$$5.3 \quad \left(\mu_\ell - \frac{\nabla V}{2\ell} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A_\ell(\hat{x}) - i\int d\hat{p}\, d\hat{\eta} \int_{-\infty}^{0} d\tau\, e^{i(\nu\dot{\theta} - \mu_\ell)\tau} \frac{\partial \bar{f}_0}{\partial \hat{\eta}} A_\ell(\hat{x}_+) = 0$$

$\dot{\theta} = 0$ because Parallel beam

$$\left(\mu_\ell - \frac{\nabla V}{2\ell} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A_\ell(\hat{x}) = i\int d\hat{p}\, d\hat{\eta} \int_{-\infty}^{0} d\tau\, e^{-\mu_\ell \tau} \frac{\partial \bar{f}_0}{\partial \hat{\eta}} A_\ell(\hat{x}_+)$$

$$\left(\mu_\ell - \frac{\nabla V}{2\ell} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A_\ell(\hat{x}) = i\int d\hat{p}\, d\hat{\eta} \int_{-\infty}^{0} d\tau\, e^{-\mu_\ell \tau} \frac{\delta(\hat{\eta})\,\delta(\hat{p})\,U(\hat{x})}{\partial \hat{\eta}} A_\ell(\hat{x}_+)$$

$$\left(\mu_\ell - \frac{\nabla V}{2\ell} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A_\ell(\hat{x}) = \cancel{i}\int \frac{d\hat{p}\, d\hat{\eta}}{\cancel{i}\mu^2} \frac{\delta(\hat{\eta})\,\delta(\hat{p})}{\partial \hat{\eta}} U(\hat{x}) A_\ell(\hat{x}_+)$$

$$\left(\mu_\ell - \frac{\nabla V}{2\ell} + \frac{1}{2}\hat{\nabla}_\perp^2\right) A_\ell(\hat{x}) = \frac{U(\hat{x}) A_\ell(\hat{x}_+)}{\mu^2}$$

$$\frac{1}{2}\hat{\nabla}_\perp^2 A_\ell(\hat{x}) + \left[\mu_\ell - \frac{\Delta V}{2\ell} - \frac{U(\hat{x})}{\mu^2}\right] A_\ell(\hat{x}) = 0$$

$$U(\hat{x}) = 1 - \frac{|\hat{x}|^2}{2\sigma_x^2} = 1 - \frac{r^2}{2\sigma_x^2}$$

Substitute!

$$\frac{1}{2}\hat{\nabla}_\perp^2 A_\ell(\hat{x}) + \left[\mu_\ell - \frac{\Delta V}{2\ell} - \frac{1}{\mu^2} + \frac{r^2}{\mu^2 2\sigma_x^2}\right] A_\ell(\hat{x}) = 0$$

Separation of variables in cylindrical coordinates:

$$A_\ell(\hat{x}) = A_{\ell,m} e^{im\varphi} \quad \text{e.g.}$$

$$\frac{1}{2}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial}{\partial \hat{r}}\left(A_{\ell,m}(\hat{r})\, e^{im\varphi}\right)\right) + \frac{1}{\hat{r}^2}\frac{\partial^2}{\partial \varphi^2}\left(A_{\ell m}(\hat{r})\, e^{im\varphi}\right)\right]$$

$$+\left[\mu_\ell - \frac{\Delta V}{2\ell} - \frac{1}{\mu^2} + \frac{r^2}{2\mu\sigma_x^2}\right] A_{\ell m}(\hat{r})\, e^{i\varphi m} = 0$$

$$\frac{1}{2}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial}{\partial \hat{r}}(A_{\ell,m}(\hat{r}))\cancel{e^{i\varphi m}}\right) + \frac{1}{\hat{r}^2}(-m^2) A_{\ell,m}(\hat{r})\cancel{e^{im\varphi}}\right] + \left[\mu - \frac{\Delta V}{2\ell} - \frac{1}{\mu^2} + \frac{r^2}{\mu^2 2\sigma_x^2}\right] A_{\ell,m}(\hat{r})\cancel{e^{i\varphi m}} = 0$$