§2.3 Homogeneous Equations:

Def: A ftn. $f(x,y)$ is said to be a homogeneous ftn. if it can be written as

$$f(tx, ty) = t^n f(x,y).$$

for some real No. n. In this case we say f is Homogeneous ftn. of degree n.

Exples

(1) $f(x,y) = x^2 - 3xy + 5y^2$

Homo : Degree 2

(2) $f(x,y) = \sqrt[3]{x^2+y^2}$

Homo : Degree $\frac{2}{3}$

(3) $f(x,y) = x^3 + y^3 + 1$

Non-Homo :

(4) $f(x,y) = \frac{x}{2y} + 4$

Homo : Degree 0

Defn. A Diff. Eqn. of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is Homogeneous if both coefficients M & N are Homogeneous Functions of the same Degree.

Method of Soln: Let $y = ux$ or $x = vy$

→ Convert to separable.
→ Solve
→ Backward Subst.

* Use $x = vy$ if M is simpler than N

2.3 Exercises

(29) $(x^2+xy-y^2)dx + xy\,dy = 0$

Soln. Here Coefficient of dy is simpler than that of dx, so $\therefore$ we let

$$\boxed{y = ux} \longrightarrow (1)$$

$$\Rightarrow \boxed{dy = u\,dx + x\,du} \longrightarrow (2)$$

$\therefore$ Eqn. becomes

$$(x^2+ux^2-u^2x^2)dx + ux^2(u\,dx + x\,du) = 0$$

$$(1+u-u^2)x^2\,dx + u^2x^2\,dx + ux^3\,du = 0$$

$$(1+u-\cancel{u^2}+\cancel{u^2})x^2\,dx + u\,x^3\,du = 0$$

$$(1+u)\,x^2\,dx + u\,x^3\,du = 0$$

$$\frac{x^2}{x^3}\,dx + \frac{u}{1+u}\,du = 0$$

$$\frac{1}{x}\,dx + \frac{u}{1+u}\,du = 0$$

$$1+u \overline{)\,u\,}\; 1,\quad u+1,\quad -1,\quad 1-\frac{1}{1+u}$$

$$\ln|x| + \int\left(1-\frac{1}{1+u}\right)du = 0$$

$$\ln|x| + u - \ln|1+u| = C \Rightarrow \ln\frac{x}{1+u} = C-u \Rightarrow \boxed{\ln\left|\frac{x}{1+y/x}\right| = C-\frac{y}{x}}$$

43 $\left(x+\sqrt{y^2-xy}\right)\frac{dy}{dx}=y$, $y(1/2)=1$ (3)

Soln. Eqn. implies

$$\left(x+\sqrt{y^2-xy}\right)dy = y\,dx$$

Here M & N are both Homo. fn. of Degree 1

$M=y$ is simpler, so $\therefore$

Let $x = vy \longrightarrow (1)$

$$\Rightarrow dx = v\,dy + y\,dv \rightarrow (2)$$

$$\left(vy+\sqrt{y^2-vy^2}\right)dy = y\,(v\,dy + y\,dv)$$

$$\cancel{vy\,dy} + y\sqrt{1-v}\,dy = \cancel{vy\,dy} + y^2\,dv$$

$$\frac{y}{y^2}\,dy = \frac{1}{\sqrt{1-v}}\,dv$$

$$\frac{1}{y}\,dy = -(1-v)^{-1/2}(-1)\,dv$$

$$\ln|y| = -\frac{\sqrt{1-v}}{1/2} + \ln c$$

$$\ln|y| = -2\sqrt{1-v} + \ln c$$

$$\ln\frac{y}{c} = -2\sqrt{1-v}$$

$$\frac{y}{c} = e^{-2\sqrt{1-v}}$$

④

$$Y = c e^{-2\sqrt{1-V}}$$

$$y = c e^{-2\sqrt{1-\frac{x}{y}}}$$

$$\boxed{y = c e^{-2\sqrt{\frac{1-x}{y}}}} \rightarrow (B)$$

Now

$$y(1/2) = 1$$

$$\Rightarrow \quad 1 = c e^{-2\sqrt{\frac{1-1/2}{1}}}$$

$$1 = c e^{-2\sqrt{\frac{1}{2}}}$$

$$c = \frac{1}{e^{-\frac{2}{\sqrt{2}}}} = e^{\frac{1}{\sqrt{2}}}$$

$\therefore$

$$\boxed{Y = e^{\frac{1}{\sqrt{2}}} \, e^{-2\sqrt{\frac{1-x}{y}}}}$$

Backward Subst. than N