Chapter 2 First-Order Differential Eqns.

§ 2.1 : Preliminary Theory :.

I.V.P : The problem

Solve : $\frac{dy}{dx} = f(x,y)$ (1)

Subject to : $y(x_0) = y_0$ (2)

$\rightarrow$ (*)

is called an I.V.P.

The condition (2) is known as Initial Condition.

Geometrically we are seeking at least one soln. of the D.E defined on some interval I s.t the graph of the sol. passes through a prescribed pt. (x_0, y_0) fig (a)

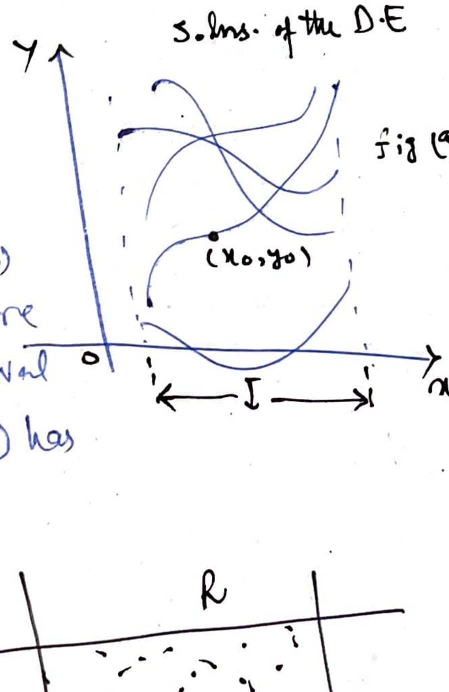


fig (a)

Existence of a Unique Soln.* :.

Let R be a region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in it's interior. If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous on R, then $\exists$ an interval I centered at x_0 s.t I.V.P (*) has a unique soln.

Examples ①: $y' = y$, $y(0) = 3$

Theorem* guarantees that $\exists$ an interval about $x=0$ on which $y = 3e^{x}$ is the only soln. of I.V.P.

This is because of the fact that

$$f(x,y) = y$$
$$\&$$
$$\frac{\partial f}{\partial y} = 1$$

are continuous throughout the entire xy-plane.

Example ②: $\frac{dy}{dx} = x^2 + y^2$

Here
$$f(x,y) = x^2 + y^2$$
$$\& \ \frac{\partial f}{\partial y} = 2y,$$

both are continuous throughout the entire xy-plane.

Therefore, through any given point (x_0, y_0) there passes one and only one soln. of the D.E.

Example ③: $y' = x y^{1/2}$, $y(0) = 0$

Has at least two solns. $y = 0$ & $y = x^4/16$, whose graph passes through $(0,0)$.

The ftns. $f(x,y) = x y^{1/2}$ & $\frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$

are continuous in the upper half-plane defined by $y > 0$

$\therefore$ We conclude from Theorem * that through any point (x_0, y_0), $y_0 > 0$ (say $(0,1)$ or $(1,3)$ etc.) there is some interval around x_0 on which ODE has a unique soln.

2.1 Exercises

(11) Determine by inspection at least two Solns. of the I.V.P

$$y' = 3y^{2/3}, \quad y(0) = 0$$

Soln.

First : $y = 0$

2nd : $y = x^3$

$\Downarrow$

$y' = 3x^2 = 3(x^3)^{\frac{2}{3}} = 3y^{2/3}$

(12). Same as Q.11

$$x y' = 2y, \quad y(0) = 0$$

First : $y = 0$

2nd : $y = x^2$

(13) $y' = y^3, \quad y(0) = 0.$

Find by inspection a Soln.

Is it Unique ?

Soln.

i) $y = 0$ is a Soln,

ii) $y = 0$ is the only Soln b/c

$f(x,y) = y^3$ & $\frac{\partial f}{\partial y} = 3y^2$

are continuous throughout the entire xy-plane

$\therefore$ I.V.P (13) has a Unique Soln.

Q.14 By inspection find a soln. on

$$y' = |y-1|, \quad y(0) = 1$$

& State why the condition of theorem (*) isn't applicable for ODE?

Note*: soln. of I.V.P is unique (Don't prove by *)

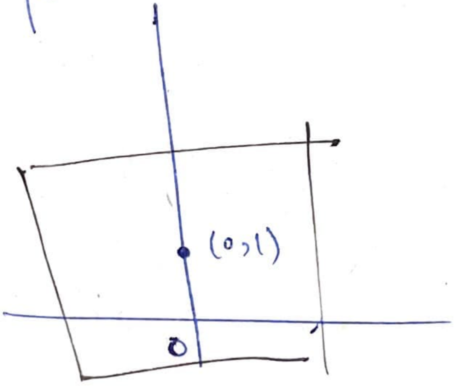
Soln.

By inspection, one soln. is

$$\boxed{y = 1}$$

(*)

Theorem (*) isn't applicable b/c $f(x,y) = |y-1|$ is not differentiable at $(0,1)$, so we can't say / apply (*). B/c for (*)



$f(x,y)$ & $\frac{\partial f}{\partial y}$ both must be continuous on some region containing $(0,1)$. but here $\frac{\partial f}{\partial y}$ Does't exists on $(0,1)$.

Q. 15. a) Verify $y = cx$ is a sol. of $xy' = y$ for every c.

Sol. Done ✓

b) Find at least two solns. of the I.V.P

$$xy' = y, \quad y(0) = 0$$

Sol.

$$xy' = y, \quad y(0) = 0$$

$$\Rightarrow y' = \frac{y}{x}$$

$$\boxed{f(x,y) = \frac{y}{x}, \quad \frac{\partial f}{\partial y} = \frac{1}{x}}$$

One soln. $\quad y = 0$

2nd soln. $\quad y = x$

$$\Downarrow$$

$$y' = 1$$

$$xy' = x$$

$$\boxed{xy' = y}, \quad y(0) = 0$$

c) Observe that the piecewise defined ftn.

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the D.E & Condition $y(0) = 0$.

Is it a soln. of I.V.P?

Ans. No, It's not a soln. of I.V.P, b/c

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$ isn't

Differentiable on $x = 0$.

Q 16. a) $y' = 1 + y^2$

Determine a region of the XY-plane for which the ODE has a unique soln. Through a pt. (x_0, y_0) in the region

Ans: $f(x,y) = 1+y^2$, $\frac{\partial f}{\partial y} = 2y$

f & $\frac{\partial f}{\partial y}$ are continuous throughout the entire XY-plane $\therefore$ Region is $(-\infty, \infty)$ or entire xy-plane.

b) Formally show that $y = \tan x$ satisfies ODE & the condition. $y(0) = 0$

Ans: $y' = \sec^2 x$
$= 1 + \tan^2 x$
$\boxed{y' = 1 + y^2}$

Showed.

$y(0) = \tan(0)$
$y(0) = 0$

c) Explain why $y = \tan x$ is not a soln. of I.V.P $y' = 1+y^2$, $y(0) = 0$ on $(-2, 2)$?

Ans: B|C $\exists$ a point $x = \frac{\pi}{2} \in (-2, 2)$ at which $y = \tan x$ isn't continuous, so not differentiable, $\therefore$ $y = \tan x$ isn't a soln. of I.V.P on $(-2, 2)$.

D) Explain why $y = \tan x$ is a soln. of I.V.P in (C) on $(-1, 1)$

Ans On $(-1,1)$, $y = \tan x$ satisfies I.V.P & is continuous too. Therefore it is soln. on $(-1, 1)$

17-20: Determine whether theorem (*) guarantees that the D.E. $y' = \sqrt{y^2-9}$ has a unique soln. through the given point?

(17) (1, 4).

$$f(x,y) = \sqrt{y^2-9}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2-9}}$$

f is continuous : if $y^2 - 9 \geq 0$.

$y^2 \geq 9$

$|y| \geq 3$

$y \geq 3$ or $y \leq -3$

$f \rightarrow \boxed{(-\infty, -3] \cup [3, \infty)}$

$\frac{\partial f}{\partial y}$ is continuous : if $\boxed{(-\infty, -3) \cup (3, \infty)}$

(17) (1, 4) Yes → ($y > 3$)

(18) (5, 3) No → No (interval ??)

(19) (2, -3) No

(20) (-1, 1) No