Contents

Preface

Chapter-1
FORMATION OF INTEGRAL EQUATIONS

1-1	Classification of Integral Equations	1
1-2	Formation of Volterra Integral Equations	4
1-3	Formation of Fredholm Integral Equations	23
	Exercise 1	42

Chapter-2
INTRODUCTION TO SETS

2-1	Separable Kernel Method	43
	Exercise 2-1	78
2-2	Eigenvalues and Eigenfunctions	79
	Exercise 2-2	123

Chapter-3
APPROXIMATION SOLUTIONS

3-1	Iterative Scheme	125
	Exercise 3	156
	Solutions	158
	Answers	227
	Index	230

Chapter 1 FORMATION OF INTEGRAL EQUATIONS

1-1 Classification of Integral Equations

1-1.1 Definition: An equation which involves an unknown function under one or more integral signs is called an *integral equation.*

For example, for $s,t \in [a,b]$, the equations

$$g(s) = \int_a^b K(s,t)g(t)dt \qquad \ldots(1)$$

$$g(s) = f(s) + \int_a^b K(s,t)g(t)dt \qquad \ldots(2)$$

are integral equations, where the function $g(s)$ is the unknown function while all the other functions are known.

The functions involved in the integral equations may be complex-valued functions of the real variables s and t.

1-1.2 Definition: An integral equation is called *linear integral equation* if the unknown function appears in it linearly.

For example,

$$g(s) = f(s) + \int_a^b K(s,t)g(t)dt$$

is a linear integral equation as the unknown function $g(s)$ in it appears linearly.

But the integral equation

$$g(s) = f(s) + \int_a^b K(s,t)[g(t)]^2 dt$$

is not a linear integral equation as the unknown function $g(s)$ in it does not appear linearly.

The most general linear integral equation is of the form

$$h(s)g(s) = f(s) + \lambda \int_a K(s,t)g(t)dt$$

where the upper limit may be either variable or fixed. The functions f, h, and K are known functions while g is unknown function; λ is a nonzero real or complex parameter. The function $K(s,t)$ is called the *kernel*.

1-1.3 Definition: An integral equation in which both the lower and upper limits of integration are fixed is called *Fredholm integral equation*. For example,

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is Fredholm integral equation because both the lower and upper limits of integration are fixed.

1-1.4 Definition: The Fredholm integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is called the *Fredholm integral equation* of the *first kind* if $h(s) = 0$. For example, the integral equation

$$f(s) + \lambda \int_a^b K(s,t)g(t)dt = 0$$

is the Fredholm integral equation of the first kind.

1-1.5 Definition: The Fredholm integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is called the *Fredholm integral equation* of the *second kind* if $h(s) = 1$. For example, the integral equation

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is the Fredholm integral equation of the second kind.

1-1.6 Definition: The Fredholm integral equation of the second kind

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is called the *homogeneous Fredholm integral equation* of the *second kind* if $f(s) = 0$.

For example, the integral equation

$$g(s) = \lambda \int_a^b K(s,t)g(t)dt$$

is the homogeneous Fredholm integral equation of the second kind.

1-1.7 Definition: The Fredholm integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is called the *Fredholm integral equation* of the *third kind* if it is neither of the first kind nor of the second kind, i.e., neither $h(s)=0$ nor $h(s)=1$. For example, the integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is the Fredholm integral equation of the third kind if $h(s) \neq 0$ and $h(s) \neq 1$.

1-1.8 Definition: An integral equation in which the lower limit of integration is fixed but the upper limit of integration is the variable s called *Volterra integral equation*.

For example,

$$h(s)g(s) = f(s) + \lambda \int_a^s K(s,t)g(t)dt$$

is Volterra integral equation because the lower limit of integration a is fixed while the upper limit s is variable.

1-1.9 Definition: The Volterra integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^s K(s,t)g(t)dt$$

is called the *Volterra integral equation* of the *first kind* if $h(s)=0$. For example, the integral equation

$$f(s) + \lambda \int_a^s K(s,t)g(t)dt = 0$$

is the Volterra integral equation of the first kind.

1-1.10 Definition: The Volterra integral equation

$$h(s)g(s) = f(s) + \lambda \int_a^s K(s,t)g(t)dt$$

is called the *Volterra integral equation* of the *second kind* if $h(s)=1$.

For example, the integral equation

$$g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is the Volterra integral equation of the second kind.

1-1.11 Definition: The Volterra integral equation of the second kind

$$g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is called the *homogeneous Volterra integral equation* of the *second kind* if $f(s)=0$. For example, the integral equation

$$g(s)=\lambda\int_a^s K(s,t)g(t)dt$$

is the homogeneous Volterra integral equation of the second kind.

1-1.12 Definition: The Volterra integral equation

$$h(s)g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is called the *Volterra integral equation* of the *third kind* if it is neither of the first kind nor of the second kind, i.e., neither $h(s)=0$ nor $h(s)=1$. For example, the integral equation

$$h(s)g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is the Volterra integral equation of the third kind if $h(s)\neq 0$ and $h(s)\neq 1$.

1-1.13 Definition: The integral equation is called the *singular integral equation* when one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration. For example, the integral equations

$$g(s)=f(s)+\lambda\int_{-\infty}^{\infty} e^{-|s-t|}g(t)dt$$

and

$$f(s)=\lambda\int_0^s\left[\frac{1}{(s-t)^{\alpha}}\right]g(t)dt,\qquad 0<\alpha<1$$

are singular integral equations.

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1-2 Formation of Volterra Integral Equations

In this section, we shall form the Volterra integral equations from the initial value boundary value problems. For this formation, we have an important Lemma which we prove in the following.

1-2.1 Lemma:

$$\int_a^s \int_a^{s_n} \cdots \int_a^{s_3} \int_a^{s_2} F(s_1)ds_1 ds_2 \ldots ds_{n-1} ds_n = \frac{1}{(n-1)!}\int_a^s (s-t)^{n-1} F(t)dt$$

Proof: For any function $f(s,t)$, the formula for differentiation of an integral involving a parameter is

$$\frac{d}{ds}\int_{A(s)}^{B(s)} f(s,t)dt = \int_{A(s)}^{B(s)} \frac{\partial f(s,t)}{\partial s}dt + f[s,B(s)]\frac{dB}{ds} - f[s,A(s)]\frac{dA}{ds} \quad \ldots(1)$$

For any positive integer n and a constant a, let us define

$$I_n(s) = \int_a^s (s-t)^{n-1} F(t)dt \quad \ldots(2)$$

Further assume that

$$f(s,t) = (s-t)^{n-1} F(t) \quad \ldots(3)$$

Making this substitution in (2), we have

$$I_n(s) = \int_a^s f(s,t)dt$$

$$\Rightarrow \frac{dI_n}{ds} = \frac{d}{ds}\int_a^s f(s,t)dt$$

$$= \int_a^s \frac{\partial f(s,t)}{\partial s}dt + f(s,s)\frac{ds}{ds} - f(s,a)\frac{da}{ds} \quad \text{(using (1))}$$

$$= \int_a^s \frac{\partial f(s,t)}{\partial s}dt + f(s,s) - 0 \qquad \boxed{\because \frac{da}{ds} = 0}$$

$$\Rightarrow \frac{dI_n}{ds} = \int_a^s \frac{\partial}{\partial s}\left[(s-t)^{n-1}F(t)\right]dt + (s-s)^{n-1}F(s) \quad \text{(using (3))}$$

$$\Rightarrow \frac{dI_n}{ds} = \int_a^s (n-1)(s-t)^{n-2}F(t)dt + 0 \qquad n>1$$

$$\Rightarrow \frac{dI_n}{ds} = (n-1)\boxed{\int_a^s (s-t)^{n-2}F(t)dt} \qquad n>1$$

$$= (n-1)I_{n-1} \qquad n>1 \qquad \boxed{\because I_{n-1} = \int_a^s (s-t)^{n-2}F(t)dt}$$

$$\Rightarrow \frac{dI_n}{ds} = (n-1)I_{n-1} \qquad n>1 \quad \ldots(4)$$

$$\Rightarrow \frac{d^2 I_n}{ds^2} = (n-1)(n-2)I_{n-2}$$

$$\Rightarrow \frac{d^3 I_n}{ds^3} = (n-1)(n-2)(n-3)I_{n-3}$$

Continuing in this way, we have

$$\frac{d^k I_n}{ds^k} = (n-1)(n-2)(n-3)\ldots(n-k)I_{n-k} \qquad n > k \quad \ldots(5)$$

For $k = n-1$, we have

$$\frac{d^{n-1} I_n}{ds^{n-1}} = (n-1)(n-2)(n-3)\ldots 1 I_1$$

$$\Rightarrow \frac{d^{n-1} I_n}{ds^{n-1}} = (n-1)! I_1 \qquad \ldots(6)$$

$$\Rightarrow \frac{d^n I_n}{ds^n} = (n-1)! \frac{dI_1}{ds} \qquad \ldots(7)$$

For $n = 1$, we have directly from (2)

$$I_1(s) = \int_a^s (s-t)^{1-1} F(t)dt$$

$$\Rightarrow I_1(s) = \int_a^s F(t)dt \qquad \ldots(8)$$

$$\Rightarrow \frac{dI_1(s)}{ds} = F(s) \qquad \ldots(9)$$

Putting this value in (7), we have

$$\frac{d^n I_n}{ds^n} = (n-1)! F(s) \qquad \ldots(10)$$

From (2), we note that

$$I_1(a) = \int_a^a (a-t)^{1-1} F(t)dt$$

$$= 0$$

$$I_2(a) = 0$$

$$\vdots \qquad \vdots$$

$$I_n(a) = 0$$

Using these in (5) and (6), we note that the first $n-1$ derivatives of $I_n(s)$ all vanish at $s = a$. Hence from (9), we have

$$I_1(s) = \int_a^s F(s_1)ds_1 \qquad \ldots(11)$$

From (10), we have

$$\frac{d^{n-1} I_n}{ds^{n-1}} = (n-1)! \int_a^s F(s_1)ds_1$$

$$\Rightarrow \frac{d^{n-2}I_n}{ds^{n-2}} = (n-1)!\int_a^s\int_a^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow \frac{d^{n-3}I_n}{ds^{n-3}} = (n-1)!\int_a^s\int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3$$

$$\vdots \qquad\qquad \vdots$$

$$I_n(s) = (n-1)!\int_a^s\int_a^{s_n}\int_a^{s_{n-1}} \cdots \int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3\ldots ds_{n-1}ds_n \qquad \ldots(12)$$

Equating (2) and (11), we have

$$(n-1)!\int_a^s\int_a^{s_n}\int_a^{s_{n-1}} \cdots \int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3\ldots ds_{n-1}ds_n = \int_a^s (s-t)^{n-1}F(t)dt$$

$$\Rightarrow \int_a^s\int_a^{s_n}\int_a^{s_{n-1}} \cdots \int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3\ldots ds_{n-1}ds_n = \frac{1}{(n-1)!}\int_a^s (s-t)^{n-1}F(t)dt \ldots(13)$$

This completes the proof of the identity.

From (13), we can also find the following deductions:

Putting $n=2$ in (13), we have

$$\int_a^s\int_a^{s_2} F(s_1)ds_1ds_2 = \frac{1}{(2-1)!}\int_a^s (s-t)^{2-1}F(t)dt$$

$$\boxed{\int_a^s\int_a^{s_2} F(s_1)ds_1ds_2 = \int_a^s (s-t)F(t)dt}$$

Similarly, putting $n=3$ in (13), we have

$$\int_a^s\int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3 = \frac{1}{(3-1)!}\int_a^s (s-t)^{3-1}F(t)dt$$

$$\boxed{\int_a^s\int_a^{s_3}\int_a^{s_2} F(s_1)ds_1ds_2ds_3 = \frac{1}{2}\int_a^s (s-t)^2F(t)dt}$$

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1-2.2 Example: *Transform the initial value boundary value problem*

$$y''+y=0,\ y(0)=0,\ y'(0)=1$$

into Volterra integral equation. **(P.U. 2003, M.Sc. Math)**

Solution: The given differential equation is

$$y''(s)+y(s)=0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = 0 \qquad \because y'(0) = 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - \int_0^s ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1ds_2 = 0$$

$$\Rightarrow |y(s_2)|_0^s - |s_2|_0^s + \int_0^s (s-t)y(t)dt = 0$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) - s + \int_0^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 0 - s + \int_0^s (s-t)y(t)dt = 0 \qquad \because y(0) = 0$$

$$\Rightarrow y(s) = s + \int_0^s (t-s)y(t)dt$$

which is the required Volterra integral equation.

1-2.3 Example: *Transform the initial value problem*

$$y'' + y = 0,\ y(0) = 2,\ y'(0) = 1$$

into Volterra integral equation.

Solution: The given differential equation is

$$y''(s) + y(s) = 0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = 0 \qquad \because y'(0) = 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - \int_0^s ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = 0$$

$$\Rightarrow |y(s_2)|_0^s - |s_2|_0^s + \int_0^s (s-t)y(t)dt = 0$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) - s + \int_0^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 2 - s + \int_0^s (s-t)y(t)dt = 0 \qquad \because y(0) = 2$$

$$\Rightarrow y(s) = 2 + s + \int_0^s (t-s)y(t)dt$$

which is the required Volterra integral equation.

1-2.4 Example: *Transform the initial value boundary value problem*

$$y'' + y = \cos s, \; y(0) = 0, \; y'(0) = 1$$

into Volterra integral equation.

(P.U. 1988, M.Sc. Math)

Solution: The given differential equation is

$$y''(s) + y(s) = \cos s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s \cos s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = |\sin s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = \sin s$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = \sin s \qquad \because y'(0) = 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - \int_0^s ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s \sin s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s - |s_2|_0^s + \int_0^s (s-t)y(t)dt = |-\cos s_2|_0^s$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) - s + \int_0^s (s-t)y(t)dt = -\cos s + \cos 0$$

$$\Rightarrow y(s) - 0 - s + \int_0^s (s-t)y(t)dt = -\cos s + 1 \quad \because y(0) = 0, \cos 0 = 1$$

$$\Rightarrow y(s) = 1 + s - \cos s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

✓**1-2.5 Example:** *Transform the initial value boundary value problem*

$$y'' + y = \sin s, \; y(0) = 1, \; y'(0) = 2$$

into Volterra integral equation.

Solution: The given differential equation is

$$y''(s) + y(s) = \sin s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s \sin s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = -|\cos s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = -\cos s + \cos 0$$

$$\Rightarrow y'(s) - 2 + \int_0^s y(s_1)ds_1 = -\cos s + 1 \quad \because y'(0) = 2$$

$$\Rightarrow y'(s) + \int_0^s y(s_1)ds_1 = 3 - \cos s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = 3\int_0^s ds_2 - \int_0^s \cos s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s + \int_0^s (s-t)y(t)dt = 3|s_2|_0^s - |\sin s_2|_0^s$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) + \int_0^s (s-t)y(t)dt = 3s - \sin s$$

$$\Rightarrow y(s) - 1 + \int_0^s (s-t)y(t)dt = 3s - \sin s \qquad \because y(0) = 1$$

$$\Rightarrow y(s) = 1 + 3s - \sin s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

✓ **1-2.6 Example:** *Transform the initial value boundary value problem*

$$y'' + y = 2\tan s \sec^2 s, \; y(0) = 0, \; y'(0) = 1$$

into Volterra integral equation.

Solution: The given differential equation is

$$y''(s) + y(s) = 2\tan s \sec^2 s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s 2\tan s_1 \sec^2 s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = |\tan^2 s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = \tan^2 s$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = \tan^2 s \qquad \because y'(0) = 1$$

$$\Rightarrow y'(s) = 1 + \tan^2 s - \int_0^s y(s_1)ds_1$$

$$\Rightarrow y'(s) = \sec^2 s - \int_0^s y(s_1)ds_1 \qquad \because 1 + \tan^2 s = \sec^2 s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 = \int_0^s \sec^2 s_2 ds_2 - \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2$$

$$\Rightarrow |y(s_2)|_0^s = |\tan s_2|_0^s - \int_0^s (s-t)y(t)dt$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) = \tan s - \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - 0 = \tan s - \int_0^s (s-t)y(t)dt \qquad \because y(0) = 0$$

$$\Rightarrow y(s) = \tan s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

✓ **1-2.7 Example:** *Transform the initial value boundary value problem*

$$y'' + y = e^s, \; y(0) = 0, \; y'(0) = 1$$

into Volterra integral equation.

Solution: The given differential equation is

$$y''(s) + y(s) = e^s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s e^{s_1} ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = |e^{s_1}|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = e^s - 1$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = e^s - 1 \qquad \because y'(0) = 1$$

$$\Rightarrow y'(s) = e^s - \int_0^s y(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 = \int_0^s e^{s_2} ds_2 - \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2$$

$$\Rightarrow |y(s_2)|_0^s = |e^{s_2}|_0^s - \int_0^s (s-t)y(t)dt$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) = e^s - 1 - \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - 0 = e^s - 1 - \int_0^s (s-t)y(t)dt \qquad \because y(0) = 0$$

$$\Rightarrow y(s) = e^s - 1 - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

1-2.8 Example: *Transform the initial value boundary value problem*

$$y'' - y = 0, \; y(1) = 1 = y(-1)$$

into Volterra integral equation. **(P.U. 1987, M.Sc. Math)**

Solution: The given differential equation is

$$y'' - y = 0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have

$$\int_1^s y''(s_1)ds_1 - \int_1^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_1^s - \int_1^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(1) - \int_1^s y(s_1)ds_1 = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have

$$\int_1^s y'(s_2)ds_2 - y'(1)\int_1^s ds_2 - \int_1^s \int_1^{s_2} y(s_1)ds_1ds_2 = 0$$

$$\Rightarrow |y(s_2)|_1^s - y'(1)|s_2|_1^s - \int_1^s (s-t)y(t)dt = 0$$

$$\because \int_1^s \int_1^{s_2} y(s_1)ds_1ds_2 = \int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(1) - (s-1)y'(1) - \int_1^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 1 - (s-1)y'(1) - \int_1^s (s-t)y(t)dt = 0 \qquad \because y(1) = 1$$

$$\Rightarrow y(s) = 1 + (s-1)y'(1) + \int_1^s (s-t)y(t)dt \qquad \ldots(2)$$

In order to find the value of $y'(1)$, put $s = -1$ in (2), i.e.,

$$y(-1) = 1 + (-1-1)y'(1) + \int_1^{-1} (-1-t)y(t)dt$$

$$\Rightarrow 1 = 1 - 2y'(1) + \int_1^{-1} (-1-t)y(t)dt \qquad \because y(-1) = 1$$

$$\Rightarrow 2y'(1) = -\int_1^{-1} (1+t)y(t)dt$$

$$\Rightarrow y'(1) = \frac{1}{2}\int_{-1}^{1} (1+t)y(t)dt$$

Putting this value in (2), we have

$$y(s) = 1 + \frac{s-1}{2}\int_{-1}^{1} (1+t)y(t)dt + \int_1^s (s-t)y(t)dt$$

or $$y(s) = f(s) + \int_1^s (s-t)y(t)dt$$

where $$f(s) = 1 + \frac{s-1}{2}\int_{-1}^{1} (1+t)y(t)dt$$

1-2.9 Example: *Derive the Volterra integral equation corresponding to the initial value problem*

$$y''(s) + x(s)y'(s) + y'(s) = 0, \; y(0) = 1, \; y'(0) = 0$$

(P.U. 1985; 1984, M.Sc. Math

Solution: The given differential equation is

$$y''(s) + x(s)y'(s) + y'(s) = 0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s x(s_1)y'(s_1)ds_1 + \int_0^s y'(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s x(s_1)y'(s_1)ds_1 + |y(s_1)|_0^s = 0$$

$$\Rightarrow y'(s) - y'(0) + |x(s_1)y(s_1)|_0^s - \int_0^s x'(s_1)y(s_1)ds_1 + y(s) - y(0) = 0$$

$$\Rightarrow y'(s) + x(s)y(s) - x(0)y(0) - \int_0^s x'(s_1)y(s_1)ds_1 + y(s) - 1 = 0$$

$$\Rightarrow y'(s)+x(s)y(s)-x(0)-\int_0^s x'(s_1)y(s_1)ds_1+y(s)-1=0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2+\int_0^s x(s_2)y(s_2)ds_2-x(0)\int_0^s ds_2-\int_0^s\int_0^{s_2} x'(s_2)y(s_2)ds_1ds_2$$

$$+\int_0^s y(s_2)ds_2-\int_0^s ds_2=0$$

$$\Rightarrow |y(s_2)|_0^s+\int_0^s x(s_2)y(s_2)ds_2-x(0)s-\int_0^s (s-t)x'(t)y(t)dt+\int_0^s y(t)dt-s=0$$

$$\Rightarrow y(s)-y(0)+\int_0^s x(t)y(t)dt-[x(0)+1]s-\int_0^s (s-t)x'(t)y(t)dt+\int_0^s y(t)dt=0$$

$$\Rightarrow y(s)-1-[x(0)+1]s+\int_0^s x(t)y(t)dt-\int_0^s (s-t)x'(t)y(t)dt+\int_0^s y(t)dt=0$$

$$\Rightarrow y(s)=1+[x(0)+1]s+\int_0^s (s-t)x'(t)y(t)dt-\int_0^s x(t)y(t)dt-\int_0^s y(t)dt$$

$$\Rightarrow y(s)=1+[x(0)+1]s+\int_0^s [(s-t)x'(t)y(t)-x(t)y(t)-y(t)]dt$$

$$\Rightarrow y(s)=1+[x(0)+1]s+\int_0^s [(s-t)x'(t)-x(t)-1]y(t)dt$$

which is the required Volterra integral equation.

1-2.10 Example: *Transform the initial value boundary value problem*

$$y''+y=\cos s,\ y(0)=0=y'(0)$$

into Volterra integral equation.

(P.U. 1992; 1991; 1989, M.Sc. Math)

Solution: The given differential equation is

$$y''(s)+y(s)=\cos s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\int_0^s y(s_1)ds_1=\int_0^s \cos s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s+\int_0^s y(s_1)ds_1=|\sin s_1|_0^s$$

$$\Rightarrow y'(s)-y'(0)+\int_0^s y(s_1)ds_1=\sin s$$

$$\Rightarrow y'(s) - 0 + \int_0^s y(s_1)ds_1 = \sin s \qquad \because y'(0) = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s \sin s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s + \int_0^s (s-t)y(t)dt = |-\cos s_2|_0^s$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) + \int_0^s (s-t)y(t)dt = -\cos s + \cos 0$$

$$\Rightarrow y(s) - 0 + \int_0^s (s-t)y(t)dt = -\cos s + 1 \qquad \because y(0) = 0, \cos 0 = 1$$

$$\Rightarrow y(s) = 1 - \cos s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

1-2.11 Example: *Convert the initial value problem*

$$y''(s) + A(s)y'(s) + B(s)y(s) = F(s),\ y(a) = y_0,\ y'(a) = y_1$$

into a Volterra integral equation.

(P.U. 2005; 1994, M.Sc. Math)

Solution: The given differential equation is

$$y''(s) + A(s)y'(s) + B(s)y(s) = F(s) \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from a to s, we have

$$\int_a^s y''(s_1)ds_1 + \int_a^s A(s_1)y'(s_1)ds_1 + \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$|y'(s_1)|_a^s + |A(s_1)y(s_1)|_a^s - \int_a^s A'(s_1)y(s_1)ds_1 + \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$y'(s) - y'(a) + A(s)y(s) - A(a)y(a) - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+ \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$y'(s) - y_1 + A(s)y(s) - A(a)y_0 - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$\Rightarrow y'(s) - y_1 - A(a)y_\circ + A(s)y(s) - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from a to s, we have

$$\int_a^s y'(s_2)ds_2 - [y_1 + A(a)y_\circ]\int_a^s ds_2 + \int_a^s A(s_2)y(s_2)ds_2$$

$$-\int_a^s\int_a^{s_2} A'(s_1)y(s_1)ds_1ds_2 + \int_a^s\int_a^{s_2} B(s_1)y(s_1)ds_1ds_2 = \int_a^s\int_a^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_a^s - [y_1 + A(a)y_\circ]s_2|_a^s + \int_a^s A(s_2)y(s_2)ds_2 - \int_a^s (s-t)A'(t)y(t)dt$$

$$+\int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y(a) - [y_1 + A(a)y_\circ](s-a) + \int_a^s A(s_2)y(s_2)ds_2$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y_\circ - [y_1 + A(a)y_\circ](s-a) + \int_a^s A(s_2)y(s_2)ds_2$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y_1 + A(a)y_\circ](s-a) - \int_a^s A(t)y(t)dt$$

$$+\int_a^s (s-t)A'(t)y(t)dt - \int_a^s (s-t)B(t)y(t)dt + \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y_1 + A(a)y_\circ](s-a)$$

$$-\int_a^s [A(t)y(t) - (s-t)A'(t)y(t) + (s-t)B(t)y(t) - (s-t)F(t)]dt \quad \ldots(2)$$

Let $\quad f(s) = y_\circ + [y_1 + A(a)y_\circ](s-a), \ \lambda = -1$

And $\quad K(s,t) = A(t)y(t) - (s-t)A'(t)y(t) + (s-t)B(t)y(t) - (s-t)F(t)$

Then (2) takes the form

$$y(s)=f(s)+\lambda\int_a^s K(s,t)dt$$

which is the required Volterra integral equation of the second kind.

1-2.12 Example: *Reduce the initial value problem*

$$y''(s)+\lambda\, y(s)=F(s),\ y(0)=1,\ y'(0)=0$$

into a Volterra integral equation.

(P.U. 1996; 1991, M.Sc. Math)

Solution: The given differential equation is

$$y''(s)+\lambda\, y(s)=F(s) \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\lambda\int_0^s y(s_1)ds_1=\int_0^s F(s_1)ds_1$$

$$\Rightarrow |y'(s_1)|_0^s+\lambda\int_0^s y(s_1)ds_1=\int_0^s F(s_1)ds_1$$

$$\Rightarrow y'(s)-y'(0)+\lambda\int_0^s y(s_1)ds_1=\int_0^s F(s_1)ds_1$$

$$\Rightarrow y'(s)+\lambda\int_0^s y(s_1)ds_1=\int_0^s F(s_1)ds_1 \qquad \because y'(0)=0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2+\lambda\int_0^s\int_0^{s_2} y(s_1)ds_1ds_2=\int_0^s\int_0^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_0^s+\lambda\int_0^s (s-t)y(t)dt=\int_0^s (s-t)F(t)dt$$

$$\Rightarrow y(s)-y(0)+\lambda\int_0^s (s-t)y(t)dt=\int_0^s (s-t)F(t)dt$$

$$\Rightarrow y(s)-1+\lambda\int_0^s (s-t)y(t)dt=\int_0^s (s-t)F(t)dt \qquad \because y(0)=1$$

$$\Rightarrow y(s)=1+\int_0^s (s-t)F(t)dt-\lambda\int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s)=1+\int_0^s (s-t)F(t)dt+\lambda\int_0^s (t-s)y(t)dt \qquad \ldots(2)$$

Let $\quad f(s)=1+\int_0^s (s-t)F(t)dt$

And $K(s,t)=t-s$

Making these substitutions in (2), we have

$$y(s)=f(s)+\lambda\int_0^s K(s,t)y(t)dt$$

which is the required Volterra integral equation.

1-2.13 Example: *Reduce the initial value problem*

$$y''(s)+\lambda\, y(s)=0,\ \ y(0)=0,\ y'(0)=1$$

into a Volterra integral equation.

Solution: The given differential equation is

$$y''(s)+\lambda y(s)=0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\lambda\int_0^s y(s_1)ds_1=0$$

$$\Rightarrow |y'(s_1)|_0^s+\lambda\int_0^s y(s_1)ds_1=0$$

$$\Rightarrow y'(s)-y'(0)+\lambda\int_0^s y(s_1)ds_1=0$$

$$\Rightarrow y'(s)-1+\lambda\int_0^s y(s_1)ds_1=0 \quad \because y'(0)=1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2-\int_0^s ds_2+\lambda\int_0^s\int_0^{s_2} y(s_1)ds_1ds_2=0$$

$$\Rightarrow |y(s_2)|_0^s-|s_2|_0^s+\lambda\int_0^s (s-t)y(t)dt=0$$

$$\because \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2=\int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s)-y(0)-s+\lambda\int_0^s (s-t)y(t)dt=0$$

$$\Rightarrow y(s)-0-s+\lambda\int_0^s (s-t)y(t)dt=0 \quad \because y(0)=0$$

$$\Rightarrow y(s)=s+\lambda\int_0^s (t-s)y(t)dt$$

which is the required Volterra integral equation.

1-2.14 Example: *Transform the following equation into an integral equation*

$$y''(s)+n^2(s)y(s)=0,\ y(0)=0,\ y'(0)=1$$

(P.U. 1986; 1985, M.Sc. Math)

Solution: The given differential equation is ...(1)

$$y''(s)+n^2(s)y(s)=0$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\int_0^s n^2(s_1)y(s_1)ds_1=0$$

$$\Rightarrow |y'(s_1)|_0^s+\int_0^s n^2(s_1)y(s_1)ds_1=0$$

$$\Rightarrow y'(s)-y'(0)+\int_0^s n^2(s_1)y(s_1)ds_1=0$$

$$\Rightarrow y'(s)-1+\int_0^s n^2(s_1)y(s_1)ds_1=0 \qquad \because y'(0)=1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2-\int_0^s ds_2+\int_0^s\int_0^{s_2} n^2(s_1)y(s_1)ds_1ds_2=0$$

$$\Rightarrow |y(s_2)|_0^s-|s_2|_0^s+\int_0^s (s-t)n^2(t)y(t)dt=0$$

$$\because \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2=\int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s)-y(0)-s+\int_0^s (s-t)n^2(t)y(t)dt=0$$

$$\Rightarrow y(s)-0-s+\int_0^s (s-t)n^2(t)y(t)dt=0 \qquad \because y(0)=0$$

$$\Rightarrow y(s)=s-\int_0^s (s-t)n^2(t)y(t)dt$$

which is the required Volterra integral equation.

1-2.15 Example: *Derive the Volterra integral equation corresponding to the initial value problem*

$$y''(s)+x(s)y'(s)+y(s)=0,\ y(0)=1,\ y'(0)=0$$

Solution: The given differential equation is

$$y''(s)+x(s)y'(s)+y(s)=0 \qquad ...(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s x(s_1)y'(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s x(s_1)y'(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(0) + |x(s_1)y(s_1)|_0^s - \int_0^s x'(s_1)y(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) + x(s)y(s) - x(0)y(0) - \int_0^s x'(s_1)y(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) + x(s)y(s) - x(0) - \int_0^s x'(s_1)y(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 + \int_0^s x(s_2)y(s_2)ds_2 - x(0)\int_0^s ds_2 - \int_0^s\int_0^{s_2} x'(s_2)y(s_2)ds_1ds_2$$

$$+\int_0^s\int_0^{s_2} y(s_1)ds_1ds_2 = 0$$

$$|y(s_2)|_0^s + \int_0^s x(s_2)y(s_2)ds_2 - x(0)s - \int_0^s (s-t)x'(t)y(t)dt + \int_0^s (s-t)y(t)dt = 0$$

$$y(s) - y(0) + \int_0^s x(s_2)y(s_2)ds_2 - x(0)s - \int_0^s (s-t)[x'(t)-1]y(t)dt = 0$$

$$y(s) = 1 + x(0)s - \int_0^s x(s_2)y(s_2)ds_2 + \int_0^s (s-t)[x'(t)-1]y(t)dt$$

which is the required Volterra integral equation.

1-2.16 Example: *Transform the following equation into an integral equation*

$$y''(s) + \omega^2 y(s) = 0,\ y(0) = 0,\ y'(0) = 1$$

(P.U. 1990, M.Sc. Math)

Solution: The given differential equation is

$$y''(s) + \omega^2 y(s) = 0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \omega^2\int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \omega^2 \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(0) + \omega^2 \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - 1 + \omega^2 \int_0^s y(s_1)ds_1 = 0 \qquad \because y'(0) = 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - \int_0^s ds_2 + \omega^2 \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = 0$$

$$\Rightarrow |y(s_2)|_0^s - |s_2|_0^s + \omega^2 \int_0^s (s-t)y(t)dt = 0$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) - s + \omega^2 \int_0^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 0 - s + \omega^2 \int_0^s (s-t)y(t)dt = 0 \quad \because y(0) = 0$$

$$\Rightarrow y(s) = s - \omega^2 \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

1-2.17 Example: *Convert the initial value problem*

$$y'' + y = F(s), \qquad 0 < s < 1$$

$$y(0) = 0 = y'(0)$$

into an integral equation. **(P.U. 1993, M.Sc. Math)**

Solution: The given differential equation is

$$y'' + y = F(s) \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s F(s_1)ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = \int_0^s F(s_1)ds_1$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = \int_0^s F(s_1)ds_1$$

$$\Rightarrow y'(s)+\int_0^s y(s_1)ds_1=\int_0^s F(s_1)ds_1 \qquad \because y'(0)=0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2+\int_0^s\int_0^{s_2} y(s_1)ds_1ds_2=\int_0^s\int_0^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow \int_0^s y'(s_2)ds_2=\int_0^s\int_0^{s_2} F(s_1)ds_1ds_2-\int_0^s\int_0^{s_2} y(s_1)ds_1ds_2$$

$$\Rightarrow \int_0^s y'(s_2)ds_2=\int_0^s\int_0^{s_2} [F(s_1)-y(s_1)]ds_1ds_2$$

$$\Rightarrow |y(s_2)|_0^s=\int_0^s (s-t)[F(t)-y(t)]dt$$

$$\because \int_0^s\int_0^{s_2} [F(s_1)-y(s_1)]ds_1ds_2=\int_0^s (s-t)[F(t)-y(t)]dt$$

$$\Rightarrow y(s)-y(0)=\int_0^s (s-t)[F(t)-y(t)]dt$$

$$\Rightarrow y(s)=\int_0^s (s-t)[F(t)-y(t)]dt \qquad \because y(0)=0$$

$$\Rightarrow y(s)=\int_0^s (s-t)[F(t)-y(t)]dt \qquad \because y(0)=0$$

which is the required Volterra integral equation.

1-3 Formation of Fredholm Integral Equations

In the last section, we formed the Volterra integral equations from the initial value boundary value problems. In this section we shall form the Fredholm integral equations.

1-3.1 Example: *Convert the initial value problem*

$$y''(s)+A(s)y'(s)+B(s)y(s)=F(s),\ y(a)=y_0,\ y(b)=y_1$$

into a Fredholm integral equation.

Solution: The given differential equation is

$$y''(s)+A(s)y'(s)+B(s)y(s)=F(s) \qquad ...(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from a to s, we have

$$\int_a^s y''(s_1)ds_1+\int_a^s A(s_1)y'(s_1)ds_1+\int_a^s B(s_1)y(s_1)ds_1=\int_a^s F(s_1)ds_1$$

$$|y'(s_1)|_a^s + |A(s_1)y(s_1)|_a^s - \int_a^s A'(s_1)y(s_1)ds_1 + \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$y'(s) - y'(a) + A(s)y(s) - A(a)y(a) - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$y'(s) - y'(a) + A(s)y(s) - A(a)y_\circ - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from a to s, we have

$$\int_a^s y'(s_2)ds_2 - y'(a)\int_a^s ds_2 + \int_a^s A(s_2)y(s_2)ds_2 - A(a)y_\circ \int_a^s ds_2$$

$$-\int_a^s\int_a^{s_2} A'(s_1)y(s_1)ds_1ds_2 + \int_a^s\int_a^{s_2} B(s_1)y(s_1)ds_1ds_2 = \int_a^s\int_a^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_a^s - y'(a)|s_2|_a^s + \int_a^s A(s_2)y(s_2)ds_2 - A(a)y_\circ|s_2|_a^s$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y(a) - y'(a)(s-a) + \int_a^s A(s_2)y(s_2)ds_2 - A(a)y_\circ(s-a)$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y_\circ - [y'(a) + A(a)y_\circ](s-a) + \int_a^s A(s_2)y(s_2)ds_2$$

$$= \int_a^s (s-t)F(t)dt + \int_a^s (s-t)A'(t)y(t)dt - \int_a^s (s-t)B(t)y(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y'(a) + A(a)y_\circ](s-a) + \int_a^s (s-t)F(t)dt$$

$$+\int_a^s (s-t)A'(t)y(t)dt - \int_a^s (s-t)B(t)y(t)dt - \int_a^s A(t)y(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y'(a) + A(a)y_\circ](s-a)$$
$$+\int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt \qquad \ldots(2)$$

Replacing s by b in (2), we have

$$y(b) = y_\circ + [y'(a) + A(a)y_\circ](b-a)$$
$$+\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$

$$\Rightarrow y_1 = y_\circ + [y'(a) + A(a)y_\circ](b-a)$$
$$+\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt \qquad \because y(b) = y_1$$

$$\Rightarrow [y'(a) + A(a)y_\circ](b-a) = y_1 - y_\circ$$
$$-\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$

$$y'(a) + A(a)y_\circ = \frac{y_1 - y_\circ}{b-a}$$
$$-\frac{1}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$

Putting this value of $y'(a) + A(a)y_\circ$ in (2), we have

$$y(s) = y_\circ + \frac{s-a}{b-a}[y_1 - y_\circ]$$
$$-\frac{s-a}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$
$$+\int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$

$$\Rightarrow y(s) = y_\circ + \frac{s-a}{b-a}[y_1 - y_\circ]$$
$$+\int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$
$$-\frac{s-a}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$

$$\Rightarrow y(s) = y_\circ + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[y_1 - y_\circ - \int_a^b (b-t)F(t)dt\right]$$

$$-\int_a^s [A(t)-(s-t)\{A'(t)-B(t)\}]y(t)dt$$

$$+\frac{s-a}{b-a}\int_a^s [A(t)-(b-t)\{A'(t)-B(t)\}y(t)]dt$$

$$+\frac{s-a}{b-a}\int_s^b [A(t)-(b-t)\{A'(t)-B(t)\}]y(t)dt$$

$$y(s)=y_\circ+\int_a^s (s-t)F(t)dt+\frac{s-a}{b-a}\left[y_1-y_\circ-\int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s\left[\left(\frac{s-a}{b-a}-1\right)A(t)+\left((s-t)-\frac{(s-a)(b-t)}{b-a}\right)\{A'(t)-B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t)-(b-t)\{A'(t)-B(t)\}]y(t)dt$$

$$y(s)=y_\circ+\int_a^s (s-t)F(t)dt+\frac{s-a}{b-a}\left[y_1-y_\circ-\int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s\left[\left(\frac{s-a}{b-a}-1\right)A(t)+\left(\frac{bs-bt-as+at-bs+ab+ts-at}{b-a}\right)\{A'(t)-B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t)-(b-t)\{A'(t)-B(t)\}]y(t)dt$$

$$y(s)=y_\circ+\int_a^s (s-t)F(t)dt+\frac{s-a}{b-a}\left[y_1-y_\circ-\int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s\left[\left(\frac{s-a}{b-a}-1\right)A(t)+\left(\frac{-bt-as+ab+ts}{b-a}\right)\{A'(t)-B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t)-(b-t)\{A'(t)-B(t)\}]y(t)dt$$

$$y(s)=y_\circ+\int_a^s (s-t)F(t)dt+\frac{s-a}{b-a}\left[y_1-y_\circ-\int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s\left[\left(\frac{s-a}{b-a}-1\right)A(t)-\left(\frac{(t-a)(b-s)}{b-a}\right)\{A'(t)-B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t)-(b-t)\{A'(t)-B(t)\}]y(t)dt \qquad \ldots(3)$$

Let $\quad f(s)=y_\circ+\int_a^s (s-t)F(t)dt+\frac{s-a}{b-a}\left[y_1-y_\circ-\int_a^b (b-t)F(t)dt\right]$

and

$$K(s,t)=\begin{cases}\left(\dfrac{s-a}{b-a}-1\right)A(t)-\left(\dfrac{(t-a)(b-s)}{b-a}\right)\{A'(t)-B(t)\}, & t<s\\ \dfrac{s-a}{b-a}[A(t)-(b-t)\{A'(t)-B(t)\}] & t>s\end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_a^b K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.2 Example: *Convert the initial value problem*

$$y''(s)+A(s)y'(s)+B(s)y(s)=F(s),\ y(1)=y_\circ,\ y(2)=y_1$$

into a Fredholm integral equation.

Solution: The given differential equation is

$$y''(s)+A(s)y'(s)+B(s)y(s)=F(s) \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have

$$\int_1^s y''(s_1)ds_1+\int_1^s A(s_1)y'(s_1)ds_1+\int_1^s B(s_1)y(s_1)ds_1=\int_1^s F(s_1)ds_1$$

$$\Rightarrow |y'(s_1)|_1^s+|A(s_1)y(s_1)|_1^s-\int_1^s A'(s_1)y(s_1)ds_1+\int_1^s B(s_1)y(s_1)ds_1=\int_1^s F(s_1)ds_1$$

$$\Rightarrow y'(s)-y'(1)+A(s)y(s)-A(1)y(1)-\int_1^s A'(s_1)y(s_1)ds_1$$

$$+\int_1^s B(s_1)y(s_1)ds_1=\int_1^s F(s_1)ds_1$$

$$\Rightarrow y'(s)-y'(1)+A(s)y(s)-A(1)y_\circ-\int_1^s A'(s_1)y(s_1)ds_1$$

$$+\int_1^s B(s_1)y(s_1)ds_1=\int_1^s F(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have

$$\int_1^s y'(s_2)ds_2-y'(1)\int_1^s ds_2+\int_1^s A(s_2)y(s_2)ds_2-A(1)y_\circ\int_1^s ds_2$$

$$-\int_1^s\int_1^{s_2} A'(s_1)y(s_1)ds_1ds_2+\int_1^s\int_1^{s_2} B(s_1)y(s_1)ds_1ds_2=\int_1^s\int_1^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_1^s-y'(1)|s_2|_1^s+\int_1^s A(s_2)y(s_2)ds_2-A(1)y_\circ|s_2|_1^s$$

$$-\int_1^s (s-t)A'(t)y(t)dt + \int_1^s (s-t)B(t)y(t)dt = \int_1^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y(1) - y'(1)(s-1) + \int_1^s A(s_2)y(s_2)ds_2 - A(1)y_\circ(s-1)$$

$$-\int_1^s (s-t)A'(t)y(t)dt + \int_1^s (s-t)B(t)y(t)dt = \int_1^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y_\circ - [y'(1) + A(1)y_\circ](s-1) + \int_1^s A(s_2)y(s_2)ds_2$$

$$= \int_1^s (s-t)F(t)dt + \int_1^s (s-t)A'(t)y(t)dt - \int_1^s (s-t)B(t)y(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y'(1) + A(1)y_\circ](s-1) + \int_1^s (s-t)F(t)dt$$

$$+\int_1^s (s-t)A'(t)y(t)dt - \int_1^s (s-t)B(t)y(t)dt - \int_1^s A(t)y(t)dt$$

$$\Rightarrow y(s) = y_\circ + [y'(1) + A(1)y_\circ](s-1)$$

$$+\int_1^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt \qquad \ldots(2)$$

Replacing s by 2 in (2), we have

$$y(2) = y_\circ + [y'(1) + A(1)y_\circ](2-1)$$

$$+\int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

$$y_1 = y_\circ + [y'(1) + A(1)y_\circ] + \int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

$$[y'(1) + A(1)y_\circ] = y_1 - y_\circ - \int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

$$y'(1) + A(1)y_\circ = y_1 - y_\circ - \int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

Putting this value of $y'(1) + A(1)y_\circ$ in (2), we have

$$y(s) = y_\circ + (s-1)[y_1 - y_\circ]$$

$$-(s-1)\int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

$$+\int_1^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$

$$y(s) = y_\circ + (s-1)[y_1 - y_\circ]$$
$$+\int_1^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$
$$-(s-1)\int_1^2 [\{F(t) + A'(t)y(t) - B(t)y(t)\}(2-t) - A(t)y(t)]dt$$

$$y(s) = y_\circ + \int_1^s (s-t)F(t)dt + (s-1)\left[y_1 - y_\circ - \int_1^2 (2-t)F(t)dt\right]$$
$$-\int_1^s [A(t) - (s-t)\{A'(t) - B(t)\}]y(t)dt$$
$$+(s-1)\int_1^2 [A(t) - (2-t)\{A'(t) - B(t)\}y(t)]dt$$

$$y(s) = y_\circ + \int_1^s (s-t)F(t)dt + (s-1)\left[y_1 - y_\circ - \int_1^2 (2-t)F(t)dt\right]$$
$$-\int_1^s [A(t) - (s-t)\{A'(t) - B(t)\}]y(t)dt$$
$$+(s-1)\int_1^s [A(t) - (2-t)\{A'(t) - B(t)\}y(t)]dt$$
$$+(s-1)\int_s^2 [A(t) - (2-t)\{A'(t) - B(t)\}]y(t)dt$$

$$y(s) = y_\circ + \int_1^s (s-t)F(t)dt + (s-1)\left[y_1 - y_\circ - \int_1^2 (2-t)F(t)dt\right]$$
$$+\int_1^s [((s-1)-1)A(t) + ((s-t)-(s-1)(2-t))\{A'(t) - B(t)\}]y(t)dt$$
$$+\int_s^2 (s-1)[A(t) - (2-t)\{A'(t) - B(t)\}]y(t)dt$$

$$y(s) = y_\circ + \int_1^s (s-t)F(t)dt + (s-1)\left[y_1 - y_\circ - \int_1^2 (2-t)F(t)dt\right]$$
$$+\int_1^s [(s-2)A(t) + (s-t-2s+2+st-t)\{A'(t) - B(t)\}]y(t)dt$$
$$+\int_s^2 (s-1)[A(t) - (2-t)\{A'(t) - B(t)\}]y(t)dt$$

$$y(s)=y_0+\int_1^s (s-t)F(t)dt+(s-1)\left[y_1-y_0-\int_1^2 (2-t)F(t)dt\right]$$

$$+\int_1^s [(s-2)A(t)-(t-1)(2-s)\{A'(t)-B(t)\}]y(t)dt$$

$$+\int_s^2 (s-1)[A(t)-(2-t)\{A'(t)-B(t)\}]y(t)dt \qquad \ldots(3)$$

Let

$$f(s)=y_0+\int_1^s (s-t)F(t)dt+(s-1)\left[y_1-y_0-\int_1^2 (2-t)F(t)dt\right]$$

and

$$K(s,t)=\begin{cases}(s-2)A(t)-(t-1)(2-s)\{A'(t)-B(t)\}, & t<s\\ (s-1)[A(t)-(2-t)\{A'(t)-B(t)\}] & t>s\end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_1^2 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.3 Example: *Transform the boundary value problem*

$$y''+y=s,\ y(0)=1,\ y'(1)=0$$

into an integral equation. **(P.U. 1993, M.Sc. Math)**

Solution: The given differential equation is

$$y''+y=s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have

$$\int_1^s y''(s_1)ds_1+\int_1^s y(s_1)ds_1=\int_1^s s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_1^s+\int_1^s y(s_1)ds_1=\left|\frac{s_1^2}{2}\right|_1^s$$

$$\Rightarrow y'(s)-y'(1)+\int_1^s y(s_1)ds_1=\frac{s^2-1}{2}$$

$$\Rightarrow y'(s)-0+\int_1^s y(s_1)ds_1=\frac{s^2-1}{2} \qquad \because y'(1)=0$$

$$\Rightarrow y'(s)+\int_1^s y(s_1)ds_1=\frac{s^2-1}{2}.$$

Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have

$$\int_1^s y'(s_2)ds_2 + \int_1^s \int_1^{s_2} y(s_1)ds_1 ds_2 = \frac{1}{2}\int_1^s (s_2^2 - 1)ds_2$$

$$\Rightarrow |y(s_2)|_1^s + \int_1^s (s-t)y(t)dt = \frac{1}{2}\left|\frac{s_2^3}{3} - s_2\right|_1^s$$

$$\Rightarrow y(s) - y(1) + \int_1^s (s-t)y(t)dt = \frac{1}{2}\left|\frac{s_2^3 - 3s_2}{3}\right|_1^s$$

$$\Rightarrow y(s) - y(1) + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6} \qquad \ldots(2)$$

Replacing s by 0 in (2), we have

$$y(0) - y(1) + \int_1^0 (0-t)y(t)dt = \frac{2}{6}$$

$$\Rightarrow 1 - y(1) - \int_1^0 ty(t)dt = \frac{1}{3} \qquad \because y(0) = 1$$

$$\Rightarrow y(1) = 1 - \frac{1}{3} - \int_1^0 ty(t)dt$$

Putting this value of $y(1)$ in (2), we have

$$y(s) - \left[1 - \frac{1}{3} - \int_1^0 ty(t)dt\right] + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6}$$

$$\Rightarrow y(s) - 1 + \frac{1}{3} + \int_1^0 ty(t)dt + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6}$$

$$\Rightarrow y(s) = 1 - \frac{1}{3} + \frac{s^3 - 3s + 2}{6} - \int_1^0 ty(t)dt - \int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s) = 1 + \frac{s^3 - 3s}{6} + \int_0^1 ty(t)dt - \int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s) = 1 + \frac{s^3 - 3s}{6} + \int_0^1 ty(t)dt + \int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s) = 1 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 ty(t)dt + \int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s) = 1 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 [t + (s-t)]y(t)dt$$

$$\Rightarrow y(s) = 1 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 sy(t)dt \qquad \ldots(3)$$

Let
$$f(s)=1+\frac{s^3-3s}{6}$$
and
$$K(s,t)=\begin{cases} t, & t<s \\ s, & t>s \end{cases}$$
Making these substitutions in (3), we have
$$y(s)=f(s)+\int_0^1 K(s,t)y(t)dt$$
which is the required Fredholm integral equation.

1-3.4 Example: *Transform the boundary value problem*
$$y''+y=1,\ y(0)=1,\ y'(1)=0$$
into an integral equation.

Solution: The given differential equation is
$$y''+y=1 \qquad \ldots(1)$$
Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have
$$\int_1^s y''(s_1)ds_1+\int_1^s y(s_1)ds_1=\int_1^s ds_1$$
$$\Rightarrow |y'(s_1)|_1^s+\int_1^s y(s_1)ds_1=|s_1|_1^s$$
$$\Rightarrow y'(s)-y'(1)+\int_1^s y(s_1)ds_1=s-1$$
$$\Rightarrow y'(s)-0+\int_1^s y(s_1)ds_1=s-1 \qquad \because y'(1)=0$$
$$\Rightarrow y'(s)+\int_1^s y(s_1)ds_1=s-1$$
Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have
$$\int_1^s y'(s_2)ds_2+\int_1^s\int_1^{s_2} y(s_1)ds_1ds_2=\int_1^s (s_2-1)ds_2$$
$$\Rightarrow |y(s_2)|_1^s+\int_1^s (s-t)y(t)dt=\left|\frac{(s_2-1)^2}{2}\right|_1^s$$
$$\Rightarrow y(s)-y(1)+\int_1^s (s-t)y(t)dt=\frac{(s-1)^2}{2} \qquad \ldots(2)$$
Replacing s by 0 in (2), we have

$$y(0)-y(1)+\int_1^0 (0-t)y(t)dt=\frac{1}{2}$$

$$\Rightarrow 1-y(1)-\int_1^0 ty(t)dt=\frac{1}{2} \qquad \because y(0)=1$$

$$\Rightarrow y(1)=1-\frac{1}{2}-\int_1^0 ty(t)dt$$

$$\Rightarrow y(1)=\frac{1}{2}+\int_0^1 ty(t)dt$$

Putting this value of $y(1)$ in (2), we have

$$y(s)-\left[\frac{1}{2}+\int_0^1 ty(t)dt\right]+\int_1^s (s-t)y(t)dt=\frac{(s-1)^2}{2}$$

$$\Rightarrow y(s)-\frac{1}{2}-\int_0^1 ty(t)dt+\int_1^s (s-t)y(t)dt=\frac{(s-1)^2}{2}$$

$$\Rightarrow y(s)=\frac{1}{2}+\frac{(s-1)^2}{2}+\int_0^1 ty(t)dt-\int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{1}{2}+\frac{(s-1)^2}{2}+\int_0^1 ty(t)dt+\int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{1}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 ty(t)dt+\int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{1}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 [t+(s-t)]y(t)dt$$

$$\Rightarrow y(s)=\frac{1}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 sy(t)dt \qquad \ldots(3)$$

Let

$$f(s)=\frac{1}{2}+\frac{(s-1)^2}{2}$$

and

$$K(s,t)=\begin{cases} t, & t<s \\ s, & t>s \end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.5 Example: *Transform the boundary value problem*

$$y'' + y = \sin s,\ y(0) = 1,\ y'\left(\frac{\pi}{2}\right) = 0$$

into an integral equation.

Solution: The given differential equation is

$$y'' + y = \sin s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from $\frac{\pi}{2}$ to s, we have

$$\int_{\frac{\pi}{2}}^{s} y''(s_1)ds_1 + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = \int_{\frac{\pi}{2}}^{s} \sin s_1 ds_1$$

$$\Rightarrow \left|y'(s_1)\right|_{\frac{\pi}{2}}^{s} + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = -\left|\cos s_1\right|_{\frac{\pi}{2}}^{s}$$

$$\Rightarrow y'(s) - y'\left(\frac{\pi}{2}\right) + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = -\cos s$$

$$\Rightarrow y'(s) - 0 + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = -\cos s \qquad \because y'\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow y'(s) + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = -\cos s$$

Replacing s by s_2 and integrating w.r.t. s_2 from $\frac{\pi}{2}$ to s, we have

$$\int_{\frac{\pi}{2}}^{s} y'(s_2)ds_2 + \int_{\frac{\pi}{2}}^{s}\int_{\frac{\pi}{2}}^{s_2} y(s_1)ds_1 ds_2 = -\int_{\frac{\pi}{2}}^{s} \cos s_2 ds_2$$

$$\Rightarrow \left|y(s_2)\right|_{\frac{\pi}{2}}^{s} + \int_{\frac{\pi}{2}}^{s} (s-t)y(t)dt = -\left|\sin s_2\right|_{\frac{\pi}{2}}^{s}$$

$$\Rightarrow y(s) - y\left(\frac{\pi}{2}\right) + \int_{\frac{\pi}{2}}^{s} (s-t)y(t)dt = -\left[\sin s - \sin\left(\frac{\pi}{2}\right)\right]$$

$$\Rightarrow y(s) - y\left(\frac{\pi}{2}\right) + \int_{\frac{\pi}{2}}^{s} (s-t)y(t)dt = -(\sin s - 1)$$

$$\Rightarrow y(s)-y\left(\frac{\pi}{2}\right)+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=1-\sin s \qquad \ldots(2)$$

Replacing s by 0 in (2), we have

$$y(0)-y\left(\frac{\pi}{2}\right)+\int_{\frac{\pi}{2}}^{0}(0-t)y(t)dt=1-\sin 0$$

$$\Rightarrow 1-y\left(\frac{\pi}{2}\right)-\int_{\frac{\pi}{2}}^{0}ty(t)dt=1-0 \quad \because y(0)=1$$

$$\Rightarrow y\left(\frac{\pi}{2}\right)=-\int_{\frac{\pi}{2}}^{0}ty(t)dt=\int_{0}^{\frac{\pi}{2}}ty(t)dt$$

Putting this value of $y\left(\frac{\pi}{2}\right)$ in (2), we have

$$y(s)-\int_{0}^{\frac{\pi}{2}}ty(t)dt+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=1-\sin s$$

$$\Rightarrow y(s)=1-\sin s+\int_{0}^{\frac{\pi}{2}}ty(t)dt-\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt$$

$$\Rightarrow y(s)=1-\sin s+\int_{0}^{\frac{\pi}{2}}ty(t)dt+\int_{s}^{\frac{\pi}{2}}(s-t)y(t)dt$$

$$\Rightarrow y(s)=1-\sin s+\int_{0}^{s}ty(t)dt+\int_{s}^{\frac{\pi}{2}}ty(t)dt+\int_{s}^{\frac{\pi}{2}}(s-t)y(t)dt$$

$$\Rightarrow y(s)=1-\sin s+\int_{0}^{s}ty(t)dt+\int_{s}^{\frac{\pi}{2}}[t+(s-t)]y(t)dt$$

$$\Rightarrow y(s)=1-\sin s+\int_{0}^{s}ty(t)dt+\int_{s}^{\frac{\pi}{2}}sy(t)dt \qquad \ldots(3)$$

Let $\qquad f(s)=1-\sin s$

And $\qquad K(s,t)=\begin{cases} t, & t<s \\ s, & t>s \end{cases}$

Making these substitutions in (3), we have

$$y(s) = f(s) + \int_0^{\frac{\pi}{2}} K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.6 Example: *Transform the boundary value problem*

$$y'' + y = \cos s,\ y(0) = 1,\ y'\left(\frac{\pi}{2}\right) = 0$$

into an integral equation.

Solution: The given differential equation is

$$y'' + y = \cos s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from $\frac{\pi}{2}$ to s, we have

$$\int_{\frac{\pi}{2}}^{s} y''(s_1)ds_1 + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = \int_{\frac{\pi}{2}}^{s} \cos s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_{\frac{\pi}{2}}^{s} + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = |\sin s_1|_{\frac{\pi}{2}}^{s}$$

$$\Rightarrow y'(s) - y'\left(\frac{\pi}{2}\right) + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = \sin s - \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow y'(s) - 0 + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = \sin s - 1 \qquad \because y'\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow y'(s) + \int_{\frac{\pi}{2}}^{s} y(s_1)ds_1 = \sin s - 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from $\frac{\pi}{2}$ to s, we have

$$\int_{\frac{\pi}{2}}^{s} y'(s_2)ds_2 + \int_{\frac{\pi}{2}}^{s}\int_{\frac{\pi}{2}}^{s_2} y(s_1)ds_1 ds_2 = \int_{\frac{\pi}{2}}^{s} (\sin s_2 - 1)ds_2$$

$$\Rightarrow |y(s_2)|_{\frac{\pi}{2}}^{s} + \int_{\frac{\pi}{2}}^{s} (s-t)y(t)dt = |-\cos s_2 - s_2|_{\frac{\pi}{2}}^{s}$$

$$\Rightarrow y(s)-y\left(\frac{\pi}{2}\right)+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=-\cos s-s+\cos\left(\frac{\pi}{2}\right)+\frac{\pi}{2}$$

$$\Rightarrow y(s)-y\left(\frac{\pi}{2}\right)+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=-\cos s-s+\frac{\pi}{2} \qquad \ldots(2)$$

Replacing s by 0 in (2), we have

$$y(0)-y\left(\frac{\pi}{2}\right)+\int_{\frac{\pi}{2}}^{0}(0-t)y(t)dt=-\cos 0-0+\frac{\pi}{2}$$

$$\Rightarrow 1-y\left(\frac{\pi}{2}\right)-\int_{\frac{\pi}{2}}^{0}ty(t)dt=-1+\frac{\pi}{2} \qquad \because y(0)=1$$

$$\Rightarrow y\left(\frac{\pi}{2}\right)=2-\frac{\pi}{2}-\int_{\frac{\pi}{2}}^{0}ty(t)dt$$

$$\Rightarrow y\left(\frac{\pi}{2}\right)=2-\frac{\pi}{2}+\int_{0}^{\frac{\pi}{2}}ty(t)dt$$

Putting this value of $y\left(\frac{\pi}{2}\right)$ in (2), we have

$$\Rightarrow y(s)-\left[2-\frac{\pi}{2}+\int_{0}^{\frac{\pi}{2}}ty(t)dt\right]+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=-\cos s-s+\frac{\pi}{2}$$

$$\Rightarrow y(s)-2+\frac{\pi}{2}-\int_{0}^{\frac{\pi}{2}}ty(t)dt+\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt=-\cos s-s+\frac{\pi}{2}$$

$$\Rightarrow y(s)=2-\cos s-s+\int_{0}^{\frac{\pi}{2}}ty(t)dt-\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt$$

$$\Rightarrow y(s)=2-\cos s-s+\int_{0}^{s}ty(t)dt+\int_{s}^{\frac{\pi}{2}}ty(t)dt-\int_{\frac{\pi}{2}}^{s}(s-t)y(t)dt$$

$$\Rightarrow y(s)=2-\cos s-s+\int_{0}^{s}ty(t)dt+\int_{s}^{\frac{\pi}{2}}ty(t)dt+\int_{s}^{\frac{\pi}{2}}(s-t)y(t)dt$$

$$\Rightarrow y(s) = 2 - \cos s - s + \int_0^s ty(t)dt + \int_s^{\frac{\pi}{2}} [t + (s-t)]y(t)dt$$

$$\Rightarrow y(s) = 2 - \cos s - s + \int_0^s ty(t)dt + \int_s^{\frac{\pi}{2}} sy(t)dt \qquad \ldots(3)$$

Let $$f(s) = 2 - \cos s - s$$

and

$$K(s,t) = \begin{cases} t, & t < s \\ s, & t > s \end{cases}$$

Making these substitutions in (3), we have

$$y(s) = f(s) + \int_0^{\frac{\pi}{2}} K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.7 Example: *Reduce the boundary value problem*

$$y'' + sy = 1, \; y(0) = 0 = y(1)$$

into a Fredholm integral equation.

(P.U. 1991, M.Sc. Math)

Solution: The given differential equation is

$$y'' + sy = 1 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s s_1 y(s_1)ds_1 = \int_0^s ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s s_1 y(s_1)ds_1 = |s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s s_1 y(s_1)ds_1 = s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - y'(0)\int_0^s ds_2 + \int_0^s \int_0^{s_2} s_1 y(s_1)ds_1 ds_2 = \int_0^s s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s - y'(0)|s_2|_0^s + \int_0^s (s-t)ty(t)dt = \left|\frac{s_2^2}{2}\right|_0^s$$

$$\Rightarrow y(s) - y(0) - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2}$$

$$\Rightarrow y(s) - 0 - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2} \qquad \because y(0) = 0$$

$$\Rightarrow y(s) - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2} \qquad \ldots(2)$$

Replacing s by 1 in (2), we have

$$y(1) - y'(0)(1) + \int_0^1 (1-t)ty(t)dt = \frac{1}{2}$$

$$\Rightarrow y'(0) = -\frac{1}{2} + y(1) + \int_0^1 (1-t)ty(t)dt$$

$$\Rightarrow y'(0) = -\frac{1}{2} + 0 + \int_0^1 (1-t)ty(t)dt \qquad \because y(1) = 0$$

$$\Rightarrow y'(0) = -\frac{1}{2} + \int_0^1 (1-t)ty(t)dt$$

Putting this value of $y'(0)$ in (2), we have

$$y(s) - \left[-\frac{1}{2} + \int_0^1 (1-t)ty(t)dt\right]s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2}$$

$$\Rightarrow y(s) + \frac{s}{2} - s\int_0^1 (1-t)ty(t)dt + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2}$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^1 (1-t)sty(t)dt - \int_0^s (s-t)ty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s (1-t)sty(t)dt + \int_s^1 (1-t)sty(t)dt - \int_0^s (s-t)ty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s (1-t)sty(t)dt - \int_0^s (s-t)ty(t)dt + \int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s [(1-t)st - (s-t)t]y(t)dt + \int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s (st - st^2 - st + t^2)y(t)dt + \int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s (1-s)t^2y(t)dt + \int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^2}{2} - \frac{s}{2} + \int_0^s (1-s)t^2y(t)dt + \int_s^1 (1-t)sty(t)dt \qquad \ldots(3)$$

Let

$$f(s)=\frac{s^2}{2}-\frac{s}{2}$$

and

$$K(s,t)=\begin{cases}(1-s)t^2, & t<s\\ (1-t)st, & t>s\end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

1-3.8 Example: *Reduce the boundary value problem*

$$y''+sy=s^2+3s+2,\ y(0)=0=y(1)$$

into a Fredholm integral equation.

Solution: The given differential equation is

$$y''+sy=s^2+3s+2 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\int_0^s s_1y(s_1)ds_1=\int_0^s(s_1^2+3s_1+2)ds_1$$

$$\Rightarrow |y'(s_1)|_0^s+\int_0^s s_1y(s_1)ds_1=\left|\frac{s_1^3}{3}+\frac{3s_1^2}{2}+2s_1\right|_0^s$$

$$\Rightarrow y'(s)-y'(0)+\int_0^s s_1y(s_1)ds_1=\frac{s^3}{3}+\frac{3s^2}{2}+2s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2-y'(0)\int_0^s ds_2+\int_0^s\int_0^{s_2} s_1y(s_1)ds_1ds_2=\int_0^s\left(\frac{s_2^3}{3}+\frac{3s_2^2}{2}+2s_2\right)ds_2$$

$$\Rightarrow |y(s_2)|_0^s-y'(0)|s_2|_0^s+\int_0^s(s-t)ty(t)dt=\left|\frac{s_2^4}{12}+\frac{s_2^3}{2}+s_2^2\right|_0^s$$

$$\Rightarrow y(s)-y(0)-y'(0)s+\int_0^s(s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)-0-y'(0)s+\int_0^s(s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2 \quad \because y(0)=0$$

$$\Rightarrow y(s)-y'(0)s+\int_0^s(s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2 \qquad \ldots(2)$$

Replacing s by 1 in (2), we have

$$y(1)-y'(0)(1)+\int_0^1(1-t)ty(t)dt=\frac{1}{12}+\frac{1}{2}+1$$

$$\Rightarrow 0-y'(0)(1)+\int_0^1(1-t)ty(t)dt=\frac{1+6+12}{12} \qquad \because y(1)=0$$

$$\Rightarrow y'(0)=-\frac{19}{12}+\int_0^1(1-t)ty(t)dt$$

Putting this value of $y'(0)$ in (2), we have

$$y(s)-\left[-\frac{19}{12}+\int_0^1(1-t)ty(t)dt\right]s+\int_0^s(s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)+\frac{19s}{12}-s\int_0^1(1-t)ty(t)dt+\int_0^s(s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)=\frac{s^4}{12}+\frac{s^3}{2}+s^2-\frac{19s}{12}+\int_0^1(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt$$

$$y(s)=\frac{(s^3+6s^2+12s-19)s}{12}+\int_0^s(1-t)sty(t)dt+\int_s^1(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt$$

$$y(s)=\frac{(s^3+6s^2+12s-19)s}{12}+\int_0^s(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt+\int_s^1(1-t)sty(t)dt$$

$$y(s)=\frac{(s^3+6s^2+12s-19)s}{12}+\int_0^s[(1-t)st-(s-t)t]y(t)dt+\int_s^1(1-t)sty(t)dt$$

$$y(s)=\frac{(s^3+6s^2+12s-19)s}{12}+\int_0^s(st-st^2-st+t^2)y(t)dt+\int_s^1(1-t)sty(t)dt$$

$$y(s)=\frac{(s^3+6s^2+12s-19)s}{12}+\int_0^s(1-s)t^2y(t)dt+\int_s^1(1-t)sty(t)dt \qquad \ldots(3)$$

Let

$$f(s)=\frac{(s^3+6s^2+12s-19)s}{12}$$

and

$$K(s,t)=\begin{cases}(1-s)t^2, & t<s\\ (1-t)st, & t>s\end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_0^1K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

EXERCISE 1

Q.1 Distinguish between Fredholm and Voletrra integral equations and cite their examples. **(P.U. 2000, M.Sc. Math)**

Q.2 Transform the initial value problem

$$y'' + y = 0,\ y(0) = 5,\ y'(0) = 6$$

into Volterra integral equation.

Q.3 Transform the initial value boundary value problem

$$y'' - y = 3\sin^2 s\cos s,\ y(0) = 1,\ y'(0) = 1$$

into Volterra integral equation.

Q.4 Transform the initial value boundary value problem

$$y'' - y = 5e^{5s},\ y(0) = 6,\ y'(0) = 7$$

into Volterra integral equation.

Q.5 Transform the initial value boundary value problem

$$y'' - y = 0,\ y(2) = 1 = y(-2)$$

into Volterra integral equation.

Q.6 Derive the Volterra integral equation corresponding to the initial value problem

$$y''(s) + x(s)y'(s) + y'(s) = 0,\ y(0) = 2,\ y'(0) = 1$$

Q.7 Transform the initial value boundary value problem

$$y'' + y = \sin s,\ y(0) = 0 = y'(0)$$

into Volterra integral equation.

Q.8 Convert the initial value problem

$$y''(s) + A(s)y'(s) + B(s)y(s) = F(s),\ y(a) = 7,\ y(b) = 6$$

into a Fredholm integral equation.

Q.9 Transform the boundary value problem

$$y'' + y = s,\ y(0) = 5,\ y'(1) = 0$$

into a an integral equation.

Q.10 Transform the boundary value problem

$$y'' + y = 1,\ y(0) = 8,\ y'(1) = 0$$

into a an integral equation.

Q.11 Reduce the boundary value problem

$$y'' + sy = 1,\ y(0) = 4 = y(1)$$

into a Fredholm integral equation.

Q.12 Reduce the boundary value problem

$$y'' + sy = s^2 + 3s + 2,\ y(0) = 3 = y(1)$$

into a Fredholm integral equation.

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SOLUTIONS

EXERCISE 1

Q.1 **Distinguish between Fredholm and Voletrra integral equations and cite their examples.** **(P.U. 2000, M.Sc. Math)**

Sol. **Fredholm Integral Equation:** An integral equation in which both the lower and upper limits of integration are fixed is called Fredholm integral equation. For example,

$$h(s)g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt \qquad \ldots(1)$$

is Fredholm integral equation because both the lower and upper limits of integration are fixed.

The Fredholm integral equation (1) is called the Fredholm integral equation of the first kind if $h(s) = 0$. For example, the integral equation

$$f(s) + \lambda \int_a^b K(s,t)g(t)dt = 0$$

is the Fredholm integral equation of the first kind.

The Fredholm integral equation (1) is called the Fredholm integral equation of the second kind if $h(s) = 1$. For example, the integral equation

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is the Fredholm integral equation of the second kind.

The Fredholm integral equation of the second kind

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

is called the homogeneous Fredholm integral equation of the second kind if $f(s) = 0$. For example, the integral equation

$$g(s) = \lambda \int_a^b K(s,t)g(t)dt$$

is the homogeneous Fredholm integral equation of the second kind. The Fredholm integral equation (1) is called the Fredholm integral equation of the third kind if it is neither of the first kind nor

of the second kind, i.e., neither $h(s)=0$ nor $h(s)=1$. For example, the integral equation

$$h(s)g(s)=f(s)+\lambda\int_a^b K(s,t)g(t)dt$$

is the Fredholm integral equation of the third kind if $h(s)\neq 0$ and $h(s)\neq 1$.

Volterra Integral Equation: An integral equation in which the lower limit of integration is fixed but the upper limit of integration is the variable s called Volterra integral equation. For example,

$$h(s)g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt \qquad ...(2)$$

is Volterra integral equation because the lower limit of integration a is fixed while the upper limit s is variable.

The Volterra integral equation (2) is called the Volterra integral equation of the first kind if $h(s)=0$. For example, the integral equation

$$f(s)+\lambda\int_a^s K(s,t)g(t)dt=0$$

is the Volterra integral equation of the first kind.

The Volterra integral equation (2) is called the Volterra integral equation of the second kind if $h(s)=1$. For example, the integral equation

$$g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is the Volterra integral equation of the second kind.

The Volterra integral equation of the second kind is called the homogeneous Volterra integral equation of the second kind if $f(s)=0$. For example, the integral equation

$$g(s)=\lambda\int_a^s K(s,t)g(t)dt$$

is the homogeneous Volterra integral equation of the second kind.

The Volterra integral equation (2) is called the Volterra integral equation of the third kind if it is neither of the first kind nor of the second kind, i.e., neither $h(s)=0$ nor $h(s)=1$. For example, the integral equation

$$h(s)g(s)=f(s)+\lambda\int_a^s K(s,t)g(t)dt$$

is Volterra integral equation of third kind if $h(s)\neq 0$ and $h(s)\neq 1$.

Q.2 Transform the initial value problem

$$y'' + y = 0,\ y(0) = 5,\ y'(0) = 6$$

into Volterra integral equation.

Sol. The given differential equation is

$$y''(s) + y(s) = 0 \quad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - 6 + \int_0^s y(s_1)ds_1 = 0 \qquad \because y'(0) = 6$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - 6\int_0^s ds_2 + \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2 = 0$$

$$\Rightarrow |y(s_2)|_0^s - 6|s_2|_0^s + \int_0^s (s-t)y(t)dt = 0 \because \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) - 6s + \int_0^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 5 - 6s + \int_0^s (s-t)y(t)dt = 0 \qquad \because y(0) = 5$$

$$\Rightarrow y(s) = 5 + 6s + \int_0^s (t-s)y(t)dt$$

which is the required Volterra integral equation.

Q.3 Transform the initial value boundary value problem

$$y'' - y = 3\sin^2 s\cos s,\ y(0) = 1,\ y'(0) = 1$$

into Volterra integral equation.

Sol. The given differential equation is

$$y'' - y = 3\sin^2 s\cos s \quad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 - \int_0^s y(s_1)ds_1 = \int_0^s 3\sin^2 s_1\cos s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s - \int_0^s y(s_1)ds_1 = |\sin^3 s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = \sin^3 s$$

$$\Rightarrow y'(s) - 1 + \int_0^s y(s_1)ds_1 = \sin^3 s \qquad \because y'(0) = 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - \int_0^s ds_2 + \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2 = \int_0^s \sin^3 s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s - |s_2|_0^s + \int_0^s (s-t)y(t)dt = \int_0^s \sin^2 s_2 \sin s_2 ds_2$$

$$\Rightarrow y(s) - y(0) - s + \int_0^s (s-t)y(t)dt = \int_0^s (1-\cos^2 s_2)\sin s_2 ds_2$$

$$\Rightarrow y(s) - 1 - s + \int_0^s (s-t)y(t)dt = \int_0^s \sin s_2 ds_2 + \int_0^s \cos^2 s_2(-\sin s_2)ds_2$$

$$= -|\cos s_2|_0^s + \left|\frac{\cos^3 s_2}{3}\right|_0^s$$

$$= -(\cos s - \cos 0) + \frac{\cos^3 s - \cos^3 0}{3}$$

$$\Rightarrow y(s) - 1 - s + \int_0^s (s-t)y(t)dt = -\cos s + 1 + \frac{1}{3}\cos^3 s - \frac{1}{3}$$

$$\Rightarrow y(s) - 1 - s + \int_0^s (s-t)y(t)dt = 1 - \frac{1}{3} + \frac{1}{3}\cos^3 s - \cos s$$

$$\Rightarrow y(s) = \frac{5}{3} + s + \frac{1}{3}\cos^3 s - \cos s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

Q.4 Transform the initial value boundary value problem

$$y'' - y = 5e^{5s}, \; y(0) = 0, \; y'(0) = 7$$

into Volterra integral equation.

Sol. The given differential equation is

$$y'' - y = 5e^{5s} \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 - \int_0^s y(s_1)ds_1 = 5\int_0^s e^{5s_1}ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = \left|e^{5s_1}\right|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = e^{5s} - 1$$

$$\Rightarrow y'(s) - 7 + \int_0^s y(s_1)ds_1 = e^{5s} - 1 \qquad \because y'(0) = 7$$

$$\Rightarrow y'(s) = 6 + e^{5s} - \int_0^s y(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 = 6\int_0^s ds_2 + \int_0^s e^{5s_2}ds_2 - \int_0^s\int_0^{s_2} y(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_0^s = 6|s_2|_0^s + \frac{1}{5}\left|e^{5s_2}\right|_0^s - \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) = 6 + \frac{e^{5s}}{5} - \frac{1}{5} - \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - 6 = 6 + \frac{e^{5s}}{5} - \frac{1}{5} - \int_0^s (s-t)y(t)dt \qquad \because y(0) = 6$$

$$\Rightarrow y(s) = \frac{59}{5} + \frac{e^{5s}}{5} - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

Q.5 Transform the initial value boundary value problem

$$y'' - y = 0, \; y(2) = 1 = y(-2)$$

into Volterra integral equation.

Sol. The given differential equation is

$$y'' - y = 0 \qquad \text{...(1)}$$

Replacing s by s_1 and integrating w.r.t. s_1 from 2 to s, we have

$$\int_2^s y''(s_1)ds_1 - \int_2^s y(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_2^s - \int_2^s y(s_1)ds_1 = 0$$

$$\Rightarrow y'(s) - y'(2) - \int_2^s y(s_1)ds_1 = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 2 to s, we have

$$\int_2^s y'(s_2)ds_2 - y'(2)\int_2^s ds_2 - \int_2^s\int_2^{s_2} y(s_1)ds_1ds_2 = 0$$

$$\Rightarrow |y(s_2)|_2^s - y'(2)|s_2|_2^s - \int_2^s (s-t)y(t)dt = 0$$

$$\because \int_2^s \int_2^{s_2} y(s_1)ds_1 ds_2 = \int_2^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(2) - (s-2)y'(2) - \int_2^s (s-t)y(t)dt = 0$$

$$\Rightarrow y(s) - 1 - (s-2)y'(2) - \int_2^s (s-t)y(t)dt = 0 \quad \because y(2) = 1$$

$$\Rightarrow y(s) = 1 + (s-2)y'(2) + \int_2^s (s-t)y(t)dt \qquad \ldots(2)$$

In order to find the value of $y'(2)$, put $s = -2$ in (2), i.e.,

$$y(-2) = 1 + (-2-2)y'(2) + \int_2^{-2} (-2-t)y(t)dt$$

$$\Rightarrow 1 = 1 - 4y'(2) + \int_2^{-2} (-2-t)y(t)dt \qquad \because y(-2) = 1$$

$$\Rightarrow 4y'(2) = -\int_2^{-2} (2+t)y(t)dt$$

$$\Rightarrow y'(2) = \frac{1}{4}\int_{-2}^{2} (2+t)y(t)dt$$

Putting this value in (2), we have

$$y(s) = 1 + \frac{s-2}{4}\int_{-2}^{2} (2+t)y(t)dt + \int_2^s (s-t)y(t)dt$$

or $$y(s) = f(s) + \int_2^s (s-t)y(t)dt$$

where $$f(s) = 1 + \frac{s-2}{4}\int_{-2}^{2} (2+t)y(t)dt$$

Q.6 Derive the Volterra integral equation corresponding to the initial value problem

$$y''(s) + x(s)y'(s) + y'(s) = 0, \; y(0) = 2, \; y'(0) = 1$$

Sol. The given differential equation is

$$y''(s) + x(s)y'(s) + y'(s) = 0 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s x(s_1)y'(s_1)ds_1 + \int_0^s y'(s_1)ds_1 = 0$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s x(s_1)y'(s_1)ds_1 + |y(s_1)|_0^s = 0$$

$$\Rightarrow y'(s) - y'(0) + |x(s_1)y(s_1)|_0^s - \int_0^s x'(s_1)y(s_1)ds_1 + y(s) - y(0) = 0$$

$$\Rightarrow y'(s) - 1 + x(s)y(s) - x(0)y(0) - \int_0^s x'(s_1)y(s_1)ds_1 + y(s) - 2 = 0$$

$$\Rightarrow y'(s) + x(s)y(s) - 2x(0) - \int_0^s x'(s_1)y(s_1)ds_1 + y(s) - 3 = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 + \int_0^s x(s_2)y(s_2)ds_2 - 2x(0)\int_0^s s_2 ds_2$$

$$-\int_0^s \int_0^{s_2} x'(s_2)y(s_2)ds_1 ds_2 + \int_0^s y(s_2)ds_2 - 3\int_0^s ds_2 = 0$$

$$\Rightarrow |y(s_2)|_0^s + \int_0^s x(s_2)y(s_2)ds_2 - 2x(0)s - \int_0^s (s-t)x'(t)y(t)dt$$

$$+\int_0^s y(t)dt - 3s = 0$$

$$\Rightarrow y(s) - y(0) + \int_0^s x(t)y(t)dt - [2x(0)+3]s$$

$$-\int_0^s (s-t)x'(t)y(t)dt + \int_0^s y(t)dt = 0$$

$$\Rightarrow y(s) - 2 - [2x(0)+3]s + \int_0^s x(t)y(t)dt$$

$$-\int_0^s (s-t)x'(t)y(t)dt + \int_0^s y(t)dt = 0$$

$$\Rightarrow y(s) = 2 + [2x(0)+3]s + \int_0^s (s-t)x'(t)y(t)dt$$

$$-\int_0^s x(t)y(t)dt - \int_0^s y(t)dt$$

$$\Rightarrow y(s) = 2 + [2x(0)+3]s + \int_0^s [(s-t)x'(t)y(t) - x(t)y(t) - y(t)]dt$$

$$\Rightarrow y(s) = 2 + [2x(0)+3]s + \int_0^s [(s-t)x'(t) - x(t) - 1]y(t)dt$$

which is the required Volterra integral equation.

Q.7 Transform the initial value boundary value problem

$$y'' + y = \sin s, \quad y(0) = 0 = y'(0)$$

into Volterra integral equation.

Sol. The given differential equation is

$$y''(s) + y(s) = \sin s \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s y(s_1)ds_1 = \int_0^s \sin s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s y(s_1)ds_1 = -|\cos s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s y(s_1)ds_1 = -\cos s + \cos 0$$

$$\Rightarrow y'(s) - 0 + \int_0^s y(s_1)ds_1 = 1 - \cos s \qquad \because y'(0) = 0$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 + \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s ds_2 - \int_0^s \cos s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s + \int_0^s (s-t)y(t)dt = |s_2|_0^s - |\sin s_2|_0^s$$

$$\because \int_0^s \int_0^{s_2} y(s_1)ds_1 ds_2 = \int_0^s (s-t)y(t)dt$$

$$\Rightarrow y(s) - y(0) + \int_0^s (s-t)y(t)dt = s - \sin s$$

$$\Rightarrow y(s) - 0 + \int_0^s (s-t)y(t)dt = s - \sin s \quad \because y(0) = 0, \sin 0 = 0$$

$$\Rightarrow y(s) = s - \sin s - \int_0^s (s-t)y(t)dt$$

which is the required Volterra integral equation.

Q.8 Convert the initial value problem

$$y''(s) + A(s)y'(s) + B(s)y(s) = F(s), \quad y(a) = 7, \; y(b) = 6$$

into a Fredholm integral equation.

Sol. The given differential equation is

$$y''(s) + A(s)y'(s) + B(s)y(s) = F(s) \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from a to s, we have

$$\int_a^s y''(s_1)ds_1 + \int_a^s A(s_1)y'(s_1)ds_1 + \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$\Rightarrow |y'(s_1)|_a^s + |A(s_1)y(s_1)|_a^s - \int_a^s A'(s_1)y(s_1)ds_1 + \int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$\Rightarrow y'(s) - y'(a) + A(s)y(s) - A(a)y(a) - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

$$\Rightarrow y'(s) - y'(a) + A(s)y(s) - 7A(a) - \int_a^s A'(s_1)y(s_1)ds_1$$

$$+\int_a^s B(s_1)y(s_1)ds_1 = \int_a^s F(s_1)ds_1$$

Replacing s by s_2 and integrating w.r.t. s_2 from a to s, we have

$$\int_a^s y'(s_2)ds_2 - y'(a)\int_a^s ds_2 + \int_a^s A(s_2)y(s_2)ds_2 - 7A(a)\int_a^s ds_2$$

$$-\int_a^s\int_a^{s_2} A'(s_1)y(s_1)ds_1ds_2 + \int_a^s\int_a^{s_2} B(s_1)y(s_1)ds_1ds_2 = \int_a^s\int_a^{s_2} F(s_1)ds_1ds_2$$

$$\Rightarrow |y(s_2)|_a^s - y'(a)|s_2|_a^s + \int_a^s A(s_2)y(s_2)ds_2 - 7A(a)|s_2|_a^s$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - y(a) - y'(a)(s-a) + \int_a^s A(s_2)y(s_2)ds_2 - 7A(a)(s-a)$$

$$-\int_a^s (s-t)A'(t)y(t)dt + \int_a^s (s-t)B(t)y(t)dt = \int_a^s (s-t)F(t)dt$$

$$\Rightarrow y(s) - 7 - [y'(a) + 7A(a)](s-a) + \int_a^s A(s_2)y(s_2)ds_2$$

$$= \int_a^s (s-t)F(t)dt + \int_a^s (s-t)A'(t)y(t)dt - \int_a^s (s-t)B(t)y(t)dt$$

$$\Rightarrow y(s) = 7 + [y'(a) + 7A(a)](s-a) + \int_a^s (s-t)F(t)dt$$

$$+\int_a^s (s-t)A'(t)y(t)dt - \int_a^s (s-t)B(t)y(t)dt - \int_a^s A(t)y(t)dt$$

$$\Rightarrow y(s) = 7 + [y'(a) + 7A(a)](s-a)$$
$$+\int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt \quad \ldots(2)$$

Replacing s by b in (2), we have

$$y(b) = 7 + [y'(a) + 7A(a)](b-a)$$
$$+\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$
$$\Rightarrow 6 = 7 + [y'(a) + 7A(a)](b-a)$$
$$+\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt \quad \because y(b) = 6$$
$$\Rightarrow [y'(a) + 7A(a)](b-a) = -1$$
$$-\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$
$$\Rightarrow y'(a) + 7A(a) = -\frac{1}{b-a}$$
$$-\frac{1}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$

Putting this value of $y'(a) + 7A(a)$ in (2), we have

$$y(s) = 7 - \frac{s-a}{b-a} - \frac{s-a}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$
$$+\int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$
$$\Rightarrow y(s) = 7 - \frac{s-a}{b-a} + \int_a^s [\{F(t) + A'(t)y(t) - B(t)y(t)\}(s-t) - A(t)y(t)]dt$$
$$-\frac{s-a}{b-a}\int_a^b [\{F(t) + A'(t)y(t) - B(t)y(t)\}(b-t) - A(t)y(t)]dt$$
$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$
$$-\int_a^s [A(t) - (s-t)\{A'(t) - B(t)\}]y(t)dt$$
$$+\frac{s-a}{b-a}\int_a^b [A(t) - (b-t)\{A'(t) - B(t)\}y(t)]dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$-\int_a^s [A(t) - (s-t)\{A'(t) - B(t)\}]y(t)dt$$

$$+\frac{s-a}{b-a}\int_a^s [A(t) - (b-t)\{A'(t) - B(t)\}y(t)]dt$$

$$+\frac{s-a}{b-a}\int_s^b [A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s \left[\left(\frac{s-a}{b-a} - 1\right)A(t) + \left((s-t) - \frac{(s-a)(b-t)}{b-a}\right)\{A'(t) - B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s \left[\left(\frac{s-a}{b-a} - 1\right)A(t) + \left(\frac{(s-t)(b-a) - (s-a)(b-t)}{b-a}\right)\{A'(t) - B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s \left[\left(\frac{s-a}{b-a} - 1\right)A(t) + \left(\frac{bs - bt - as + at - bs + ab + ts - at}{b-a}\right)\{A'(t) - B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s \left[\left(\frac{s-a}{b-a} - 1\right)A(t) + \left(\frac{-bt - as + ab + ts}{b-a}\right)\{A'(t) - B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt$$

$$\Rightarrow y(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

$$+\int_a^s \left[\left(\frac{s-a}{b-a} - 1\right)A(t) - \left(\frac{(t-a)(b-s)}{b-a}\right)\{A'(t) - B(t)\}\right]y(t)dt$$

$$+\int_s^b \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}]y(t)dt \qquad ...(3)$$

Let $$f(s) = 7 + \int_a^s (s-t)F(t)dt + \frac{s-a}{b-a}\left[-1 - \int_a^b (b-t)F(t)dt\right]$$

and

$$K(s,t) = \begin{cases} \left(\frac{s-a}{b-a} - 1\right)A(t) - \left(\frac{(t-a)(b-s)}{b-a}\right)\{A'(t) - B(t)\}, & t < s \\ \frac{s-a}{b-a}[A(t) - (b-t)\{A'(t) - B(t)\}], & t > s \end{cases}$$

Making these substitutions in (3), we have

$$y(s) = f(s) + \int_a^b K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

Q.9 Transform the boundary value problem

$$y'' + y = s, \; y(0) = 5, \; y'(1) = 0$$

into a an integral equation.

Sol. The given differential equation is

$$y'' + y = s \qquad ...(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have

$$\int_1^s y''(s_1)ds_1 + \int_1^s y(s_1)ds_1 = \int_1^s s_1 ds_1$$

$$\Rightarrow |y'(s_1)|_1^s + \int_1^s y(s_1)ds_1 = \left|\frac{s_1^2}{2}\right|_1^s$$

$$\Rightarrow y'(s) - y'(1) + \int_1^s y(s_1)ds_1 = \frac{s^2 - 1}{2}$$

$$\Rightarrow y'(s) - 0 + \int_1^s y(s_1)ds_1 = \frac{s^2 - 1}{2} \qquad \because y'(1) = 0$$

$$\Rightarrow y'(s) + \int_1^s y(s_1)ds_1 = \frac{s^2 - 1}{2}$$

Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have

$$\int_1^s y'(s_2)ds_2 + \int_1^s \int_1^{s_2} y(s_1)ds_1 ds_2 = \frac{1}{2}\int_1^s (s_2^2 - 1)ds_2$$

$$\Rightarrow \left|y(s_2)\right|_1^s + \int_1^s (s-t)y(t)dt = \frac{1}{2}\left|\frac{s_2^3}{3} - s_2\right|_1^s$$

$$\Rightarrow y(s) - y(1) + \int_1^s (s-t)y(t)dt = \frac{1}{2}\left|\frac{s_2^3 - 3s_2}{3}\right|_1^s$$

$$\Rightarrow y(s) - y(1) + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6} \qquad \ldots(2)$$

Replacing s by 0 in (2), we have

$$y(0) - y(1) + \int_1^0 (0-t)y(t)dt = \frac{2}{6}$$

$$\Rightarrow 5 - y(1) - \int_1^0 ty(t)dt = \frac{1}{3} \qquad \because y(0) = 5$$

$$\Rightarrow y(1) = 5 - \frac{1}{3} - \int_1^0 ty(t)dt$$

Putting this value of $y(1)$ in (2), we have

$$y(s) - \left[5 - \frac{1}{3} - \int_1^0 ty(t)dt\right] + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6}$$

$$\Rightarrow y(s) - 5 + \frac{1}{3} + \int_1^0 ty(t)dt + \int_1^s (s-t)y(t)dt = \frac{s^3 - 3s + 2}{6}$$

$$\Rightarrow y(s) = 5 - \frac{1}{3} + \frac{s^3 - 3s + 2}{6} - \int_1^0 ty(t)dt - \int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s) = 5 + \frac{s^3 - 3s}{6} + \int_0^1 ty(t)dt - \int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s) = 5 + \frac{s^3 - 3s}{6} + \int_0^1 ty(t)dt + \int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s) = 5 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 ty(t)dt + \int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s) = 5 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 [t + (s-t)]y(t)dt$$

$$\Rightarrow y(s) = 5 + \frac{s^3 - 3s}{6} + \int_0^s ty(t)dt + \int_s^1 sy(t)dt \qquad \ldots(3)$$

Let

$$f(s) = 5 + \frac{s^3 - 3s}{6}$$

and

$$K(s,t) = \begin{cases} t, & t < s \\ s, & t > s \end{cases}$$

Making these substitutions in (3), we have

$$y(s) = f(s) + \int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

Q.10 Transform the boundary value problem

$$y'' + y = 1, \; y(0) = 8, \; y'(1) = 0$$

into a an integral equation.

Sol. The given differential equation is

$$y'' + y = 1 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 1 to s, we have

$$\int_1^s y''(s_1)ds_1 + \int_1^s y(s_1)ds_1 = \int_1^s ds_1$$

$$\Rightarrow |y'(s_1)|_1^s + \int_1^s y(s_1)ds_1 = |s_1|_1^s$$

$$\Rightarrow y'(s) - y'(1) + \int_1^s y(s_1)ds_1 = s - 1$$

$$\Rightarrow y'(s) - 0 + \int_1^s y(s_1)ds_1 = s - 1 \qquad \because y'(1) = 0$$

$$\Rightarrow y'(s) + \int_1^s y(s_1)ds_1 = s - 1$$

Replacing s by s_2 and integrating w.r.t. s_2 from 1 to s, we have

$$\int_1^s y'(s_2)ds_2 + \int_1^s \int_1^{s_2} y(s_1)ds_1ds_2 = \int_1^s (s_2 - 1)ds_2$$

$$\Rightarrow |y(s_2)|_1^s + \int_1^s (s-t)y(t)dt = \left|\frac{(s_2-1)^2}{2}\right|_1^s$$

$$\Rightarrow y(s) - y(1) + \int_1^s (s-t)y(t)dt = \frac{(s-1)^2}{2} \qquad \ldots(2)$$

Replacing s by 0 in (2), we have

$$y(0)-y(1)+\int_1^0 (0-t)y(t)dt=\frac{1}{2}$$

$$\Rightarrow 8-y(1)-\int_1^0 ty(t)dt=\frac{1}{2} \qquad \because y(0)=8$$

$$\Rightarrow y(1)=8-\frac{1}{2}-\int_1^0 ty(t)dt$$

$$\Rightarrow y(1)=\frac{15}{2}+\int_0^1 ty(t)dt$$

Putting this value of $y(1)$ in (2), we have

$$y(s)-\left[\frac{15}{2}+\int_0^1 ty(t)dt\right]+\int_1^s (s-t)y(t)dt=\frac{(s-1)^2}{2}$$

$$\Rightarrow y(s)-\frac{15}{2}-\int_0^1 ty(t)dt+\int_1^s (s-t)y(t)dt=\frac{(s-1)^2}{2}$$

$$\Rightarrow y(s)=\frac{15}{2}+\frac{(s-1)^2}{2}+\int_0^1 ty(t)dt-\int_1^s (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{15}{2}+\frac{(s-1)^2}{2}+\int_0^1 ty(t)dt+\int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{15}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 ty(t)dt+\int_s^1 (s-t)y(t)dt$$

$$\Rightarrow y(s)=\frac{15}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 [t+(s-t)]y(t)dt$$

$$\Rightarrow y(s)=\frac{15}{2}+\frac{(s-1)^2}{2}+\int_0^s ty(t)dt+\int_s^1 sy(t)dt \qquad \ldots(3)$$

Let $$f(s)=\frac{15}{2}+\frac{(s-1)^2}{2}$$

And $$K(s,t)=\begin{cases} t, & t<s \\ s, & t>s \end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

Q.11 Reduce the boundary value problem

$$y''+sy=1,\ y(0)=4=y(1)$$

into a Fredholm integral equation.

Sol. The given differential equation is

$$y'' + sy = 1 \qquad \ldots(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1 + \int_0^s s_1 y(s_1)ds_1 = \int_0^s ds_1$$

$$\Rightarrow |y'(s_1)|_0^s + \int_0^s s_1 y(s_1)ds_1 = |s_1|_0^s$$

$$\Rightarrow y'(s) - y'(0) + \int_0^s s_1 y(s_1)ds_1 = s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2 - y'(0)\int_0^s ds_2 + \int_0^s \int_0^{s_2} s_1 y(s_1)ds_1 ds_2 = \int_0^s s_2 ds_2$$

$$\Rightarrow |y(s_2)|_0^s - y'(0)|s_2|_0^s + \int_0^s (s-t)ty(t)dt = \left|\frac{s_2^2}{2}\right|_0^s$$

$$\Rightarrow y(s) - y(0) - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2}$$

$$\Rightarrow y(s) - 4 - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2} \qquad \because y(0) = 4$$

$$\Rightarrow y(s) - 4 - y'(0)s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2} \qquad \ldots(2)$$

Replacing s by 1 in (2), we have

$$y(1) - 4 - y'(0)(1) + \int_0^1 (1-t)ty(t)dt = \frac{1}{2}$$

$$\Rightarrow y'(0) = -4 - \frac{1}{2} + y(1) + \int_0^1 (1-t)ty(t)dt$$

$$\Rightarrow y'(0) = -4 - \frac{1}{2} + 4 + 0 + \int_0^1 (1-t)ty(t)dt \qquad \because y(1) = 4$$

$$\Rightarrow y'(0) = -\frac{1}{2} + \int_0^1 (1-t)ty(t)dt$$

Putting this value of $y'(0)$ in (2), we have

$$y(s) - 4 - \left[-\frac{1}{2} + \int_0^1 (1-t)ty(t)dt\right]s + \int_0^s (s-t)ty(t)dt = \frac{s^2}{2}$$

$$\Rightarrow y(s)-4+\frac{s}{2}-s\int_0^1(1-t)ty(t)dt+\int_0^s(s-t)ty(t)dt=\frac{s^2}{2}$$

$$\Rightarrow y(s)=\frac{s^2}{2}-\frac{s}{2}+4+\int_0^1(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s(1-t)sty(t)dt+\int_s^1(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s(1-t)sty(t)dt-\int_0^s(s-t)ty(t)dt+\int_s^1(1-t)sty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s[(1-t)st-(s-t)t]y(t)dt+\int_s^1(1-t)sty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s(st-st^2-st+t^2)y(t)dt+\int_s^1(1-t)sty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s(1-s)t^2y(t)dt+\int_s^1(1-t)sty(t)dt$$

$$\Rightarrow y(s)=\frac{s^2-s+8}{2}+\int_0^s(1-s)t^2y(t)dt+\int_s^1(1-t)sty(t)dt \qquad ...(3)$$

Let $$f(s)=\frac{s^2-s+8}{2}$$

And $$K(s,t)=\begin{cases}(1-s)t^2, & t<s\\ (1-t)st, & t>s\end{cases}$$

Making these substitutions in (3), we have

$$y(s)=f(s)+\int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

Q.12 Reduce the boundary value problem

$$y''+sy=s^2+3s+2,\ y(0)=3=y(1)$$

into a Fredholm integral equation.

Sol. The given differential equation is

$$y''+sy=s^2+3s+2 \qquad ...(1)$$

Replacing s by s_1 and integrating w.r.t. s_1 from 0 to s, we have

$$\int_0^s y''(s_1)ds_1+\int_0^s s_1y(s_1)ds_1=\int_0^s(s_1^2+3s_1+2)ds_1$$

$$\Rightarrow |y'(s_1)|_0^s+\int_0^s s_1y(s_1)ds_1=\left|\frac{s_1^3}{3}+\frac{3s_1^2}{2}+2s_1\right|_0^s$$

$$\Rightarrow y'(s)-y'(0)+\int_0^s s_1 y(s_1)ds_1=\frac{s^3}{3}+\frac{3s^2}{2}+2s$$

Replacing s by s_2 and integrating w.r.t. s_2 from 0 to s, we have

$$\int_0^s y'(s_2)ds_2-y'(0)\int_0^s ds_2+\int_0^s\int_0^{s_2} s_1 y(s_1)ds_1 ds_2=\int_0^s\left(\frac{s_2^3}{3}+\frac{3s_2^2}{2}+2s_2\right)ds_2$$

$$\Rightarrow \left|y(s_2)\right|_0^s-y'(0)\left|s_2\right|_0^s+\int_0^s (s-t)ty(t)dt=\left|\frac{s_2^4}{12}+\frac{s_2^3}{2}+s_2^2\right|_0^s$$

$$\Rightarrow y(s)-y(0)-y'(0)s+\int_0^s (s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)-3-y'(0)s+\int_0^s (s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2 \quad \because y(0)=3$$

$$\Rightarrow y(s)-3-y'(0)s+\int_0^s (s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2 \qquad \ldots(2)$$

Replacing s by 1 in (2), we have

$$y(1)-3-y'(0)(1)+\int_0^1 (1-t)ty(t)dt=\frac{1}{12}+\frac{1}{2}+1$$

$$\Rightarrow 3-3-y'(0)(1)+\int_0^1 (1-t)ty(t)dt=\frac{1+6+12}{12} \quad \because y(1)=3$$

$$\Rightarrow y'(0)=-\frac{19}{12}+\int_0^1 (1-t)ty(t)dt$$

Putting this value of $y'(0)$ in (2), we have

$$y(s)-3-\left[-\frac{19}{12}+\int_0^1 (1-t)ty(t)dt\right]s+\int_0^s (s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)-3+\frac{19s}{12}-s\int_0^1 (1-t)ty(t)dt+\int_0^s (s-t)ty(t)dt=\frac{s^4}{12}+\frac{s^3}{2}+s^2$$

$$\Rightarrow y(s)=\frac{s^4}{12}+\frac{s^3}{2}+s^2-\frac{19s}{12}+3+\int_0^1 (1-t)sty(t)dt-\int_0^s (s-t)ty(t)dt$$

$$\Rightarrow y(s)=\frac{s^4+6s^3+12s^2-19s+36}{12}+\int_0^s (1-t)sty(t)dt+\int_s^1 (1-t)sty(t)dt$$

$$-\int_0^s (s-t)ty(t)dt$$

$$\Rightarrow y(s) = \frac{s^4 + 6s^3 + 12s^2 - 19s + 36}{12} + \int_0^s (1-t)sty(t)dt$$

$$-\int_0^s (s-t)ty(t)dt + \int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^4 + 6s^3 + 12s^2 - 19s + 36}{12} + \int_0^s [(1-t)st - (s-t)t]y(t)dt$$

$$+\int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^4 + 6s^3 + 12s^2 - 19s + 36}{12} + \int_0^s (st - st^2 - st + t^2)y(t)dt$$

$$+\int_s^1 (1-t)sty(t)dt$$

$$\Rightarrow y(s) = \frac{s^4 + 6s^3 + 12s^2 - 19s + 36}{12} + \int_0^s (1-s)t^2 y(t)dt$$

$$+\int_s^1 (1-t)sty(t)dt \qquad \ldots(3)$$

Let $\qquad f(s) = \dfrac{s^4 + 6s^3 + 12s^2 - 19s + 36}{12}$

And $\qquad K(s,t) = \begin{cases} (1-s)t^2, & t < s \\ (1-t)st, & t > s \end{cases}$

Making these substitutions in (3), we have

$$y(s) = f(s) + \int_0^1 K(s,t)y(t)dt$$

which is the required Fredholm integral equation.

EXERCISE 2-1

Q.1 Solve the Fredholm integral equation of the second kind

$$g(s) = s + \lambda \int_0^2 (st + s^2t^2)g(t)dt$$

Sol. The given integral equation is

$$g(s) = s + \lambda \int_0^2 (st + s^2t^2)g(t)dt \qquad \ldots(1)$$

Here the Kernel