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RESEARCH ARTICLE

# Linguistic Z-number fuzzy soft sets and its application on multiple attribute group decision making problems

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**Abstract**

In this study, the concept of linguistic Z-number fuzzy soft set (*LZnFSS*) is proposed to describe multiple uncertainties in practical decision making problems. *LZnFSS* combines the concepts of fuzzy soft set, linguistic Z-number, and soft set, which could reflect both of the uncertainty in structure and the uncertainty in detailed evaluations. As an initial idea, the set operations on *LZnFSSs* are put forward, the properties of such operations are also discussed. With traditional soft set based decision procedure and fuzzy soft set based decision procedure, a novel linguistic Z-number fuzzy soft set based group decision procedure is developed to solve multiattribute group decision making with linguistic Z-numbers. Wherein an extended technique for order preference by similarity to ideal solution is also developed. Finally, a numerical example is shown to illustrate the practicality and effectiveness of the given method.

KEYWORDS

linguistic Z-number, linguistic Z-number fuzzy soft set, multiattribute group decision making, soft set, TOPSIS

## 1 | INTRODUCTION

Uncertainty is existed in many real applications. To reflect and model the uncertainties, several mathematical models have been developed. As a traditional tool, probability theory was introduced to study the quantitative law of random phenomenon.[1] In probability theory, a random event has two situations when making an observation, that is, may or may not occur. Besides, such a random event could be repeatedly observed. But there existed many uncertainties that are not randomized. For instance, the definition of youth for human beings, is there a significant distinction between men/women in 20 and 21? To describe such uncertainties, Zadeh[2] introduced the concept of fuzzy set theory. By using the membership function, the fuzzy attribute of uncertainty could be considered. In 1999, Molodtsov[3] put forward the notion of soft set, which provided a nonparametric way for handling vagueness and uncertainties. Maji et al.[4] put forward the concept of the fuzzy soft set to enrich the expression of classical soft set. Since then, types of fuzzy soft sets include intuitionistic fuzzy soft sets,[5,6] interval-valued fuzzy soft sets,[7] interval-valued intuitionistic fuzzy soft set,[8] uncertain linguistic fuzzy soft sets[9] and 2-Tuple linguistic soft set.[10] The applications of diverse soft sets have also been extended to decision making[9-12] and supplier selection.[13]

Zadeh[14] introduced the concept of Z-number, which used both of the linguistic expression (measured by probability function) and possibility function to describe practical uncertainties like that (*Walking from teaching building to dinning room, about 5 miniutes, very likely*). The advantages of Z-number could be summarized in two aspects: For one aspect, by transforming linguistic variables to other formats of fuzzy information,[15-18] Z-number provides an extension of traditional fuzzy sets using the possibility function. For another aspect, Z-number shows a tool to deal with imprecise and/or incomplete information in real applications. Up until now, Z-number has been applied to areas including earned value management,[19] evolutionary games,[20] renewable energy selection[21] and decision making[1,22-25] and approximate reasoning.[26] As a special case, the concept of linguistic Z-number[24] could provide a structured evaluation on practical objects, thus the form of Z-numbers is set in linguistic Z-number in this paper.

Noting that the soft set provides a technological tool to extend the concept of Zadeh's fuzzy set,[2] while Z-number is a conceptual extension of a fuzzy set. Motivated by the ideas of fuzzy soft sets,[4,5,7,9-11] a direct extension would be combining soft set and Z-number, that is, the linguistic Z-number fuzzy soft set. By considering such novel fuzzy representation model, the following theoretical contributions are planned to be made:

1) A novel type of fuzzy soft set named as linguistic Z-number fuzzy soft set will be obtained, in which the uncertainties of events from both the macro-angle and the micro-angle could be presented. For given objects with certain attributes, the form of the soft set can be used to describe the uncertainty that not all the objects satisfy all attributes, while the application of Z-number can show detail cognitive information of one object satisfying the attribute.
2) The applications of both fuzzy soft sets and Z-number are enlarged. As mentioned above, both of fuzzy soft sets and Z-number have been widely applied to realistic areas, but the combination of Z-number and fuzzy soft set shows the theoretical applications of the two models.
3) Novel solutions of multiattribute group decision making (MAGDM) with linguistic Z-numbers will be developed. In this paper, we will apply the novel representation model to MAGDM problems with linguistic Z-numbers, that is, novel angle to solve such practical decision making problems would be given.

To realize the contributions mentioned above, the rest of this paper is structured as follows: In Section 2, we mainly review some backgrounds about fuzzy soft sets and Z-number for further consideration. Main results on the combination of fuzzy soft sets and linguistic Z-number are shown in Section 3, operations on the linguistic Z-number fuzzy soft sets are also presented in this section. While in Section 4, the application of linguistic Z-number fuzzy soft sets to MAGDM with linguistic Z-numbers is put forward. Section 5 shows a numerical study to illustrate the developed decision model. In Section 6, some conclusions and further possible works are summarized.

# 2 | PRELIMINARIES

## 2.1 | Fuzzy soft set

Given that $U$ and $E$ are initial universe set and parameter set, respectively. Molodtsov[3] introduced the following concept of soft set:

**Definition 2.1.** A pair $(F, E)$ is said to be a soft set over $U$ if and only if $F$ is a mapping from $E$ to the set of all subsets of $U$.

By Definition 2.1, it can be seen that a soft set is a parameterized family of subsets of $U$. For any $\epsilon \in E$, the mapping $F(\epsilon)$ is called the set of $\epsilon-$approximate elements ($\epsilon-$elements for short) of $(F, E)$.

By combining the concept of fuzzy set,[2] the following fuzzy soft set[4] is given.

**Definition 2.2.** Assume that $\tilde{P}(U)$ is the fuzzy power set of $U$, a pair is called a fuzzy soft set over $U$, in which the mapping $\tilde{F}$ is given according to $\tilde{F}: A \to \tilde{P}(U)$.

As can be seen in Definition 2.2, fuzzy soft set is a generalization of soft set because the approximate functions are fuzzy subsets of $U$. The following example shows an application of fuzzy soft set.

**Example 2.1.** Credit assessment is a certain assurance for the security of capital or investment. To make credit assessment for an enterprise, several attributes including character ($e_1$), capacity ($e_2$), capital ($e_3$), collateral ($e_4$) and condition ($e_5$) are considered. Suppose that $U = \{u_1, u_2, ..., u_6\}$ is an universal set which is composed of respondents. If $A = \{e_1, e_4, e_5\} \subseteq E$ is a subset that evaluated by the first working group, their assessments (fuzzy approximations) can be shown as follows:

$$\tilde{F}(e_1) = \left\{\frac{0.80}{u_1}, \frac{0.40}{u_2}, \frac{0.90}{u_3}, \frac{0.60}{u_4}, \frac{0.70}{u_5}, \frac{0.80}{u_6}\right\},$$
$$\tilde{F}(e_4) = \left\{\frac{0.40}{u_1}, \frac{0.60}{u_2}, \frac{0.50}{u_3}, \frac{0.90}{u_4}, \frac{0.30}{u_5}, \frac{1.00}{u_6}\right\},$$
$$\tilde{F}(e_5) = \left\{\frac{0.70}{u_1}, \frac{0.50}{u_2}, \frac{0.60}{u_3}, \frac{0.80}{u_4}, \frac{0.20}{u_5}, \frac{0.90}{u_6}\right\}.$$

The tabular form of the given fuzzy soft set is shown in Table 1.

As mentioned above, different from the traditional soft set, the fuzzy soft set can be seen as a fuzzy information system, which is presented by a data table with entries in the unit interval [0, 1].

**TABLE 1** Tabular representation of the fuzzy soft set

| Enterprise | $e_1$ | $e_4$ | $e_5$ |
|---|---|---|---|
| $u_1$ | 0.8 | 0.4 | 0.7 |
| $u_2$ | 0.4 | 0.6 | 0.5 |
| $u_3$ | 0.9 | 0.5 | 0.6 |
| $u_4$ | 0.6 | 0.9 | 0.8 |
| $u_5$ | 0.7 | 0.3 | 0.2 |
| $u_6$ | 0.8 | 1.0 | 0.9 |

### 2.2 | Linguistic Z-number

**Definition 2.3** (Zadeh[14]). A Z-number is an ordered pair of fuzzy numbers denoted as $Z = (\tilde{A}, \tilde{B})$. It is associated with a real-valued uncertain variable $X$. The first component $\tilde{A}$ is a fuzzy restriction on the values that $X$ can take, and the second component $\tilde{B}$ is a measure of reliability of $\tilde{A}$.

Typically, $\tilde{A}$ and $\tilde{B}$ are described in a natural language. For example, (about 45 million, quite sure).

The ordered triple $(X, \tilde{A}, \tilde{B})$ is referred to as a Z-valuation, which an equivalent to an assignment statement, $X$ is $(\tilde{A}, \tilde{B})$.

Given a linguistic term set $S = \{s_\alpha | \alpha = 0, 1, ..., g\}$ (g is an even number, where $s_\alpha$ represents a possible value of a linguistic variable[27-30]), Wang et al[24] introduced the concept of linguistic Z-number, that is, $A$ and $B$ are given by two linguistic term sets $S = \{s_0, s_1, ..., s_g\}$ and $S' = \{s'_0, s'_1, ..., s'_{g'}\}$ ($g'$ is an even numbers).

## 3 | LINGUISTIC Z-NUMBER FUZZY SOFT SET

### 3.1 | Set operations on linguistic Z-numbers

For further consideration, the following set operations on Z-numbers are forward.

Let $Z_i = \{(A_1^i, B_1^i), (A_2^i, B_2^i), ..., (A_n^i, B_n^i)\} (i = 1, 2)$ be two sets of Z-numbers, the set operations on Z-numbers can be summarized as below:

1) Logic negation operation[14]: If $Z = (A, B)$, then the complement of $Z$, denoted as $Z'$, can be shown as $Z' = (A', 1 - B)$, where $A'$ is the complement of $A$ and $1 - B$ is the logic negation of B. For instance, if $Z = (A, \textit{likely})$, then $Z' = (\textit{not} \quad A, \textit{unlikely})$.
2) Inclusion relation: By the inclusion of fuzzy sets, the inclusion relation between $Z_1$ and $Z_2$ is defined as

$$Z_1 \subseteq Z_2 \quad \text{if} \quad \left(A_j^1, B_j^1\right) \leq \left(A_j^2, B_j^2\right),$$

where the comparison could be given according to $(I(B_j^1)/g')A_j^1 \leq (I(B_j^2)/g')A_j^2$, that is, $I(B_j^1) \cdot I(A_j^1) \leq I(B_j^2) \cdot I(A_j^2)$ and $I(*)$ is the subscript index of $*$.

3) Union operation: By extending the union operation of fuzzy sets, the union operation between $Z_1$ and $Z_2$ can be defined according to

$$Z_1 \cup Z_2 = \left\{ \max\left(A_j^1, A_j^2\right), \max\left(B_j^1, B_j^2\right) \middle| j = 1, 2, \dots, n \right\}.$$

4) Intersection operation: Similarly, the intersection operation between $Z_1$ and $Z_2$ can be defined by

$$Z_1 \cap Z_2 = \left\{ \max\left(A_j^1, A_j^2\right), \max\left(B_j^1, B_j^2\right) \middle| j = 1, 2, \dots, n \right\}.$$

Noting that the concept of linguistic Z-number is an extension of traditional fuzzy set, when two linguistic Z-numbers with the same level of fuzzy restriction (the part of $A$ in $Z$), the higher level of the reliability (the part of $B$ in $Z$), the larger the linguistic Z-number would be. Thus, the set operational laws of linguistic Z-numbers mentioned above are generalized rules of fuzzy sets.

By the given set operations of linguistic Z-numbers, we have the following theorem:

**Theorem 3.1.** *Given that $Z_i = \{(A_1^i, B_1^i), (A_2^i, B_2^i), \dots, (A_n^i, B_n^i)\} (i = 1, 2)$ are two sets of linguistic Z-numbers, then the following De Morgan's laws are valid*

1) $(Z_1 \cup Z_2)^C = Z_1^C \cap Z_2^C$;
2) $(Z_1 \cap Z_2)^C = Z_1^C \cup Z_2^C$.

*Proof.* 1) According to the set operations, it can be obtained that

$$\begin{aligned}(Z_1 \cup Z_2)^C &= \left\{ \left( \max\left(A_j^1, A_j^2\right), \max\left(B_j^1, B_j^2\right) \right) \middle| j = 1, 2, \dots, n \right\}^C \\ &= \left\{ \left( \left( \max\left(A_j^1, A_j^2\right) \right)', 1 - \max\left(B_j^1, B_j^2\right) \right) \middle| j = 1, 2, \dots, n \right\} \\ &= \left\{ \left( \left( \max\left(A_j^1, A_j^2\right) \right)', \min\left(1 - B_j^1, 1 - B_j^2\right) \right) \middle| j = 1, 2, \dots, n \right\}.\end{aligned}$$

While

$$\begin{aligned}Z_1^C \cap Z_2^C &= \left( \left(A_j^1\right)', 1 - B_j^1 \right) \cap \left( \left(A_j^2\right)', 1 - B_j^2 \right) \Big|\ j = 1, 2, \dots, n \\ &= \left\{ \left( \min\left( \left(A_j^1\right)', \left(A_j^2\right)' \right), \min\left(1 - B_j^1, 1 - B_j^2\right) \right) \middle|\ j = 1, 2, \dots, n \right\}.\end{aligned}$$

Next, we just prove that $(\max(A_j^1, A_j^2)', \min(A_j^1, A_j^2)'), j = 1, 2, \dots, n$. The following two cases are direct:

Because $(A_j^1)'$ and $(A_j^2)'$ are complement of $A_j^1$ and $A_j^2 (j = 1, 2, ..., n)$, then for any fixed $j \in \{1, 2, ..., n\}$, the larger the initial linguistic assessment is, the smaller the complement will be, that is, the conclusion that $(\max(A_j^1, A_j^2))' = \min((A_j^1)', (A_j^2)')$ is valid.

Therefore, it can be obtained that $(Z_1 \cup Z_2)^C = Z_1^C \cap Z_2^C$.

Similarly, the second conclusion is also valid. □

## 3.2 | Linguistic Z-number fuzzy soft set

By combing the concepts of linguistic Z-number and fuzzy soft set, the concept of a linguistic Z-number fuzzy soft set will be introduced.

**Definition 3.1.** Given that $U$ is a universe set, $E$ a set of parameters and $\bar{A} \subseteq E$. Define $F$: $\bar{A} \to LZnF^U$, where $LZnF^U$ is the collection of linguistic Z-numbers of $U$. Then $(F, \bar{A})$ is called a linguistic Z-number fuzzy soft set ($LZnFSS$) over $U$, which can be detailed as below:

$$(F, \bar{A}) = \{(x, (x \quad satisfies \quad parameter \quad \epsilon, A_\epsilon(x), B_\epsilon(x))): \forall\, x \in U \quad and \quad \epsilon \in \bar{A}\}, \tag{1}$$

where $A_\epsilon(x)$ is fuzzy restriction on the values that $U$ can take and $B_\epsilon(x)$ is the reliability of $A_\epsilon(x)$.

*Remark* 3.1. The form of $LZnFSS$ can also be described according to

$$(F, \bar{A}) = \{(x, (A_\epsilon(x), B_\epsilon(x))) : \forall\, x \in U \quad \text{and} \quad \epsilon \in \bar{A}\}, \tag{2}$$

or

$$(F, \bar{A}) = \left\{\frac{(A_\epsilon(x), B_\epsilon(x))}{x} : \forall\, x \in U \quad \text{and} \quad \epsilon \in \bar{A}\right\}. \tag{3}$$

The following example will show the application of $LZnFSS$ in practical issue.

**Example 3.1.** Let $U = \{Ent_1, Ent_2, ..., Ent_5\}$ be the set of enterprises for further risk analysis, and
$E = \{\epsilon_1 = management \quad risk, \epsilon_2 = technical \quad risk, \epsilon_3 = financial \quad risk, \epsilon_4$ be the set of
$= legal \quad risk, \epsilon_5 = policy \quad risk, \epsilon_6 = counter - guarantee \quad measures\}$
parameters and $\bar{A} = \{\epsilon_1, \epsilon_3, \epsilon_5, \epsilon_6\} \subseteq E$. Then

$$(F,\bar{A}) = \left\{\begin{array}{l} F(\epsilon_1) = \left\{\begin{array}{l}(Ent_1,(High,likely)),(Ent_2,(Medium,very\quad likely)),\\ (Ent_3,(High,likely)),(Ent_4,(Very\quad High,very\quad likely)),\\ (Ent_5,(Medium,medium))\end{array}\right\}, \\ F(\epsilon_3) = \left\{\begin{array}{l}(Ent_1,(Very\quad Low,very\quad likely)),(Ent_2,(Low,likely)),\\ (Ent_3,(Medium,likely)),(Ent_4,(Low,very\quad likely)),\\ (Ent_5,(High,medium))\end{array}\right\}, \\ F(\epsilon_5) = \left\{\begin{array}{l}(Ent_1,(Low,likely)),(Ent_2,(Very\quad Low,likely)),\\ (Ent_3,(Medium,very\quad likely)),(Ent_4,(High,likely)),\\ (Ent_5,(High,very\quad likely))\end{array}\right\}, \\ F(\epsilon_6) = \left\{\begin{array}{l}(Ent_1,(Medium,likely)),(Ent_2,(Low,very\quad likely)),\\ (Ent_3,(Very\quad Low,likely)),(Ent_4,(High,likely)),\\ (Ent_5,(High,very\quad likely))\end{array}\right\} \end{array}\right\}$$

By considering special cases of *LZnFSS*, the following definitions can be developed:

**Definition 3.2.** A *LZnFSS* is called a null linguistic Z-number fuzzy soft set if for all $\epsilon \in \bar{A}, F(\bar{A}) = \emptyset$.

**Definition 3.3.** A *LZnFSS* is said to be an absolute linguistic Z-number fuzzy soft set if for all $\epsilon \in \bar{A}, F(\bar{A}) = ZnF^U$.

## 3.3 | Linguistic Z-number fuzzy soft relations

As a novel type of information representation, relations on linguistic Z-number fuzzy soft sets (*LZnFSSs*) are fundamental knowledge that need to be considered.[31,32]

With the developed set operations on linguistic Z-numbers, the following linguistic Z-number fuzzy soft relations can be defined:

**Definition 3.4.** Assume that $\bar{A}, \bar{B} \subseteq E$ and $(F,\bar{A}), (G,\bar{B})$ are two *LZnFSSs* over $U$, then $(F,\bar{A})$ is said to be a linguistic Z-number fuzzy soft subset of $(G,\bar{B})$ if and only if

1) $\bar{A} \subseteq \bar{B}$;
2) For $\forall\, \epsilon \in \bar{A}, F(\epsilon)$ is a linguistic Z-number fuzzy subset of $G(\epsilon)$, that is, for $\forall\, x \in U, Zn_{F(\epsilon)}(x) \subseteq Zn_{G(\epsilon)}(x)$, which can be equivalently written by $(F,\bar{A}) \subset (G,\bar{B})$.

In other words, $(G,\bar{B})$ is called a linguistic Z-number fuzzy soft superset of $(F,\bar{A})$, denoted as $(G,\bar{B}) \supset (F,\bar{A})$.

According to Definition 3.4, the following definition is direct.

**Definition 3.5.** Given that $\bar{A}, \bar{B} \subseteq E$ and $(F,\bar{A}), (G,\bar{B})$ are two *LZnFSSs* over $U$, then $(F,\bar{A})$ and $(G,\bar{B})$ are said to be linguistic Z-number fuzzy soft equal if and only if (a)

$(F, \bar{A})$ is linguistic Z-number fuzzy soft subset of $(G, \bar{B})$, and (b) $(G, \bar{B})$ is linguistic Z-number fuzzy soft subset of $(F, \bar{A})$, which can be denoted as $(F, \bar{A}) = (G, \bar{B})$.

By the set operations of linguistic Z-numbers and the definitions mentioned above, the following conclusions can be derived:

**Proposition 3.1.** *The linguistic Z-number fuzzy soft inclusion relation* $\subset$ *(or* $\supset$*) is a binary relation between two LZnFSSs, which has the properties of Reflexive, Antisymmetric, and Transitive.*

**Proposition 3.2.** *The linguistic Z-number fuzzy soft equal relation is a binary relation between two LZnFSSs, which has the properties of Reflexive, Symmetric, and Transitive.*

With the complement of linguistic Z-number, the complement of linguistic Z-number fuzzy soft set can be defined as below:

**Definition 3.6.** Suppose that $\bar{A} \subseteq E, (F, \bar{A})$ is a *LZnFSS*, the complement of $(F, \bar{A})$ is defined according to $F^C : A \to ZnF^U$, which is denoted by $(F^C, \bar{A})$. Herein, $F^C(\epsilon) = (F(\epsilon))^C, \forall\ \epsilon \in \bar{A}$.

Similarly, we have the following proposition.

**Proposition 3.3.** *The linguistic Z-number fuzzy soft complement relation is a binary relation between two LZnFSSs, which has the properties of Irreflexive and Symmetric.*

To illustrate the definitions mentioned above, the following example is given.

**Example 3.2.** Following Example 3.1, let $\bar{B} = \{\epsilon_1, \epsilon_5\}, \bar{B} \subseteq \bar{A}$, and

$$(G, \bar{B}) = \left\{ \begin{array}{l} G(\epsilon_1) = \left\{ \begin{array}{l} (Ent_1, (Medium, likely)), (Ent_2, (Low, very\quad likely)), \\ (Ent_3, (High, likely)), (Ent_4, (High, likely)), \\ (Ent_5, (Low, medium)) \end{array} \right\}, \\ G(\epsilon_5) = \left\{ \begin{array}{l} (Ent_1, (Very\quad Low, likely)), (Ent_2, (Very\quad Low, likely)), \\ (Ent_3, (Medium, likely)), (Ent_4, (Medium, likely)), \\ (Ent_5, (High, likely)) \end{array} \right\} \end{array} \right\}$$

According to $(F, \bar{A})$ in Example 3.1 and $(G, \bar{B})$ in Example 3.2, it can be easily obtained that $(G, \bar{B}) \subseteq (F, \bar{A})$.

## 3.4 | Set operations on *LZnFSSs*

As another basic operations on soft sets,[9-11,33,34] "AND" and "OR" operations need to be considered. Correspondingly, we give the "AND" and "OR" operations on *LZnFSSs*.

**Definition 3.7.** Suppose that $(F, \bar{A})$ and $(G, \bar{B})$ are two *LZnFSSs* over $U$, then the "AND" operation defined on $(F, \bar{A})$ and $(G, \bar{B})$ can be given according to

$$(F, \bar{A}) \wedge (G, \bar{B}) = (\bar{H}, \bar{A} \times \bar{B}), \tag{4}$$

where $\bar{H}(\epsilon, \delta) = F(\epsilon) \cap G(\delta), \forall (\epsilon, \delta) \in \bar{A} \times \bar{B}$.

**Definition 3.8.** Assume that $(F, \bar{A})$ and $(G, \bar{B})$ are two *LZnFSSs* over $U$, then the "OR" operation defined on $(F, \bar{A})$ and $(G, \bar{B})$ can be given by

$$(F, \bar{A}) \vee (G, \bar{B}) = (\tilde{H}, \bar{A} \times \bar{B}), \tag{5}$$

where $\tilde{H}(\epsilon, \delta) = F(\epsilon) \cup G(\delta), \forall (\epsilon, \delta) \in \bar{A} \times \bar{B}$.

By Definitions 3.7 and 3.8, the "AND" and "OR" operations show the way obtaining the approximate elements when combining the parameters in $A$ and $B$.

**Example 3.3.** Following Examples 3.1 and 3.2, we have

$$\begin{aligned}\bar{H}(F(\epsilon_1), G(\epsilon_1)) &= F(\epsilon_1) \cap G(\epsilon_1)\\ &= \left\{\begin{matrix}\dfrac{(Medium, likely)}{Ent_1}, \dfrac{(Low, very \quad likely)}{Ent_2}, \dfrac{(High, likely)}{Ent_3},\\ \dfrac{(High, likely)}{Ent_4}, \dfrac{(Low, medium)}{Ent_5}\end{matrix}\right\},\end{aligned}$$

and

$$\begin{aligned}\tilde{H}(F(\epsilon_1), G(\epsilon_1)) &= F(\epsilon_1) \cup G(\epsilon_1)\\ &= \left\{\begin{matrix}\dfrac{(High, likely)}{Ent_1}, \dfrac{(Medium, very \quad likely)}{Ent_2}, \dfrac{(High, likely)}{Ent_3},\\ \dfrac{(Very \quad High, very \quad likely)}{Ent_4}, \dfrac{(Medium, medium)}{Ent_5}\end{matrix}\right\}.\end{aligned}$$

By using the comparison laws of linguistic Z-numbers, the results of "AND" and "OR" operations corresponding to $(F, \bar{A})$ in Example 3.1 and $(G, \bar{B})$ in Example 3.2 can be obtained and listed in Table 2.

By Table 2, in the results produced by "AND" or "OR" operation, the elements in the universe set would be considered by complex factors. Thus, more detailed information could be referenced when making an analysis.

With the set operations of linguistic Z-numbers and the two set operations mentioned above, the following conclusions can be derived:

**Theorem 3.2.** *Let $(F, \bar{A})$ and $(G, \bar{B})$ be two LZnFSSs, then*

**TABLE 2** Results produced by "AND" and "OR" operations of $(F, \bar{A})$ and $(G, \bar{B})$

| | $U$ | $(\epsilon_1, \epsilon_1)$ | $(\epsilon_1, \epsilon_5)$ | $(\epsilon_3, \epsilon_1)$ | $(\epsilon_3, \epsilon_5)$ | $(\epsilon_5, \epsilon_1)$ | $(\epsilon_5, \epsilon_5)$ | $(\epsilon_6, \epsilon_1)$ | $(\epsilon_6, \epsilon_5)$ |
|---|---|---|---|---|---|---|---|---|---|
| AND | $Ent_1$ | (M, L) | (VL, L) | (VL, L) | (VL, L) | (L, L) | (VL, L) | (VL, L) | (VL, L) |
| | $Ent_2$ | (L, VL) | (VL, L) | (L, L) | (VL, L) | (VL, L) | (VL, L) | (L, VL) | (VL, L) |
| | $Ent_3$ | (H, L) | (M, L) | (M, L) | (M, L) | (M, L) | (M, L) | (VL, L) | (VL, L) |
| | $Ent_4$ | (H, L) | (M, L) | (L, L) | (L, L) | (H, L) | (M, L) | (H, L) | (M, L) |
| | $Ent_5$ | (L, M) | (M, M) | (L, M) | (H, M) | (L, M) | (H, L) | (L, M) | (H, L) |
| OR | $Ent_1$ | (H, L) | (H, L) | (M, VL) | (L, L) | (M, L) | (L, L) | (M, L) | (VL, L) |
| | $Ent_2$ | (M, VL) | (M, VL) | (L, VL) | (VL, L) | (L, VL) | (VL, L) | (L, VL) | (L, VL) |
| | $Ent_3$ | (H, L) | (H, L) | (H, L) | (M, VL) | (H, VL) | (M, VL) | (H, L) | (M, L) |
| | $Ent_4$ | (VH, VL) | (VH, VL) | (H, VL) | (H, L) | (H, L) | (H, L) | (H, L) | (H, L) |
| | $Ent_5$ | (M, M) | (H, L) | (H, M) | (H, VL) | (H, VL) | (H, VL) | (H, VL) | (H, VL) |

1) $((F, \bar{A}) \wedge (G, \bar{B}))^C = (F, \bar{A})^C \vee (G, \bar{B})^C$;
2) $((F, \bar{A}) \vee (G, \bar{B}))^C = (F, \bar{A})^C \wedge (G, \bar{B})^C$.

*Proof.* (1) According to Definition 3.1, for $\forall\, \epsilon \in \bar{A}$ and $\delta \in \bar{B}$, then $F(\epsilon) = \{(A_\epsilon(x), B_\epsilon(x)) | x \in U\}$ and $G(\delta) = \{(A_\delta(x), B_\delta(x)) | x \in U\}$, then for any $(\epsilon, \delta) \in \bar{A} \times \bar{B}$, we have

$$\begin{aligned}
((F, \bar{A}) \wedge (G, \bar{B}))^C &= (F(\epsilon) \cap G(\delta))^C \\
&= \{(\min(A_\epsilon(x), A_\delta(x)), \min(B_\epsilon(x), B_\delta(x)))^C : x \in U\} \\
&= \{((\min(A_\epsilon(x), A_\delta(x)))', 1 - \min(B_\epsilon(x), B_\delta(x))) : x \in U\} \\
&= \{((\max((A_\epsilon(x))', (A_\delta(x))')), \max(1 - B_\epsilon(x), 1 - B_\delta(x))) : x \in U\} \\
&= ((A_\epsilon(x))', 1 - B_\epsilon(x)) \cup ((A_\delta(x))', 1 - B_\delta(x)) : x \in U \\
&= (F, \bar{A})^C \vee (G, \bar{B})^C.
\end{aligned}$$

Thus, the first conclusion is valid.

Similarly, the second conclusion can also be proved. □

**Theorem 3.3.** *Assume that $(F, \bar{A}), (G, \bar{B})$ and $(H, \bar{C})$ are three LZnFSSs, then*

1) *Associative law*

$$((F, \bar{A}) \wedge (G, \bar{B})) \wedge (H, \bar{C}) = (F, \bar{A}) \wedge ((G, \bar{B}) \wedge (H, \bar{C}));$$
$$((F, \bar{A}) \vee (G, \bar{B})) \vee (H, \bar{C}) = (F, \bar{A}) \vee ((G, \bar{B}) \vee (H, \bar{C}));$$

2) *Distribution law*

$$(F, \bar{A}) \wedge ((G, \bar{B}) \vee (H, \bar{C})) = ((F, \bar{A}) \wedge (G, \bar{B})) \vee ((F, \bar{A}) \wedge (H, \bar{C}));$$
$$(F, \bar{A}) \vee ((G, \bar{B}) \wedge (H, \bar{C})) = ((F, \bar{A}) \vee (G, \bar{B})) \wedge ((F, \bar{A}) \vee (H, \bar{C})).$$

*Proof.* (1) According to Definition 3.1, for $\forall\, \epsilon \in \bar{A}, \delta \in \bar{B}$ and $\theta \in \bar{C}$, then $F(\epsilon) = \{(A_\epsilon(x), B_\epsilon(x)) | x \in U\}$, $G(\delta) = \{(A_\delta(x), B_\delta(x)) | x \in U\}$ and $H(\theta) = \{(A_\theta(x), B_\theta(x)) | x \in U\}$, then

$$
\begin{aligned}
&((F,\bar{A}) \wedge (G,\bar{B})) \wedge (H,\bar{C}) \\
&\quad = \{(\min(A_\epsilon(x), A_\delta(x)), \min(B_\epsilon(x), B_\delta(x))) \wedge (H,\bar{C}) : x \in U\} \\
&\quad = \{(\min\{\min(A_\epsilon(x), A_\delta(x)), A_\theta(x)\}, \min\{\min(B_\epsilon(x), B_\delta(x)), B_\theta(x)\}) x \in U\} \\
&\quad = \{(\min(A_\epsilon(x), A_\delta(x), A_\theta(x)), \min(B_\epsilon(x), B_\delta(x), B_\theta(x))) : x \in U\} \\
&\quad = \{(\min\{A_\epsilon(x), \min(A_\delta(x), A_\theta(x))\}, \min\{B_\epsilon(x), \min(B_\delta(x), B_\theta(x))\}) x \in U\} \\
&\quad = (F,\bar{A}) \wedge (\min(A_\delta(x), A_\theta(x)), \min(B_\delta(x), B_\theta(x))) \\
&\quad = (F,\bar{A}) \wedge ((G,\bar{B}) \wedge (H,\bar{C})).
\end{aligned}
$$

Thus, the first associative law for "AND" operation holds. Similarly, the second associative law for "OR" operation also holds.

With the notations mentioned above, then

$$
\begin{aligned}
&(F,\bar{A}) \wedge ((G,\bar{B}) \vee (H,\bar{C})) \\
&\quad = (F,\bar{A}) \wedge \{(\max(A_\delta(x), A_\theta(x)), \max(B_\delta(x), B_\theta(x))): x \in U\} \\
&\quad = \{(\min(A_\epsilon(x), \max(A_\delta(x), A_\theta(x))), \min(B_\epsilon(x), \max(B_\delta(x), B_\theta(x)))): x \in U\} \\
&\quad = \left\{ \begin{array}{l} (\max\{\min(A_\epsilon(x), A_\delta(x)), \min(A_\epsilon(x), A_\theta(x))\}, \\ \max\{\min(B_\epsilon(x), B_\delta(x)), \min(B_\epsilon(x), B_\theta(x))\}): x \in U \end{array} \right\} \\
&\quad = ((F,\bar{A}) \wedge (G,\bar{B})) \vee ((F,\bar{A}) \wedge (H,\bar{C})).
\end{aligned}
$$

Thus, the first distribution law holds. Similarly, the second distribution law also holds. □

# 4 | MAGDM BASED ON *LZnFSSs*

## 4.1 | Multiple attribute group decision making with linguistic Z-information

Given that $U = \{u_1, u_2, ..., u_m\}$ and $E = \{e_1, e_2, ..., e_n\}$ are the sets of alternatives and attributes. In MAGDM problems, there would be a set of decision makers denoted as $DM = \{D_1, D_2, ..., D_l\}$, who will provide their assessments of alternatives with respect to attributes. To reflect the uncertainties in practical issues, let the decision information be given by Z-numbers, denoted as $\left(A_{ij}^k, B_{ij}^k\right)$, that is, the coincidence level that the $i$-th alternative satisfies the $j$th attribute given by the $k$-th decision maker is $(A_{ij}^k, B_{ij}^k)(i = 1, 2, ..., m; j = 1, 2, ..., n; k = 1, 2, ..., l)$.

To measure the importance of decision makers, let $\Omega = (\omega_1, \omega_2, ..., \omega_l)$ be the set of weighting vector of decision makers, which satisfying $\omega_k \in [0, 1]$ and $\sum_{k=1}^{l} \omega_k = 1$.

According to the group decision making problem, it can be equivalently expressed as follows.

With the notations listed above, the decision information provided by the $k$-th decision maker can be transformed to be a *LZnFSS*, that is,

$$
(F_k, A_k) = \left\{ \left(u_i, \left(A_{ij}^k, B_{ij}^k\right)\right) \middle| \; i \in I, j \in J \right\}, \tag{6}
$$

where $I = \{1, 2, ..., m\}, J = \{1, 2, ..., n\}$ and $K = \{1, 2, ..., l\}$.

Thus, MAGDM with linguistic Z-numbers can be rewritten as an information fusion process for *LZnFSSs*.

## 4.2 | Extended comparison score of *LZnFSSs* and its application

For classical soft set theory based decision making, Maji et al[4] developed the following choice value based approach.

**Algorithm I** (Maji et al[4]):

Step 1 Input the soft set $(F, A)$ and represent it in tabular form;
Step 2 Calculate the choice value of $u_i$, denoted as $c_i = \sum_j h_{ij}$, where $h_{ij} = 1$ if $u_i \in F(\epsilon_j)$ and $h_{ij} = 0$ if otherwise;
Step 3 Select the best alternative $u_{best}$ with the condition that $c_{best} = \max_j c_j$. Besides, if more than one alternative own the same maximum choice value, then any of them should be chosen.

When coming to the case of fuzzy soft set $(F, A)$, Roy and Maji[33] the following comparison score method:

**Algorithm II** (Roy and Maji[33]):

Step 1 Input the fuzzy soft set $(F, A)$ and represent it in tabular form;
Step 2 Compute the comparison table of $(F, A)$ and obtain $r_i, t_i$ for each $u_i, i \in \{1, 2, \ldots, m\}$, where $r_i = \sum_{j=1}^{n} c_{ij}$, $t_j = \sum_{i=1}^{n} c_{ij}$ and $c_{ij} = |\{\epsilon \in A: F(\epsilon)(u_i) \geq F(\epsilon)(u_j)\}|$;
Step 3 Calculate the comparison score $s_i$ of $u_i$;
Step 4 Select the best alternative $u_{best}$ with the condition that $s_{best} = \max_j s_j$, where $s_j = r_i - t_i$. Besides, if more than one alternative own the same maximum comparison score, then any of them should be chosen.

Kong et al[35] pointed out that the results produced by Algorithm II would be contradictory to the fuzzy choice value method,[36] thus the calculation of $c_{ij}$ should be redesigned as $c_{ij} = \sum_{k=1}^{m}(f_{ik} - f_{jk})$, where $f_{ik}$ represents the membership of $u_i$ for the $k$th parameter $\epsilon_k$.

Next, we will put forward a comparison score method for *LZnFSSs* based MAGDM problem.

Noting that the core issue in comparison score method is the ranking of linguistic Z-numbers, thus we first introduce an order relation for linguistic Z-numbers.

By using the technique for order preference by similarity to ideal solution (TOPSIS),[37] the ranking of alternatives can be obtained. But it's a common sense that the points located on the median line between the maximum value and the minimum value can't be distinguished. Although wide extensions of TOPSIS methods have been developed,[38,39] the proposed problem has merely been studied. Thus, we introduce a modified TOPSIS method for linguistic Z-numbers:

To calculate the closeness of $Z$, let $\mathbf{0}$ (or $\mathbf{1}$) be the minimum value (or maximum value) corresponding to transformed type of fuzzy number, then the closeness of $Z$ can be given according to

$$Cl_Z = \frac{1}{2}\left[\frac{d^2(I(A), \mathbf{0})}{d^2(I(A), \mathbf{0}) + d^2(I(A), \mathbf{1})} + \frac{d^2(I(B), 0)}{d^2(I(B), 0) + d^2(I(B), g)}\right], \tag{7}$$

where $I(X)$ is the numerical index of linguistic label $X$, $d(*,*)$ is the Euclidean distance measure and $g$ is the granularity of the linguistic term set that the second component $B$ is located.

By Equation (8), it can be seen that the modified closeness is the average of traditional closenesses corresponding to two components in $Z$. According to the meaning of traditional closeness, it can be derived that the larger of the two components in $Z$, the greater the modified closeness will be. Besides, by combining the locations of multiple points, the initial problem of undistinguished points on the median line can be solved.

With the modified closeness, the following MAGDM based on *LZnFSS* is developed.

**Algorithm III**:

Step 1 Input the fuzzy soft sets $(F_k, \bar{A}_k)$, $k \in \{1, 2, ..., l\}$ and represent them in tabular form;

Step 2 Calculate modified closeness of each decision information $(A_{ij}^k, B_{ij}^k)$, $i \in I, j \in J$ by using Eq. (8) and transform $(F_k, \bar{A}_k)$ to fuzzy soft set $(\tilde{F}_k, \tilde{A}_k) = \{(u, Cl_{ij}^k) | u \in U\}$;

Step 3 Determine the ranking of alternatives for each decision maker (or *LZnFSS*), the following two ways could be used:

**By using Algorithm I**: Let $\alpha$ be the satisfactory level of decision maker, and

$$reZ_{ij}^k = \begin{cases} 1, & Cl_{ij}^k \geq \alpha; \\ 0, & Cl_{ij}^k < \alpha. \end{cases} \tag{8}$$

By Equation (10), $(\tilde{F}_k, \tilde{A}_k)$ is further transformed to be a soft set $(F^\alpha, A^*)$, which is said to be the $\alpha$-level soft set (level soft set for short[40]). Then the ranking of alternatives could be determined by using Algorithm I.

**By using Algorithm II**: Obtain the ranking of alternatives for each $(\tilde{F}_k, \tilde{A}_k)$ by using Algorithm II directly. Therefore, the ranking of alternatives corresponding to each *LZnFSS* (or decision maker) has been obtained.

Step 4 Compute the final ranking of alternative with the ranking positions obtained in Step 3 by the distance-based aggregation procedure.[41] According to Step 3, we obtain $l$ groups of the ranking positions of alternatives with weighting vector $\Omega$, thus it's actually of rank weighted aggregation problem ([41]). By introducing the weighted distance measure for ranking positions, a final ranking can be determined.

Step 5 By Step 4, select the best alternative(s).

Step 6 The End.

The idea of Algorithm III can be summarized as in Figure 1.

In Algorithm III, two submodels including soft set based decision procedure and rank aggregation are utilized, which would simplify initial computing process of linguistic Z-numbers.

# 5 | ILLUSTRATIVE EXAMPLE

## 5.1 | Illustration of the proposed method

To evaluate the project performance of four types of supply chain model in an enterprise (adopt from[22]), three decision makers provide their evaluations with respect to four attributes, the decision information is listed in Tables 3 to 5.

Step 1 The Z-number fuzzy soft representations of the decision information can be shown as below:

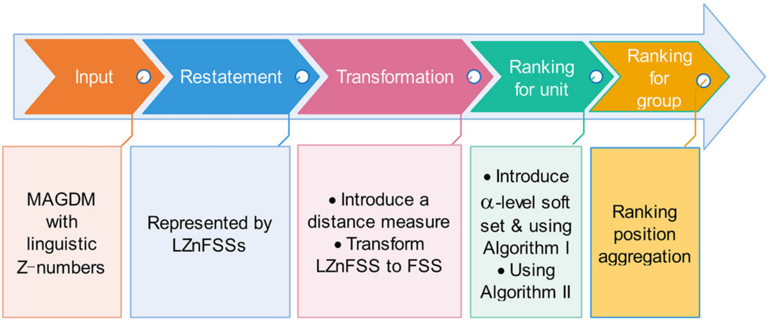


**FIGURE 1** Procedure of multiattribute group decision making (MAGDM) based on linguistic Z-number fuzzy soft set (*LZnFSS*)

**TABLE 3** Evaluations given by $d_1$

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $u_1$ | $(s_3, s_6')$ | $(s_5, s_5')$ | $(s_5, s_3')$ | $(s_6, s_6')$ |
| $u_2$ | $(s_3, s_5^s)$ | $(s_6, s_4')$ | $(s_3, s_5')$ | $(s_4, s_5')$ |
| $u_3$ | $(s_5, s_3')$ | $(s_4, s_3')$ | $(s_4, s_6')$ | $(s_4, s_5')$ |
| $u_4$ | $(s_4, s_5')$ | $(s_3, s_6')$ | $(s_6, s_3')$ | $(s_5, s_6')$ |

$$(F_1, \bar{A}_1) = \left\{ \begin{array}{l} F(e_1) = \left\{ \left(u_1, (s_3, s_6')\right), \left(u_2, (s_3, s_5')\right), \left(u_3, (s_5, s_3')\right), \left(u_4, (s_4, s_5')\right) \right\}, \\ F(e_2) = \left\{ \left(u_1, (s_5, s_5')\right), \left(u_2, (s_6, s_4')\right), \left(u_3, (s_4, s_3')\right), \left(u_4, (s_3, s_6')\right) \right\}, \\ F(e_3) = \left\{ \left(u_1, (s_5, s_3')\right), \left(u_2, (s_3, s_5')\right), \left(u_3, (s_4, s_6')\right), \left(u_4, (s_6, s_3')\right) \right\}, \\ F(e_4) = \left\{ \left(u_1, (s_6, s_6')\right), \left(u_2, (s_4, s_5')\right), \left(u_3, (s_4, s_5')\right), \left(u_4, (s_5, s_6')\right) \right\}. \end{array} \right\}$$

$$(F_2, \bar{A}_2) = \left\{ \begin{array}{l} F(e_1) = \left\{ \left(u_1, (s_3, s_6')\right), \left(u_2, (s_5, s_3')\right), \left(u_3, (s_4, s_5')\right), \left(u_4, (s_5, s_4')\right) \right\}, \\ F(e_2) = \left\{ \left(u_1, (s_5, s_6')\right), \left(u_2, (s_4, s_4')\right), \left(u_3, (s_5, s_4')\right), \left(u_4, (s_3, s_5')\right) \right\}, \\ F(e_3) = \left\{ \left(u_1, (s_4, s_3')\right), \left(u_2, (s_5, s_4')\right), \left(u_3, (s_3, s_4')\right), \left(u_4, (s_4, s_5')\right) \right\}, \\ F(e_4) = \left\{ \left(u_1, (s_5, s_4')\right), \left(u_2, (s_4, s_3')\right), \left(u_3, (s_5, s_6')\right), \left(u_4, (s_6, s_5')\right) \right\}. \end{array} \right\}$$

$$(F_3, \bar{A}_3) = \left\{ \begin{array}{l} F(e_1) = \left\{ \left(u_1, (s_4, s_4')\right), \left(u_2, (s_3, s_5')\right), \left(u_3, (s_5, s_4')\right), \left(u_4, (s_4, s_5')\right) \right\}, \\ F(e_2) = \left\{ \left(u_1, (s_3, s_4')\right), \left(u_2, (s_5, s_4')\right), \left(u_3, (s_4, s_3')\right), \left(u_4, (s_3, s_5')\right) \right\}, \\ F(e_3) = \left\{ \left(u_1, (s_3, s_6')\right), \left(u_2, (s_4, s_2')\right), \left(u_3, (s_4, s_5')\right), \left(u_4, (s_5, s_5')\right) \right\}, \\ F(e_4) = \left\{ \left(u_1, (s_4, s_3')\right), \left(u_2, (s_4, s_4')\right), \left(u_3, (s_3, s_5')\right), \left(u_4, (s_5, s_6')\right) \right\}. \end{array} \right\}$$

**TABLE 4** Evaluations given by $d_2$

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $u_1$ | $(s_3, s'_6)$ | $(s_5, s'_6)$ | $(s_4, s'_3)$ | $(s_5, s'_4)$ |
| $u_2$ | $(s_5, s'_3)$ | $(s_4, s'_4)$ | $(s_5, s'_4)$ | $(s_4, s'_3)$ |
| $u_3$ | $(s_4, s'_5)$ | $(s_5, s'_4)$ | $(s_3, s'_4)$ | $(s_5, s'_6)$ |
| $u_4$ | $(s_5, s'_4)$ | $(s_3, s'_5)$ | $(s_4, s'_5)$ | $(s_6, s'_5)$ |

**TABLE 5** Evaluations given by $d_3$

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $u_1$ | $(s_4, s'_4)$ | $(s_3, s'_4)$ | $(s_3, s'_6)$ | $(s_4, s'_3)$ |
| $u_2$ | $(s_3, s'_5)$ | $(s_5, s'_4)$ | $(s_4, s'_2)$ | $(s_4, s'_4)$ |
| $u_3$ | $(s_5, s'_4)$ | $(s_4, s'_3)$ | $(s_4, s'_5)$ | $(s_3, s'_5)$ |
| $u_4$ | $(s_4, s'_5)$ | $(s_3, s'_5)$ | $(s_5, s'_5)$ | $(s_5, s'_6)$ |

Step 2 For the qualitative linguistic Z-numbers, according to Equation (8), the closeness of linguistic Z-number in $(F_k, \bar{A}_k)$, $k = 1, 2, 3$ can be directly defined by

$$Cl_{z_{ij}^k} = \frac{1}{2}\left[\frac{I^2(A_{ij}^k)}{I^2(A_{ij}^k) + (I(A_{ij}^k) - 6)^2} + \frac{I^2(B_{ij}^k)}{I^2(B_{ij}^k) + (I(B_{ij}^k) - 6)^2}\right].$$

Then the corresponding fuzzy soft set $(\tilde{F}_k, \tilde{A}_k)$, $k = 1, 2, 3$ are

$$(\tilde{F}_1, \tilde{A}_1) = \left\{\begin{array}{l} \tilde{F}(e_1) = \{(u_1, 0.7500), (u_2, 0.7308), (u_3, 0.7308), (u_4, 0.8808)\}, \\ \tilde{F}(e_2) = \{(u_1, 0.9615), (u_2, 0.9000), (u_3, 0.6500), (u_4, 0.7500)\}, \\ \tilde{F}(e_3) = \{(u_1, 0.7308), (u_2, 0.7308), (u_3, 0.9000), (u_4, 0.7500)\}, \\ \tilde{F}(e_4) = \{(u_1, 1.0000), (u_2, 0.8808), (u_3, 0.8808), (u_4, 0.9808)\}. \end{array}\right\}$$

$$(\tilde{F}_2, \tilde{A}_2) = \left\{\begin{array}{l} \tilde{F}(e_1) = \{(u_1, 0.7500), (u_2, 0.7308), (u_3, 0.8808), (u_4, 0.8808)\}, \\ \tilde{F}(e_2) = \{(u_1, 0.9808), (u_2, 0.8000), (u_3, 0.8808), (u_4, 0.7308)\}, \\ \tilde{F}(e_3) = \{(u_1, 0.6500), (u_2, 0.8808), (u_3, 0.6500), (u_4, 0.8808)\}, \\ \tilde{F}(e_4) = \{(u_1, 0.8808), (u_2, 0.6500), (u_3, 0.9808), (u_4, 0.9808)\}. \end{array}\right\}$$

$$(\tilde{F}_3, \tilde{A}_3) = \left\{\begin{array}{l} \tilde{F}(e_1) = \{(u_1, 0.8000), (u_2, 0.7308), (u_3, 0.8808), (u_4, 0.8808)\}, \\ \tilde{F}(e_2) = \{(u_1, 0.6500), (u_2, 0.8808), (u_3, 0.6500), (u_4, 0.7308)\}, \\ \tilde{F}(e_3) = \{(u_1, 0.7500), (u_2, 0.5000), (u_3, 0.8808), (u_4, 0.9615)\}, \\ \tilde{F}(e_4) = \{(u_1, 0.6500), (u_2, 0.8000), (u_3, 0.7308), (u_4, 0.9808)\}. \end{array}\right\}$$

Step 3 To derive the ranking of alternatives corresponding to each decision maker, both of Algorithm I and Algorithm II will be utilized in this step.
**By Algorithm I**: Let $\alpha = 0.80$, then the corresponding $\alpha$-level soft set are

$$\left(F_1^{0.8}, A_1^*\right) = \begin{Bmatrix} \tilde{F}(e_1) = \{(u_1, 0), (u_2, 0), (u_3, 0), (u_4, 1)\}, \\ \tilde{F}(e_2) = \{(u_1, 1), (u_2, 1), (u_3, 0), (u_4, 0)\}, \\ \tilde{F}(e_3) = \{(u_1, 0), (u_2, 0), (u_3, 1), (u_4, 0)\}, \\ \tilde{F}(e_4) = \{(u_1, 1), (u_2, 1), (u_3, 1), (u_4, 1)\}. \end{Bmatrix}$$

$$\left(F_2^{0.8}, A_2^*\right) = \begin{Bmatrix} \tilde{F}(e_1) = \{(u_1, 0), (u_2, 0), (u_3, 1), (u_4, 1)\}, \\ \tilde{F}(e_2) = \{(u_1, 1), (u_2, 1), (u_3, 1), (u_4, 0)\}, \\ \tilde{F}(e_3) = \{(u_1, 0), (u_2, 1), (u_3, 0), (u_4, 1)\}, \\ \tilde{F}(e_4) = \{(u_1, 1), (u_2, 0), (u_3, 1), (u_4, 1)\}. \end{Bmatrix}$$

$$\left(F_3^{0.8}, A_3^*\right) = \begin{Bmatrix} \tilde{F}(e_1) = \{(u_1, 1), (u_2, 0), (u_3, 1), (u_4, 1)\}, \\ \tilde{F}(e_2) = \{(u_1, 0), (u_2, 1), (u_3, 0), (u_4, 0)\}, \\ \tilde{F}(e_3) = \{(u_1, 0), (u_2, 0), (u_3, 1), (u_4, 1)\}, \\ \tilde{F}(e_4) = \{(u_1, 0), (u_2, 1), (u_3, 0), (u_4, 1)\}. \end{Bmatrix}$$

According to $(F_k^{0.8}, A_k^*)$, $k = 1, 2, 3$, the choice values of $(F_k^{0.8}, A_k^*)$, $k = 1, 2, 3$ can be listed in Tables 6 to 8. Thus, the single ranking given by 3 *LZnFSSs* (or decision makers) can be summarized in Table 9. For the reason that $u_1$ and $u_2$ have the same value, a clonal virtual $d_1$ is set and his/her attitude toward $u_1$ and $u_2$ are opposed.

**By Algorithm II**: When using Algorithm II, the ranking of alternatives given by three decision makers are listed in Table 10.

Step 4 According to Tables 9 and 10, by using the rank aggregation method given by Contreras,[41] the final group rankings are given in Table 11. In Table 11, the weighting factors composed of $\sum_{s=1}^{n} \omega_s \cdot t_{js}$ are related with the ordinal position that every alternative occupies in the group ranking, where $\omega = \{2^{n-1}/(2^n - 1), 2^{n-2}/(2^n - 1), \ldots, 2/(2^n - 1), 1/(2^n - 1)\}$ (provides a greater importance to the distance derived from the alternatives ranked in the top position) and $t_{js} = 1$ if the $j$th alternative is ranked with the $s$th position. While the sum $\sum_{k=1}^{l} |r_{kj} - r_{Gj}|$ represents the sum of the individual disagreements of the DMs.

Step 5 According to Table 11, the best choice will be $u_3$ if using Algorithm I while the best choice will be $u_4$ when using Algorithm II.

Step 6 The End.

## 5.2 | Comparison of ranking results

The comparisons of different rankings would be analyzed according to the following two aspects.

First, by setting the satisfactory level $\alpha$, Algorithm I provides a final ranking with precondition(s). The results would be changed according to different levels of $\alpha$, thus more softies are included. While when utilizing Algorithm II, the information provided by diverse decision makers are more sufficiently be used.

As analyzed in Peng and Wang,[22] by applying diverse approaches,[1,42,43] the rankings are $u_4 \succ u_3 \succ u_1 \succ u_2$, $u_3 \succ u_4 \succ u_2 \succ u_1$ and $u_4 \succ u_1 \succ u_3 \succ u_2$, respectively. While the ranking given by Peng and Wang[22] is $u_3 \succ u_2 \succ u_1 \succ u_4$. It can be seen that our developed group decision procedure based on *LZnFSSs* provides an alternative method to solve MAGDM with linguistic Z-numbers. The differences produced by diverse methods might be produced by

**TABLE 6** A weighted soft set $(F_1^{0.8}, A_1^*)$ with choice values

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | Choice value |
|---|---|---|---|---|---|
| $u_1$ | 0 | 1 | 0 | 1 | 0.4990 |
| $u_2$ | 0 | 1 | 0 | 1 | 0.4990 |
| $u_3$ | 0 | 0 | 1 | 1 | 0.4617 |
| $u_4$ | 1 | 0 | 0 | 1 | 0.4885 |

**TABLE 7** A weighted soft set $(F_2^{0.8}, A_2^*)$ with choice values

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | Choice value |
|---|---|---|---|---|---|
| $u_1$ | 0 | 1 | 0 | 1 | 0.4990 |
| $u_2$ | 0 | 1 | 1 | 0 | 0.5115 |
| $u_3$ | 1 | 1 | 0 | 1 | 0.7602 |
| $u_4$ | 1 | 0 | 1 | 1 | 0.7283 |

**TABLE 8** A weighted soft set $(F_3^{0.8}, A_3^*)$ with choice values

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | Choice value |
|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 0 | 0 | 0.2612 |
| $u_2$ | 0 | 1 | 0 | 1 | 0.4990 |
| $u_3$ | 1 | 0 | 1 | 0 | 0.5010 |
| $u_4$ | 1 | 0 | 1 | 1 | 0.7283 |

**TABLE 9** Single rankings given by three decision makers by Algorithm I

| | $d_1$ | $d'_1$ | $d_2$ | $d_3$ |
|---|---|---|---|---|
| $u_1$ | 1(0.4990) | 2(0.4990) | 4(0.4990) | 4(0.2612) |
| $u_2$ | 2(0.4990) | 1(0.4990) | 3(0.5115) | 3(0.4990) |
| $u_3$ | 4(0.4617) | 4(0.4617) | 1(0.7602) | 2(0.5010) |
| $u_4$ | 3(0.4885) | 3(0.4885) | 2(0.7283) | 1(0.7283) |

**TABLE 10** Single rankings given by 3 decision makers by Algorithm II

| | $d_1$ | $d_2$ | $d_3$ |
|---|---|---|---|
| $u_1$ | 1(0.5615) | 3(−0.1423) | 4(−1.0577) |
| $u_2$ | 3(−0.2385) | 4(−0.9423) | 3(−0.8115) |
| $u_3$ | 4(−0.5615) | 2(0.3808) | 2(0.1115) |
| $u_4$ | 2(0.2385) | 1(0.7038) | 1(1.7577) |

**TABLE 11** Group ranking according to Contreras[41]

| Alternative | $R_G^I$ | $\sum_{k=1}^{4} \lvert r_{kj} - r_{Gj} \rvert$ | Weighted value |
|---|---|---|---|
| $u_1$ | 3 | 3 | 0.4000 |
| $u_2$ | 4 | 5 | 0.3333 |
| $u_3$ | 1 | 4 | 2.1333 |
| $u_4$ | 2 | 2 | 0.5333 |
| Total | | 14 | 3.3999 |
| **Alternative** | $R_G^{II}$ | $\sum_{k=1}^{4} \lvert r_{kj} - r_{Gj} \rvert$ | **Weighted value** |
| $u_1$ | 4 | 4 | 0.2667 |
| $u_2$ | 3 | 1 | 0.1333 |
| $u_3$ | 2 | 2 | 0.5333 |
| $u_4$ | 1 | 1 | 0.5333 |
| Total | | 8 | 1.4666 |

the attitudes of different decision makers or the agreements among the importance of the alternatives ranked in the top position.

## 6 | CONCLUSION

To address multiattribute group decision making with linguistic Z-numbers, and to extend the applications of Z-numbers and soft sets, this study introduces the concept of linguistic Z-number fuzzy soft set (*LZnFSS*). Similar to the other types of soft set theories, set operations on *LZnFSSs* are developed, the properties of such operational laws are also developed. Next, a novel *LZnFSSs* based multiattribute group decision procedure is presented, in which no algebra operations on Z-numbers are needed. By using the rank aggregation method, the disagreement among decision makers are the lowest.

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