CSCE 222 (Carlisle), Discrete Structures for Computing
Spring 2022
Homework 4

Type your name below the pledge to sign
On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work.
HUY QUANG LAI

Instructions:

- The exercises are from the textbook. You are encouraged to work extra problems to aid in your learning; remember, the solutions to the odd-numbered problems are in the back of the book.
- Grading will be based on correctness, clarity, and whether your solution is of the appropriate length.
- Always justify your answers.
- Don't forget to acknowledge all sources of assistance in the section below, and write up your solutions on your own.
- *Turn in .pdf file to Gradescope by the start of class on Monday, February 14, 2022.* It is simpler to put each problem on its own page using the LaTeX clearpage command.

Help Received:

- Rosen, Kenneth H. *Discrete Mathematics and Its Applications*. McGraw-Hill, 2019.

Exercises for Section 2.3:

8 (a,c,e,g): (2 points)
$\lfloor 1.1 \rfloor = 1$
$\lfloor -0.1 \rfloor = -1$
$\lceil 2.99 \rceil = 3$
$\left\lfloor \frac{1}{2} + \left\lceil \frac{3}{2} \right\rceil \right\rfloor = 2$

20(a-d): (2 points)
Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is

1. one-to-one but not onto.
   $f(x) = 2x$

2. onto but not one-to-one.
   $f(x) = \begin{cases} 1 & x = 1 \\ x - 1 & x \neq 1 \end{cases}$

3. both onto and one-to-one (but different from the identity function).
   $f(x) = \begin{cases} 2 & x = 1 \\ 1 & x = 2 \\ x & x > 2 \end{cases}$

4. neither one-to-one nor onto.
   $f(x) = 1$

34(a): (2 points)
Suppose that g is a function from A to B and f is a function from B to C. Prove each of these statements.
If $f \circ g$ is onto, then f must also be onto. Since $f \circ g$ is surjective, then for every $c \in C$ there exists $a \in A$ such that $f(g(a)) = c$.
Let $b = g(a)$.
Then $b \in B$ and $f(b) = f(g(a)) = c$.
Thus $f(b) = c$. Hence f is surjective.

38: (2 points)
Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.
$f \circ g = (x + 2)^2 + 1 = x^2 + 2x^2 + 5$
$g \circ f = x^2 + 1 + 2 = x^2 + 3$

58: (2 points)
Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a \leq n \leq b$.
$a \leq n \leftrightarrow \lceil a \rceil \leq n$ and $n \leq b \leftrightarrow n \leq \lfloor b \rfloor$
Because of this, $\lceil a \rceil \leq n \leq \lfloor b \rfloor$.
The number of integers satisfying the inequality will be $\lfloor b \rfloor - \lceil a \rceil + 1$.

Exercises for Section 2.4:

4c: (1 point)
What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals $7 + 4^n$.
$a_0 = 8, a_1 = 11, a_2 = 23, a_3 = 71$

10d: (1 point)
Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.
$a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
$a_0 = -1, a_1 = 0, a_2 = 1, a_3 = 3, a_4 = 13, a_5 = 74$

14f: (1 points)
For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)
$a_n = n^2 + n$
$a_0 = 0$
$a_n = a_{n-1} + 2n$

18(a-c): (2 points)
A person deposits \$1000 in an account that yields 9% interest compounded annually.
Set up a recurrence relation for the amount in the account at the end of n years.
$a_0 = 1000$
$a_n = 1.09 \cdot a_{n-1}$

Find an explicit formula for the amount in the account at the end of n years.
$a_n = 1000 \cdot 1.09^n$e

How much money will the account contain after 100 years?
$a_100 = \$5529040.79$

22(a-c): (2 points)

An employee joined a company in 2017 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

Set up a recurrence relation for the salary of this employee n years after 2017.

$a_0 = 50000$

$a_n = 1.05a_{n+1} + 1000$

What will the salary of this employee be in 2025?

$2017, a_0 = 50000$

$2018, a_1 = 50000 \cdot 1.05 + 1000 = 53500$

$2019, a_2 = 53500 \cdot 1.05 + 1000 = 57175$

$2020, a_3 = 57175 \cdot 1.05 + 1000 = 61033.75$

$2021, a_4 = 61033.75 \cdot 1.05 + 1000 = 65065.4375$

$2022, a_5 = 65065.4375 \cdot 1.05 + 1000 = 69318.709375$

$2023, a_6 = 69318.709375 \cdot 1.05 + 1000 = 73784.6448437$

$2024, a_7 = 73784.6448437 \cdot 1.05 + 1000 = 78473.8770859$

$2025, a_8 = 78473.8770859 \cdot 1.05 + 1000 = 83397.5709402$

\$83397.57

Find an explicit formula for the salary of this employee n years after 2017.

$a_0 = 50000, k = 1.05, d = 1000$

$a_1 = ka_0 + d$

$a_2 = ka_1 + d = k(ka_0 + d) + d = k^2 a_0 + (k+1)d$

$a_3 = ka_2 + d = k(k^2 a_0 + (k+1)d) + d = k^3 a_0 + (k^2 + k + 1)d$

From the coefficient for d, $\displaystyle\sum_{i=0}^{n} k^i = \frac{k^n - 1}{k - 1}$

From the pattern, $a_n = a_0 \cdot k^n + \dfrac{k^n - 1}{k - 1} d$

Substituting values back into variables:

$$a_n = 50000 \cdot 1.05^n + \frac{1.05^n - 1}{0.05} d$$

24(a-b): (2 points)
Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [Hint: Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $\frac{r}{12}$.]
$B(k) = \left(1 + \dfrac{r}{12}\right) B(k-1) - P$

Determine what the monthly payment P should be so that the loan is paid off after T months.
Let $m = 1 + \frac{r}{12}$. Then,

$$\begin{aligned}
B(k) &= mB(k-1) - P \\
&= m(mB(k-2) - P) - P \\
&= m^2B(k-2) - (m+1)P \\
&= m^2(mB(k-3) - P) - (m+1)P \\
&= m^3B(k-3) - (m^2+m+1)P \\
&= m^kB(0) - \frac{m^k-1}{m-1}P
\end{aligned}$$

Let $k = T$ such that $B(T) = 0$
$m^TB(0) - \dfrac{m^T-1}{m-1}P = 0$
$P = \dfrac{m^TB(0)(m-1)}{m^T-1}$
Substituting for variables:

$$P = \frac{\left(\frac{r}{12}+1\right)^T B(0)\frac{r}{12}}{\left(\frac{r}{12}+1\right)^T - 1}$$

40: (1 points)
$\displaystyle\sum_{k=99}^{200} k^3$
$\displaystyle= \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3$
$= \dfrac{200^2(201)^2}{4} - \dfrac{98^2(99^2)}{4}$
$= 404010000 - 23532201$
$= 380477799$