CSCE 222 (Carlisle), Discrete Structures for Computing
Spring 2022
Homework 7

Type your name below the pledge to sign

On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work.

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Instructions:

- The exercises are from the textbook. You are encouraged to work extra problems to aid in your learning; remember, the solutions to the odd-numbered problems are in the back of the book.

- Grading will be based on correctness, clarity, and whether your solution is of the appropriate length.

- Always justify your answers.

- Don't forget to acknowledge all sources of assistance in the section below, and write up your solutions on your own.

- *Turn in .pdf file to Gradescope by the start of class on Monday, March 7, 2022.* It is simpler to put each problem on its own page using the LaTeX clearpage command.

Help Received:

- Rosen, Kenneth H. *Discrete Mathematics and Its Applications*. McGraw-Hill, 2019.

Exercises for Section 5.1:

6: (2 point).

Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a non-negative integer.

Basis Step:
$P(n) = n \cdot n!$
$P(0) = 0$
$(0+1)! - 1 = 1 - 1 = 0$
$n \cdot n! = (n+1)! - 1$ when $n = 0$

Induction Hypothesis:
$$\sum_{i=0}^{n}(n \cdot n!) = (n+1)! - 1$$

Induction Step:
$$\sum_{i=0}^{n+1}(n \cdot n!)$$
$$= \sum_{i=0}^{n}(n \cdot n!) + (n+1)(n+1)!$$
$$= (n+1)! - 1 + (n+1)(n+1)!$$
$$= (n+1)!(1 - 1 + (n+1))$$
$$= (n+1)(n+1)!$$

18(a-f): (3 points).
Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

1. What is the statement $P(2)$?
   $P(2) \equiv 2! < 2^2$

2. Show that $P(2)$ is true, completing the basis step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
   $2 < 4$
   $P(2) \equiv$ True

3. What is the inductive hypothesis of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
   Let $P(k)$ be True for $k \geq 2$.
   $k! < k^k$

4. What do you need to prove in the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1?
   Prove that $P(k+1)$ is also True.
   $(k+1)! < (k+1)^{k+1}$

5. Complete the inductive step of a proof by mathematical induction that $P(n)$ is true for all integers n greater than 1.
   $(k+1)! = (k+1)k$
   $(k+1)k! < (k+1)^k < (k+1)(k+1)^k$
   $(k+1)(k+1)^k = (k+1)^{k+1}$
   Therefore, $(k+1)! < (k+1)^{k+1}$

6. Explain why these steps show that this inequality is true whenever n is an integer greater than 1.
   Since we have completed the base and inductive steps, by the principle of mathematical induction, the inequality is true for any $n \geq 2$. If we had shown $P(3)$ as our basis step, then the inequality would only be proven for $n \geq 3$.

32:(3 points).
Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Base Case:
If $n = 1$, $1^3 + 2(1) = 3$. So it is divisible by 3.

Inductive Step:
$(n + 1)^3 + 2(n + 1)$
$= n^3 + 3n^2 + 3n + 1 + 2n + 2$
$= (n^3 + 2n) + (3n^2 + 3n + 3)$
$= (n^3 + 2n) + 3(n^2 + n + 1)$

$3(n^2 + n + 1)$ is divisible by 3.
Therefore, $n^3 + 2n$ is divisible by 3 whenever n is a positive integer.

Exercises for Section 5.2:

6(a-c): (3 points)

1. Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.
   We can form the following amounts of postage as indicated:
   $3 = 3, 6 = 3 + 3, 9 = 3 + 3 + 3, 10 = 10, 12 = 3 + 3 + 3 + 3, 13 = 10 + 3,$
   $15 = 3+3+3+3+3, 16 = 10+3+3, 18 = 3+3+3+3+3+3, 19 = 10+3+3+3, 20 = 10+10.$
   By having considered all the combinations, we know that the gaps in this list cannot be filled.
   We claim that we can form all amounts of postage greater than or equal to 18 cents using just 3-cent and 10-cent stamps.

2. Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

   Let $P(n)$ be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps.
   We want to prove that $P(n)$ is true for all $n \geq 18$.
   The basis step, $n = 18$, is handled above. $(18 = 3 + 3 + 3 + 3 + 3 + 3)$

   The Inductive Hypothesis
   Assume that we can form k cents of postage.
   Show how to form $k + 1$ cents of postage.

   The Inductive Step
   If the k cents included two 10-cent stamps, then replace them by seven 3-cent stamps $(7 \cdot 3 = 2 \cdot 10 + 1)$.
   Otherwise, k cents was formed either from just 3-cent stamps, or from one 10-cent stamp and $k - 10$ cents in 3-cent stamps.
   Because $k \geq 18$, there must be at least three 3-cent stamps involved in either case. Replace three 3-cent stamps by one 10-cent stamp, and we have formed $k + 1$ cents in postage $(10 = 3 \cdot 3 + 1)$.

3. Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

$P(n)$ is the same as in part (b).

To prove that $P(n)$ is true for all $n \geq 18$, we note for the basis step that from part (a), $P(n)$ is true for $n = 18, 19, 20$.

Assume the inductive hypothesis, that $P(j)$ is true for all j with $18 \leq j \leq k$, where $k \geq 20$

We want to show that $P(k+1)$ is true.

Because $k - 2 \geq 18$, we know that $P(k-2)$ is true.

Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

10: (3 points)

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Let $P(n)$ be the statement "A chocolate bar can be broken in $n - 1$ breaks".

Basis Step: $n = 1$
Consider a chocolate bar that consists of 1 square.
The chocolate bar is then already completely broken into pieces and thus requires $0 = 1-1 = n-1$ breaks to completely break the chocolate bar.
Therefore, $P(1)$ is true.

Inductive Step:
Assume that $P(1), P(2), \cdots, P(k)$ are all true, such that the chocolate bar consisting of i squares requires i breaks, where $1 \leq i \leq k$.

From this, prove that $P(k + 1)$ is true.
Consider a chocolate bar with $k + 1$ squares.
Let m and q be positive integers such that $k + 1 = mg$ and such that the squares of the chocolate bar form a rectangle with m rows and q columns.
Let the first break be breaking off the last column of the rectangle.
Then divide the chocolate bar in a $(m - 1) \times q$ rectangle and in a $1 \times q$ rectangle.
Since $P((m - 1)q)$ is true, we require $(m - 1)g - 1$ breaks to break the $(m - 1) \times q$ rectangle.
Since $P(q)$ is true, we require $q - 1$ breaks to break the $1 \times q$ rectangle.
The chocolate bar require $(m - 1)q - 1 + (q - 1) + 1$ breaks to complete break the chocolate bar consisting of $k + 1$ squares.

$$(m - 1)q - 1 + (q - 1) + 1 = mq - q - 1 + q - 1 + 1 = mq - 1$$
From this, $(k + 1) - 1$ breaks to is required to break a chocolate bar with $k + 1$ squares and thus $P(k + 1)$ is true.

By the principle of strong induction, $P(n)$ is true for all positive integers n. The chocolate bar requires $n - 1$ breaks to break.

12: (3 points)

Use strong induction to show that every positive integer can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on. [Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $\dfrac{k + 1}{2}$ is an integer.]

Base Case:

For $n = 1$, we can write $1 = 2^0$.

Inductive Step:

Suppose that for $1 \leq k \leq n$ we can write k as a sum of distinct powers of 2.
We want to show that $n + 1$ can be written as a sum of distinct powers of two.

If $n + 1$ is even, then $\dfrac{n+1}{2}$ is an integer, and $1 \leq \dfrac{n+1}{2} \leq n$, so by the inductive hypothesis we can write $\dfrac{n+1}{2} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_j}$, where $a_1, \cdots, a_j$ are all distinct.
Multiplying both sides by 2, $n+1 = 2 \times (2^{a_1} + 2^{a_2} + \cdots + 2^{a_j}) = 2^{a_1+1} + 2a_2 + 1 + \cdots + 2^{a_j+1}$.
Because $a_1, a_2, \cdots, a_j$ were distinct, $a_1 + 1, a_2 + 1, \cdots, a_j + 1$ are distinct.

If $n + 1$ is odd, then by by inductive hypothesis we can write $n = 2^{a_1} + ... + 2^{a_j}$, where $a_1, \cdots, a_j$ are distinct powers.
Taking both sides mod 2, all non-zero powers of 2 drop out on the right hand side, and the left hand side is 0 (because n is even).
Because of this, $0 \notin a_1, a_2, \cdots a_j$.
From this, $n + 1 = 2^0 + 2^{a_1} + 2^{a_2} + \cdots + 2^{a_j}$.

32: (3 points)

Find the flaw with the following "proof" that every postage of three cents or more can be formed using just 3-cent and 4-cent stamps.

Basis Step: We can form postage of three cents with a single 3-cent stamp and we can form postage of four cents using a single 4-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all non-negative integers j with $j \leq k$ using just 3-cent and 4-cent stamps. We can then form postage of $k + 1$ cents by replacing one 3-cent stamp with a 4-cent stamp or by replacing two 4-cent stamps by three 3-cent stamps.

Forming a 5-cent stamp would be impossible with only 3 and 4-cent stamps. Because of this, the Basis Step would be incorrect.