1. Expand $f(u) = \frac{1}{\sqrt{1-u}}$, then subs. $u = x^2$

$$f(u) = 1 + u\cdot\left(\tfrac{1}{2}\right) + \frac{u^2}{2!}\cdot\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right) + \frac{u^3}{3!}\cdot\left(\tfrac{1}{2}\right)\cdot\tfrac{3}{2}\cdot\tfrac{5}{2} + \cdots$$

$$= \cancel{1\cdot 2\cdot 3\cdot}\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \cancel{x}\, u^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} x^{2n}$$

$$1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{2\cdot 4\cdot 6\cdots(2n)}$$

$$2\cdot 4\cdot 6\cdots 2n = 2^n\cdot n!$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^{2n}$$

2. $$\int f(x)\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \theta\, d\theta = \theta = \operatorname{asin} x$$

$$\operatorname{asin} x = \int \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{(2n+1)}$$

3. $$\operatorname{asin}(x) = P_n(x) + R_n(x)$$

$$P_n(x) = \sum_{0}^{n} \frac{(2n)!}{2^{2n}(n!)^2} \frac{x^{2n+1}}{(2n+1)}$$

$$R_n(x) = \sum_{n+1}^{\infty} \frac{(2n+2)!}{2^{2n+2}\,((n+1)!)^2} \frac{x^{2n+3}}{2n+3}$$

$$\operatorname{asin}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}, \quad x = \tfrac{1}{2}$$

$$6\,(P_n(x) + R_n(x)) = \pi$$

$$6\,R_n(x) < 0.001$$

Let $T_n = \frac{(2n)!}{2^{2n}(n!)^2} \frac{(0.5)^{2n}}{2n+1}$

$T_{n+1} = \frac{(2n+2)!}{2^{2n+2}(n+1)!^2} \frac{(0.5)^{2n+2}}{2n+3}$

$\frac{T_{n+1}}{T_n} = \frac{(2n+1)(2n+2)}{2^2(n+1)(n+1)} \frac{2n+1}{2n+3} 0.5^2$

$= \frac{(2n+1)\cancel{(2n+2)}}{(2n+2)\cancel{(2n+2)}} \frac{2n+1}{2n+3} 0.5^2 < 0.5^2$

$\sum_n^\infty T_n = a + 0.5^2 a + 0.5^4 a + 0.5^6 a + \dots 0.5^\infty a$

= Geometric series , $r = 0.25$

$\sum_n^\infty T_n = \frac{a(1-r^\infty)}{1-r} = \frac{a}{1-r} > R_n$

$R_n < 0.001$ $\Rightarrow$ $a < \cancel{1.875 \times 10^{-4}}$ 1.25×10^{-4}

$\frac{a}{1-r} < 0.001$

$a = \frac{(2n+2)!}{2^{2n+2}(n+1)!^2} \frac{0.5^{2n+3}}{2n+3} < 0.5^{2n+3}$

$0.5^{2n+3} < \cancel{1.875 \times 10^{-4}}$ 1.25×10^{-4}

$n \approx 5$, $0.5^{13} = 1.22 \times 10^{-4}$

Approx. 5 term is needed.

4. $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$, let $r = -t^2$

$$\frac{1}{1+t^2} = \sum_{k=0}^{n} (-1)^k t^{2k} + \frac{(-t^2)^{n+1}}{1+t^2}$$

$$\operatorname{atan} x = \int_0^x \frac{1}{1+t^2}\,dt = \underbrace{\sum_{k=0}^{n} (-1)^k (2k+1)^{-1} x^{2k+1}}_{P_n(x)} + \underbrace{(-1)^{n+1}\int_0^x \frac{t^{2n+2}}{1+t^2}\,dt}_{R_n(x)}$$

$$R_n(x) = \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt < \int_0^x t^{2n+2}\,dt = \left[\frac{t^{2n+3}}{2n+3}\right]_0^x$$

$\operatorname{atan}(1) = \frac{\pi}{4}$, $4\operatorname{atan}(1) = \pi$, $x = 1$

$4 R_n < 0.001$

$4 \frac{1}{2n+3} < 0.001$

$n > 1998.3 \approx 2000$

5. asin(x) need 5 term to converge, but atan x need 2000 term to converge to desired precision.