligned}
f_{U_1U_2}(u_1, u_2) &= \left(\frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}(u_1 - \frac{\sqrt{n}(\mu-\mu_0)}{\sigma})^2} \right) \left(\frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} u_2^{\frac{n-1}{2}-1} e^{-\frac{u_2}{2}} \right) \\
&= \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} u_2^{\frac{n-1}{2}-1} e^{-\frac{u_2}{2}} e^{-\frac{1}{2}(u_1 - \frac{\sqrt{n}(\mu-\mu_0)}{\sigma})^2} \\
&= \frac{1}{\pi^{\frac{1}{2}} 2^{\frac{n}{2}} \Gamma(\frac{n-1}{2})} u_2^{\frac{n-1}{2}-1} e^{-\frac{u_2}{2} - \frac{1}{2}(u_1 - \frac{\sqrt{n}(\mu-\mu_0)}{\sigma})^2} \\
&= C u_2^k e^{-\frac{1}{2}u_2 - \frac{1}{2}u_1^2 + \alpha u_1 - \frac{\alpha^2}{2}},
\end{aligned}$$

where $C = \frac{1}{\pi^{\frac{1}{2}} 2^{\frac{n}{2}} \Gamma(\frac{n-1}{2})}$, $k = \frac{n-1}{2} - 1$, and $\alpha = \frac{\sqrt{n}(\mu-\mu_0)}{\sigma}$.

Thus, applying the change of variables formula yields:

$$\begin{aligned}
f_{V_1V_2}(v_1, v_2) &= C v_2^k e^{-\frac{1}{2}v_2 - \frac{1}{2}\frac{1}{n-1}v_2 v_1^2 + \alpha \frac{1}{\sqrt{n-1}} v_2^{\frac{1}{2}} v_1 - \frac{\alpha^2}{2}} J \\
&= C^* v_2^k e^{-\frac{1}{2}v_2 - \frac{1}{2}\frac{1}{n-1}v_2 v_1^2 + \alpha \frac{1}{\sqrt{n-1}} v_2^{\frac{1}{2}} v_1} v_2^{\frac{1}{2}} \\
&= C^* v_2^{k+\frac{1}{2}} e^{-\frac{1}{2}(1 + \frac{1}{n-1}v_1^2)v_2 + \alpha \frac{1}{\sqrt{n-1}} v_2^{\frac{1}{2}} v_1}
\end{aligned}$$

where $C^* = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{n-1}} C$. We want to find the marginal pdf

$$f_{V_1}(v_1) = C^* \int_0^\infty v_2^{k+\frac{1}{2}} e^{-\frac{1}{2}(1 + \frac{1}{n-1}v_1^2)v_2 + \alpha \frac{1}{\sqrt{n-1}} v_2^{\frac{1}{2}} v_1} dv_2.$$

Let us first make the substitution $s = \sqrt{v_2}$, which we are free to do since v_2 is always non-negative. Differentiating the expression $s^2 = v_2$ gives

$$2sds = dv_2.$$

so that now we have

$$
\begin{aligned}
f_{V_1}(v_1) &= C^* \int_0^\infty s^{2k+1} e^{-\frac{1}{2}(1+\frac{1}{n-1}v_1^2)s^2+\frac{\alpha}{\sqrt{n-1}}v_1 s}(2s)ds \\
&= 2C^* \int_0^\infty s^{2k+2} e^{-\frac{1}{2}(1+\frac{1}{n-1}v_1^2)s^2+\frac{\alpha}{\sqrt{n-1}}v_1 s} ds
\end{aligned}
$$

The expression $-\frac{1}{2}(1+\frac{1}{n-1}v_1^2)s^2+\frac{\alpha}{\sqrt{n-1}}v_1 s$ can be written in the form $a(s-b)^2+c$ since it is a quadratic polynomial. In that case, we have $a=-\frac{1}{2}(1+\frac{1}{n-1}v_1^2)$, $-2ab=\frac{\alpha}{\sqrt{n-1}}v_1$, and $c=-ab^2$. Solving for b and c, we have

$$
\begin{aligned}
a &= -\frac{1}{2}(1+\frac{1}{n-1}v_1^2) \\
b &= \frac{\alpha}{\sqrt{n-1}}\frac{v_1}{(1+\frac{1}{n-1}v_1^2)} \\
c &= -\frac{1}{2}\frac{\alpha^2}{n-1}\frac{v_1^2}{(1+\frac{1}{n-1}v_1^2)}
\end{aligned}
$$

The integral becomes

$$
f_{V_1}(v_1) = 2C^* e^c \int_0^\infty s^{2k+2} e^{a(s-b)^2} ds
$$

Let $r=-a(s-b)^2$ with $dr=-2as\,ds$. We have $s=b+\sqrt{-\frac{r}{a}}$, which is a real number since $a<0$. In this case r will run from $-ab^2=c$ to $-\infty$.

We can write

$$
\begin{aligned}
f_{V_1}(v_1) &= -\frac{C^* e^c}{a} \int_c^{-\infty} \left(b + \sqrt{-\frac{r}{a}} \right)^{2k+1} e^{-r} dr \\
&= \frac{C^* e^c}{a} \int_{-\infty}^{c} \left(b + \sqrt{-\frac{r}{a}} \right)^{2k+1} e^{-r} dr \\
&= \frac{C^* e^c}{a} \sum_{i=0}^{2k+1} \binom{2k+1}{i} b^{2k+1-i} \int_{-\infty}^{c} \left(\sqrt{-\frac{r}{a}} \right)^{i} e^{-r} dr \\
&= \frac{C^* e^c}{a} \sum_{i=0}^{2k+1} \binom{2k+1}{i} \frac{b^{2k+1-i}}{(-a)^{\frac{i}{2}}} \int_{-\infty}^{c} r^{\frac{i}{2}} e^{-r} dr \\
&= \frac{C^* e^c}{a} \sum_{i=0}^{2k+1} \binom{2k+1}{i} \frac{b^{2k+1-i}}{(-a)^{\frac{i}{2}}} \int_{-\infty}^{c} r^{\frac{i}{2}} e^{-r} dr
\end{aligned}
$$