§2.2: Separable Variables:

Defn. A D.E of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

or

$$\frac{dy}{dx} = \frac{h(y)}{g(x)}$$

is said to be separable, or to have separable variables.

Note: Separable Eqn. can be written as

$$h(y)\,dy = g(x)\,dx$$

or

$$\frac{dy}{h(y)} = \frac{dx}{g(x)} \quad \text{etc.}$$

Method of Soln.: 1) Separate variables.

2) Integrate the Eqn.

2.2 Exercises:

Q. (1) $\frac{dy}{dx} = \sin 5x$

Soln. $dy = \sin 5x \, dx$

$\int dy = \int \sin 5x \, dx$

$y = -\frac{\cos 5x}{5} + C$

Q(12). $\frac{dx}{dy} = \frac{1+2y^2}{y \sin x}$

Soln. $\sin x \, dx = \frac{1+2y^2}{y} dy$

$\Rightarrow \int \sin x \, dx = \int \frac{1}{y} dy + \int \frac{2y^2}{y} dy$

$-\cos x = \ln|y| + 2\frac{y^2}{2} + C$

$\ln|y| + y^2 + \cos x = C^*$

Q 27. $e^y \sin 2x \, dx + (\cos x)(e^{2y} - y) dy = 0$

$\frac{e^y \sin 2x \, dx + \cos x (e^{2y} - y) dy = 0}{}$

$\frac{\sin 2x}{\cos x} dx + \frac{e^{2y} - y}{e^y} dy = 0$

$2 \sin x \, dx + (e^y - y e^{-y}) dy = 0$

$\int 2 \sin x \, dx + \int e^y dy - \int y e^{-y} dy = 0$

Q. 39. $\left(e^{x}+e^{-x}\right)\frac{dy}{dx}=y^{2}$

Soln.

$$\frac{dy}{y^2}=\frac{dx}{e^x+e^{-x}}$$

$$\frac{dy}{y^2}=\frac{dx}{\frac{e^{2x}+1}{e^x}}$$

$$y^{-2}\,dy=\frac{e^x}{1+e^{2x}}\,dx$$

$$\int y^{-2}\,dy=\int\frac{e^x}{1+e^{2x}}\,dx$$

$$\frac{y^{-1}}{-1}=\int\frac{e^x}{1+(e^x)^2}\,dx$$

$$-\frac{1}{y}=\int\frac{1}{1+u^2}\,du \qquad u=e^x \quad du=e^x\,dx$$

$$-\frac{1}{y}=\tan^{-1}u+C$$

$$-\frac{1}{y}=\tan^{-1}(e^x)+C$$

$$-\frac{1}{y}=\tan^{-1}(e^x)+C$$

$$\boxed{-\frac{1}{y}=\tan^{-1}(e^x)+C}$$

47) $x^2 y' = y - xy$; $y(-1) = -1$

Soln.

$$x^2 \frac{dy}{dx} = y(1-x)$$

$$\frac{dy}{y} = \frac{1-x}{x^2}\, dx$$

$$\int \frac{dy}{y} = \int x^{-2}\, dx - \int \frac{1}{x}\, dx$$

$$\ln|y| = -\frac{1}{x} - \ln|x| + C$$

$$\ln|xy| = -\frac{1}{x} + C \longrightarrow (1)$$

$y(-1) = -1$

$$\Rightarrow \ln|(-1)(-1)| = \frac{-1}{-1} + C$$

$$\ln 1 = 1 + C$$

$$C = -1$$

$$xy = e^{-\frac{1}{x} + C}$$

or

$y(-1) = -1$

$$\Rightarrow 1 = e^{1+C}$$

$$1 + C = \ln 1$$

$$\boxed{C = \ln 1 - 1}$$

$$xy = e^{-\frac{1}{x} + \ln 1 - 1}$$

$$xy = e^{-\frac{1}{x} - 1} = e^{-(1+1/x)}$$

$$\boxed{xy = e^{-(1+1/x)}}$$

257. $\frac{dy}{dx} = f(ax+by+c)$, $b \neq 0$

Use $u = ax + by + c$

$u \left< \begin{matrix} x \\ y \end{matrix} \right.$?

$\frac{du}{dx}$ $\quad y = u - x - 1$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

(57) $\frac{dy}{dx} = (x+y+1)^2$

Soln. Let $\boxed{u = x+y+1}$

$\Rightarrow$ ~~$du = dx + dy$~~

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{du}{dx} - 1}$

$\Rightarrow \frac{du}{dx} - 1 = u^2$?

$\frac{du}{dx} = 1 + u^2$

$\frac{du}{1+u^2} = dx$

$\tan^{-1} u = x + C$

$\boxed{\tan^{-1}(x+y+1) = x + C}$

$x+y+1 = \tan(x+C)$

Ans $\boxed{y = -x - 1 + \tan(x+C)}$

$\boxed{du = dx + dy}$ ✓

$dy = du - dx$

$du - dx = u^2 dx$

$du = (1+u^2)dx$

$\frac{du}{1+u^2} = dx$

$\tan^{-1}$

(61) $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$ $\longrightarrow$ (*)

$dy = 2dx + \sqrt{(y - 2x + 3)}\, dx$

Let $u = y - 2x + 3 \longrightarrow (1)$

$du = dy - 2dx \longrightarrow (2)$

$\Rightarrow dy = du + 2dx$

$dy = 2dx + \sqrt{u}\, dx$

$\cancel{2dx} + du = \cancel{2dx} + \sqrt{u}\, dx$

$\Rightarrow u^{-\frac{1}{2}} du = dx$

$2\sqrt{u} = x + C$

$\boxed{2\sqrt{y - 2x + 3} = x + C} \Rightarrow \boxed{4(y - 2x + 3) = (x+C)^2}$

Q57. $\frac{dy}{dx} = f(ax+by+c)$, $b \neq 0$

Use $u = ax+by+c$

$u \langle^{x}_{y}$?

(57) $\frac{dy}{dx} = (x+y+1)^2$

Soln. Let $\boxed{u = x+y+1}$

$\Rightarrow \cancel{du = dx + dy}$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{du}{dx} - 1}$

$\Rightarrow \frac{du}{dx} - 1 = u^2$

$\frac{du}{dx}$

$y = u - x - 1$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

?

$\frac{du}{dx} = 1 + u^2$

$\frac{du}{1+u^2} = dx$

$\tan^{-1} u = x + C$

$\boxed{\tan^{-1}(x+y+1) = x + C}$

$x+y+1 = \tan(x+C)$

Ans $\boxed{y = -x - 1 + \tan(x+c)}$

$\boxed{du = dx + dy}$ ✓

$dy = du - dx$

$du - dx = u^2 dx$

$du = (1+u^2)dx$

$\frac{du}{1+u^2} = dx$

$\tan^{-1}$

(61) $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$ → (*)

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Let $u = y - 2x + 3$ → (1)

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$dy = 2dx + \sqrt{u}\, dx$

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$\boxed{2\sqrt{y-2x+3} = x + C}$ $\Rightarrow$ $\boxed{4(y-2x+3) = (x+C)^2}$