Kinematics Analysis of Four-bar mechanism

Homework support notes

2022-2023

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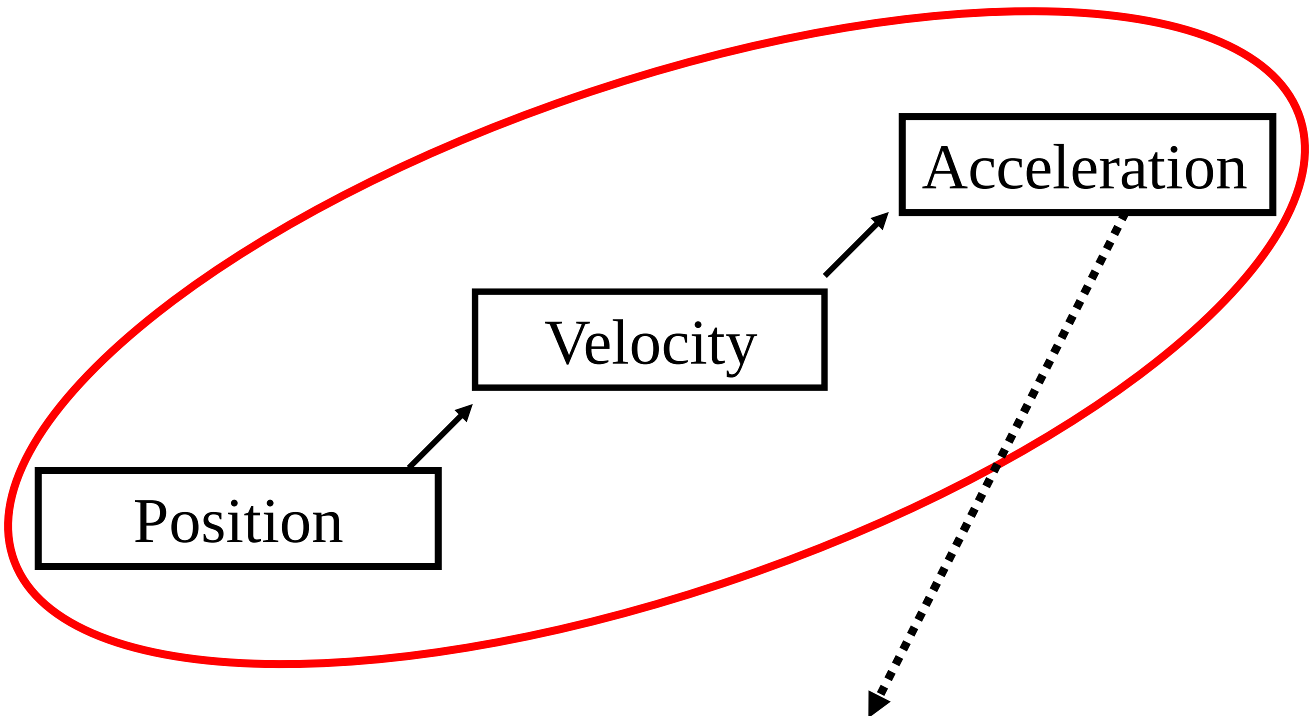
Kinematics analysis roadmap

Statics

Kinematics

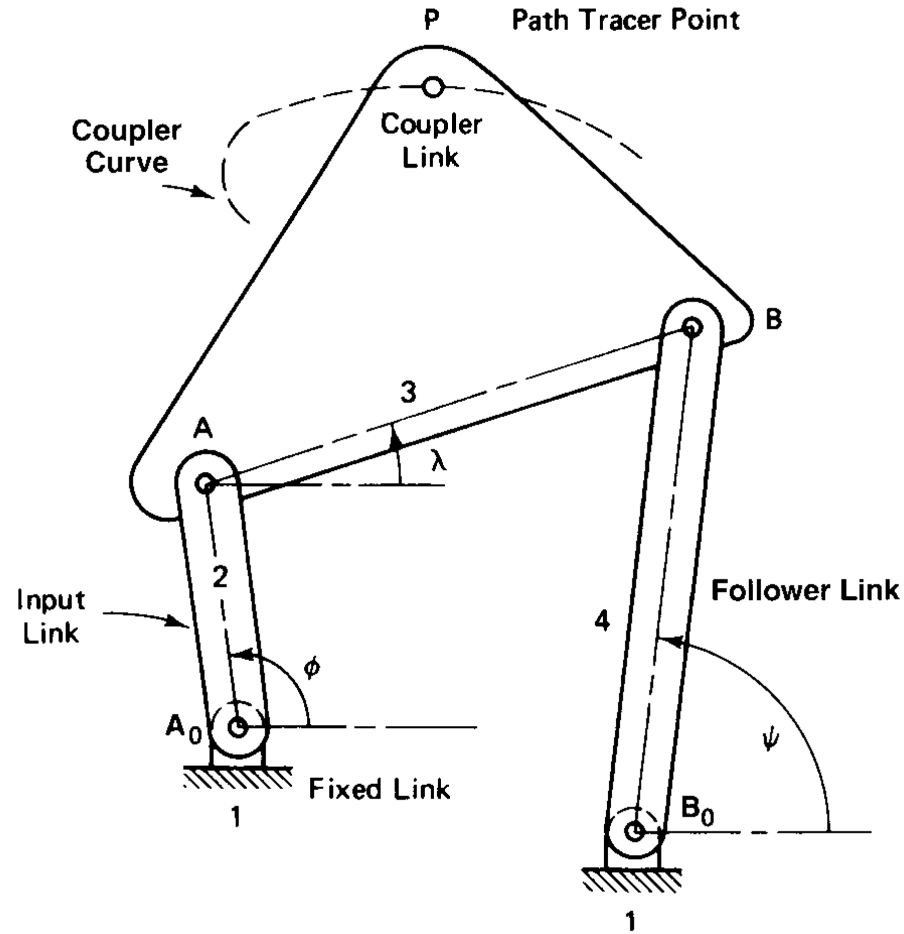


Kinetostatics

Dynamics

Four-Bar Linkage

- Simplest closed-loop linkage; consists of three **moving** links, one **fixed** link (1), and four revolute (pin) joints.
- Primary links are called: the **input** link (connected to power source) denoted by (2), the **output** or **follower** link (4), and **coupler** or **floating** link (3). The latter "couples" the input to the output link.
- Points as *P* on the coupler link generally trace out **sixth order** algebraic coupler curves.

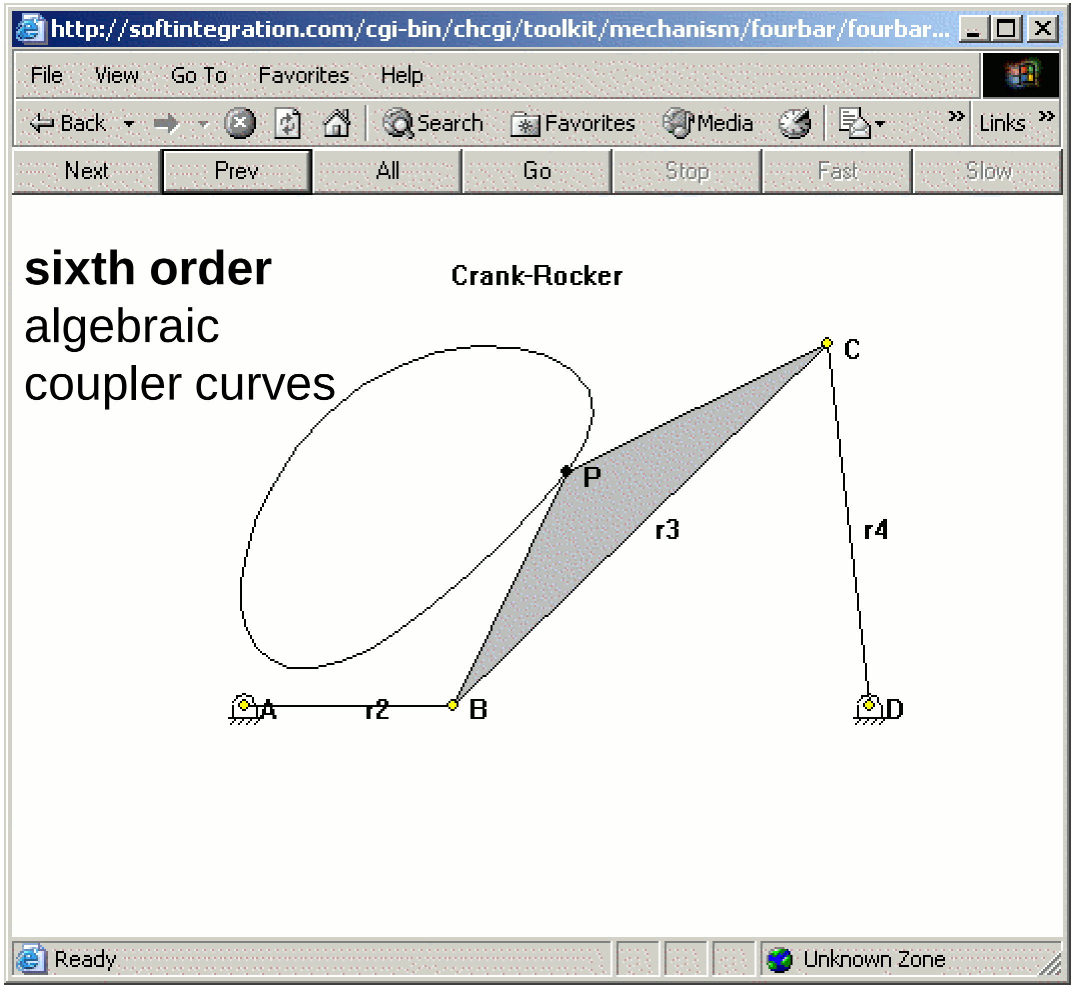


$$M_3 = 3\,m - 2\,l_p - h_p = 3\text{x}3 - 2\text{x}4 - 0 = 1$$

Number of independent loops: $N = l_p - n + 1 = l_p - m = 4 - 3 = 1$

Observation: *n* is the total number of elements

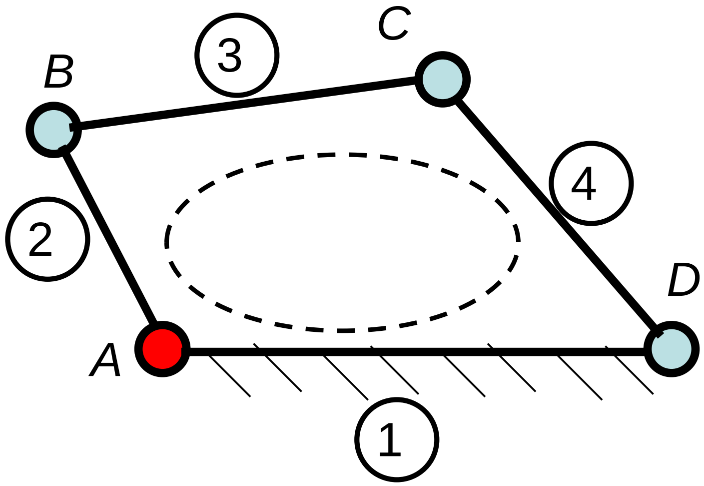
Four-bar mechanism



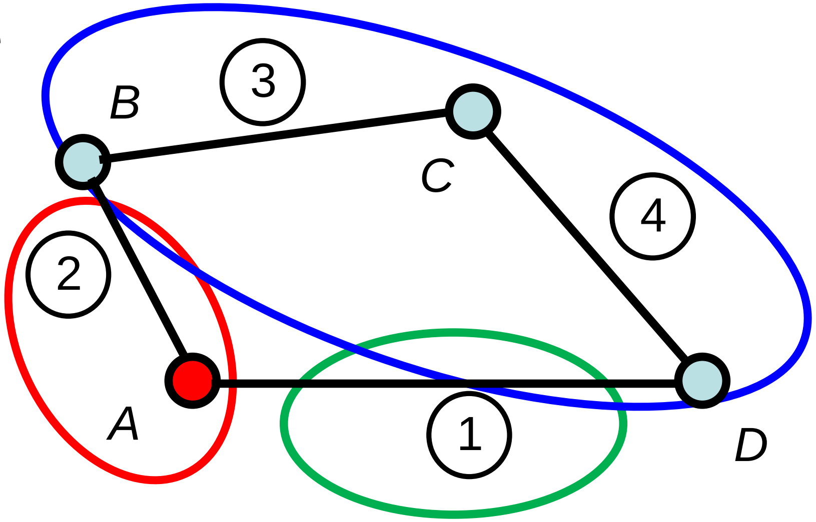
Simulator software: https://www.desmos.com/calculator/iuprdl6sxk

Four-bar mechanism

Skeleton Outline

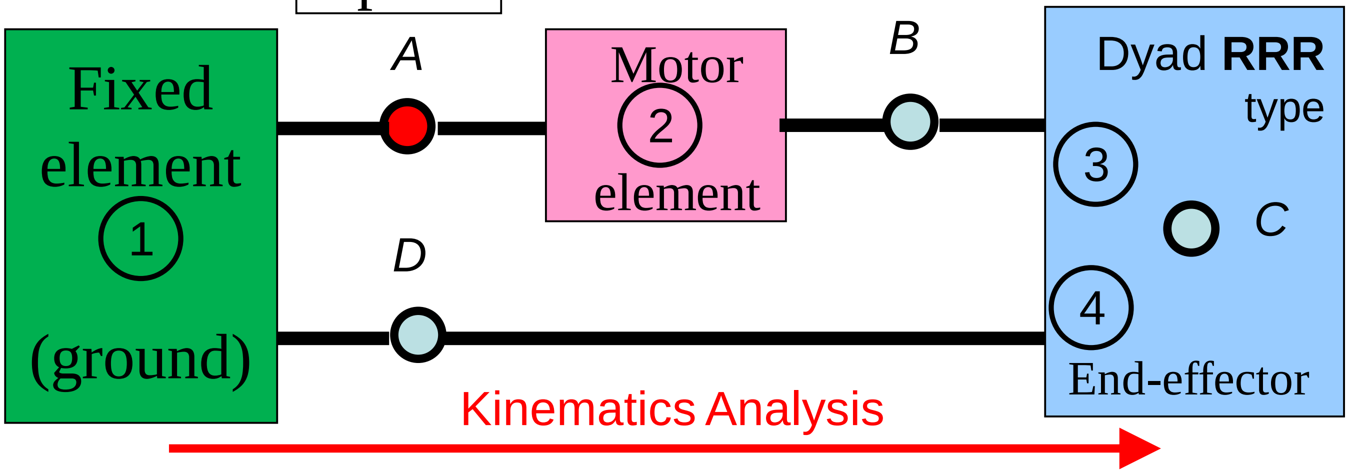


Structural scheme



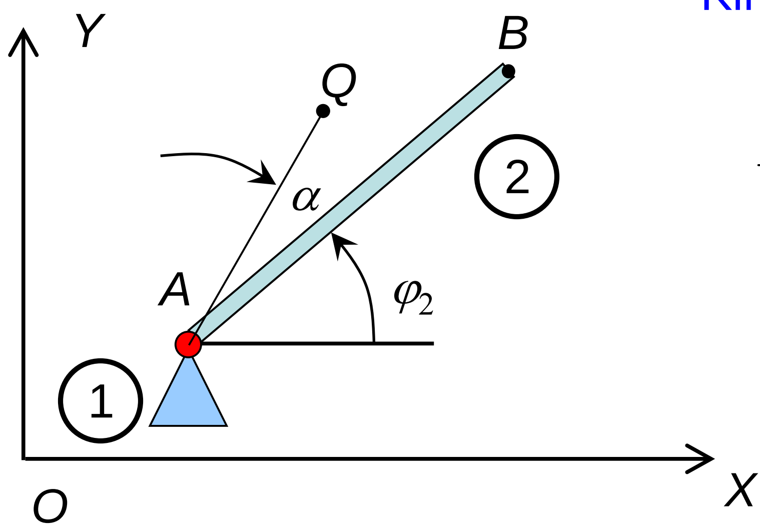
Active pair

Multipolar scheme (graph)



Kinematic analysis of motor element

Kinematic parameters of crank φ_2, ω_2 and ε_2 are known



$$\begin{cases} X_B = X_A + AB\cos\varphi_2 \\ Y_B = Y_A + AB\sin\varphi_2 \end{cases} \qquad \begin{cases} v_{BX} = -AB\,\omega_2\sin\varphi_2 \\ v_{BY} = AB\,\omega_2\cos\varphi_2 \end{cases}$$

$$\begin{cases} a_{BX} = -AB\,(\varepsilon_2\sin\varphi_2 + \omega_2^2\cos\varphi_2) \\ a_{BY} = AB\,(\varepsilon_2\cos\varphi_2 - \omega_2^2\sin\varphi_2) \end{cases}$$

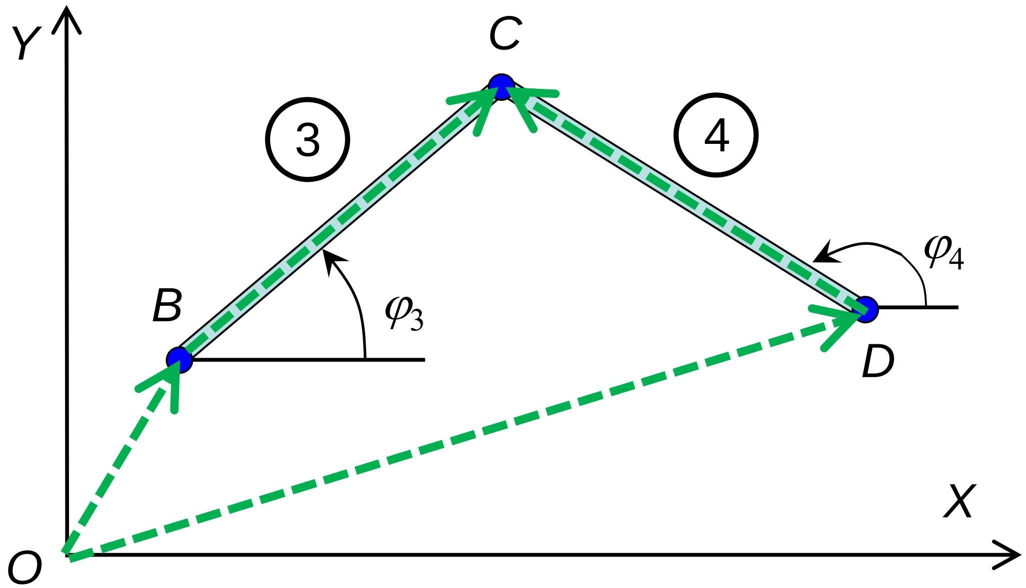
$$\begin{cases} \omega_2 = \text{angular speed of crank ②} \\ \varepsilon_2 = \text{angular acceleration of crank ②} \end{cases}$$

For other point of crank ② $Q(AQ, \alpha)$, outside direction AB, we have:

$$\begin{cases} X_Q = X_A + AQ\cos(\varphi_2 + \alpha) \\ Y_Q = Y_A + AQ\sin(\varphi_2 + \alpha) \end{cases} \qquad \begin{cases} v_{QX} = -AQ\,\omega_2\sin(\varphi_2 + \alpha) \\ v_{QY} = AQ\,\omega_2\cos(\varphi_2 + \alpha) \end{cases}$$

$$\begin{cases} a_{QX} = -AQ\,(\varepsilon_2\sin(\varphi_2 + \alpha) + \omega_2^2\cos(\varphi_2 + \alpha)) \\ a_{QY} = AQ\,(\varepsilon_2\cos(\varphi_2 + \alpha) - \omega_2^2\sin(\varphi_2 + \alpha)) \end{cases}$$

Kinematic analysis of Dyad - **RRR** type



Knowns: X_B, Y_B, X_D, Y_D, BC si CD

v_{BX}, v_{BY}, v_{DX}, v_{DY}, a_{BX}, a_{BY}, a_{DX}, a_{DY}

Unknowns: φ_3, φ_4, ω_3, ω_4, ε_3 and ε_4

$$N = l_p - m = 3 - 2 = 1$$

One single vectorial equation:

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OD} + \overrightarrow{DC}$$

which is projected on the reference axes resulting 2 analytic equations:

$$\begin{cases} X_B + BC\cos\varphi_3 = X_D + DC\cos\varphi_4 \\ Y_B + BC\sin\varphi_3 = Y_D + DC\sin\varphi_4 \end{cases}$$

non-linear equations ⇒ Numerical Methods

Newton-Raphson Method to solve non-linear equations is based on Taylor series

https://study.com/academy/lesson/newton-raphson-method-for-nonlinear-systems-of-equations.html

Positions (configurations) of Dyad **RRR** type

$$\begin{cases} BC\cos\varphi_3 - DC\cos\varphi_4 + X_B - X_D = 0 \\ BC\sin\varphi_3 - DC\sin\varphi_4 + Y_B - Y_D = 0 \end{cases}$$

$$\mathbf{\Phi} = \begin{bmatrix} \varphi_3 \\ \varphi_4 \end{bmatrix}$$ Unknown vector

$$\Rightarrow \quad \mathbf{F}(\mathbf{\Phi}) - \mathbf{B} = 0 \quad \text{where} \quad \mathbf{F}(\mathbf{\Phi}) = \begin{bmatrix} BC\cos\varphi_3 - DC\cos\varphi_4 \\ BC\sin\varphi_3 - DC\sin\varphi_4 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} X_D - X_B \\ Y_D - Y_B \end{bmatrix}$$

Newton-Raphson method for solving non-linear equations:

$$\mathbf{J}_{2x2} = \begin{bmatrix} -BC\sin\varphi_3 & DC\sin\varphi_4 \\ BC\cos\varphi_3 & -DC\cos\varphi_4 \end{bmatrix} \Rightarrow \quad \mathbf{J}\,\Delta\mathbf{\Phi} = \mathbf{B} - \mathbf{F} \quad \text{with} \quad \Delta\mathbf{\Phi} = \begin{bmatrix} \Delta\varphi_3 \\ \Delta\varphi_4 \end{bmatrix}$$

det($\mathbf{J}$) $\neq 0$ or $\varphi_3 \neq \varphi_4 + k\pi$

$$\boxed{\Delta\mathbf{\Phi} = \mathbf{J}^{-1}(\mathbf{B} - \mathbf{F})}$$

Initial values:

(from drawing !)

$$\mathbf{\Phi}^{(0)} = \begin{bmatrix} \varphi_3^{(0)} \\ \varphi_4^{(0)} \end{bmatrix} \Rightarrow \Delta\mathbf{\Phi}^{(1)} = \begin{bmatrix} \Delta\varphi_3^{(1)} \\ \Delta\varphi_4^{(1)} \end{bmatrix}$$

Solutions are: $\begin{cases} \varphi_3^{(j)} = \varphi_3^{(j-1)} + \Delta\varphi_3^{(j)} \\ \varphi_4^{(j)} = \varphi_4^{(j-1)} + \Delta\varphi_4^{(j)} \end{cases}$ while $\begin{cases} |\Delta\varphi_3^{(j)}| > e_1 \\ |\Delta\varphi_4^{(j)}| > e_2 \end{cases}$ $j \geq 1$ (number of iteration)

Errors $e_1 = e_2 < 0.001$ [radians] !

Velocities and accelerations of **RRR** Dyad

Velocities equations are:

$$\begin{cases} (-\,BC \sin \varphi_3)\, \omega_3 + (DC \sin \varphi_4)\, \omega_4 + v_{BX} - v_{DX} = 0 \\ (BC \cos \varphi_3)\, \omega_3 + (-\,DC \cos \varphi_4)\, \omega_4 + v_{BY} - v_{DY} = 0 \end{cases}$$

linear in $\mathbf{\Omega} = \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}$

$\Rightarrow$ $\mathbf{J}\,\mathbf{\Omega} - \mathbf{C} = 0$ where $\mathbf{C} = \begin{bmatrix} v_{DX} - v_{BX} \\ v_{DY} - v_{BY} \end{bmatrix}$ $\Rightarrow$ $\mathbf{\Omega} = \mathbf{J}^{-1}\,\mathbf{C}$ and $\begin{cases} v_{DX} = 0 \\ v_{DY} = 0 \end{cases}$

$\mathbf{J}^{-1}\,\mathbf{J} = \mathbf{I}_2$ where $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ linear equations with 4 equations and 4 unknowns (elements of $\mathbf{J}^{-1}$)

Accelerations equation in matrix form:

$\mathbf{J}\,\mathbf{E} + \mathbf{J}'\,\mathbf{\Omega}^2 - \mathbf{D} = 0$ linear in unknowns $\mathbf{E} = \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$

where $\mathbf{J}'_{\mathbf{2x2}} = \begin{bmatrix} -\,BC \cos \varphi_3 & DC \cos \varphi_4 \\ -\,BC \sin \varphi_3 & DC \sin \varphi_4 \end{bmatrix}$ and $\mathbf{\Omega}^2 = \begin{bmatrix} \omega_3^{\,2} \\ \omega_4^{\,2} \end{bmatrix}$

$\Rightarrow$ $\mathbf{E} = \mathbf{J}^{-1}(\mathbf{D} - \mathbf{J}'\,\mathbf{\Omega}^2)$ where $\mathbf{D} = \begin{bmatrix} a_{DX} - a_{BX} \\ a_{DY} - a_{BY} \end{bmatrix}$ and $\begin{cases} a_{DX} = 0 \\ a_{DY} = 0 \end{cases}$

Kinematic analysis of Dyad - **RRR** type

Co-ordinates of intermediary joint C are:

$$\begin{cases} X_C = X_B + BC\cos\varphi_3 \\ Y_C = Y_B + BC\sin\varphi_3 \end{cases}$$

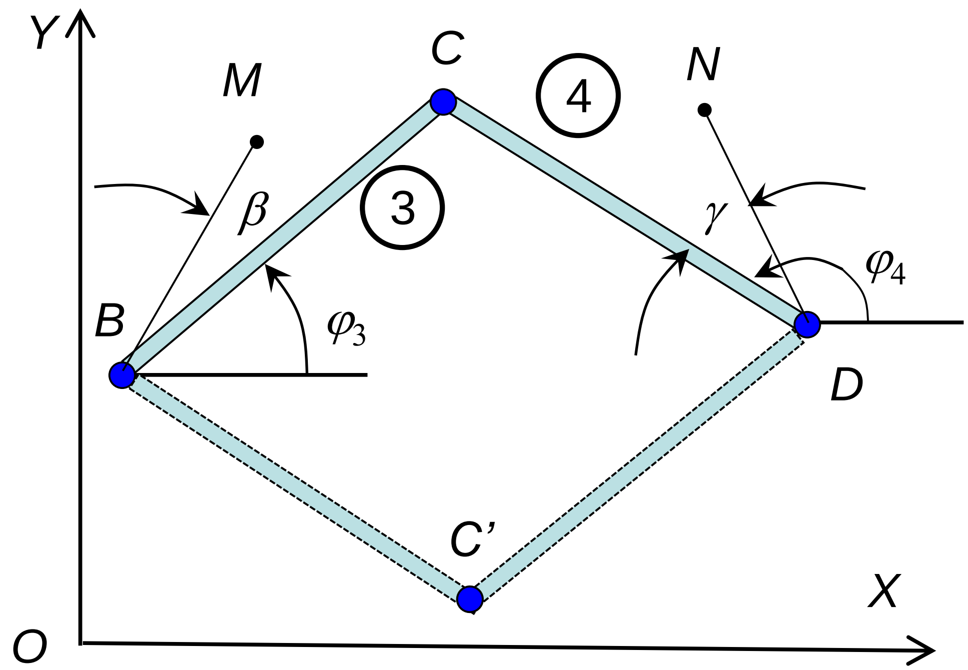
or

$$\begin{cases} X_C = X_D + DC\cos\varphi_4 \\ Y_C = Y_D + DC\sin\varphi_4 \end{cases}$$

$$\begin{cases} v_{CX} = v_{BX} - BC\,\omega_3\sin\varphi_3 \\ v_{CY} = v_{BY} + BC\,\omega_3\cos\varphi_3 \end{cases}$$

or

$$\begin{cases} v_{CX} = v_{DX} - DC\,\omega_4\sin\varphi_4 \\ v_{CY} = v_{DY} + DC\,\omega_4\cos\varphi_4 \end{cases}$$



$$\begin{cases} a_{CX} = a_{BX} - BC\,(\varepsilon_3\sin\varphi_3 + \omega_3^2\cos\varphi_3) \\ a_{CY} = a_{BY} + BC\,(\varepsilon_3\cos\varphi_3 - \omega_3^2\sin\varphi_3) \end{cases}$$

or

$$\begin{cases} a_{CX} = a_{DX} - DC\,(\varepsilon_4\sin\varphi_4 + \omega_4^2\cos\varphi_4) \\ a_{CY} = a_{DY} + DC\,(\varepsilon_4\cos\varphi_4 - \omega_4^2\sin\varphi_4) \end{cases}$$

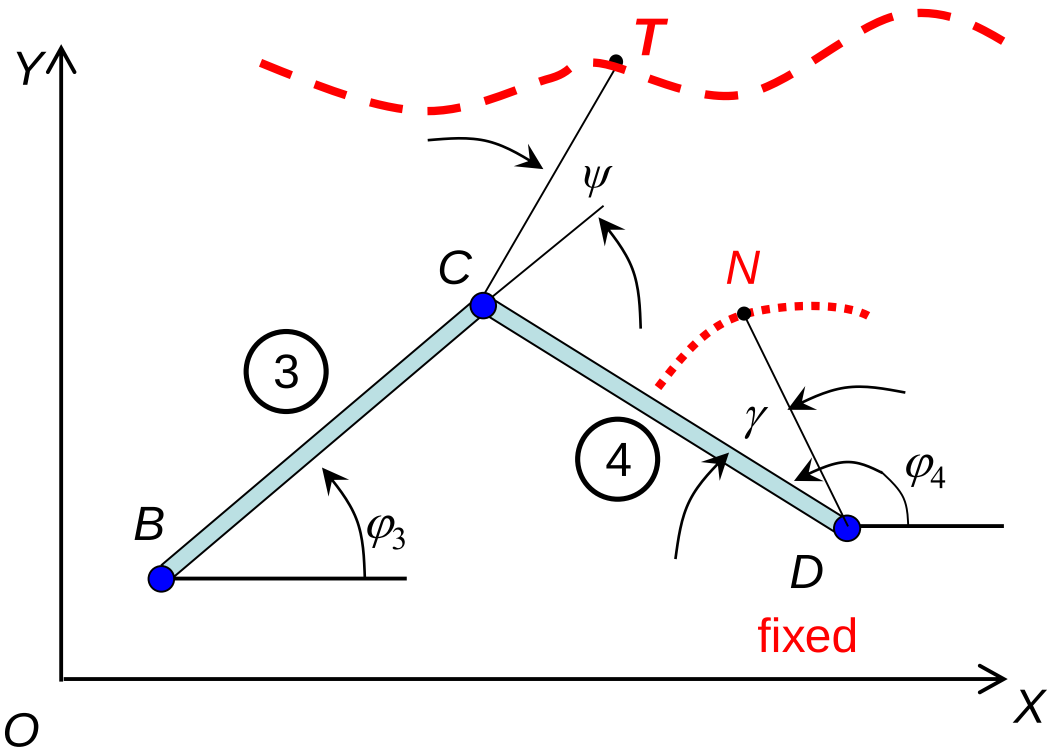
For other points of **RRR** dyad $M(CM,\ \beta)$ on element ③, outside direction BC we have:

$$\begin{cases} X_M = X_B + BM\cos(\varphi_3 + \beta) \\ Y_M = Y_B + BM\sin(\varphi_3 + \beta) \end{cases}$$

$$\begin{cases} v_{MX} = v_{BX} - BM\,\omega_3\sin(\varphi_3 + \beta) \\ v_{MY} = v_{BY} + BM\,\omega_3\cos(\varphi_3 + \beta) \end{cases}$$

$$\begin{cases} a_{MX} = a_{BX} - BM\,(\varepsilon_3\sin(\varphi_3 + \beta) + \omega_3^2\cos(\varphi_3 + \beta)) \\ a_{MY} = a_{BY} + BM\,(\varepsilon_3\cos(\varphi_3 + \beta) - \omega_3^2\sin(\varphi_3 + \beta)) \end{cases}$$

Kinematic analysis of Dyad - **RRR** type



For other points of **RRR** dyad $T(CT,\ \psi)$ on element ③, outside direction BC, and $N(DN,\ \gamma)$ on element ④, outside direction DC, we have:

$$\begin{cases} X_T = X_C + CT\cos(\varphi_3 + \psi) \\ Y_T = Y_C + CT\sin(\varphi_3 + \psi) \end{cases}$$

$$\begin{cases} X_N = X_D + DN\cos(\varphi_4 - \gamma) \\ Y_N = Y_D + DC\sin(\varphi_4 - \gamma) \end{cases}$$

$$\begin{cases} v_{TX} = v_{CX} - CT\ \omega_3 \sin(\varphi_3 + \psi) \\ v_{TY} = v_{CY} + CT\ \omega_3 \cos(\varphi_3 + \psi) \end{cases}$$

$$\begin{cases} a_{TX} = a_{CX} - CT\ (\varepsilon_3 \sin(\varphi_3 + \psi) + \omega_3^2 \cos(\varphi_3 + \psi)) \\ a_{TY} = a_{CY} + CT\ (\varepsilon_3 \cos(\varphi_3 + \psi) - \omega_3^2 \sin(\varphi_3 + \psi)) \end{cases}$$

$$\begin{cases} v_{NX} = v_{DX} - DN\ \omega_4 \sin(\varphi_4 - \gamma) \\ v_{NY} = v_{DY} + DN\ \omega_4 \cos(\varphi_4 - \gamma) \end{cases}$$

$$\begin{cases} a_{NX} = a_{DX} - DN\ (\varepsilon_4 \sin(\varphi_4 - \gamma) + \omega_4^2 \cos(\varphi_4 - \gamma)) \\ a_{NY} = a_{DY} + DN\ (\varepsilon_4 \cos(\varphi_4 - \gamma) - \omega_4^2 \sin(\varphi_4 - \gamma)) \end{cases}$$

Important observation

- Angles are measured counter-clockwise from positive direction of a vector or from reference *X* axis, to the direction of another vector or to another axis, and are expressed in radians