Problem Set I

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An Aggie does not lie, cheat or steal.
Nor does an Aggie tolerate those who do.

Problem 1

A. Prove that $\log_c(ab) = \log_c a + \log_c b$

Let $x = \log_c a$, $y = \log_c b$
Using the definition of the logarithm, $c^x = a$,$c^y = b$.
$ab = c^x \cdot c^y$
$ab = c^{x+y}$
$\log_c ab = x + y$

Substituting back for x and y,
$\log_c ab = \log_c a + \log_c b$
$\square$

B. Prove that $\log_c(a^b) = b \log_c a$

Let $x = \log_c(a)$
Using the definition of the logarithm, $c^x = a$

Substituting for a:
$\log_c(a^b) = \log_c((c^x)^b)$
$= \log_c(c^{xb})$
$= xb$

Substituting x back for its definition
$xb = b \log_c(a)$
$\therefore \log_c(a^b) = b \log_c(a)$
$\square$

C. Prove that $\log_b a = \dfrac{\log_c a}{\log_c b}$

Let $x = \log_b(a)$
Using the definition of the logarithm, $b^x = a$
Taking a logarithm to both sides $\log_c(b^x) = \log_c a$

Using the product rule, $x \log_c b = \log_c a$
$x = \dfrac{\log_c a}{\log_c b}$
Substituting x for its definition,
$\log_b a = \dfrac{\log_c a}{\log_c b}$
$\square$

Problem 2

Proving the Geometric Series Identity
Let $S_n = a^0 + a^1 + a^2 + \cdots + a^{n-1}$
$aS_n = a^1 + a^2 + a^3 + \cdots + a^n$
$aS_n - S_n = a^1 + a^2 + \cdots + a^n - (a^0 + a^1 + \cdots a^{n-1})$
$S_n(a-1) = a^n - a^0$
$S_n = \dfrac{a^n - 1}{a - 1}, a \neq 1$

Let $S_n = 0 + \dfrac{1}{4} + \dfrac{2}{4^2} + \cdots + \dfrac{n-1}{4^{n-1}}$
$4S_n = \dfrac{1}{4^0} + \dfrac{2}{4^2} + \cdots + \dfrac{n-1}{4^{n-2}}$
$4S_n - S_n = \dfrac{1}{4^0} + \dfrac{2}{4^1} - \dfrac{1}{4^1} + \dfrac{3}{4^2} - \dfrac{2}{4^2} + \cdots + \dfrac{n-1}{4^{n-1}} - \dfrac{n-2}{4^{n-1}}$
$3S_n = \dfrac{1}{4^0} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \cdots + \dfrac{1}{4^{n-1}}$
$3S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{4^k}$

Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{4^k}$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{4^k} &= \lim_{k\to\infty} \frac{\left(\frac{1}{4}\right)^n - 1}{-\frac{3}{4}} \\
&= \lim_{k\to\infty} -\frac{4}{3}\left[\left(\frac{1}{4}\right)^n - 1\right] \\
&= -\frac{4}{3}\lim_{k\to\infty}\left[\left(\frac{1}{4}\right)^k - 1\right] \\
&= \frac{4}{3}
\end{aligned}$$

Substitute this value back into the original series

$3S_n = \sum_{k=0}^{n} \frac{1}{4^k}$

$3S_\infty = \sum_{k=0}^{\infty} \frac{1}{4^k}$

$3S_\infty = \frac{4}{3}$

$S_\infty = \frac{4}{9}$

$$\sum_{i=0}^{\infty} \frac{i}{4^i} = \frac{4}{9}$$

Problem 3

$\phi^1 = \phi$
$\phi^2 = \phi + 1$
$\phi^3 = \phi \cdot \phi^2 = \phi(\phi + 1) = \phi^2 + \phi = 2\phi + 1$
$\phi^4 = (\phi^2)^2 = (\phi + 1)^2 = \phi^2 + 2\phi + 1 = 3\phi + 2$
$\phi^5 = \phi^2\phi^3 = (\phi + 1)(2\phi + 1) = 2\phi^2 + 3\phi + 1 = 5\phi + 3$
$\phi^6 = (\phi^3)^2 = (2\phi + 1)^2 = 4\phi^2 + 2\phi + 1 = 8\phi + 5$

Using this pattern
$\phi^n = f_n\phi + f_{n-1}$
$\because f_{n-1} > 0, \forall n \geq 2$
$\therefore f_n\phi + f_{n-1}$ is also always positive.
$\therefore$, $f_n \leq \phi^n$
$\square$

Problem 4

Prove that the sum of the first n odd integers equals n^2

Let $P(n) := 1 + 3 + 5 + \cdots + (2n - 1)$

Base case:
$P(1) = 1 \equiv 1^2$

Inductive case:
Assume $P(n) = n^2$

$$\begin{aligned} P(n+1) &= 1 + 3 + 5 + \cdots + 2n - 1 + 2(n+1) - 1 \\ &= 1 + 3 + 5 + \cdots + 2n - 1 + 2n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

$\therefore P(n) = n^2$
$\square$