Question #27:

Find the general solution of the given differential equation

$y''' + 3y'' + 3y' + y = 0$

Solution

Given differential equation is $y''' + 3y'' + 3y' + y = 0 \rightarrow (1)$

This is homogeneous differential equation

Auxiliary equation of eq(1) is

$$m^3 + 3m^2 + 3m + 1 = 0$$

$$\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\Rightarrow m^3 + 3m^2(1) + 3m(1)^2 + (1)^3 = 0$$

$$(m+1)^3 = 0$$

$$(m+1)(m+1)(m+1) = 0$$

$$m = -1, \ m = -1, \ m = -1$$

As complementary function $(y_c(x))$ for repeated real roots,

$$\because y_c(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_2 x^2 e^{m_1 x}$$

$$= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

Question #41:

Solve the given differential equation subject to the indicated initial conditions

$2y'' - 2y' + y = 0 \quad , \ y(0) = -1 \ , y'(0) = 0$

Solution

Given $2y'' - 2y' + y = 0 \rightarrow (1)$

Auxiliary equation of eq(1) is

$$2m^2 - 2m + 1 = 0$$

By comparing with quadratic equation $am^2 + bm + c = 0$

$$a = 2, \ b = -2, \ c = 1$$

$$\because \quad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By putting values

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$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4-8}}{4}$$

$$m = \frac{2 \pm \sqrt{-4}}{4}$$

$m_1 = \frac{2 + \sqrt{4}\,i}{4}$, $m_2 = \frac{2 - \sqrt{4}\,i}{4}$ $\Rightarrow m_1 = \frac{1}{2} + \frac{i}{2}$, $m_2 = \frac{1}{2} - \frac{i}{2}$

For complex roots, general solution is,

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$= e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \to ②$$

$\because m = \alpha + i\beta$, when $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$

eq② $\Rightarrow$ $y(x) = e^{x/2} (c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}) \to ③$

when $y(0) = -1$

eq③ $\Rightarrow$ $y(0) = e^{0/2} (c_1 \cos \frac{0}{2} + c_2 \sin \frac{0}{2})$

$$-1 = 1\,(c_1(1) + c_2(0))$$

$$-1 = c_1$$

$$\Rightarrow \boxed{c_1 = -1}$$

eq③ $y(x) = c_1 e^{x/2} \cos \frac{x}{2} + c_2 e^{x/2} \sin \frac{x}{2}$

Taking derivation on b.s

$$y'(x) = c_1 \frac{d}{dx} e^{x/2} \cos \frac{x}{2} + c_2 \frac{d}{dx} e^{x/2} \sin \frac{x}{2}$$

Use the product rule.

$$y'(x) = c_1 \left[e^{x/2} \frac{d}{dx} \cos \frac{x}{2} + \cos \frac{x}{2} \frac{d}{dx} e^{\frac{x}{2}} \right] + c_2 \left[e^{\frac{x}{2}} \frac{d}{dx} \sin \frac{x}{2} + \sin \frac{x}{2} \frac{d}{dx} e^{\frac{x}{2}} \right]$$

$$= c_1 \left[e^{\frac{x}{2}} \cdot -\frac{1}{2} \sin \frac{x}{2} + \cos \frac{x}{2} \cdot \frac{1}{2} e^{\frac{x}{2}} \right] + c_2 \left[e^{x/2} \cdot \frac{1}{2} \cos \frac{x}{2} + \sin \frac{x}{2} \cdot \frac{1}{2} e^{x/2} \right]$$

$$y'(x) = -\frac{1}{2} c_1 e^{x/2} \sin \frac{x}{2} + \frac{1}{2} c_1 e^{\frac{x}{2}} \cos \frac{x}{2} + \frac{1}{2} c_2 e^{x/2} \cos \frac{x}{2} + \frac{1}{2} c_2 e^{x/2} \sin \frac{x}{2}$$

when $y'(0) = 0$

$$0 = -\frac{1}{2} c_1 e^0 \sin 0 + \frac{1}{2} c_1 e^0 \cos 0 + \frac{1}{2} c_2 e^0 \cos 0 + \frac{1}{2} c_2 e^0 \sin 0$$

$$0 = 0 + \frac{1}{2} c_1 + \frac{1}{2} c_2 + 0$$

$\because c_1 = -1$

$\Rightarrow \quad 0 = \frac{1}{2}(-1) + \frac{1}{2}c_2$

$\cancel{\frac{1}{2}} c_2 = \cancel{\frac{1}{2}}$

$\boxed{c_2 = 1}$

eq(2) $\Rightarrow \quad y(x) = e^{x/2}\left(-1\cos\frac{x}{2} + 1\sin\frac{x}{2}\right)$

$= e^{\frac{x}{2}}\left(\sin\frac{1}{2}x - \cos\frac{1}{2}x\right)$

Question # 10:

Solve the given differential equation by undetermined coefficients.

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

Solution

Given D.E is

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

Non-homogenous because right hand side $\neq 0$

so, associated homogeneous differential equation is

$$y'' + 2y' = 0 \rightarrow (1)$$

Auxiliary equation of eq(1) is

$$m^2 + 2m = 0$$

$$m(m+2) = 0$$

$\Rightarrow m_1 = 0 \quad$ or $\quad m_2 = -2$

Roots are real distinct, so complementary function will be

$$\because y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y_c(x) = c_1 e^{0x} + c_2 e^{-2x}$$

$$= c_1 + c_2 e^{-2x} \rightarrow (2)$$

As $g(x) = 2x + 5 - e^{-2x}$

By method of undetermined co-efficients we take $y_p = Ax + B + Ce^{-2x}$

As in $y_c(x)$, exponential term and in y_p, it is same. In order to create difference between them multiply

x with $Ax+B+Ce^{-2x}$

Hence, $y_p = Ax^2+Bx+Cxe^{-2x} \rightarrow (3)$

$$y_p' = 2Ax+B-2Cxe^{-2x}+Ce^{-2x}$$
$$= 2Ax+B+Ce^{-2x}(-2x+1)$$

$$y_p'' = 2A+4Cxe^{-2x}-2Ce^{-2x}-2Ce^{-2x}$$
$$= 2A+2Ce^{-2x}(2x-1-1)$$

Put y_p and y''_p in Given differential equation

$$2A+2Ce^{-2x}(2x-2)+2(2Ax+B+Ce^{-2x}(-2x+1)) = 2x+5-e^{-2x}$$
$$2A+4C\cancel{xe^{-2x}}-4Ce^{-2x}+4Ax+2B-4C\cancel{xe^{-2x}}+2Ce^{-2x} = 2x+5-e^{-2x}$$
$$4Ax+2A+2B+-2Ce^{-2x} = 2x+5-e^{-2x}$$

By comparing

$\Rightarrow 4A=2$, $\Rightarrow 2A+2B=5$, $-2C=-1$

$\boxed{A=\frac{1}{2}}$, $\because A=\frac{1}{2}$, $\boxed{C=\frac{1}{2}}$

$\Rightarrow 2(\frac{1}{2})+2B=5$

$2B=5-1$

$\boxed{B=2}$

eq (3) $\Rightarrow y_p = \frac{1}{2}x^2+2x+\frac{1}{2}xe^{-2x}$

General solution of given differential equation is

$$y(x) = y_c(x)+y_p(x)$$
$$y(x) = c_1+c_2e^{-2x}+\frac{1}{2}x^2+2x+\frac{1}{2}xe^{-2x} \text{ Ans}$$

Question # 30:

Solve the given differential equation subject to th indicated initial conditions.

$$2y''+3y'-2y = 14x^2-4x-11, \quad y(0)=0, y'(0)=0$$

Solution

$$2y''+3y'-2y = 14x^2-4x-11 \rightarrow (1)$$

Associated homogenous differential equation is

$$2y''+3y'-2y=0$$

Auxiliary equation is

$$2m^2+3m-2=0$$
$$2m^2+4m-m-2=0$$
$$2m(m+2)-1(m+2)=0$$
$$(2m-1)(m+2)=0$$
$$2m-1=0 \quad , \quad m+2=0$$

$m_1 = \frac{1}{2}$, $m_2 = -2$

Roots are real distinct, so

$$y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$
$$= c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} \rightarrow ②$$

As $g(x) = 14x^2 - 4x - 11$

By method of undetermined co-efficients we take

$y_P = Ax^2 + Bx + C \rightarrow ③$

$$y'_P = 2Ax + B$$
$$y''_P = 2A$$

eq① $\Rightarrow$ $2(2A) + 3(2Ax+B) - 2(Ax^2+Bx+C) = 14x^2 - 4x - 11$

$$4A + 6Ax + 3B - 2Ax^2 - 2Bx - 2C = 14x^2 - 4x - 11$$
$$-2Ax^2 + 6Ax - 2Bx + 4A + 3B - 2C = 14x^2 - 4x - 11$$

By comparing co-efficients

$-2A = 14$

$\boxed{A = -7}$

, $6A - 2B = -4$

$\because A = -7$

, $\Rightarrow 6(-7) - 2B = -4$

$-2B = -4 + 42$

$B = \frac{38}{-2}$

$\boxed{B = -19}$

, $4A + 3B - 2C = -11$

$\because A = -7$, $B = -19$

, $\Rightarrow 4(-7) + 3(-19) - 2C = -$

, $-2C = 74$

, $\boxed{C = -37}$

eq③ $\Rightarrow$ $y_P = -7x^2 - 19x - 37$

General solution is : $y(x) = y_c(x) + y_p(x)$

$$y(x) = c_1 e^{x/2} + c_2 e^{-2x} - 7x^2 - 19x - 37 \rightarrow ④$$

When $y(0) = 0$, eq④ becomes

$$y(0) = c_1 e^0 + c_2 e^0 - 7(0)^2 - 19(0) - 37$$
$$0 = c_1 + c_2 - 37$$
$$c_1 + c_2 = 37 \rightarrow (i)$$

From eq④

$$y'(x) = \frac{1}{2} c_1 e^{x/2} - 2c_2 e^{-2x} - 14x - 19 \rightarrow ⑤$$

When $y'(0) = 0$

eq⑤ $\Rightarrow$ $y'(0) = \frac{1}{2} c_1 e^0 - 2c_2 e^0 - 14(0) - 19$

$$0 = \frac{1}{2} c_1 - 2c_2$$
$$\frac{1}{2} c_1 - 2c_2 = 19 \rightarrow (ii)$$

Xing eq(i) with 2 and ting in eq(ii)

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$$\frac{1}{2}c_1 - 2c_2 = 19$$

$$2c_1 + 2c_2 = 74$$

$$\frac{5}{2}c_1 = 93$$

$$\boxed{c_1 = \frac{186}{5}}$$

$$eq(i) \Rightarrow \quad c_2 + \frac{186}{5} = 37$$

$$c_1 = \frac{185}{5} - \frac{186}{5}$$

$$\boxed{c_2 = -\frac{1}{5}}$$

$$eq(4) \Rightarrow \quad y(x) = +\frac{186}{5}e^{x/2} - \frac{1}{5}e^{-2x} - 7x^2 - 19x - 37$$