Assignment 2

AI1110: Probability and Random Variables
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12.13.6.14: If each element of a $2 * 2$ determinant is either zero or one. What is the probability that the value of the determinant is positive ?
(Assume that the individual entries of the determinant are chosen independently each value being assumed with probability $\frac{1}{2}$)

Solution: Let us assume the $2 * 2$ determinant as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

(a) Given determinant of the taken matrix is positive,Hence

$$ad - bc > 0 \tag{1}$$
$$a > \frac{bc}{d} \tag{2}$$

(b) Probability of the determinant to be positive is

$$\Pr\left(a > \frac{bc}{d}\right) = 1 - \Pr\left(a \leq \frac{bc}{d}\right) \tag{3}$$

we have to find $\Pr\left(a \leq \frac{bc}{d}\right)$

$$F_A\left(\frac{bc}{d}\right) = \Pr\left(a \leq \frac{bc}{d}\right) \tag{4}$$

where $F_A(x)$ represents cdf of a. Now finding cdf of a,

$$F_A(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } 0 \leqslant x < 1, \\ 1 & \text{if } 1 \leqslant x < \infty \end{cases} \tag{5}$$

(c) Taking expectation of $F_A\left(\frac{bc}{d}\right)$ with respect to d we get,

$$E_d\left(F_A\left(\frac{bc}{d}\right)\right) = \frac{1}{2}F_A(bc) + \frac{1}{2}F_A(\infty) \tag{6}$$
$$= \frac{1}{2}F_A(bc) + \frac{1}{2} \tag{7}$$

Expectation of the above with respect to b we get,

$$E_b\left(\frac{1}{2}F_A(bc) + \frac{1}{2}\right) = \frac{1}{2}E_b(F_A(bc)) + \frac{1}{2} \tag{8}$$
$$= \frac{1}{2}\left(\frac{1}{2}F_A(0) + \frac{1}{2}F_A(c)\right) \tag{9}$$
$$= \frac{1}{2} + \frac{1}{8} + \frac{1}{4}F_A(c) \tag{10}$$
$$= \frac{5}{8} + \frac{1}{4}F_A(c) \tag{11}$$

Expectation of this with respect to c will be,

$$E_c\left(\frac{5}{8}+\frac{1}{4}F_A(c)\right)=\frac{5}{8}+\frac{1}{4}E_c\left(F_A(c)\right) \tag{12}$$

$$=\frac{5}{8}+\frac{1}{4}\left(\frac{1}{2}F_A(0)+\frac{1}{2}F_A(1)\right) \tag{13}$$

$$=\frac{5}{8}+\frac{1}{4}\left(\frac{1}{4}+\frac{1}{2}\right) \tag{14}$$

$$=\frac{5}{8}+\frac{3}{16} \tag{15}$$

$$=\frac{13}{16} \tag{16}$$

(d) Now required probability is

$$\Pr\left(a>\frac{bc}{d}\right)=1-E_b,c,d\left(F_A\left(\frac{bc}{d}\right)\right) \tag{17}$$

$$=1-\frac{13}{16} \tag{18}$$

$$=\frac{3}{16} \tag{19}$$