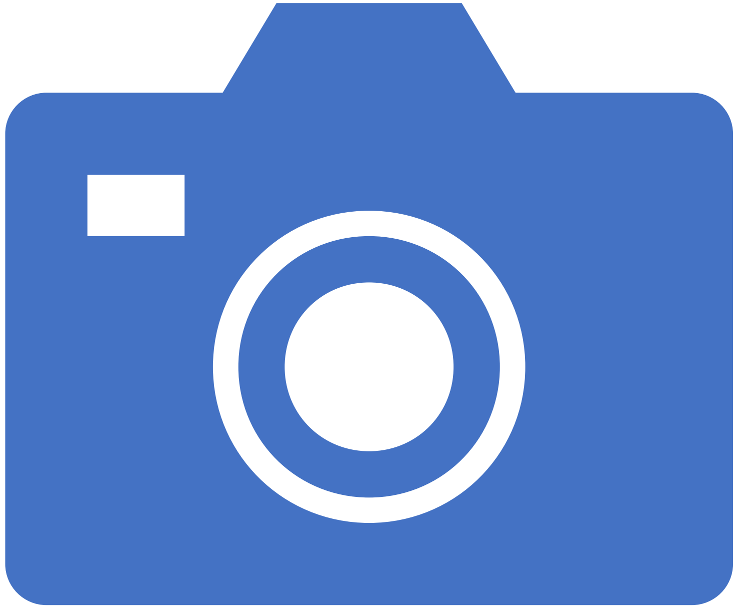


Digital Image Processing

Dr. Mubashir Ahmad (Ph.D.)

The Discrete Fourier Transform (DFT)

The *Discrete Fourier Transform* of $f(x, y)$, for x = 0, 1, 2…M-1 and y = 0,1,2…N-1, denoted by $F(u, v)$, is given by the equation:

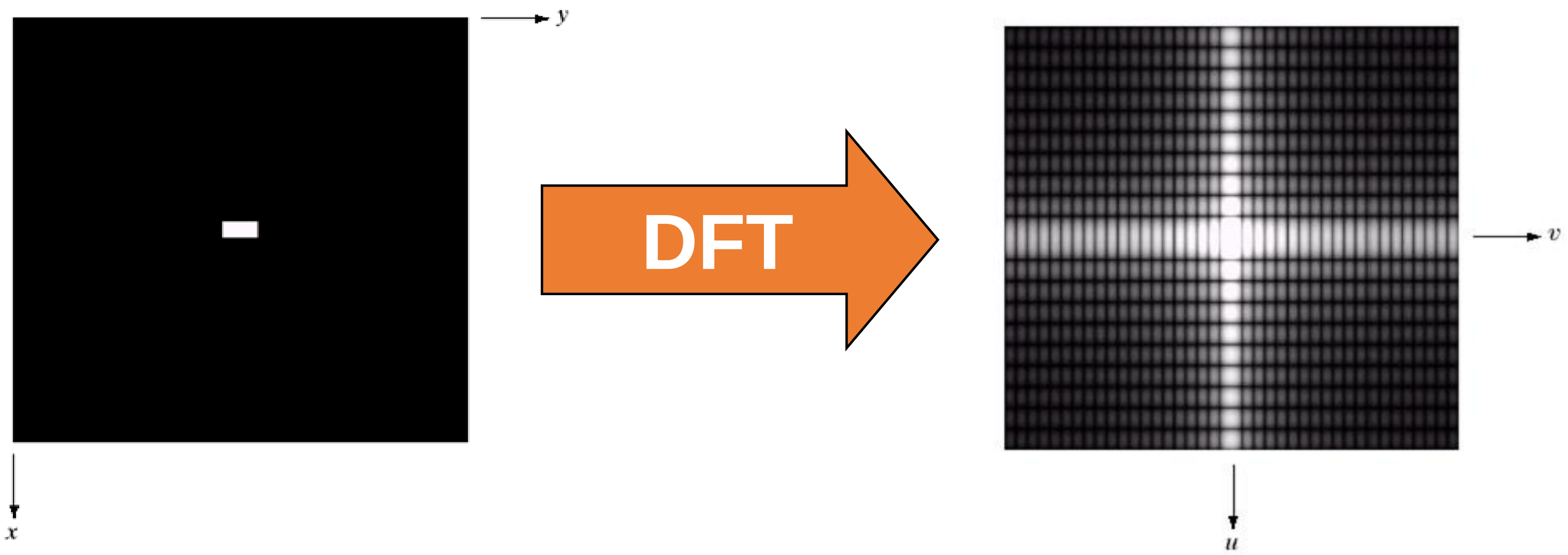
f(x,y)(-1)^$^{x+y}$ (to centre the transform)

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

for u = 0, 1, 2…M-1 and v = 0, 1, 2…N-1.

DFT & Images

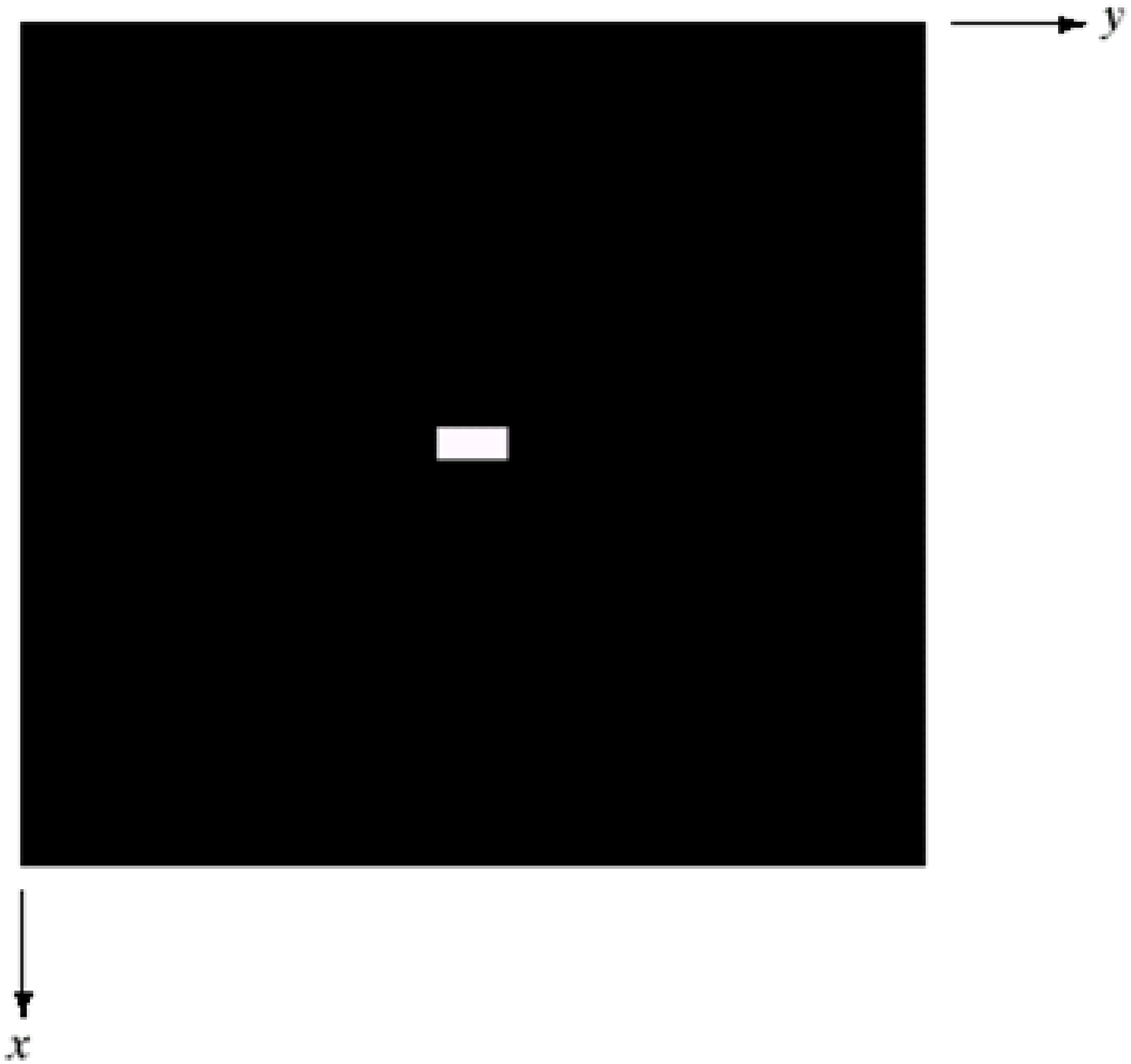
The DFT of a two dimensional image can be visualised by showing the spectrum of the images component frequencies



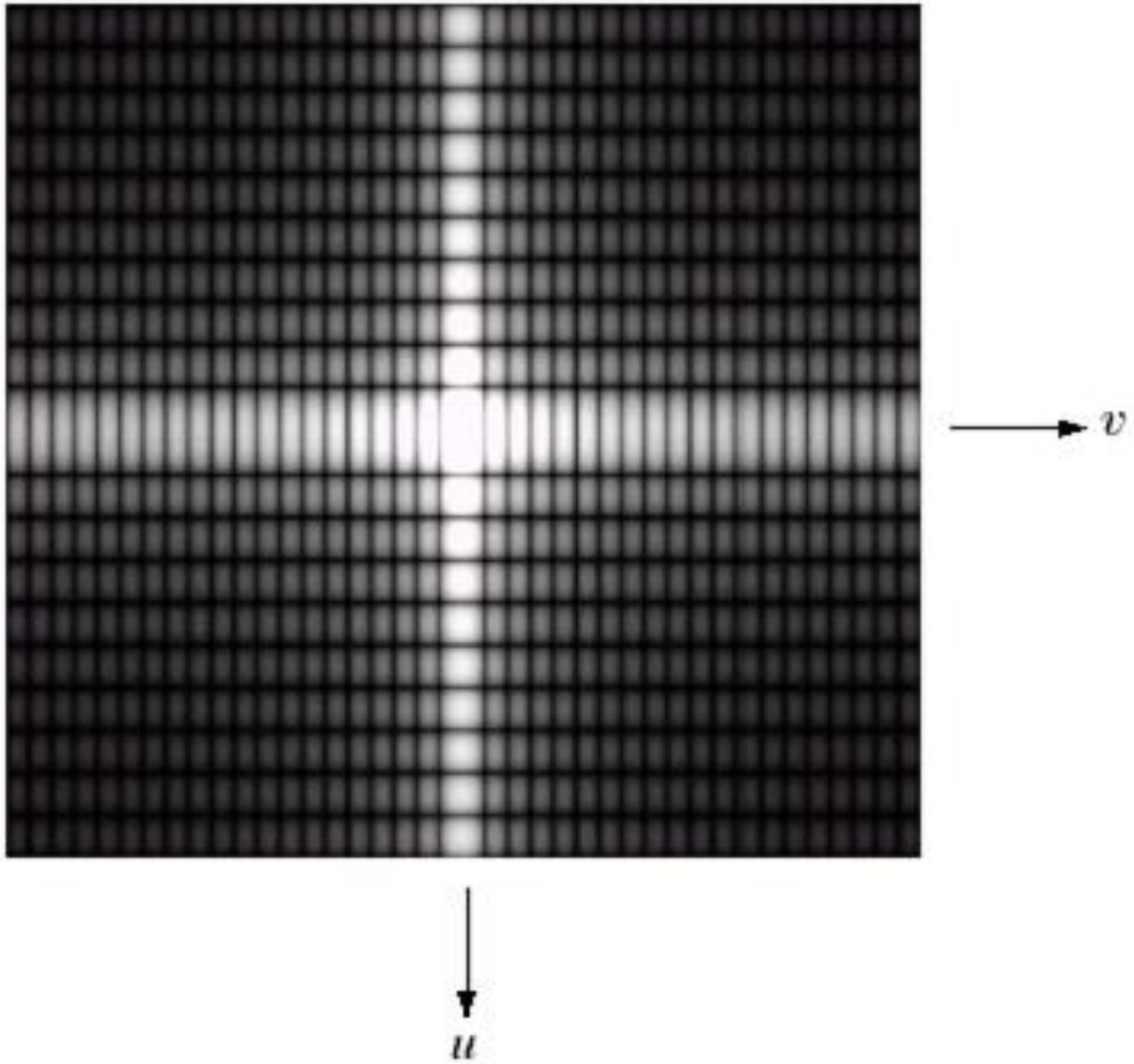
Images taken from Gonzalez & Woods, Digital Image Processing (2002)

DFT & Images

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



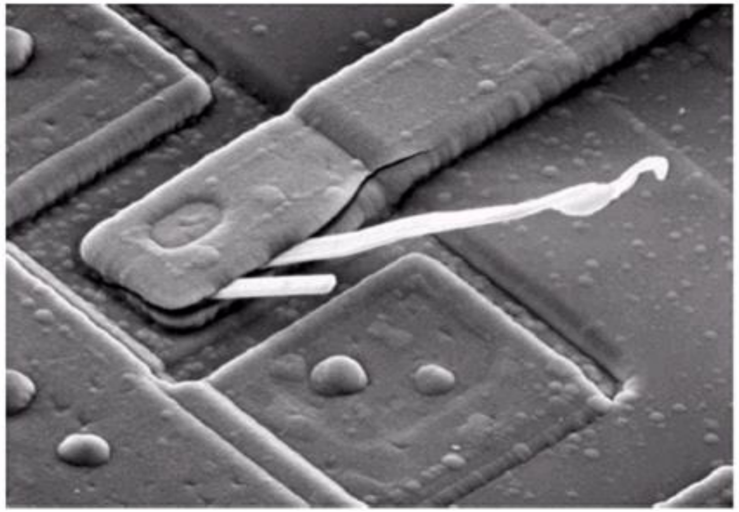
DFT & Images



Images taken from Gonzalez & Woods, Digital Image Processing (2002)

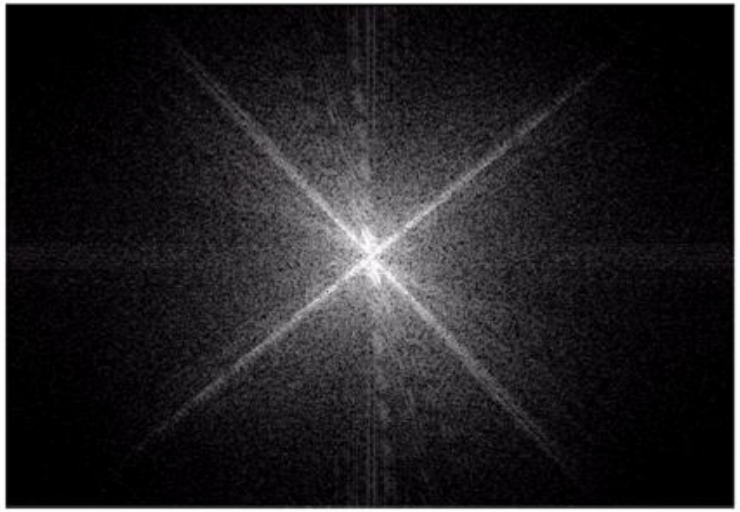
DFT & Images (cont...)

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



Scanning electron microscope image of an integrated circuit magnified

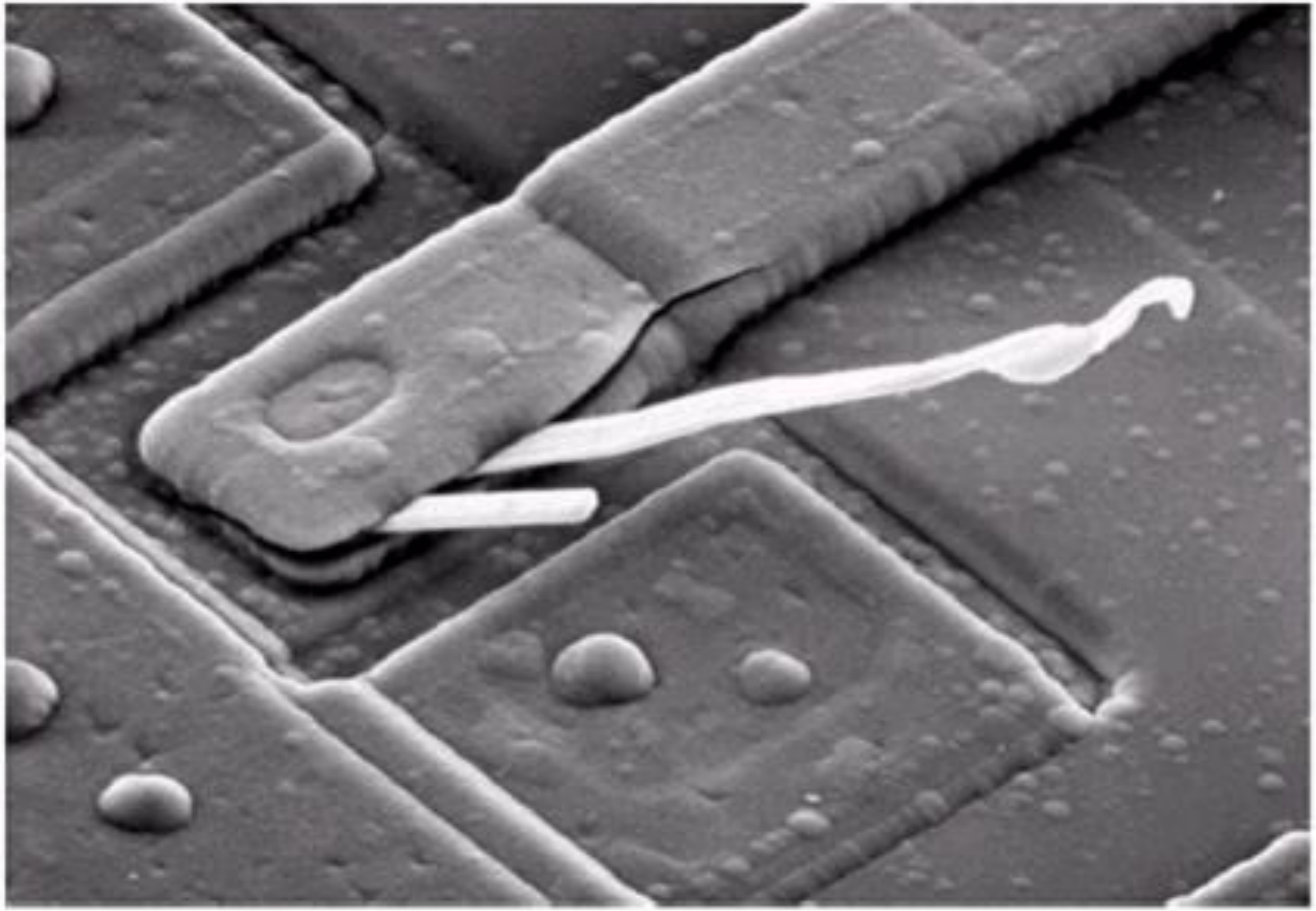
DFT



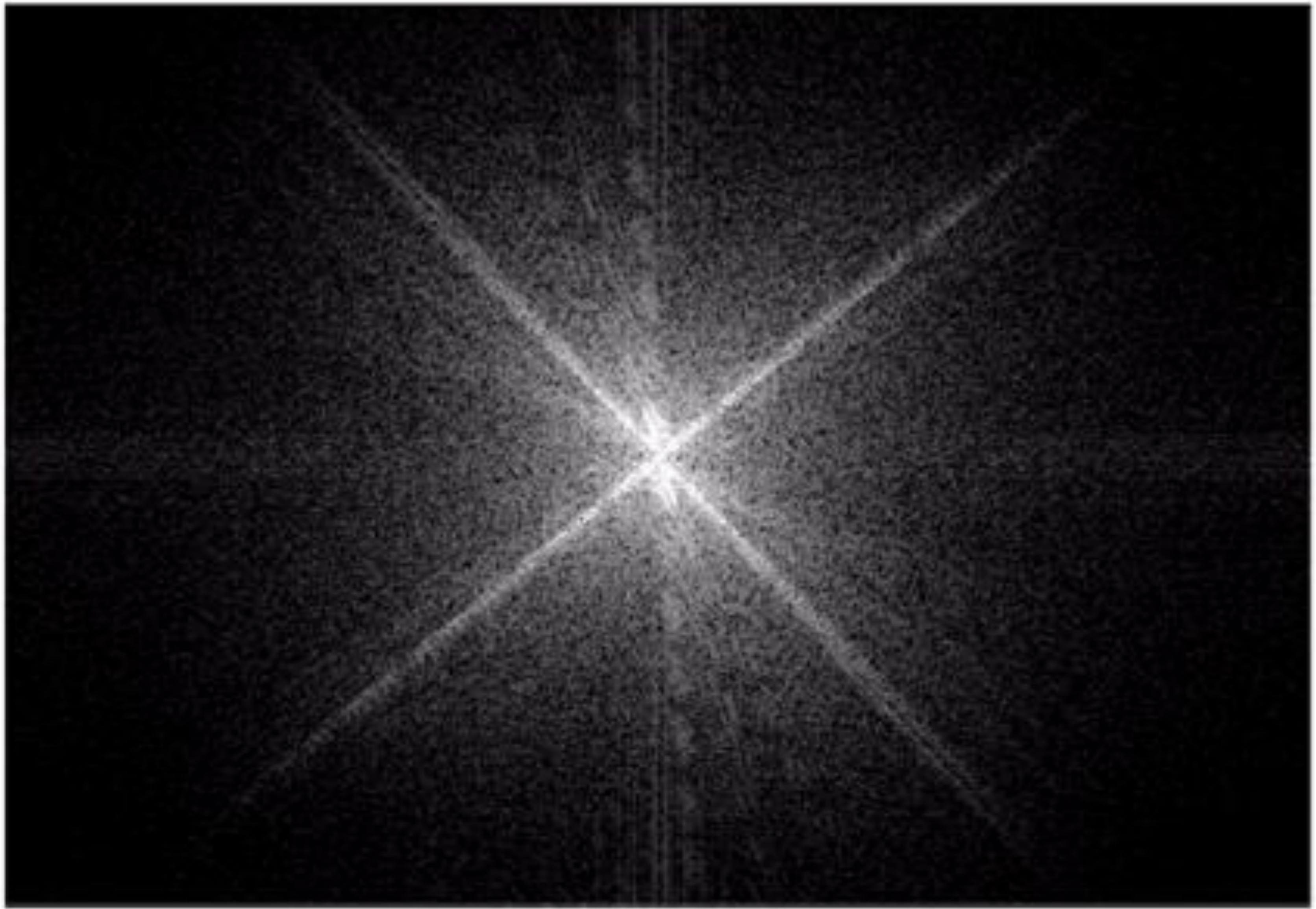
Fourier spectrum of the image

DFT & Images (cont…)

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



DFT & Images (cont…)



Images taken from Gonzalez & Woods, Digital Image Processing (2002)

The Inverse DFT

It is really important to note that the Fourier transform is completely **reversible**

The inverse DFT is given by:

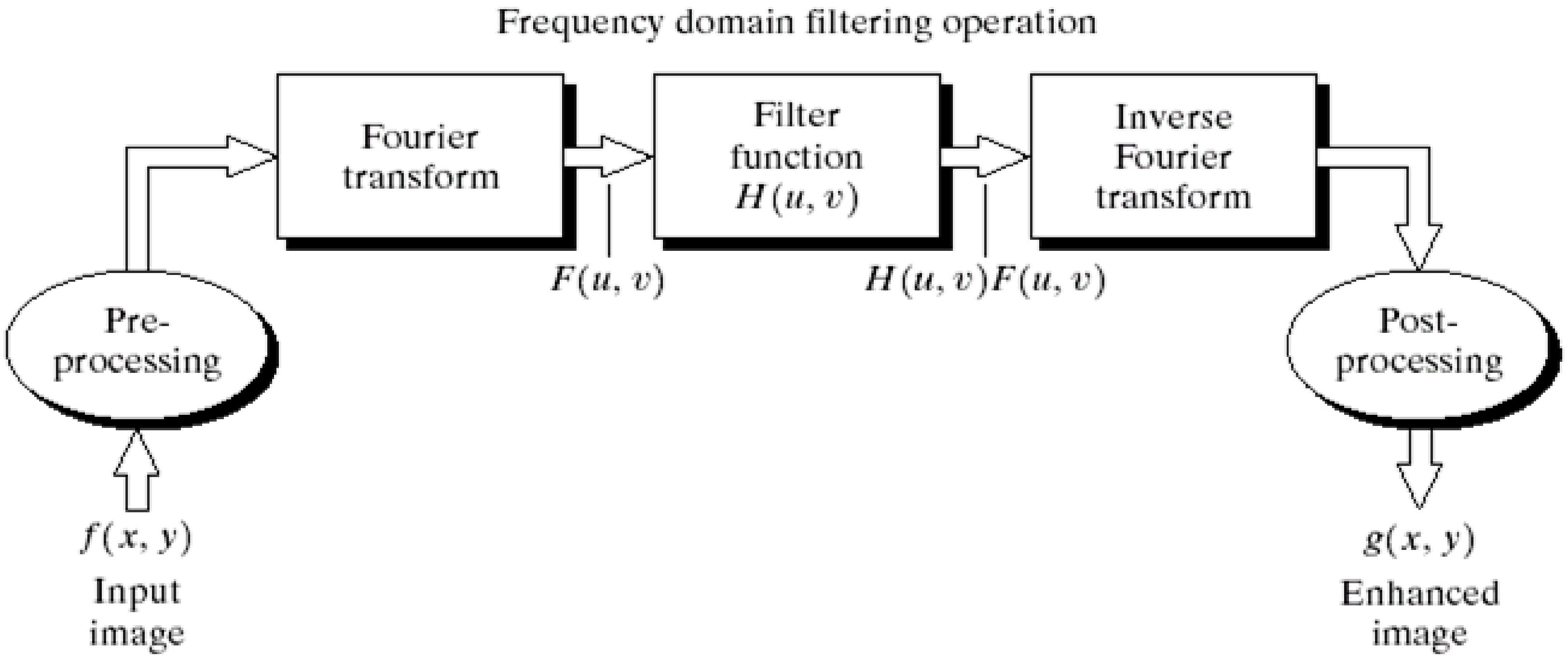
$$F(u,v) = 1/MN * \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{j2\pi(ux/M + vy/N)}$$

for x = 0, 1, 2…M-1 and y = 0, 1, 2…N-1

The DFT and Image Processing

To filter an image in the frequency domain:

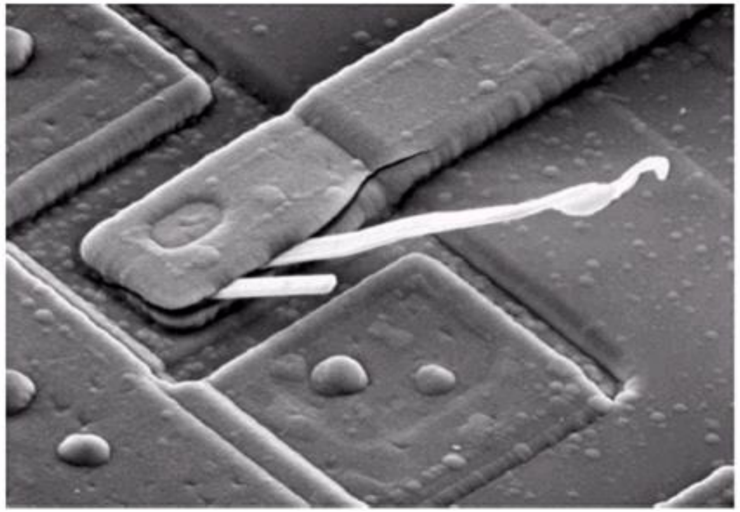
1. Compute $F(u,v)$ the DFT of the image
2. Multiply $F(u,v)$ by a filter function $H(u,v)$
3. Compute the inverse DFT of the result



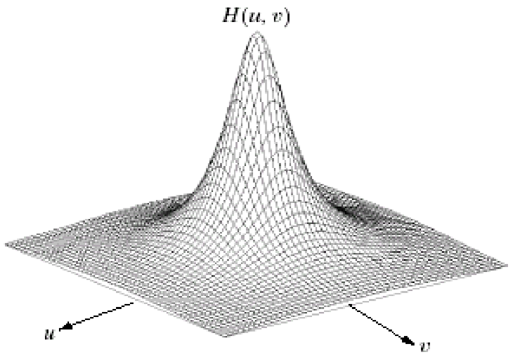
Images taken from Gonzalez & Woods, Digital Image Processing (2002)

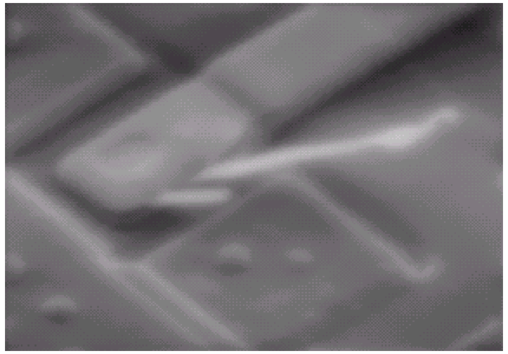
Some Basic Frequency Domain Filters

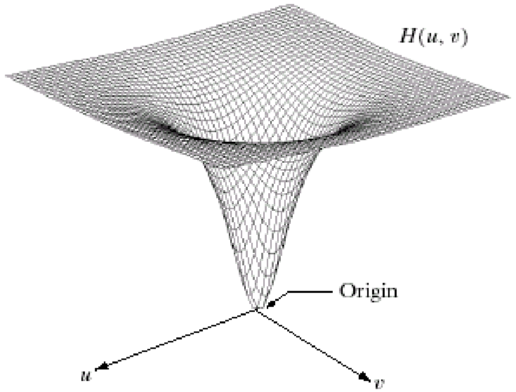
Images taken from Gonzalez & Woods, Digital Image Processing (2002)

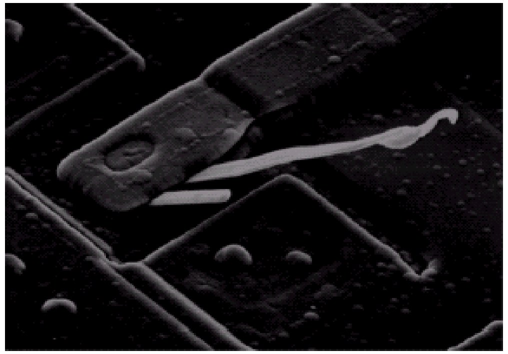


Low Pass Filter





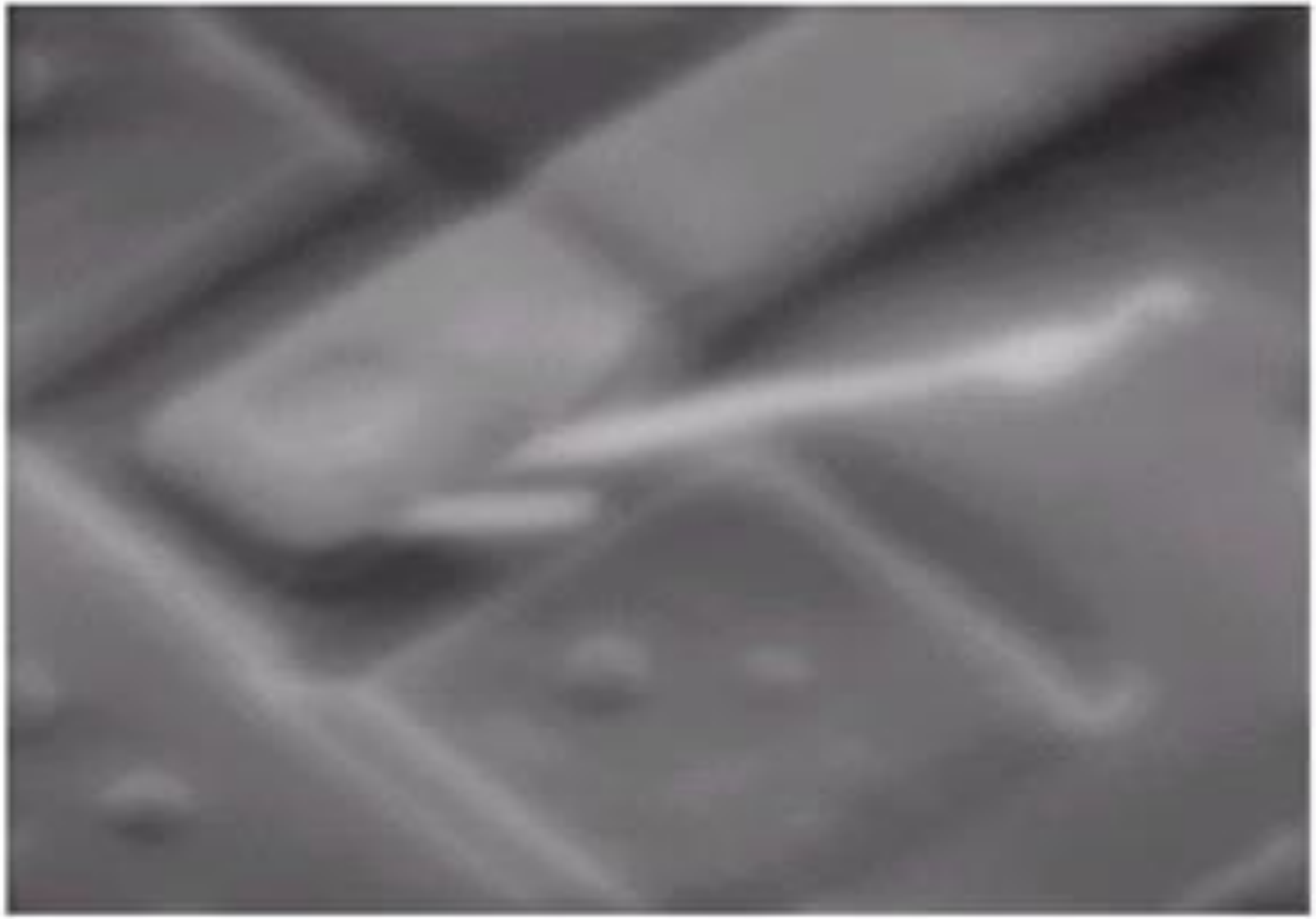




High Pass Filter

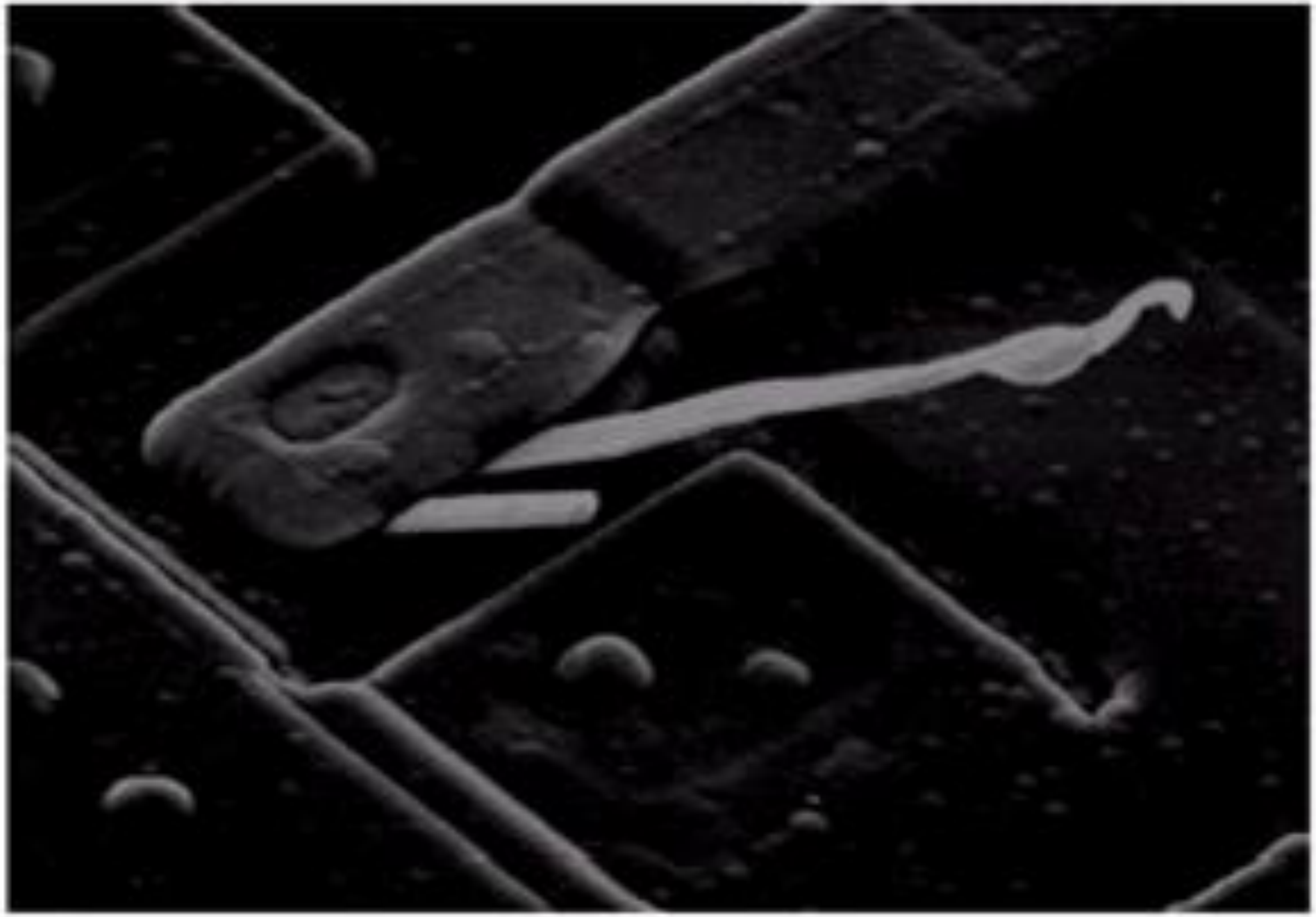
Some Basic Frequency Domain Filters
(Response of Low Pass filtering of an image)

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



Some Basic Frequency Domain Filters (Response of High Pass filtering of an image)

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



Smoothing Frequency Domain Filters

- Smoothing is achieved in the frequency domain by dropping out the high frequency components
- The basic model for filtering is:

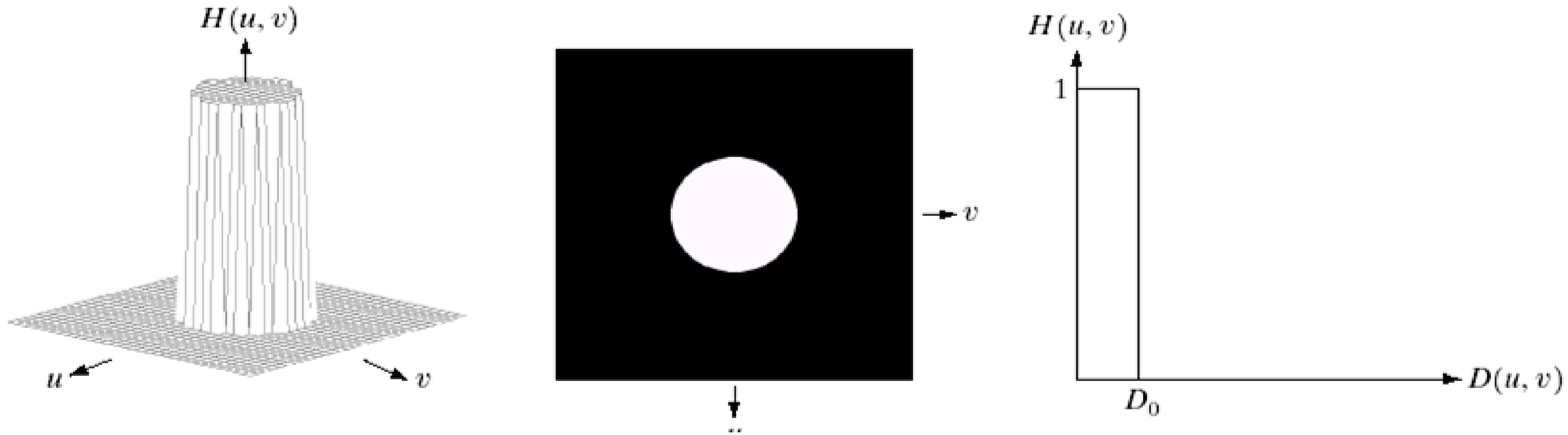
$$G(u,v) = H(u,v)F(u,v)$$

where $F(u,v)$ is the Fourier transform of the image being filtered and $H(u,v)$ is the filter transfer function

- *Low pass filters* – only pass the low frequencies, drop the high frequencies which result in noise reduction due to blurring/smoothing of the image

Ideal Low Pass Filter

Simply cut off all high frequency components that are at specified distance D_0 from the origin of the transform.



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Low Pass Filter (cont…)

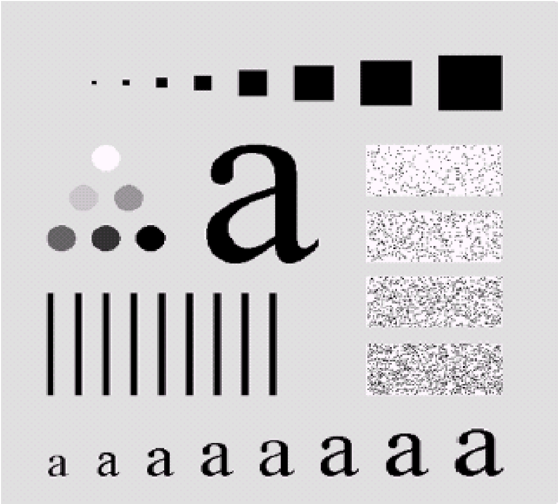
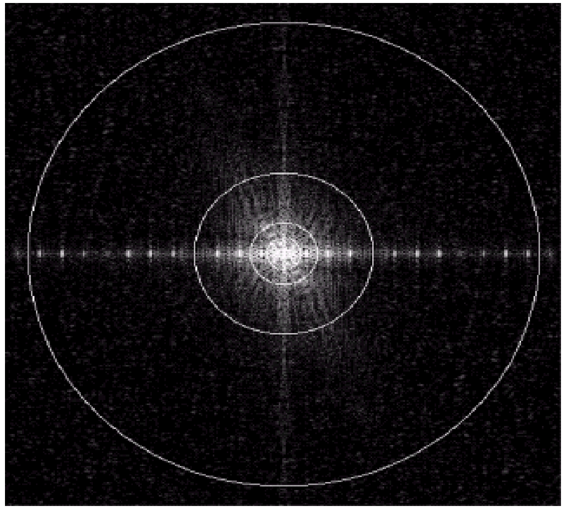
The transfer function for the ideal low pass filter can be given as:

$$H(u,v) = \begin{cases} 1 \text{ if } D(u,v) \leq D_0 \\ 0 \text{ if } D(u,v) > D_0 \end{cases}$$

where $D(u , v)$ is given as:

$$\boxed{D(u,v) = [(u)^2 + (v)^2]^{1/2}}$$

Ideal Low Pass Filter (cont…)

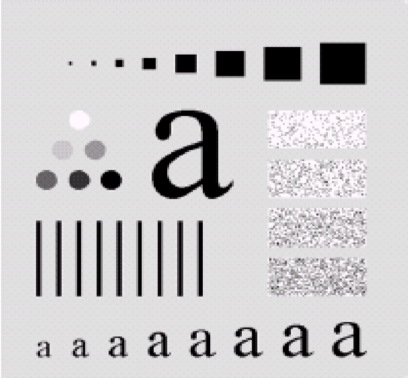



Above we show an image, it's Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it

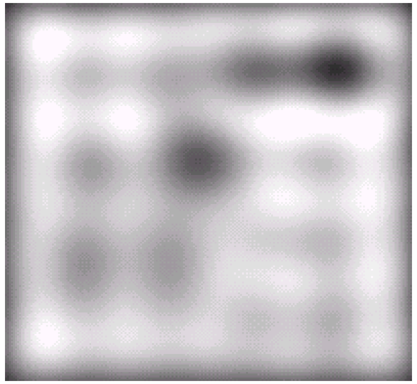
Images taken from Gonzalez & Woods, Digital Image Processing (2002)

Ideal Low Pass Filter (cont…)

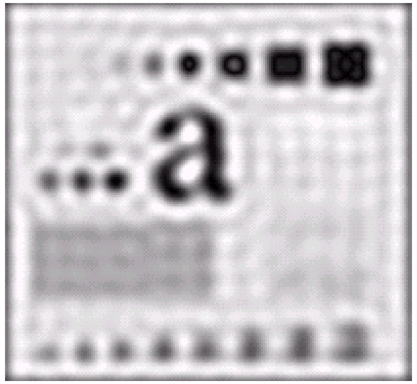
Images taken from Gonzalez & Woods, Digital Image Processing (2002)



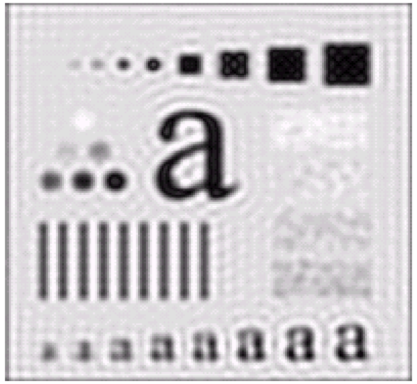
Original image



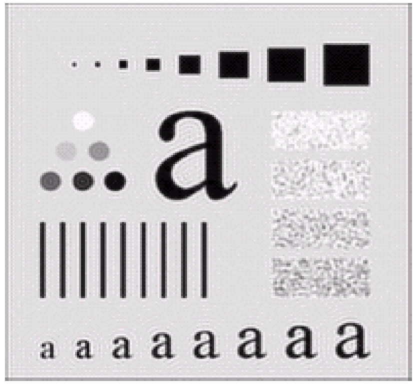
Result of filtering with ideal low pass filter of radius 5



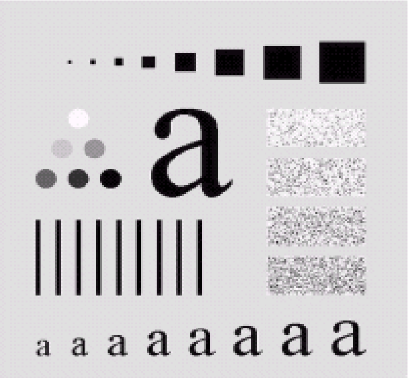
Result of filtering with ideal low pass filter of radius 15



Result of filtering with ideal low pass filter of radius 30



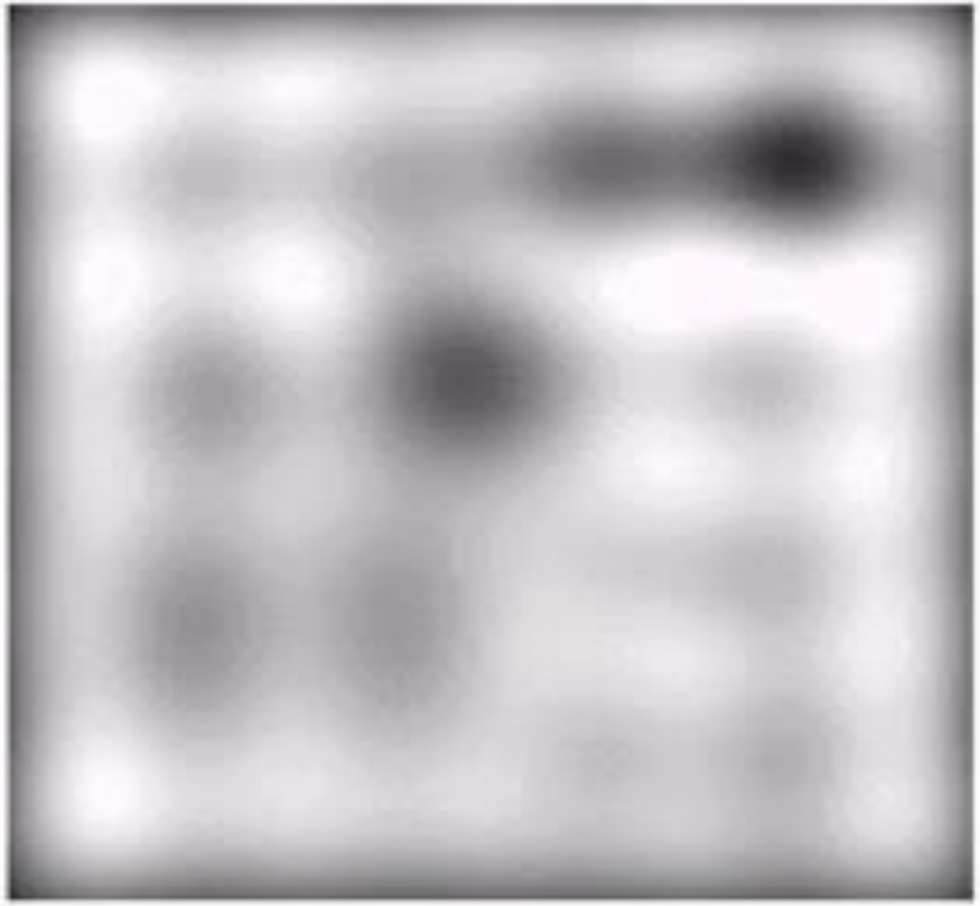
Result of filtering with ideal low pass filter of radius 80



Result of filtering with ideal low pass filter of radius 230

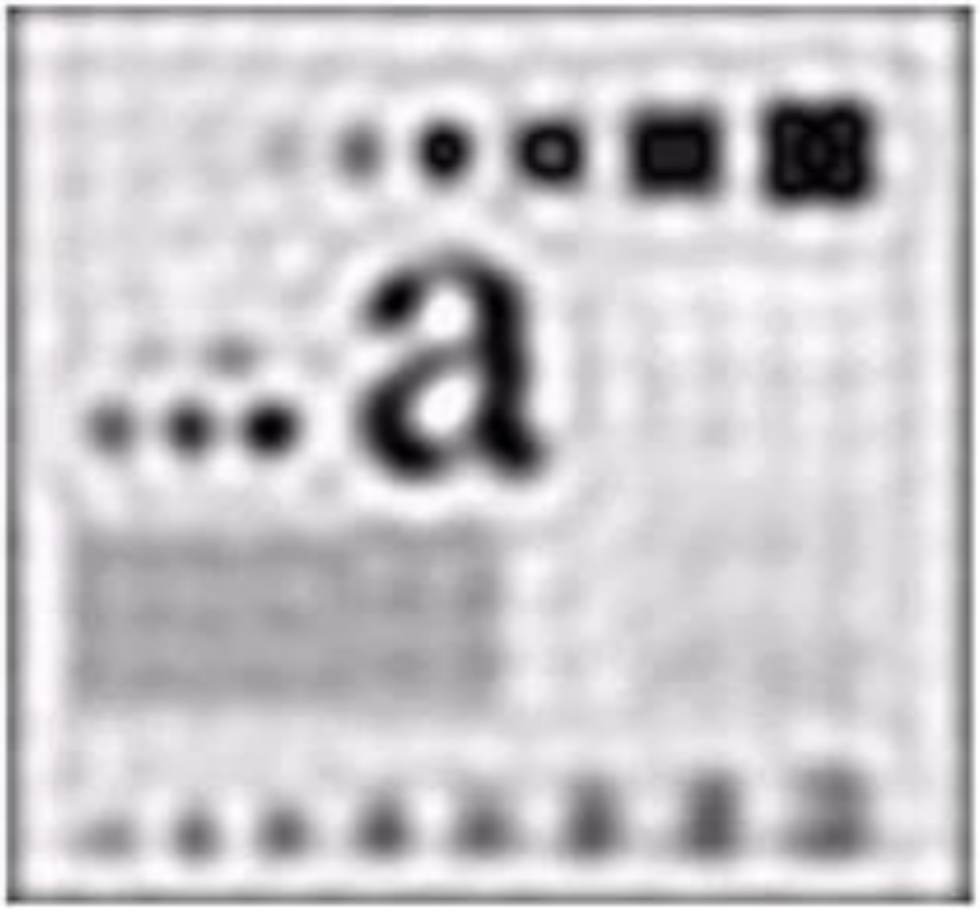
Ideal Low Pass Filter (cont…)

Images taken from Gonzalez & Woods, Digital Image Processing (2002)



Result of filtering with ideal low pass filter of radius 5

Ideal Low Pass Filter (cont…)



Result of filtering with ideal low pass filter of radius 15

Images taken from Gonzalez & Woods, Digital Image Processing (2002)

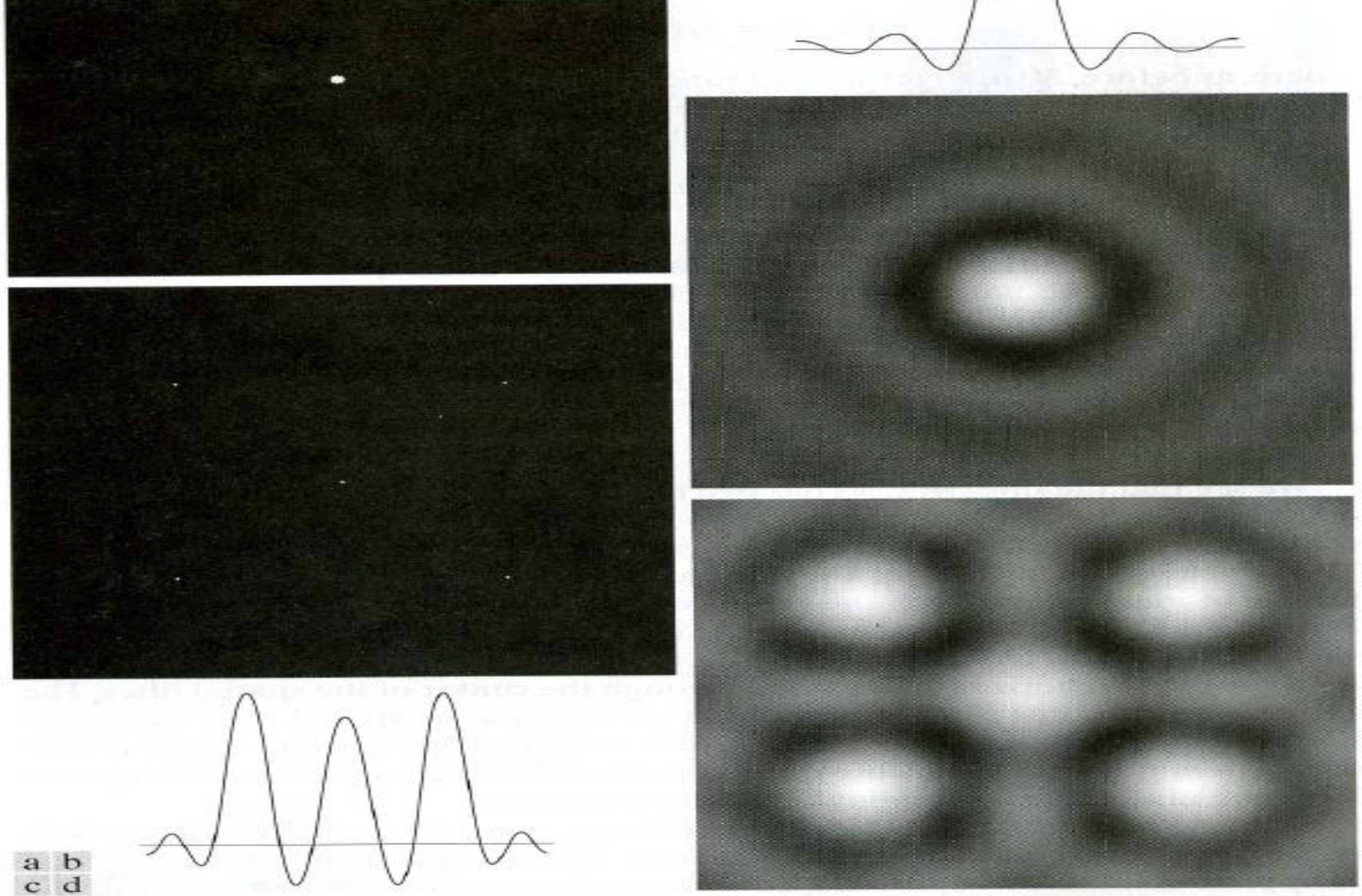


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Calculate the spatial domain to the frequency domain to perform Ideal Low pass filter, Where cutoff frequency = 0.5

- Input image f(x, y) in a spatial domain.
- Multiply the input image with (-1)^x+y to move the transform in the center.

$$F(x, y) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$(-1)^{x+y} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Ideal Low pass filter

- After the transform center arrangement we got the results as follows.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

Ideal Low pass filter: Compute the DFT of the image

- Compute the DFT of the image.
- F(u, v) = kernel * f(x, y) * kernel ^Transpose

$$f(u,v) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \times \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

Ideal Low pass filter: Compute the DFT of the image

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Distance measure

$$U = \begin{pmatrix} -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 1 \end{pmatrix} \qquad V = \begin{bmatrix} -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$D(U,V) = \begin{pmatrix} (-2,-2) & (-1,-2) & (0,-2) & (1,-2) \\ (-2,-1) & (-1,-1) & (0,-1) & (1,-1) \\ (-2,0) & (-1,0) & (0,0) & (1,0) \\ (-2,1) & (-1,1) & (0,1) & (1,1) \end{pmatrix}$$

Ideal Low pass filter:

- Now compute the distances
- D(u, v) = sqrt(u.^2 + v.^2)
- D(-2, -2) = (– 2)^2 + (– 2)^2 = (8) ^1/2 = 2.82
- D(-2, -2) = (-1)^2 + (–2)^2 = (5) ^1/2 = 2.23
- D(0, 2) = (0)^2 + (– 2)^2 = (4) ^1/2 = 2
- D(1, -2) = (1)^2 + (– 2)^2 = 2.23
- D(-2, -1) = (– 2)^2 + (– 1)^2 = 1.14
- D(-1, -1) = (-1)^2 + (-1)^2 = 2.23

Ideal Low pass filter: Distance measure

- D(0, -1) = (0)^2 + (-1)^2 = 1
- D(1, -1) = (1)^2 + (-1)^2 = 1.41
- D(-2, 0) = (– 2)^2 + (0)^2 = 2
- D(-1, 0) = (-1)^2 + (0)^2 = 1
- D(0, 0) = (0)^2 + (0)^2 = 0
- D(1, 0) = (1)^2 + (0)^2 = 1
- D(-2, 1) = (– 2)^2 + (1)^2 = 2.23
- D(-1, 1) = (-1)^2 + (1)^2 = 1.41
- D(0, 1) = (0)^2 + (1)^2 = 1
- D(1, 1) = (1)^2 + (1)^2 = 1.41

Ideal Low pass filter: Distance measure

$$D(u,v) = \begin{bmatrix} 2.8284 & 2.2361 & 2 & 2.2361 \\ 2.2361 & 1.4142 & 1 & 1.4142 \\ 2 & 1 & 0 & 1 \\ 2.2361 & 1.4142 & 1 & 1.4142 \end{bmatrix} = \begin{bmatrix} (-2,-2) & (-1,-2) & (0,-2) & (1,-2) \\ (-2,-1) & (-1,-1) & (0,-1) & (1,-1) \\ (-2,0) & (-1,0) & (0,0) & (1,0) \\ (-2,1) & (-1,1) & (0,1) & (1,1) \end{bmatrix}$$

Cut off frequency D0 = 0.5

- Distance matrix will be

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Inverse Fourier transform

$$f(x,y) = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{pmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{pmatrix}$$

$$f(x,y) = \frac{1}{16}\begin{bmatrix} 8 & -8 & 8 & -8 \\ -8 & 8 & -8 & 8 \\ 8 & -8 & 8 & -8 \\ -8 & 8 & -8 & 8 \end{bmatrix}$$