to solve in terms of familiar elementary functions such as algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. Furthermore, if we happen to have a family of solutions for a nonlinear equation, it is not obvious when this family constitutes a general solution. On a practical level, then, the designation "general solution" is applied only to linear differential equations.

1.1 EXERCISES

Answers to odd-numbered problems begin on page A-1.

In Problems 1–10 state whether the given differential equations are linear or nonlinear. Give the order of each equation.

1. $(1 - x)y'' - 4xy' + 5y = \cos x$
2. $x\dfrac{d^3y}{dx^3} - 2\left(\dfrac{dy}{dx}\right)^4 + y = 0$
3. $yy' + 2y = 1 + x^2$
4. $x^2\,dy + (y - xy - xe^x)\,dx = 0$
5. $x^3y^{(4)} - x^2y'' + 4xy' - 3y = 0$
6. $\dfrac{d^2y}{dx^2} + 9y = \sin y$
7. $\dfrac{dy}{dx} = \sqrt{1 + \left(\dfrac{d^2y}{dx^2}\right)^2}$
8. $\dfrac{d^2r}{dt^2} = -\dfrac{k}{r^2}$
9. $(\sin x)y''' - (\cos x)y' = 2$
10. $(1 - y^2)\,dx + x\,dy = 0$

In Problems 11–40 verify that the indicated function is a solution of the given differential equation. Where appropriate, c_1 and c_2 denote constants.

11. $2y' + y = 0;\quad y = e^{-x/2}$
12. $y' + 4y = 32;\quad y = 8$
13. $\dfrac{dy}{dx} - 2y = e^{3x};\quad y = e^{3x} + 10e^{2x}$
14. $\dfrac{dy}{dt} + 20y = 24;\quad y = \frac{6}{5} - \frac{6}{5}e^{-20t}$
15. $y' = 25 + y^2;\quad y = 5\tan 5x$
16. $\dfrac{dy}{dx} = \sqrt{\dfrac{y}{x}};\quad y = (\sqrt{x} + c_1)^2,\ x > 0,\ c_1 > 0$
17. $y' + y = \sin x;\quad y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + 10e^{-x}$
18. $2xy\,dx + (x^2 + 2y)\,dy = 0;\quad x^2y + y^2 = c_1$
19. $x^2\,dy + 2xy\,dx = 0;\quad y = -\dfrac{1}{x^2}$
20. $(y')^3 + xy' = y;\quad y = x + 1$
21. $y = 2xy' + y(y')^2;\quad y^2 = c_1(x + \frac{1}{4}c_1)$
22. $y' = 2\sqrt{|y|};\quad y = x|x|$
23. $y' - \dfrac{1}{x}y = 1;\quad y = x\ln x,\ x > 0$
24. $\dfrac{dP}{dt} = P(a - bP);\quad P = \dfrac{ac_1e^{at}}{1 + bc_1e^{at}}$

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25. $\dfrac{dX}{dt} = (2 - X)(1 - X)$; $\ln \dfrac{2 - X}{1 - X} = t$

26. $y' + 2xy = 1$; $y = e^{-x^2} \displaystyle\int_0^x e^{t^2}\,dt + c_1 e^{-x^2}$

27. $(x^3 + y^3)\,dx + (x^2 - xy)\,dy = 0$; $c_1(x + y)^3 = xe^{y/x}$

28. $y'' + y' - 12y = 0$; $y = c_1 e^{3x} + c_2 e^{-4x}$

29. $y'' - 6y' + 13y = 0$; $y = e^{3x} \cos 2x$

30. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$; $y = e^{2x} + xe^{2x}$

31. $y'' = y$; $y = \cosh x + \sinh x$

32. $y'' + 25y = 0$; $y = c_1 \cos 5x$

33. $y'' + (y')^2 = 0$; $y = \ln|x + c_1| + c_2$

34. $y'' + y = \tan x$; $y = -\cos x \ln(\sec x + \tan x)$

35. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$; $y = c_1 + c_2 x^{-1}$

36. $x^2y'' - xy' + 2y = 0$; $y = x\cos(\ln x)$, $x > 0$

37. $x^2y'' - 3xy' + 4y = 0$; $y = x^2 + x^2 \ln x$, $x > 0$

38. $y''' - y'' + 9y' - 9y = 0$; $y = c_1 \sin 3x + c_2 \cos 3x + 4e^x$

39. $y''' - 3y'' + 3y' - y = 0$; $y = x^2e^x$

40. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 12x^2$; $y = c_1x + c_2x\ln x + 4x^2$, $x > 0$

In Problems 41 and 42 verify that the indicated piecewise-defined function is a solution of the given differential equation.

41. $xy' - 2y = 0$; $y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

42. $(y')^2 = 9xy$; $y = \begin{cases} 0, & x < 0 \\ x^3, & x \geq 0 \end{cases}$

43. Verify that a one-parameter family of solutions for

$$y = xy' + (y')^2 \quad \text{is} \quad y = cx + c^2.$$

Determine a value of k such that $y = kx^2$ is a singular solution of the differential equation.

44. Verify that a one-parameter family of solutions for

$$y = xy' + \sqrt{1 + (y')^2} \quad \text{is} \quad y = cx + \sqrt{1 + c^2}.$$

Show that the relation $x^2 + y^2 = 1$ defines a singular solution of the equation on the interval $(-1, 1)$.

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45. A one-parameter family of solutions for

$$y' = y^2 - 1 \quad \text{is} \quad y = \frac{1 + ce^{2x}}{1 - ce^{2x}}.$$

By inspection,* determine a singular solution of the differential equation.

46. On page 6 we saw that $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ are solutions of $dy/dx = -x/y$ on the interval $(-2, 2)$. Explain why

$$y = \begin{cases} \sqrt{4 - x^2}, & -2 < x < 0 \\ -\sqrt{4 - x^2}, & 0 \le x < 2 \end{cases}$$

is not a solution of the differential equation on the interval.

In Problems 47 and 48 find values of m so that $y = e^{mx}$ is a solution of each differential equation.

47. $y'' - 5y' + 6y = 0$

48. $y'' + 10y' + 25y = 0$

In Problems 49 and 50 find values of m so that $y = x^m$ is a solution of each differential equation.

49. $x^2y'' \quad y = 0$

50. $x^2y'' + 6xy' + 4y = 0$

51. Show that $y_1 = x^2$ and $y_2 = x^3$ are both solutions of

$$x^2y'' - 4xy' + 6y = 0.$$

Are the constant multiples c_1y_1 and c_2y_2, with c_1 and c_2 arbitrary, also solutions? Is the sum $y_1 + y_2$ a solution?

52. Show that $y_1 = 2x + 2$ and $y_2 = -x^2/2$ are both solutions of

$$y = xy' + (y')^2/2.$$

Are the constant multiples c_1y_1 and c_2y_2, with c_1 and c_2 arbitrary, also solutions? Is the sum $y_1 + y_2$ a solution?

53. By inspection determine, if possible, a real solution of the given differential equation.

(a) $\left|\dfrac{dy}{dx}\right| + |y| = 0$ (b) $\left|\dfrac{dy}{dx}\right| + |y| + 1 = 0$ (c) $\left|\dfrac{dy}{dx}\right| + |y| = 1$

.2 SOME MATHEMATICAL MODELS

In science, engineering, economics, and even psychology, we often wish to describe or model the behavior of some system or phenomenon in mathematical terms. This description starts with

* Translated, this means take a good guess and see if it works.

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rem do not hold, then the initial-value problem (2) may still have (a) no solution, (b) more than one solution, or (c) a unique solution. Furthermore, the continuity condition on $\partial f/\partial y$ can be relaxed somewhat without changing the conclusion of the theorem. This results in a stronger theorem but is, unfortunately, not as easy to apply as Theorem 2.1. Indeed, if we are not interested in uniqueness, then a famous theorem due to the Italian mathematician Guiseppe Peano states that the continuity of $f(x, y)$ on R is sufficient to guarantee the existence of at least one solution of $dy/dx = f(x, y)$ through a point (x_0, y_0) interior to R.

2.1 EXERCISES

Answers to odd-numbered problems begin on page A–2.

In Problems 1–10 determine a region of the xy-plane for which the given differential equation would have a unique solution through a point (x_0, y_0) in the region.

1. $\dfrac{dy}{dx} = y^{2/3}$
2. $\dfrac{dy}{dx} = \sqrt{xy}$
3. $x\dfrac{dy}{dx} = y$
4. $\dfrac{dy}{dx} - y = x$
5. $(4 - y^2)y' = x^2$
6. $(1 + y^3)y' = x^2$
7. $(x^2 + y^2)y' = y^2$
8. $(y - x)y' = y + x$
9. $\dfrac{dy}{dx} = x^3 \cos y$
10. $\dfrac{dy}{dx} = (x - 1)e^{y/(x-1)}$

In Problems 11 and 12 determine by inspection at least two solutions of the given initial-value problem.

11. $y' = 3y^{2/3}, \quad y(0) = 0$
12. $x\dfrac{dy}{dx} = 2y, \quad y(0) = 0$
13. By inspection determine a solution of the nonlinear differential equation $y' = y^3$ satisfying $y(0) = 0$. Is the solution unique?
14. By inspection find a solution of the initial-value problem

$$y' = |y - 1|, \qquad y(0) = 1.$$

State why the conditions of Theorem 2.1 do not hold for this differential equation. Although we shall not prove it, the solution to this initial-value problem is unique.

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15. Verify that $y = cx$ is a solution of the differential equation $xy' = y$ for every value of the parameter c. Find at least two solutions of the initial-value problem

$$xy' = y, \qquad y(0) = 0.$$

Observe that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Is it a solution of the initial-value problem?

16. (a) Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2.$$

Determine a region of the xy-plane for which the equation has a unique solution through a point (x_0, y_0) in the region.

(b) Formally show that $y = \tan x$ satisfies the differential equation and the condition $y(0) = 0$.

(c) Explain why $y = \tan x$ is not a solution of the initial-value problem

$$\frac{dy}{dx} = 1 + y^2, \qquad y(0) = 0,$$

on the interval $(-2, 2)$.

(d) Explain why $y = \tan x$ is a solution of the initial-value problem in part (c) on the interval $(-1, 1)$.

In Problems 17–20 determine whether Theorem 2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

17. $(1, 4)$
18. $(5, 3)$
19. $(2, -3)$
20. $(-1, 1)$

SEPARABLE VARIABLES

Note to the student: In solving a differential equation you will often have to utilize, say, integration by parts, partial fractions, or possibly a substitution. It will be worth a few minutes of your time to review some

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satisfies the preceding examples that the constant in the one-parameter family of solutions for a first-order differential equation can be relabeled when convenient. Also, it can easily happen that two individuals solving the same equation arrive at dissimilar answers. For example, by separation of variables we can show that one-parameter families of solutions for $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ are

$$\arctan x + \arctan y = c \quad \text{or} \quad \arctan x + \arctan y = \arctan c \quad \text{or} \quad \frac{x + y}{1 - xy} = c.$$

Remark We saw in some of the

As you work your way through the next several sections, bear in mind that families of solutions may be *equivalent* in the sense that one family may be obtained from another by either relabeling the constant or applying algebra and trigonometry.

2.2 EXERCISES

Answers to odd-numbered problems begin on page A–2.

In Problems 1–40 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$
2. $\dfrac{dy}{dx} = (x + 1)^2$
3. $dx + e^{3x}\,dy = 0$
4. $dx - x^2\,dy = 0$
5. $(x + 1)\dfrac{dy}{dx} = x + 6$
6. $e^x \dfrac{dy}{dx} = 2x$
7. $xy' = 4y$
8. $\dfrac{dy}{dx} + 2xy = 0$
9. $\dfrac{dy}{dx} = \dfrac{y^3}{x^2}$
10. $\dfrac{dy}{dx} = \dfrac{y + 1}{x}$
11. $\dfrac{dx}{dy} = \dfrac{x^2y^2}{1 + x}$
12. $\dfrac{dx}{dy} = \dfrac{1 + 2y^2}{y \sin x}$
13. $\dfrac{dy}{dx} = e^{3x+2y}$
14. $e^x y \dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

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15. $(4y + yx^2)\,dy - (2x + xy^2)\,dx = 0$

16. $(1 + x^2 + y^2 + x^2y^2)\,dy = y^2\,dx$

17. $2y(x + 1)\,dy = x\,dx$

18. $x^2y^2\,dy = (y + 1)\,dx$

19. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$

20. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$

21. $\dfrac{dS}{dr} = kS$

22. $\dfrac{dQ}{dt} = k(Q - 70)$

23. $\dfrac{dP}{dt} = P - P^2$

24. $\dfrac{dN}{dt} + N = Nte^{t+2}$

25. $\sec^2 x\,dy + \csc y\,dx = 0$

26. $\sin 3x\,dx + 2y \cos^3 3x\,dy = 0$

27. $e^y \sin 2x\,dx + \cos x(e^{2y} - y)\,dy = 0$

28. $\sec x\,dy = x \cot y\,dx$

29. $(e^y + 1)^2 e^{-y}\,dx + (e^x + 1)^3 e^{-x}\,dy = 0$

30. $\dfrac{y}{x}\dfrac{dy}{dx} = (1 + x^2)^{-1/2}(1 + y^2)^{1/2}$

31. $(y - yx^2)\dfrac{dy}{dx} = (y + 1)^2$

32. $2\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{2x}{y}$

33. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

34. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

35. $\dfrac{dy}{dx} = \sin x(\cos 2y - \cos^2 y)$

36. $\sec y \dfrac{dy}{dx} + \sin(x - y) = \sin(x + y)$

37. $x\sqrt{1 - y^2}\,dx = dy$

38. $y(4 - x^2)^{1/2}\,dy = (4 + y^2)^{1/2}\,dx$

39. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

40. $(x + \sqrt{x})\dfrac{dy}{dx} = y + \sqrt{y}$

In Problems 41–48 solve the given differential equation subject to the indicated initial condition.

41. $(e^{-y} + 1)\sin x\,dx = (1 + \cos x)\,dy, \quad y(0) = 0$

42. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$

43. $y\,dy = 4x(y^2 + 1)^{1/2}\,dx, \quad y(0) = 1$

44. $\dfrac{dy}{dt} + ty = y, \quad y(1) = 3$

45. $\dfrac{dx}{dy} = 4(x^2 + 1), \quad x\left(\dfrac{\pi}{4}\right) = 1$

46. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$

47. $x^2y' = y - xy, \quad y(-1) = -1$

48. $y' + 2y = 1, \quad y(0) = \dfrac{5}{2}$

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In Problems 49 and 50 find a solution of the given differential equation that passes through the indicated points.

49. $\dfrac{dy}{dx} - y^2 = -9$ (a) $(0, 0)$ (b) $(0, 3)$ (c) $\left(\dfrac{1}{3}, 1\right)$

50. $x\dfrac{dy}{dx} = y^2 - y$ (a) $(0, 1)$ (b) $(0, 0)$ (c) $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

51. Find a singular solution for the equation in Problem 37.

52. Find a singular solution for the equation in Problem 39.

Often a radical change in the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 53–56 compare the solutions of the given initial-value problems.

53. $\dfrac{dy}{dx} = (y-1)^2, \quad y(0) = 1$

54. $\dfrac{dy}{dx} = (y-1)^2, \quad y(0) = 1.01$

55. $\dfrac{dy}{dx} = (y-1)^2 + 0.01, \quad y(0) = 1$

56. $\dfrac{dy}{dx} = (y-1)^2 - 0.01, \quad y(0) = 1$

A differential equation of the form $dy/dx = f(ax + by + c)$, $b \neq 0$, can always be reduced to an equation with separable variables by means of the substitution $u = ax + by + c$. Use this procedure to solve Problems 57–62.

57. $\dfrac{dy}{dx} = (x + y + 1)^2$

58. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$

59. $\dfrac{dy}{dx} = \tan^2(x + y)$

60. $\dfrac{dy}{dx} = \sin(x + y)$

61. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

62. $\dfrac{dy}{dx} = 1 + e^{y-x+5}$

HOMOGENEOUS EQUATIONS

Before considering the concept of a **homogeneous first-order differential equation** and its method of solution, we need to examine closely the nature of a **homogeneous function**. We begin with the definition of this concept.

DEFINITION 2.2 Homogeneous Function

If a function f has the property that

$$f(tx, ty) = t^n f(x, y) \tag{1}$$

for some real number n, then f is said to be a **homogeneous function of degree n**.

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2.3 EXERCISES

Answers to odd-numbered problems begin on page A–3.

In Problems 1–10 determine whether the given function is homogeneous. If so, state the degree of homogeneity.

1. $x^3 + 2xy^2 - y^4/x$
2. $\sqrt{x+y}(4x+3y)$
3. $\dfrac{x^3y - x^2y^2}{(x+8y)^2}$
4. $\dfrac{x}{y^2 + \sqrt{x^4+y^4}}$
5. $\cos\dfrac{x^2}{x+y}$
6. $\sin\dfrac{x}{x+y}$
7. $\ln x^2 - 2\ln y$
8. $\dfrac{\ln x^3}{\ln y^3}$
9. $(x^{-1} + y^{-1})^2$
10. $(x+y+1)^2$

In Problems 11–30 solve the given differential equation by using an appropriate substitution.

11. $(x-y)\,dx + x\,dy = 0$
12. $(x+y)\,dx + x\,dy = 0$
13. $x\,dx + (y-2x)\,dy = 0$
14. $y\,dx = 2(x+y)\,dy$
15. $(y^2+yx)\,dx - x^2\,dy = 0$
16. $(y^2+yx)\,dx + x^2\,dy = 0$
17. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$
18. $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$
19. $-y\,dx + (x+\sqrt{xy})\,dy = 0$
20. $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$
21. $2x^2y\,dx = (3x^3+y^3)\,dy$
22. $(x^4+y^4)\,dx - 2x^3y\,dy = 0$
23. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$
24. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2} + 1$
25. $y\dfrac{dx}{dy} = x + 4ye^{-2x/y}$
26. $(x^2e^{-y/x} + y^2)\,dx = xy\,dy$
27. $\left(y + x\cot\dfrac{y}{x}\right)dx - x\,dy = 0.$
28. $\dfrac{dy}{dx} = \dfrac{y}{x}\ln\dfrac{y}{x}$
29. $(x^2+xy-y^2)\,dx + xy\,dy = 0$
30. $(x^2+xy+3y^2)\,dx - (x^2+2xy)\,dy = 0$

In Problems 31–44 solve the given differential equation subject to the indicated initial condition.

31. $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$
32. $(x^2+2y^2)\,dx = xy\,dy, \quad y(-1) = 1$
33. $2x^2\dfrac{dy}{dx} = 3xy + y^2, \quad y(1) = -2$

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34. $xy\,dx - x^2\,dy = y\sqrt{x^2 + y^2}\,dy, \quad y(0) = 1$

35. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$

36. $y\,dx + \left(y\cos\dfrac{x}{y} - x\right)dy = 0, \quad y(0) = 2$

37. $(y^2 + 3xy)\,dx = (4x^2 + xy)\,dy, \quad y(1) = 1$

38. $y^3\,dx = 2x^3\,dy - 2x^2y\,dx, \quad y(1) = \sqrt{2}$

39. $(x + \sqrt{xy})\dfrac{dy}{dx} + x - y = x^{-1/2}y^{3/2}, \quad y(1) = 1$

40. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

41. $y^2\,dx + (x^2 + xy + y^2)\,dy = 0, \quad y(0) = 1$

42. $(\sqrt{x} + \sqrt{y})^2\,dx = x\,dy, \quad y(1) = 0$

43. $(x + \sqrt{y^2 - xy})\dfrac{dy}{dx} = y, \quad y(\tfrac{1}{2}) = 1$

44. $\dfrac{dy}{dx} - \dfrac{y}{x} = \cosh\dfrac{y}{x}, \quad y(1) = 0$

45. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the substitution $x = vy$ reduces the equation to one with separable variables.

46. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the substitutions $x = r\cos\theta$, $y = r\sin\theta$ reduce the equation to one with separable variables.

47. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the equation has the alternative form $dy/dx = G(x/y)$.

48. If $f(x, y)$ is a homogeneous function of degree n, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

.4 EXACT EQUATIONS

Although the simple equation

$$y\,dx + x\,dy = 0$$

is both separable and homogeneous, we should recognize that it is also equivalent to the differential of the product of x and y; that is,

$$y\,dx + x\,dy = d(xy) = 0.$$

By integrating we immediately obtain the implicit solution $xy = c$.

From calculus you might remember that if $z = f(x, y)$ is a function having continuous first partial derivatives in a region R of the xy-plane, then

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is a solution of both equa...

ERCISES

Answers to odd-numbered problems begin on page A–3.

In Problems 1–24 determine whether the given equation is exact. If exact, solve.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$
2. $(2x + y)\,dx - (x + 6y)\,dy = 0$
3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$
4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$
5. $(2y^2x - 3)\,dx + (2yx^2 + 4)\,dy = 0$
6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$
7. $(x + y)(x - y)\,dx + x(x - 2y)\,dy = 0$
8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$
9. $(y^3 - y^2 \sin x - x)\,dx + (3xy^2 + 2y \cos x)\,dy = 0$
10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$
11. $(y \ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x \ln y\right)dy = 0$
12. $\dfrac{2x}{y}\,dx - \dfrac{x^2}{y^2}\,dy = 0$
13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$
14. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$
15. $\left(1 - \dfrac{3}{x} + y\right)dx + \left(1 - \dfrac{3}{y} + x\right)dy = 0$
16. $(e^y + 2xy \cosh x)y' + xy^2 \sinh x + y^2 \cosh x = 0$
17. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$
18. $(5y - 2x)y' - 2y = 0$
19. $(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$
20. $(3x \cos 3x + \sin 3x - 3)\,dx + (2y + 5)\,dy = 0$
21. $(1 - 2x^2 - 2y)\dfrac{dy}{dx} = 4x^3 + 4xy$
22. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$

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23. $(4x^3y - 15x^2 - y)\,dx + (x^4 + 3y^2 - x)\,dy = 0$

24. $\left(\frac{1}{x} + \frac{1}{x^2} - \frac{y}{x^2 + y^2}\right)dx + \left(ye^y + \frac{x}{x^2 + y^2}\right)dy = 0$

In Problems 25–30 solve the given differential equation subject to the indicated initial condition.

25. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

26. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

27. $(4y + 2x - 5)\,dx + (6y + 4x - 1)\,dy = 0, \quad y(-1) = 2$

28. $\left(\frac{3y^2 - x^2}{y^5}\right)\frac{dy}{dx} + \frac{x}{2y^4} = 0, \quad y(1) = 1$

29. $(y^2 \cos x - 3x^2y - 2x)\,dx + (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$

30. $\left(\frac{1}{1 + y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$

In Problems 31–34 find the value of k so that the given differential equation is exact.

31. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

32. $(2x - y \sin xy + ky^4)\,dx - (20xy^3 + x \sin xy)\,dy = 0$

33. $(2xy^2 + ye^x)\,dx + (2x^2y + ke^x - 1)\,dy = 0$

34. $(6xy^3 + \cos y)\,dx + (kx^2y^2 - x \sin y)\,dy = 0$

35. Determine a function $M(x, y)$ so that the following differential equation is exact:

$$M(x, y)\,dx + \left(xe^{xy} + 2xy + \frac{1}{x}\right)dy = 0.$$

36. Determine a function $N(x, y)$ so that the following differential equation is exact:

$$\left(y^{1/2}x^{-1/2} + \frac{x}{x^2 + y}\right)dx + N(x, y)\,dy = 0.$$

In Problems 37–42 solve the given differential equation by verifying that the indicated function $\mu(x, y)$ is an integrating factor.

37. $6xy\,dx + (4y + 9x^2)\,dy = 0, \quad \mu(x, y) = y^2$

38. $-y^2\,dx + (x^2 + xy)\,dy = 0, \quad \mu(x, y) = 1/x^2y$

39. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0, \quad \mu(x, y) = xy$

40. $y(x + y + 1)\,dx + (x + 2y)\,dy = 0, \quad \mu(x, y) = e^x$

41. $(2y^2 + 3x)\,dx + 2xy\,dy = 0, \quad \mu(x, y) = x$

42. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0, \quad \mu(x, y) = (x + y)^{-2}$

43. Show that any separable first-order differential equation in the form $h(y)\,dy - g(x)\,dx = 0$ is also exact.

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is continuous but not differentiable at $x = 1$.

Remark Formula (7), representing the general solution of (1), actually consists of the sum of two solutions. We define

$$y = y_c + y_p, \tag{14}$$

where

$$y_c = ce^{-\int P(x)\,dx} \quad \text{and} \quad y_p = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} f(x)\,dx.$$

The function y_c is readily shown to be the general solution of $y' + P(x)y = 0$, whereas y_p is a particular solution of $y' + P(x)y = f(x)$. As we shall see in Chapter 4, the additivity property of solutions (14) to form a general solution is an intrinsic property of linear equations of any order.

2.5 EXERCISES

Answers to odd-numbered problems begin on page A–4.

In Problems 1–40 find the general solution of the given differential equation. State an interval on which the general solution is defined.

1. $\dfrac{dy}{dx} = 5y$
2. $\dfrac{dy}{dx} + 2y = 0$
3. $3\dfrac{dy}{dx} + 12y = 4$
4. $x\dfrac{dy}{dx} + 2y = 3$
5. $\dfrac{dy}{dx} + y = e^{3x}$
6. $\dfrac{dy}{dx} = y + e^{x}$
7. $y' + 3x^2y = x^2$
8. $y' + 2xy = x^3$
9. $x^2y' + xy = 1$
10. $y' = 2y + x^2 + 5$
11. $(x + 4y^2)\,dy + 2y\,dx = 0$
12. $\dfrac{dx}{dy} = x + y$

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13. $x\,dy = (x \sin x - y)\,dx$

14. $(1 + x^2)\,dy + (xy + x^3 + x)\,dx = 0$

15. $(1 + e^x)\dfrac{dy}{dx} + e^x y = 0$

16. $(1 - x^3)\dfrac{dy}{dx} = 3x^2 y$

17. $\cos x \dfrac{dy}{dx} + y \sin x = 1$

18. $\dfrac{dy}{dx} + y \cot x = 2 \cos x$

19. $x\dfrac{dy}{dx} + 4y = x^3 - x$

20. $(1 + x)y' = xy = x + x^2$

21. $x^2 y' + x(x + 2)y = e^x$

22. $xy' + (1 + x)y = e^{-x} \sin 2x$

23. $\cos^2 x \sin x\,dy + (y \cos^3 x - 1)\,dx = 0$

24. $(1 - \cos x)\,dy + (2y \sin x - \tan x)\,dx = 0$

25. $y\,dx + (xy + 2x - ye^y)\,dy = 0$

26. $(x^2 + x)\,dy = (x^5 + 3xy + 3y)\,dx$

27. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

28. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

29. $y\,dx - 4(x + y^6)\,dy = 0$

30. $xy' + 2y = e^x + \ln x$

31. $\dfrac{dy}{dx} + y = \dfrac{1 - e^{-2x}}{e^x + e^{-x}}$

32. $\dfrac{dy}{dx} - y = \sinh x$

33. $y\,dx + (x + 2xy^2 - 2y)\,dy = 0$

34. $y\,dx = (ye^y - 2x)\,dy$

35. $\dfrac{dr}{d\theta} + r \sec \theta = \cos \theta$

36. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

37. $(x + 2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$

38. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

39. $y' = (10 - y) \cosh x$

40. $dx = (3e^y - 2x)\,dy$

In Problems 41–54 solve the given differential equation subject to the indicated initial condition.

41. $\dfrac{dy}{dx} + 5y = 20, \quad y(0) = 2$

42. $y' = 2y + x(e^{3x} - e^{2x}), \quad y(0) = 2$

43. $L\dfrac{di}{dt} + Ri = E$; L, R, and E constants, $i(0) = i_0$

44. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

45. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

46. $\dfrac{dQ}{dx} = 5x^4 Q, \quad Q(0) = -7$

47. $\dfrac{dT}{dt} = k(T - 50)$; k a constant, $T(0) = 200$

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48. $x\,dy + (xy + 2y - 2e^{-x})\,dx = 0, \quad y(1) = 0$

49. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

50. $xy' + y = e^x, \quad y(1) = 2$

51. $x(x - 3)y' + 2y = 0, \quad y(3) = 6$

52. $\sin x\dfrac{dy}{dx} + (\cos x)y = 0, \quad y\left(-\dfrac{\pi}{2}\right) = 1$

53. $\dfrac{dy}{dx} = \dfrac{y}{y - x}, \quad y(5) = 2$

54. $\cos^2 x\dfrac{dy}{dx} + y = 1, \quad y(0) = -3$

In Problems 55–58 find a continuous solution satisfying each differential equation and the given initial condition.

55. $\dfrac{dy}{dx} + 2y = f(x), \quad f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}, \quad y(0) = 0$

56. $\dfrac{dy}{dx} + y = f(x), \quad f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}, \quad y(0) = 1$

57. $\dfrac{dy}{dx} + 2xy = f(x), \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}, \quad y(0) = 2$

58. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & x \ge 1 \end{cases}, \quad y(0) = 0$

TIONS OF BERNOULLI, RICATTI, AND CLAIRAUT*

In this section we are not going to study any one particular type of differential equation. Rather, we are going to consider three classical equations that in some instances can be transformed into equations we have already studied.

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*Jakob Bernoulli (1654–1705) The Bernoullis were a Swiss family of scholars whose contributions to mathematics, physics, astronomy, and history spanned the sixteenth to the twentieth centuries. Jakob, the elder of the two sons of the patriarch Jacques Bernoulli, made many contributions to the then-new fields of calculus and probability. Originally the second of the two major divisions of calculus was called *calculus summatorius*. In 1696, at Jakob Bernoulli's suggestion, its name was changed to *calculus integralis* or, as we know it today, integral calculus.

Jacobo Francesco Ricatti (1676–1754) An Italian count, Ricatti was also a mathematician and philosopher.

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2.6 EXERCISES

Answers to odd-numbered problems begin on page A–4.

In Problems 1–6 solve the given Bernoulli equation.

1. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

2. $\dfrac{dy}{dx} - y = e^x y^2$

3. $\dfrac{dy}{dx} = y(xy^3 - 1)$

4. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

5. $x^2\dfrac{dy}{dx} + y^2 = xy$

6. $3(1 + x^2)\dfrac{dy}{dx} = 2xy(y^3 - 1)$

In Problems 7–10 solve the given differential equation subject to the indicated initial condition.

7. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \dfrac{1}{2}$

8. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

9. $xy(1 + xy^2)\dfrac{dy}{dx} = 1, \quad y(1) = 0$

10. $2\dfrac{dy}{dx} = \dfrac{y}{x} - \dfrac{x}{y^2}, \quad y(1) = 1$

In Problems 11–16 solve the given Ricatti equation; y_1 is a known solution of the equation.

11. $\dfrac{dy}{dx} = -2 - y + y^2, \quad y_1 = 2$

12. $\dfrac{dy}{dx} = 1 - x - y + xy^2, \quad y_1 = 1$

13. $\dfrac{dy}{dx} = -\dfrac{4}{x^2} - \dfrac{1}{x}y + y^2, \quad y_1 = \dfrac{2}{x}$

14. $\dfrac{dy}{dx} = 2x^2 + \dfrac{1}{x}y - 2y^2, \quad y_1 = x$

15. $\dfrac{dy}{dx} = e^{2x} + (1 + 2e^x)y + y^2, \quad y_1 = -e^x$

16. $\dfrac{dy}{dx} = \sec^2 x - (\tan x)y + y^2, \quad y_1 = \tan x$

17. Solve $\dfrac{dy}{dx} = 6 + 5y + y^2$.

18. Solve $\dfrac{dy}{dx} = 9 + 6y + y^2$.

In Problems 19–24 solve the given Clairaut equation. Obtain a singular solution.

19. $y = xy' + 1 - \ln y'$

20. $y = xy' + (y')^{-2}$

21. $y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^3$

22. $y = (x + 4)y' + (y')^2$

23. $xy' - y = e^{y'}$

24. $y = xy' - \ln y'$

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25. Show that if y_1 is a solution of (4), then the substitution $y = y_1 + u$ in (4) yields (5).

26. Show that (5) can be reduced to (6) by means of the substitution $w = u^{-1}$.

27. When $R(x) = -1$, the Ricatti equation can be written as $y' + y^2 - Q(x)y - P(x) = 0$. Show that the substitution $y = w'/w$ leads to the linear second-order equation $w'' - Q(x)w' - P(x)w = 0$. (When Q and P are also constants, there is little difficulty in solving equations of this type.)

28. An alternative definition of Clairaut's equation is any equation of the form $F(y - xy', y') = 0$.

    (a) Show that a family of solutions of the latter equation is

    $$F(y - cx, c) = 0.$$

    (b) Use the result of part (a) to solve

    $$(xy' - y)^3 = (y')^2 + 5.$$

29. Show that $y = cx + f(c)$, where c is an arbitrary constant, is a solution of (7).

30. Show that (8) is a solution of (7). [*Hint:* Differentiate both sides of (7) with respect to x and consider two cases. Use parametric differentiation to show

    $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t, \qquad f''(t) \neq 0.$$

    Note that since the slope of $y = cx + f(c)$ is constant, the singular solution cannot be obtained from it.]

2.7 SUBSTITUTIONS

In the preceding sections, we saw that sometimes a differential equation could be transformed by means of a substitution into a form that could then be solved by one of the standard methods. An equation may look different from any of those that we have just studied, but through a judicious change of variables perhaps an apparently difficult problem may be readily solved. Although we can give no firm rules on what, *if any*, substitution to use, a working axiom might be: Try something! It sometimes pays to be clever.

EXAMPLE 1

The differential equation

$$y(1 + 2xy)\,dx + x(1 - 2xy)\,dy = 0$$

is not separable, homogeneous, exact, linear, or Bernoulli. However, if we stare at the equation long enough, we might be prompted to try the substitution

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$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2}, \quad \text{or} \quad dy = -\frac{dx}{x^2 + c_1^2}$$

$$\int dy = -\int \frac{dx}{x^2 + c_1^2}$$

$$y + c_2 = -\frac{1}{c_1} \tan^{-1} \frac{x}{c_1}. \quad \blacksquare$$

2.7 EXERCISES

Answers to odd-numbered problems begin on page A–4.

In Problems 1–26 solve the given differential equation by using an appropriate substitution.

1. $xe^{2y}\dfrac{dy}{dx} + e^{2y} = \dfrac{\ln x}{x}$
2. $y' + y \ln y = ye^x$
3. $y\,dx + (1 + ye^x)\,dy = 0$
4. $(2 + e^{-x/y})\,dx + 2\left(1 - \dfrac{x}{y}\right)dy = 0$
5. $\dfrac{dy}{dx} - \dfrac{4}{x}y = 2x^5 e^{y/x^4}$
6. $\dfrac{dy}{dx} + x + y + 1 = (x + y)^2 e^{3x}$
7. $2yy' + x^2 + y^2 + x = 0$
8. $y' = y + x(y + 1)^2 + 1$
9. $2x \csc 2y \dfrac{dy}{dx} = 2x - \ln(\tan y)$
10. $x^2\dfrac{dy}{dx} + 2xy = x^4y^2 + 1$
11. $x^4y^2y' + x^3y^3 = 2x^3 - 3$
12. $xe^y y' - 2e^y = x^2$
13. $y' + 1 = e^{-(x+y)} \sin x$
14. $\sin y \sinh x\,dx + \cos y \cosh x\,dy = 0$
15. $y\dfrac{dx}{dy} + 2x \ln x = xe^y$
16. $x \sin y \dfrac{dy}{dx} + \cos y = -x^2e^x$
17. $y'' + (y')^2 + 1 = 0$
18. $xy'' = y' + x(y')^2$
19. $xy'' = y' + (y')^3$
20. $x^2y'' + (y')^2 = 0$
21. $y' - xy'' - (y'')^3 = 1$
22. $y'' = 1 + (y')^2$
23. $xy'' - y' = 0$
24. $y'' + (\tan x)y' = 0$
25. $y'' + 2y(y')^3 = 0$
26. $y^2y'' = y'$

$$\left[\textit{Hint: Let } u = y' \textit{ so that } y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u.\right]$$

alculus the curvature of a curve whose equation is $y = f(x)$ is defined

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and hence the initial-value problem (1). Indeed, it is precisely Picard's method of successive approximations that is used in proving Picard's theorem of Section 2.1. However, the proof of Theorem 2.1 uses concepts from advanced calculus and is not presented here. Our purpose in introducing this topic is twofold: to gain an appreciation for the potential of the procedure and to obtain at least a nodding acquaintance with an iterative technique. In Chapter 9 we shall consider other methods for approximating solutions of differential equations that also utilize iteration.

2.8 EXERCISES

Answers to odd-numbered problems begin on page A–5.

In Problems 1–6 use Picard's method to find y_1, y_2, y_3, y_4. Determine the limit of the sequence $\{y_n(x)\}$ as $n \to \infty$.

1. $y' = -y, \quad y(0) = 1$
2. $y' = x + y, \quad y(0) = 1$
3. $y' = 2xy, \quad y(0) = 1$
4. $y' + 2xy = x, \quad y(0) = 0$
5. $y' + y^2 = 0, \quad y(0) = 0$
6. $y' = 2e^x - y, \quad y(0) = 1$
7. **(a)** Use Picard's method to find y_1, y_2, y_3 for the problem

$$y' = 1 + y^2, \qquad y(0) = 0.$$

   (b) Solve the initial-value problem in part (a) by one of the methods of this chapter.

   (c) Compare the results of parts (a) and (b).

8. In Picard's method the initial choice $y_0(x) = y_0$ is not necessary. Rework Problem 3 with **(a)** $y_0(x) = k$ a constant and $k \neq 1$, and **(b)** $y_0(x) = x$.

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