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MTH 241 Ordinary Differential Equations

Lecture # 1

Introduction:

Differential Equations are the mathematical backbone of many areas of science and engineering. Differential equations arise from attempts to formulate, or describe certain physical systems in terms of mathematics. In other words the mathematical description or mathematical model of experiments, observations, or theories may be a differential equation. The words differential and equations certainly suggest solving some kind of equations that contains derivatives or rates of change. Practical questions such as How fast does a disease spread? How fast does a population change? involve rate of change or derivatives.

Basic Definitions and Terminology

Differential Equation:

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

Classification of Differential Equation (DE)

Classification by Type

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be an ordinary differential equation (ODE). For example

$$\frac{dy}{dx} + 5y = e^x,$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \text{ and}$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

An equation involving partial derivatives of one or more dependent variabes w.r.t two or more independent variables is called Partial differential equation (PDE). For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} , \quad \text{and}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Classification by order

The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation. For example

(i) $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$

is a second-order ordinary differential equation.

(ii) $(y-x)dx + 4x\,dy = 0 \Rightarrow 4x\frac{dy}{dx} + y = x$

is a first-order ODE.

(iii) $a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$

is a fourth-order PDE.

Classification by linearity :

An nth-order ordinary differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \longrightarrow 1$$

Two important special cases of 1 are linear first-order ($n=1$) and linear second-order ($n=2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) , \quad \text{and}$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

In additive combination on the left-hand side of equation ①, the main two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficient $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

Examples

The equations

$(y-x)\,dx + 4x\,dy = 0$,

$y'' - 2y' + y = 0$, and

$\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$

are, in turn, linear first-order, second-order, and third order ODEs.

A nonlinear ordinary differential equation is simply one that is not linear.

Remarks

(i) Nonlinear functions of the dependent variable or its derivatives such as $\sin y$ or $e^{y'}$, cannot appear in a linear ODE.

Therefore

$(1-y)y' + 2y = e^x$, coefficient depends on y

$\frac{d^2y}{dx^2} + \sin y = 0$, and nonlinear function of y

$\frac{d^4y}{dx^4} + y^2 = 0$ power of y not 1, i.e nonlinear function of y

are the examples of nonlinear first-order, second-order, and fourth-order ODEs.

Home Work

Exercise 1.1

Questions: 1-10 (All)

MTH-241 Ordinary Differential Equations

Lecture # 2

Leibniz notation & prime notation

Throughout this course, ordinary derivatives will be written by using either the Leibniz notation $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$ or the prime notation $y', y'', y''', \dots$. The prime notation is used to denote only the first three derivatives; the fourth derivative is written as $y^{(4)}$ instead y''''. In general, the nth derivative of y is written as $\frac{d^ny}{dx^n}$ or $y^{(n)}$.

Remarks:

(i) Although it is less convenient to write and to typeset, but the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example in DE:

$$\frac{d^2x}{dt^2} + 16x = 0 ,$$

x represents a dependent variable, whereas t is the independent variable.

(ii) flyspeck notation: (Newton's dot notation)

In physical sciences and engineering, Newton's dot notation or flyspeck notation is sometimes used to denote derivatives with respect to time t. Thus the differential equation $\frac{d^2s}{dt^2} = -32$ becomes $\ddot{s} = -32$.

(iii) subscript notation:

Partial derivatives are often denoted by a subscript notation indicating the independent variables. For example the PDE:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

can be written in subscript notation as

$$u_{xx} = u_{tt} - 2u_t$$

Def. Solution of an ODE:

Any function f, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth order ordinary differential equation reduces the equation to an identity, is said to be a solution of the ODE on that interval.

Interval of Definition:

Solution of an ordinary differential equation will be incomplete without thinking of interval of definition. The interval I in above definition is variously called the interval of definition, the interval of existence, the interval of validity, or the domain of the solution, and can be an open interval (a,b), a closed interval $[a,b]$, an infinite interval (a,∞), and so on.

Verification of a solution:

Solution of an ODE can be verified by substituting the solution into ODE, if this process reduces the ODE to an identity then it is done.

Remark: Not every DE that we write necessarily has a solution.

Example 1

Verify that the function $y=\frac{1}{16}x^4$ is a solution of ODE, $\frac{dy}{dx}=xy^{1/2}$ on interval $(-\infty,\infty)$.

Solution:

Given ODE is

$$\frac{dy}{dx}=xy^{1/2}$$

Left-hand side: $\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1}{16}x^4\right)=\frac{1}{16}(4x^3)=\frac{1}{4}x^3$

Right hand side: $xy^{1/2}=x\left(\frac{1}{16}x^4\right)^{1/2}=x\left(\frac{x^2}{4}\right)=\frac{1}{4}x^3$

Thus each side of the equation is the same for every real number. So $y=\frac{1}{16}x^4$ is a solution of ODE, $\frac{dy}{dx}=xy^{1/2}$ on interval $(-\infty,\infty)$.

As $y=\frac{1}{16}x^4$ and $\frac{dy}{dx}=\frac{1}{4}x^3$ are both continuous on interval $(-\infty,\infty)$.

Example 2 :

Verify that the function $y = xe^x$ is a solution of the given ODE, $y'' - 2y' + y = 0$ on the interval $(-\infty, \infty)$.

Solution :

Given ODE is

$$y'' - 2y' + y = 0$$

as $y = xe^x$

$y' = xe^x + e^x$

$y'' = xe^x + e^x + e^x$

or $y'' = xe^x + 2e^x$

Left hand side : $y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + xe^x$

$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x$

$= 0$

Right hand side : 0

We see that y, y', and y'' are continuous on all real numbers and on substitution of function y in given ODE, we have an identity, so $y = xe^x$ is the solution of given ODE on the interval $(-\infty, \infty)$.

Trivial Solution :

In examples 1 and 2, each ODE possesses the constant solution $y = 0$ on interval $(-\infty, \infty)$. A solution of a differential equation that is identically zero on an interval I is said to be a trivial solution.

Example 3 :

The first-order differential equations :

(i) $\left(\frac{dy}{dx}\right)^2 + 1 = 0$

(ii) $(y')^2 + y^2 + 4 = 0$

possess no real solution. Because for (i) if it exists, its square added with 1 must be identically zero, which is impossible. Moreover there does not exist any real number, whose square added with 1 equals zero. Identical reason can be stated for ODE (ii).

Example 4 :

The second-order ODE : $(y'')^2 + 10y^4 = 0$ possesses only one real solution i-e $y = 0$ (Trivial solution).

Solution Curve :

The graph of a solution ϕ of an ODE is called a solution curve. since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the function ϕ and the graph of the solution ϕ. In other words, the domain of the function ϕ needs not be the same as the interval of definition (or domain) of the solution ϕ.

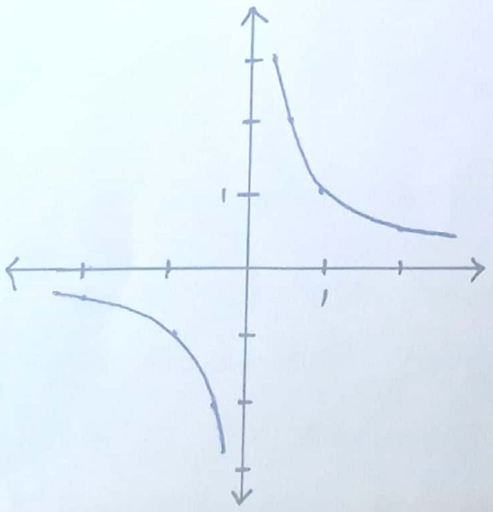
Example : Function versus Solution

consider the function:

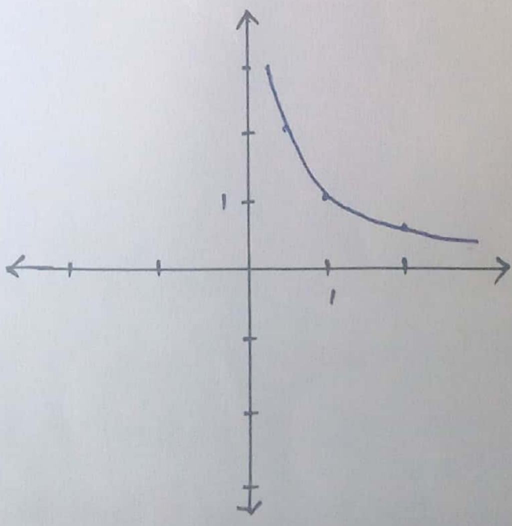
$$y = \frac{1}{x}$$

The domain of this function is the set of all real numbers x except 0. moreover this function is not differentiable at $x = 0$.

Now $y = \frac{1}{x}$ is also a solution of the Linear first-order differential equation $xy' + y' = 0$. But when we say that $y = \frac{1}{x}$ is a solution of this DE, we mean that it is a function defined on an interval I on which it is continuous, differentiable and satisfies the DE when substituted into it. It makes sense to take the interval I to be as large as possible. Thus we take I to be either $(-\infty, 0)$ or $(0, \infty)$.



a) function $y = \frac{1}{x}$, $x \neq 0$ $\downarrow$ Domain



b) solution $y = \frac{1}{x}$, $(0, \infty)$ $\downarrow$ Interval of definition

Explicit and Implicit Solutions :

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an explicit solution.

Examples

The solutions $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = \frac{1}{x}$ are, in turn, explicit Solutions of DEs $\frac{dy}{dx} = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$.

Moreover the trivial solution $y = 0$ is an explicit solution of all three equations.

Implicit Solution of an ODE :

When we attempt to solve nonlinear first-order differential equations, often we have a relation or expression $G(x,y) = 0$ that defines a solution ϕ implicitly.

Def:

A relation $G(x,y) = 0$ is said to be an implicit solution of an ODE on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well the differential equation on I.

Example :

The relation $x^2 + y^2 = 25$ is an implicit solution of DE :

$$\frac{dy}{dx} = \frac{-x}{y}$$

on the open interval $(-5, 5)$.

Explanation :

Solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$

The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$

satisfy the relation $x^2 + y^2 = 25$ as well as the DE : $\frac{dy}{dx} = \frac{-x}{y}$.

Thus $x^2 + y^2 = 25$ is an implicit solution of DE: $\frac{dy}{dx} = \frac{-x}{y}$ on open interval $(-5, 5)$.

Example:

The relation $x^2+y^2+25=0$ is not an implicit solution of DE

$\frac{dy}{dx} = \frac{-x}{y}$, despite of the fact that it satisfies the DE.

Because there exists no real valued explicit function ϕ.

Families of Solutions

When solving a first-order ODE, we usually obtain a solution containing an arbitrary constant called a one-parameter family of solutions. When solving an nth-order ODE, we seek an n-parameter family of solutions. This means that a single differential equation can possesses an infinite number of solutions corresponding to the unlimited number of choices for the parameter(s).

Particular solution:

A solution of a differential equation that is free of arbitrary parameters is called a particular solution.

Singular solution:

Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the DE, that is, a solution that cannot be obtained by specializing any of the parameters in the family of solutions. Such an extra solution is called a singular solution.

Example:

The differential equation $\frac{dy}{dx} = xy^{1/2}$ possesses the one-parameter family of solutions $y = \left(\frac{1}{4}x^2+c\right)^2$, when $c=0$ the resulting particular solution is $y = \frac{1}{16}x^4$. But notice that the trivial solution $y=0$ is a singular solution, since it is not a member of the family $y = \left(\frac{1}{4}x^2+c\right)^2$; there is no way of assigning a value to the constant c to obtain $y=0$.

Different Forms of nth-order ODE:

General Form

We can express an nth-order ODE in one dependent variable y w.r.t independent variable x by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0 \longrightarrow (1)$$

Where F is a real-valued function of $n+2$ variables: $x, y, y', \ldots, y^{(n)}$.

Normal Form

The differential equation of the form

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}),$$

where f is a real-valued continuous function, is reffered to as the normal form of ODE (1).

Examples

The normal forms of first and second order ODE's are respectively:

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2 y}{dx^2} = f(x, y, y')$$

Home Work

Exercise: 1.1

Questions: 11-38