# Digital Image Processing

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In this lecture we will look at image enhancement in the frequency domain

- Jean Baptiste Joseph Fourier
- The Fourier series & the Fourier transform
- Image Processing in the frequency domain
  - Image smoothing
  - Image sharpening
- Fast Fourier Transform

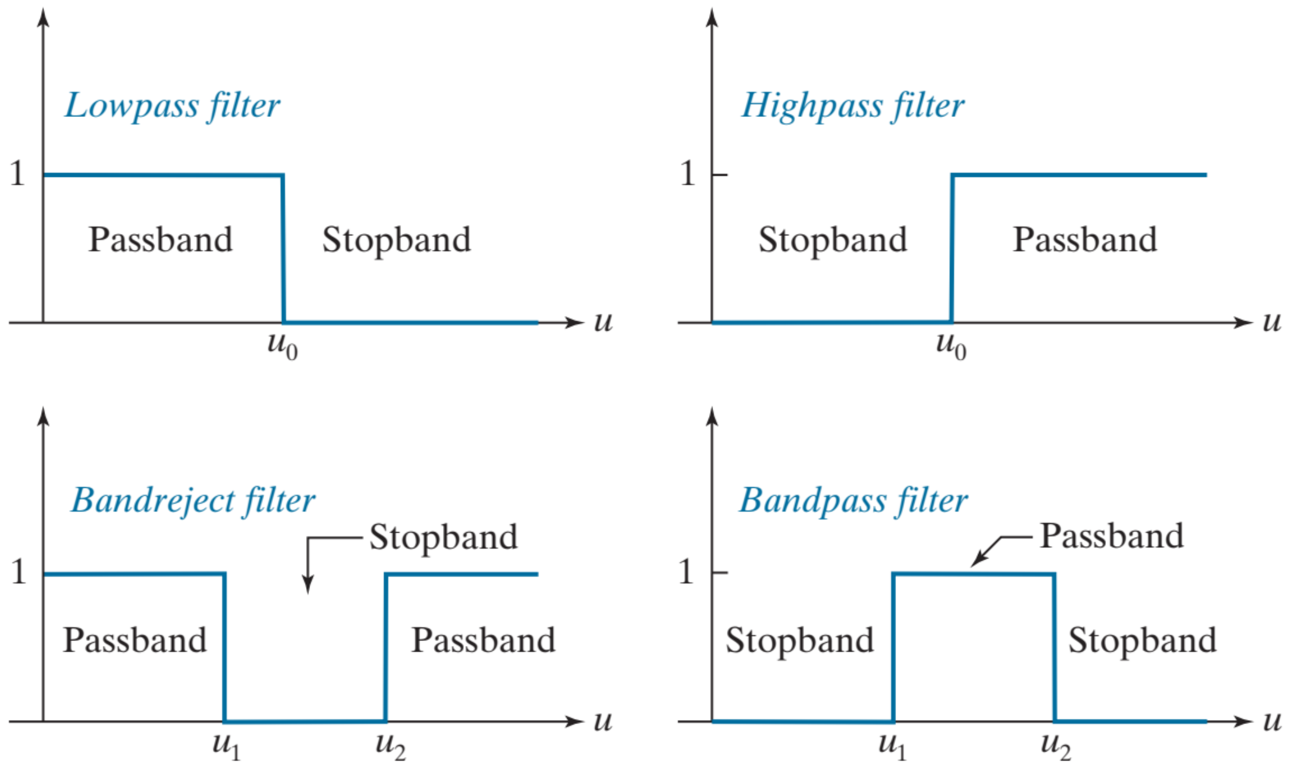
# Image enhancement using frequency domain

- Spatial and frequency-domain linear filters are classified into four broad categories: lowpass and high pass filters, and bandpass and band reject filters, which we introduce in this section.

# Image enhancement using frequency domain

a b
c d

**FIGURE 3.52**
Transfer functions of ideal 1-D filters in the frequency domain ($u$ denotes frequency).
(a) Lowpass filter.
(b) Highpass filter.
(c) Bandreject filter.
(d) Bandpass filter.
(As before, we show only positive frequencies for simplicity.)



# Jean Baptiste Joseph Fourier

Fourier was born in Auxerre,
France in 1768

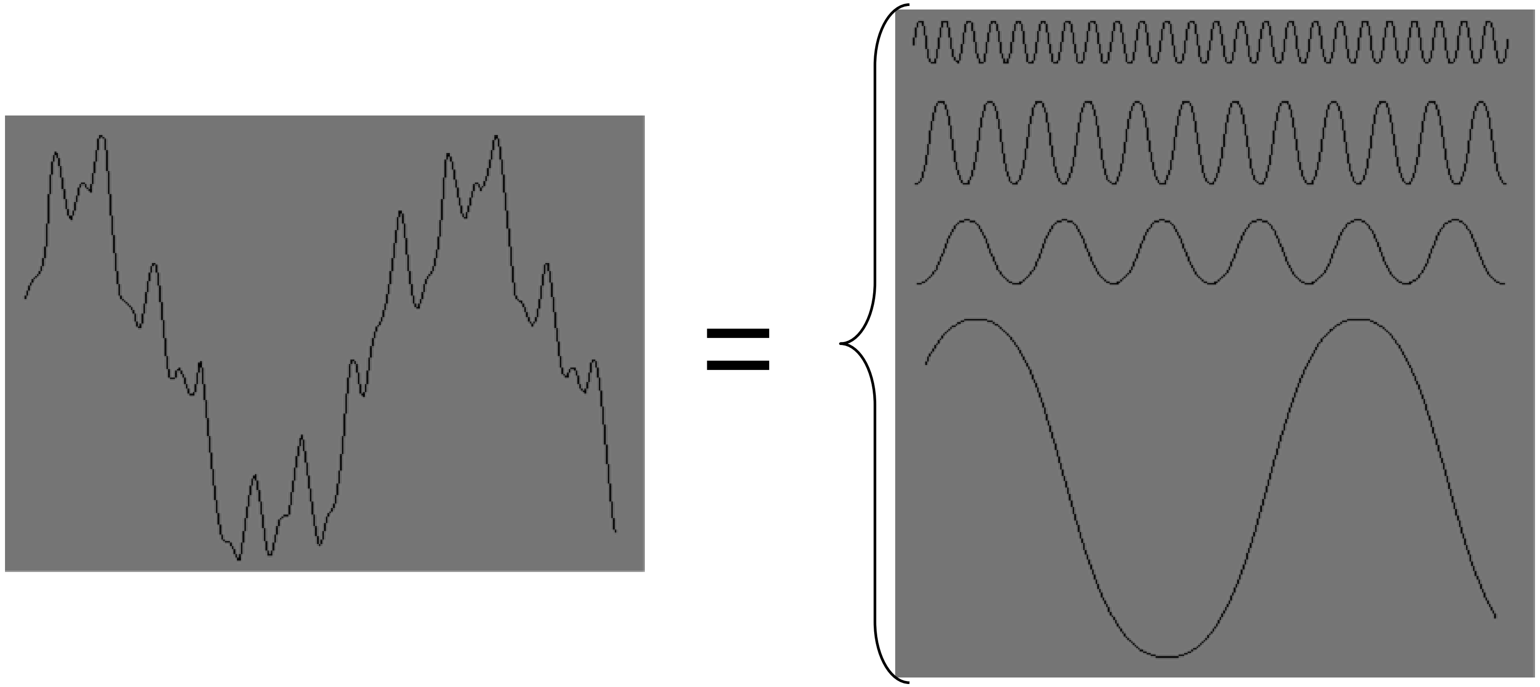
- Most famous for his work *"La Théorie Analitique de la Chaleur"* published in 1822
- Translated into English in 1878: *"The Analytic Theory of Heat"*

Nobody paid much attention when the work was first published

One of the most important mathematical theories in modern engineering

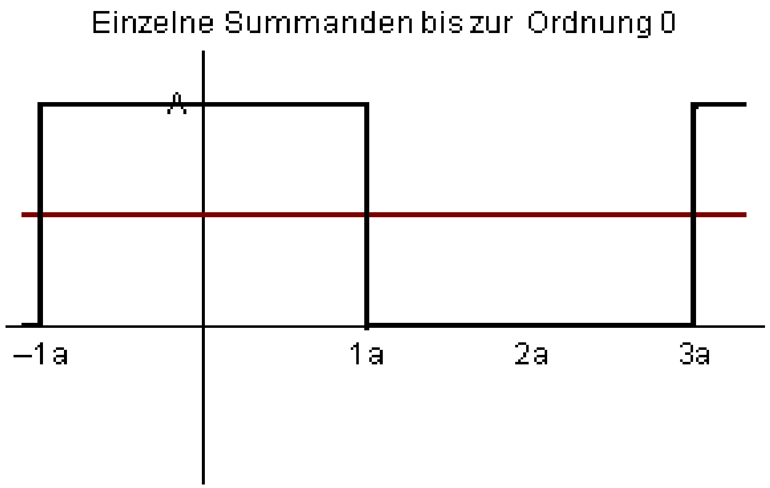
# The Big Idea

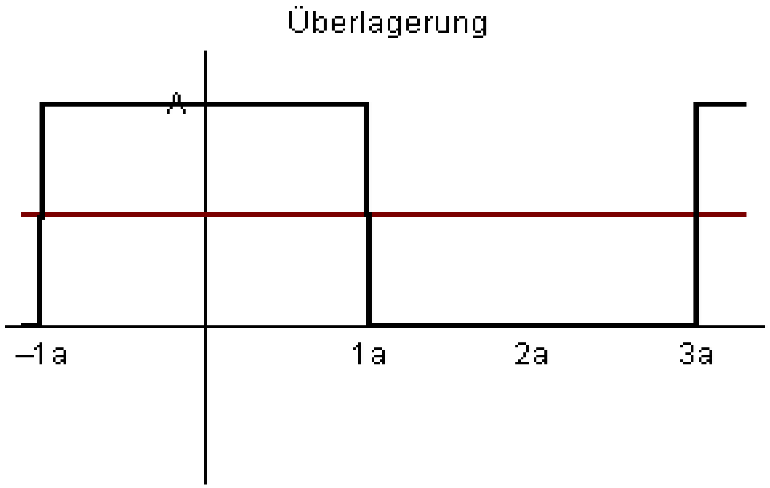
Images taken from Gonzalez & Woods, Digital Image Processing (2002)



Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

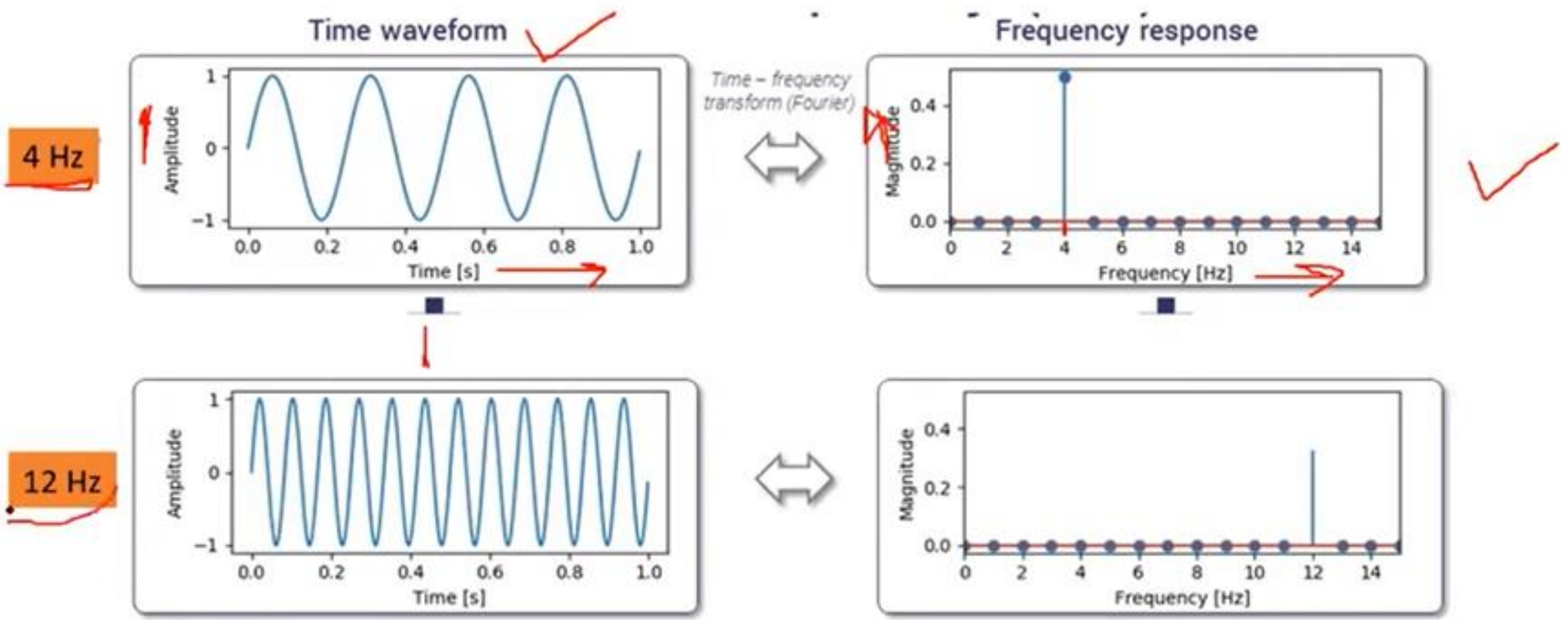
# The Big Idea (cont…)



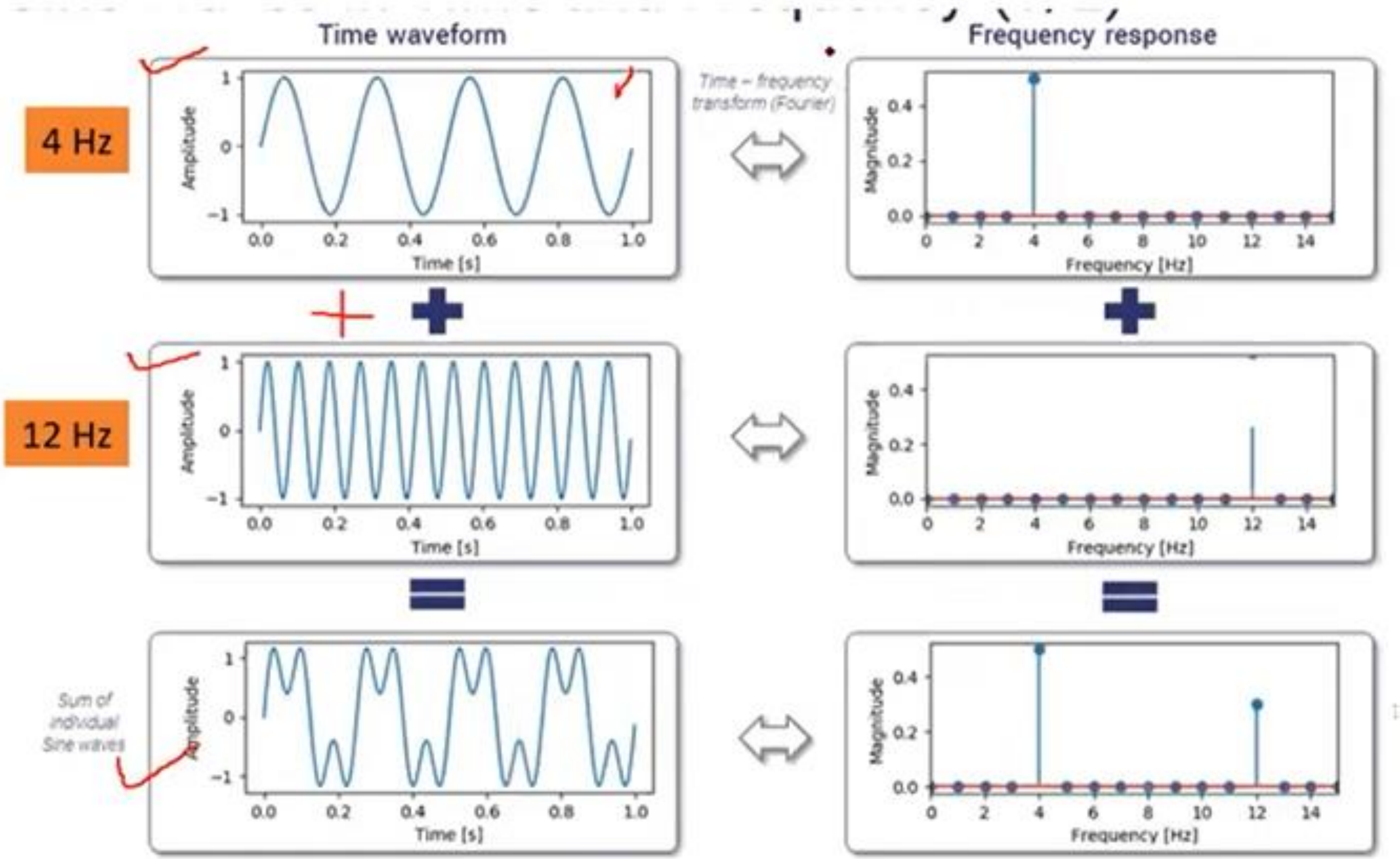

Notice how we get closer and closer to the original function as we add more and more frequencies

Taken from www.tfh-berlin.de/~schwenk/hobby/fourier/Welcome.html

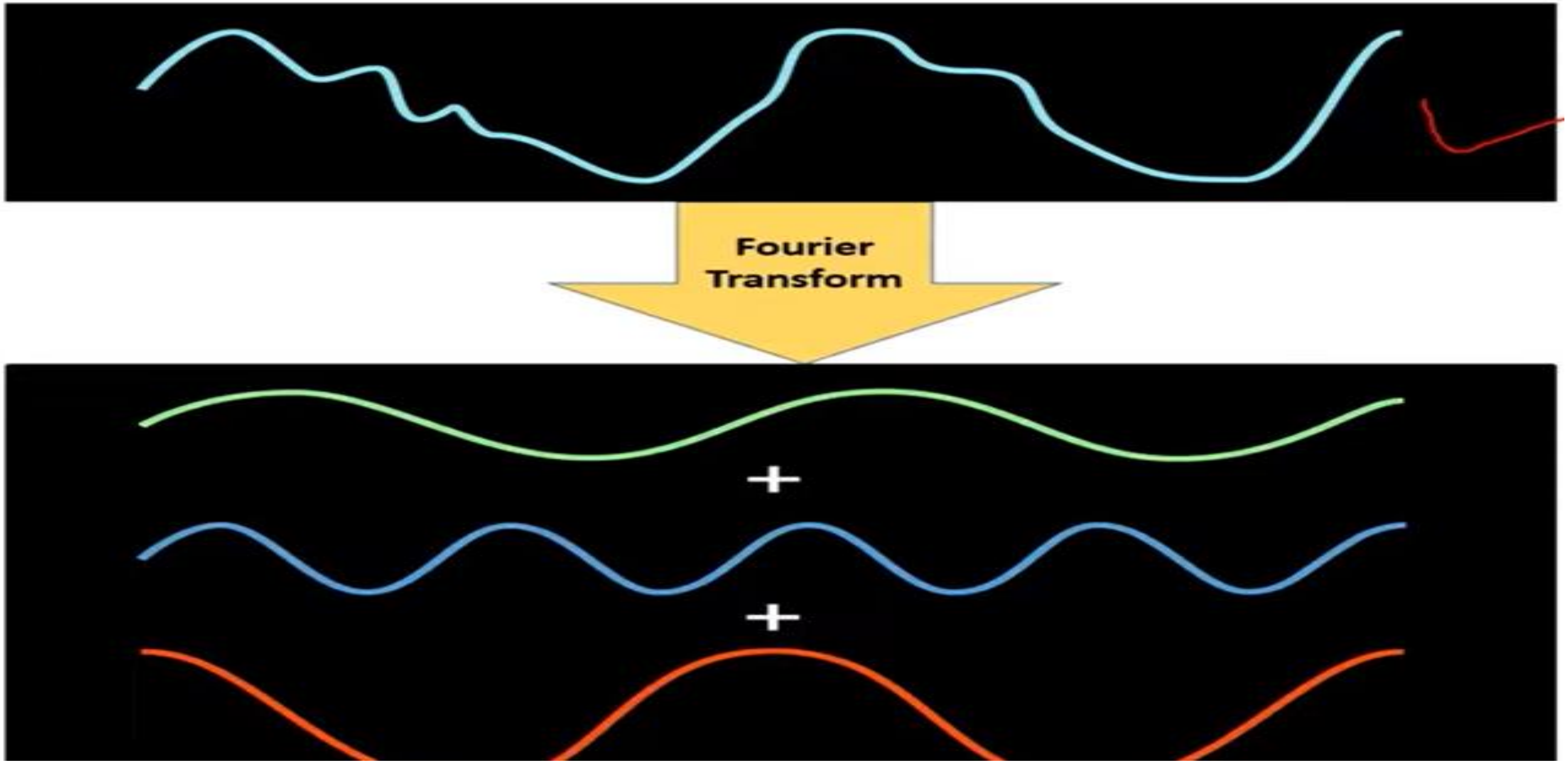


Time waveform
Frequency response
4 Hz
12 Hz
Time – frequency transform (Fourier)
Amplitude
Time [s]
Magnitude
Frequency [Hz]

# Time Domain & Frequency Domain



# If we don't know which combination of frequencies.



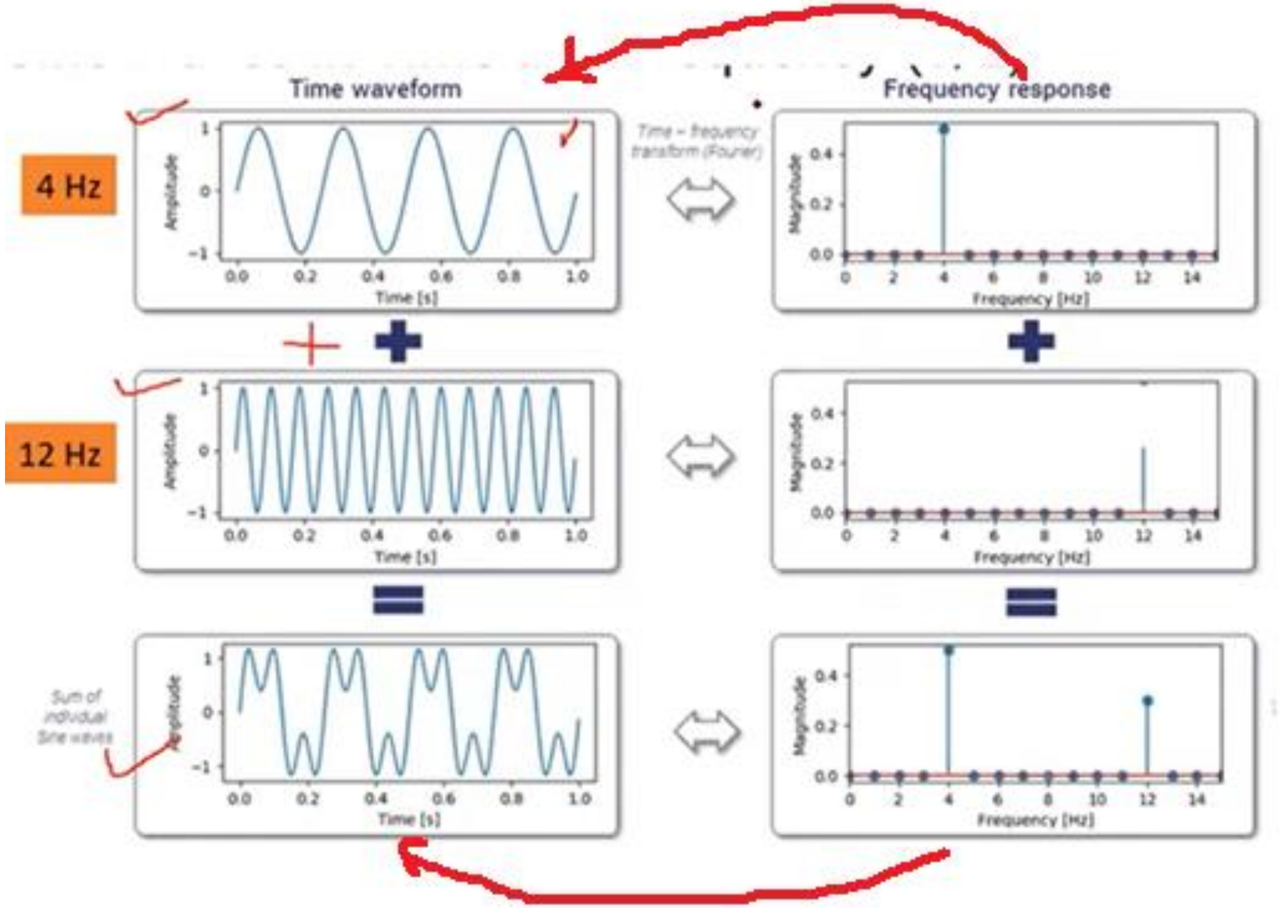
# Fourier Transform of an Image

- The Fourier Transform is an important image processing tool which is used to decompose an image into its sine and cosine components.
- As we are only concerned with digital images, we will restrict this discussion to the Discrete Fourier Transform (DFT).
- For a square image of size N×N, the two-dimensional DFT is given by:

$$F(u,v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{xu}{N}+\frac{vy}{N}\right)}$$

$$e^{-j\theta} = \cos\theta - j\,\sin\theta$$

# Inverse Fourier transform to get original image or signal

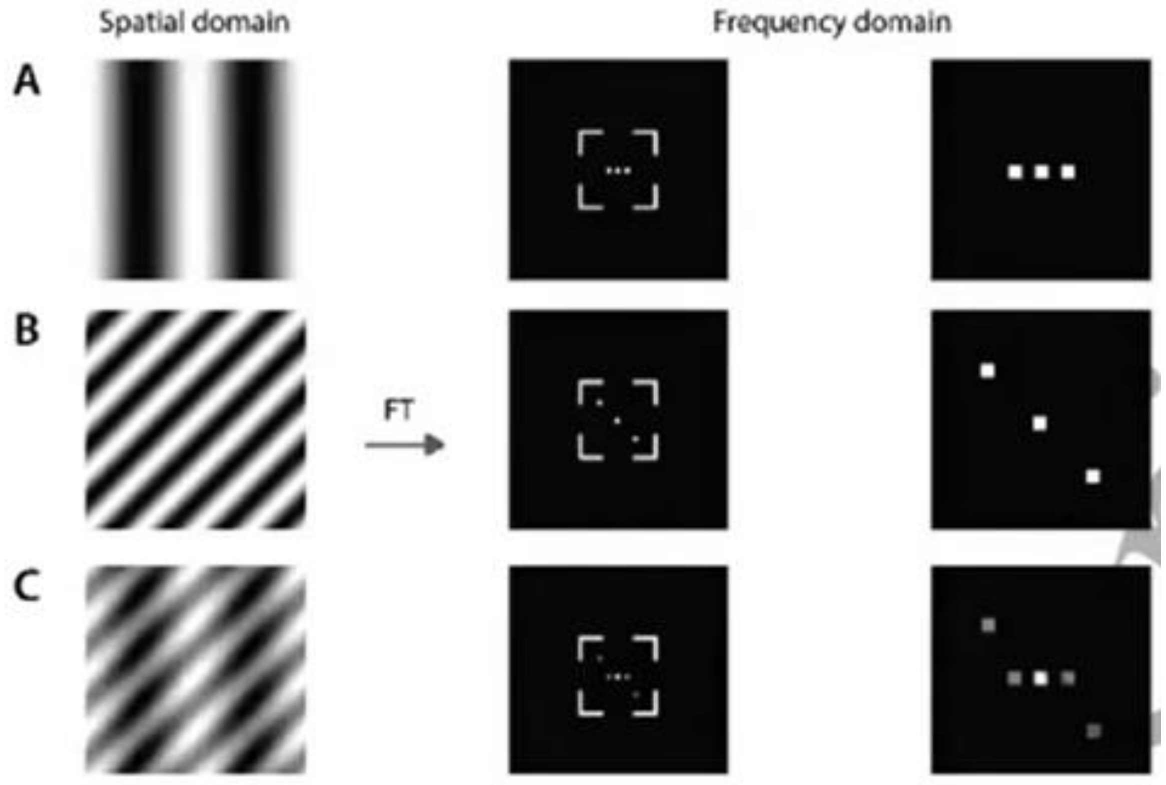



$$f(x,y) = \frac{1}{N^2}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{xu}{N}+\frac{vy}{N}\right)}$$

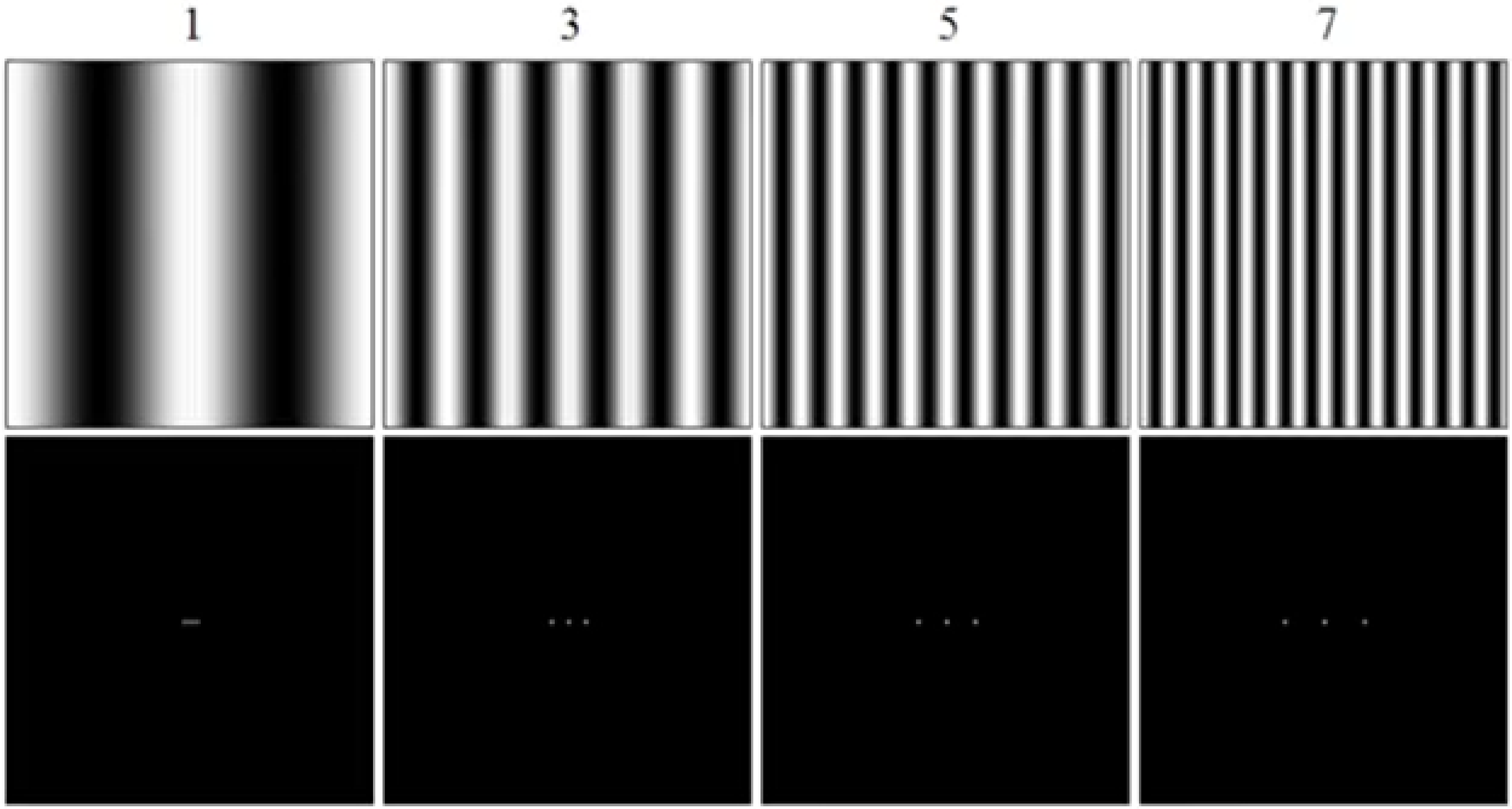
$$e^{j\theta} = cos\theta + j\ sin\theta$$

# The Discrete Fourier Transform (DFT)

- Shows variations in X and Y axis



# The Discrete Fourier Transform (DFT)



# Compute 2D DFT Problem?

- F(x, y) is the original image.
- F(u, v) is the Fourier transform.
- F(u, v) = [kernel] * [f(x, y)] * Transpose([kernel])
- The above formula will calculate the 2D Fourier transform of an image.
- We will use the following formula to calculate the kernel.
- If we need 4*4 kernel then.
- **Wn = $e^{-j2\pi/n}$**
- **So Wn = Cos(θ) – j*Sin(θ)**

# Question?

- Calculate the DFT of the following 1D matrix
- X(n) = $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

# 4*4 kernel sketch

**Wn = $e^{-j2\pi/n}$**

So **W4 = $e^{-j2\pi/4}$**

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **0** | $W_4^0$ | $W_4^0$ | $W_4^0$ | $W_4^0$ |
| **1** | $W_4^0$ | $W_4^1$ | $W_4^2$ | $W_4^3$ |
| **2** | $W_4^0$ | $W_4^2$ | $W_4^4$ | $W_4^6$ |
| **3** | $W_4^0$ | $W_4^3$ | $W_4^6$ | $W_4^9$ |

# Calculate the elements by formula

| | | | |
|---|---|---|---|
| $W_4^0$ | $e^{-j2\pi/4 *0}$ | 1 | 1 |
| $W_4^1$ | $e^{-j2\pi/4 *1}$ | $(Cos(2\pi/4) – j*Sin(2\pi/4)*1 )$ | -j |
| $W_4^2$ | $e^{-j2\pi/4 *2}$ | $(Cos(\pi) – j*Sin(\pi) )$ | -1 |
| $W_4^3$ | $e^{-j2\pi/4 *3}$ | $(Cos(3\pi/2) – j*Sin(3\pi/2) )$ | +j |
| $W_4^4$ | $e^{-j2\pi/4 *4}$ | $(Cos(2\pi) – j*Sin(2\pi) )$ | 1 |
| $W_4^6$ | $e^{-j2\pi/4 *6}$ | $(Cos(3\pi) – j*Sin(3\pi) )$ | -1 |
| $W_4^9$ | $e^{-j2\pi/4 *9}$ | $(Cos(9\pi/2) – j*Sin(9\pi/2) )$ | -j |

# Substituting the values

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **0** | 1 | 1 | 1 | 1 |
| **1** | 1 | -j | -1 | j |
| **2** | 1 | -1 | 1 | -1 |
| **3** | 1 | j | -1 | -j |

# Final f(x, y) is converted to DFT

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

# Inverse Fourier Transform

It is really important to note that the Fourier transform is completely **reversible.** The inverse DFT is given by:

$$\mathbf{Wn^* = e^{j2\pi/n}}$$

$\mathbf{Wn^* = Cos(\theta) + j{*}Sin(\theta)}$

# 4*4 kernel sketch

**$W_n = e^{j2\pi/n}$**

So **$W_4 = e^{j2\pi/4}$**

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **0** | $W^*{}_4^0$ | $W^*{}_4^0$ | $W^*{}_4^0$ | $W^*{}_4^0$ |
| **1** | $W^*{}_4^0$ | $W^*{}_4^1$ | $W^*{}_4^2$ | $W^*{}_4^3$ |
| **2** | $W^*{}_4^0$ | $W^*{}_4^2$ | $W^*{}_4^4$ | $W^*{}_4^6$ |
| **3** | $W^*{}_4^0$ | $W^*{}_4^3$ | $W^*{}_4^6$ | $W^*{}_4^9$ |

# Calculate the elements by formula

| | | | |
|---|---|---|---|
| $W^{*}{}_{4}^{0}$ | $e^{j2\pi/4 *0}$ | 1 | 1 |
| $W^{*}{}_{4}^{1}$ | $e^{j2\pi/4 *1}$ | $(Cos(2\pi/4) + j*Sin(2\pi/4)*1\ )$ | $j$ |
| $W^{*}{}_{4}^{2}$ | $e^{j2\pi/4 *2}$ | $(Cos(\pi) + j*Sin(\pi)\ )$ | -1 |
| $W^{*}{}_{4}^{3}$ | $e^{j2\pi/4 *3}$ | $(Cos(3\pi/2) + j*Sin(3\pi/2)\ )$ | $-j$ |
| $W^{*}{}_{4}^{4}$ | $e^{j2\pi/4 *4}$ | $(Cos(2\pi) + j*Sin(2\pi)\ )$ | 1 |
| $W^{*}{}_{4}^{6}$ | $e^{j2\pi/4 *6}$ | $(Cos(3\pi) + j*Sin(3\pi)\ )$ | -1 |
| $W^{*}{}_{4}^{9}$ | $e^{j2\pi/4 *9}$ | $(Cos(9\pi/2) + j*Sin(9\pi/2)\ )$ | $j$ |

# Substituting the values

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | j | -1 | -j |
| 2 | 1 | -1 | 1 | -1 |
| 3 | 1 | -j | -1 | j |

# Inverse Fourier Transform

- Apply inverse Fourier transform on [2, 1+j, 0, 1-j]
- Xn = 1/n × Wn* × Xn, where n = 4, Wn = 4 ×4 matrix, Xn = 4 ×1 matrix.

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{pmatrix}$$

# Inverse Fourier Transform

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix} \times \begin{pmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

# Inverse Fourier Transform

- So we can say that Inverse Fourier transform = f(x, y)

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$