Exercise # 3.2

Question # 1:

The population of certain community is known to increase at a rate proportional to the number of people present at any time. If the population has doubled in 5 years, how long will it take to triple? to quadruple?

Solution

Given that, $P(0) = P_0 \to ①$

$P(5) = 2P_0$

$P(t) = 3P_0 \;;\; t = ?$

$P(t) = 4P_0 \;;\; t = ?$

$\because \frac{dP}{dt} = kP$

By using separation of variable

$\frac{1}{P}dP = kdt$

Taking integral $\int$ on b.s

$\Rightarrow \int \frac{1}{P} dP = \int k dt$

$\ln|P| = kt + c_1$ $\qquad \because \int x^{-1} dx = \ln|x| + c$

Taking antilog on b.s $\qquad \because \int dx = x + c$

$P(t) = e^{kt + c_1}$

$P(t) = e^{kt} \cdot e^{c_1}$

$P(t) = ce^{kt} \to ②$

As $P(0) = P_0$, so equation ② becomes

$P(0) = ce^{k(0)}$

$P_0 = ce^0$

$\Rightarrow \boxed{c = P_0}$

so eq ②, we get

$P(t) = P_0 e^{kt} \to ③$

As $P(5) = 2P_0$, put in eq ③

eq ③ $\Rightarrow P(5) = P_0 e^{k5}$

$2\cancel{P_0} = \cancel{P_0} e^{5k}$

$\Rightarrow \ln|2| = 5k$

$\Rightarrow \frac{k5}{5} = \frac{\ln|2|}{5}$

$\boxed{k = 0.1386}$

so eq 3 $\Rightarrow$ $P(t) = P_o e^{0.1386t} \rightarrow (4)$

when $P(t) = 3P_o$, eq 4 becomes

$$3P_o = P_o e^{0.1386t}$$

$$3 = e^{0.1386t}$$

$$\ln|3| = 0.1386t$$

$$\Rightarrow t = \frac{1.0986}{0.1386}$$

$$t = 7.9 \approx 8 \text{ years}$$

when $P(t) = 4P_o$, eq 4 becomes

$$4P_o = P_o e^{0.1386t}$$

$$\ln|4| = 0.1386t$$

$$\Rightarrow t = \frac{1.3862}{0.1386}$$

$$t = 10.002 \approx 10 \text{ years}$$

Question # 3:

The population of town grows at a rate proportional to the population at any time. It's initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?

Solution

$\rightarrow$ Given that, $P(0) = P_o = 500$

$$P(10) = 15\% \times P_o$$

$$= \frac{15}{100} \times 500$$

$$= 75$$

So, $P(10) = 75 + 500$

$$= 575$$

$$P(t) = P(30) = ?$$

$$\because \frac{dP}{dt} = kP$$

By using separation of variable

$$\frac{1}{P} dP = k\,dt$$

Taking integral $\int$ on b.s

$$\int \frac{1}{P} dP = \int k\,dt$$

$\because \int x^{-1} dx = \ln|x| + c$

$\because \int dx = x + c$

$$\ln|P| = kt + c_1$$

Taking antilog on b.s

$$P(t) = e^{kt + c_1}$$

$$P(t) = e^{kt} \cdot e^{c_1}$$

$$P(t) = c e^{kt} \rightarrow (1)$$

As $P(0) = P_0 = 500$, so

eq① $\Rightarrow P(0) = ce^{k(0)}$

$\Rightarrow \boxed{c = 500}$

eq① $\Rightarrow P(t) = 500e^{kt} \rightarrow$ ②

As $P(10) = 575$, put in eq② we get,

$P(10) = 500e^{k(10)}$

$575 = 500e^{10k}$

$\frac{23}{20} = e^{10k}$

$\ln\left|\frac{23}{20}\right| = 10k$

$\Rightarrow k = \frac{0.1397}{10}$

$k = 0.01397$

so eq② $\Rightarrow P(t) = 500e^{0.01397t}$

As $P(t) = P(30)$

$\Rightarrow P(30) = 500e^{0.0139(30)}$

$P(30) = 758.7$

Question #5:

The radioactive isotope of lead, Pb-209, decay at a rate proportional to the amount present at any time and has a half-life of 3.3 hours. If 1 gram of lead is present initially, how long will it take for 90% of the lead to decay?

Solution

$A(0) = A_0 = 1$ gram

$A(3.3) = \frac{A_0}{2} = \frac{1}{2}$

$A(t) = 90\%$ decay time $t = ?$

$A(t) = \frac{10}{100} \times 1$

$= 0.1$ gram remaining

$\because \frac{dA}{dt} = kA$

By using separation of variable

$\int \frac{dA}{A} = \int k\,dt$

$\ln|A| = kt + c_1$

$A(t) = ce^{kt} \rightarrow$ ①

As given $A(0) = 1$,

eq① $\Rightarrow A(0) = c e^{k(0)}$

$1 = c$

Put value of c in eq①

$A(t) = 1e^{kt} \rightarrow ②$

As $A(3.3) = \frac{1}{2}$

eq② $\Rightarrow A(3.3) = e^{k(3.3)}$

$\frac{1}{2} = e^{3.3k}$

$\ln\left|\frac{1}{2}\right| = 3.3k$

$\Rightarrow k = -0.210$

Put this in eq②

eq② $\Rightarrow A(t) = e^{-0.210t}$

if $A(t) = 0.1$ then,

$0.1 = e^{-0.210t}$

$\ln|0.1| = -0.210\,t$

$t = 10.96 \approx 11$ hours Answer

Question# 7:

Determine the half-life of the radioactive substance described in Problem 6.

Solution

$m(0) = m_0 = 100$ milligram

$m(6) = 3\%$ of $100 = \frac{3}{100} \times 100$

$= 3$

So, $m = \text{Total} - 3$

$m(6) = 100 - 3$

$= 97$

So, by using $m = m_0 e^{kt} \rightarrow ①$

$97 = 100e^{k(t)}$

$e^{6k} = 0.97$

$6k = \ln|0.97|$

$k = -0.0051$

For half-life of radioactive, $m = 50$

so, eq① $\Rightarrow 50 = 100e^{-0.0051t}$

$\ln(0.5) = -0.0051t$

$t = 135.91$ Answer

Question #9

When a vertical beam of light passes through a transparent substance, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

Solution

Given that,

$$I(0) = I_0$$

$$I(3) = 25\% \, I_0$$

$$= \frac{25}{100} \times I_0$$

$$= 0.25 \, I_0$$

$$\because \frac{dI}{dt} = kI$$

By using separation of variable

$$\frac{1}{I} dI = k\,dt$$

Integration both sides

$$\int \frac{1}{I} dI = \int k\,dt$$

$$\ln|I| = kt + c_1$$

Take antilog on b.s

$$I(t) = e^{kt + c_1}$$

$$I(t) = e^{kt} \cdot e^{c_1}$$

$$I(t) = c e^{kt} \rightarrow ①$$

As we know, $I(0) = I_0$ given

$$\Rightarrow I(0) = c e^{k(0)}$$

$$I_0 = c\,(1)$$

$$c = I_0$$

so eq① $\Rightarrow$ $I(t) = I_0 e^{kt} \rightarrow ②$

Given $I(3) = 0.25 I_0$

so eq② $\Rightarrow$ $I(3) = I_0 e^{k3}$

$$0.25 \cancel{I_0} = \cancel{I_0} e^{k3}$$

Taking ln on b.s

$$\ln|0.25| = k(3)$$

$$k = \frac{-1.386}{3}$$

$$k = -0.462$$

eq ② $\Rightarrow$ $I(t) = I_o e^{-0.462t} \rightarrow$ ③

$I(15) = ?$

eq ③ $\Rightarrow$ $I(15) = I_o e^{-0.462(15)}$

$$= I_o e^{-6.9}$$

$$= 9.8 \times 10^{-4} I_o$$

$\Rightarrow I(15) = 0.00098 I_o$ Answer

Question #11

In a piece of burned wood, or charcoal, it was found that 85.5% of the C-14 had decayed. Use the information in Example 3 to determine the approximate age of the wood.

Solution

Let $A(t)$ be the amount of C-14 in burned word at any time t in years.

By Growth decay problem we know that $A(t)$ at any time t will be,

$$A(t) = A_o e^{kt} \rightarrow ①$$

As, half life of C-14 is 5600 years

so,

$$A(5600) = \frac{A_o}{2}$$

so eq ① $\Rightarrow$ $A(5600) = A_o e^{k(5600)}$

$$\frac{A_o}{2} = A_o e^{5600k}$$

$$\frac{1}{2} = e^{5600k}$$

Taking ln on b.s

$$\ln|0.5| = 5600k$$

$$\Rightarrow k = \frac{-0.69}{5600}$$

$$k = -0.00012378$$

so, eq ① becomes

$$A(t) = A_0 e^{-0.00012378t} \rightarrow ②$$

Given $A(t) = 14.5\%$ of A_0

$$= \frac{14.5}{100} A_0$$

$$= 0.145 A_0$$

eq ② $\Rightarrow$ $0.145 \cancel{A_0} = \cancel{A_0} e^{-0.00012378t}$

Taking ln on b.s

$$\ln|0.145| = -0.00012378t$$

$$t = \frac{\ln|0.145|}{-0.00012378}$$

$$t = 15600.43 \approx 15600 \text{ years} \quad \text{Ans.}$$

Question # 13

A thermometer is removed from a room where the air temperature is 70°F to the outside where the temperature is 10°F. After $\frac{1}{2}$ minute the thermometer reads 50°F. What is the reading at $t = 1$ minute? How long will it take for the thermometer to reach 15°F?

Solution

$$T_m = 10°C$$

$$T(0) = 70°C$$

$$T(\tfrac{1}{2}) = 50°C$$

$$T(1) = ?$$

$$T(t) = 15°C, \quad t = ?$$

$$\because \frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T - 10)$$

By using separation of variables

$$\int \frac{dT}{T-10} = k \int dt$$

$$\ln|T - 10| = kt + c_1$$

Taking antilog on b.s

$$T - 10 = e^{kt + c_1}$$

$$T(t) = e^{kt} \cdot e^{c_1} + 10$$

$$T(t) = ce^{kt} + 10 \rightarrow ①$$

For $T(0)$,

$$T(0) = 70°C$$

$$eq① \Rightarrow T(0) = ce^{k(0)} + 10$$

$$T(0) = c + 10$$

$$70 - 10 = c + \cancel{10} - \cancel{10}$$

$$\Rightarrow c = 60$$

$$eq① \Rightarrow T(t) = 10 + 60e^{kt} \rightarrow ②$$

For $T(\frac{1}{2}) = 50$

$$eq② \Rightarrow T(\tfrac{1}{2}) = 10 + 60e^{k(\frac{1}{2})}$$

$$50 - 10 = 60e^{k(\frac{1}{2})}$$

$$\frac{4\cancel{0}}{6\cancel{0}} = e^{k(\frac{1}{2})}$$

$$\frac{2}{3} = e^{k(\frac{1}{2})}$$

Taking ln on b.s

$$\ln|\tfrac{2}{3}| = k(\tfrac{1}{2})$$

$$k = -0.8109$$

$$k = -0.811$$

$$eq② \Rightarrow T(t) = 10 + 60e^{-0.811t} \rightarrow ③$$

For $T(1) = 10 + 60e^{-0.811(t)}$

$$= 36.66°C$$

$T(t) = 15$, $t = ?$

$$eq③ \Rightarrow 15 = 10 + 60e^{-0.811t}$$

$$\frac{5}{60} = e^{-0.811t}$$

Taking ln on b.s

$$\ln|\tfrac{5}{60}| = -0.811t$$

$$t = \frac{\cancel{-}2.485}{\cancel{-}0.811}$$

$t = 3.064$ min Answer.